SINGULAR CURVES AND THE ÉTALE BRAUER–MANIN OBSTRUCTION FOR SURFACES

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Abstract. We give an elementary construction of a smooth and projective surface over an arbitrary number field k that is a counterexample to the Hasse principle but has infinite étale Brauer–Manin set. Our surface has a surjective morphism to a curve with exactly one k-point such that the unique k-fibre is geometrically a union of projective lines with an adelic point and the trivial Brauer group, but no k-point.

Nous présentons une construction élémentaire d'une surface lisse et projective sur un corps de nombres quelconque k qui est un contre-exemple au principe de Hasse ayant l'ensemble de Brauer–Manin infini. La surface est munie d'un morphisme surjectif vers une courbe avec un seul k-point tel que l'unique fibre rationnelle, qui géométriquement est l'union de droites projectives, a un point adélique et le groupe de Brauer trivial, mais pas de k-points.

Introduction

For a variety X over a number field k one can study the set $X(k)$ of k-points of X by embedding it into the topological space of adelic points $X(\mathbb{A}_k)$. In 1970 Manin [10] suggested to use the pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

provided by local class field theory. The left kernel of this pairing $X(\mathbb{A}_k)^{\mathrm{Br}}$ is a closed subset of $X(\mathbb{A}_k)$, and the reciprocity law of global class field theory implies that $X(k)$ is contained in $X(\mathbb{A}_k)^{\mathrm{Br}}$. The first example of a smooth and projective variety X such that $X(k) = \emptyset$ but $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ was constructed in [16] (see [1] for a similar example; an earlier example conditional on the Bombieri–Lang conjecture was found in [14]). Later, Harari [4] found many varieties X such that $X(k)$ is not dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$. For all of these examples except for that of [14] the failure of the Hasse principle or weak approximation can be

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explained by the étale Brauer–Manin obstruction (introduced in [16], see also [12]): the closure of $X(k)$ in $X(\mathbb{A}_k)$ is contained in the étale Brauer–Manin set $X(\mathbb{A}_k)^{\text{ét,Br}} \subset X(\mathbb{A}_k)^{\text{Br}}$ which in these cases is smaller than $X(\mathbb{A}_k)^{\text{Br}}$. Recently Poonen [12] constructed threefolds (fibred into rational surfaces over a curve of genus at least 1) such that $X(k) = \emptyset$ but $X(\mathbb{A}_k)^{\text{ét,Br}} \neq \emptyset$. It is known that $X(\mathbb{A}_k)^{\text{ét,Br}}$ coincides with the set of adelic points surviving the descent obstructions defined by torsors of arbitrary linear algebraic groups (as proved in [3, 18] using [5, 19]).

In 1997 Scharaschkin and the second author independently asked the question whether $X(k) = \emptyset$ if and only if $X(\mathbb{A}_k)^{\text{Br}} = \emptyset$ when X is a smooth and projective curve. They also asked if the embedding of $X(k)$ into $X(\mathbb{A}_k)^{\text{Br}}$ defines a bijection between the closure of $X(k)$ in $X(\mathbb{A}_k)$ and the set of connected components of $X(\mathbb{A}_k)^{\text{Br}}$. Despite some evidence for these conjectures, it may be prudent to consider also their weaker analogues with $X(\mathbb{A}_k)^{\text{ét,Br}}$ in place of $X(\mathbb{A}_k)^{\text{Br}}$.

In this note we give an elementary construction of a smooth and projective surface X over an arbitrary number field k that is a counterexample to the Hasse principle and has infinite étale Brauer–Manin set (Section 3). Even simpler is our counterexample to weak approximation (Section 2). This is a smooth and projective surface X over k with a unique k-point and infinite étale Brauer–Manin set $X(\mathbb{A}_k)^{\text{ét,Br}}$; moreover, already the subset $Y(\mathbb{A}_k) \cap X(\mathbb{A}_k)^{\text{ét,Br}}$ is infinite, where Y is the Zariski open subset of X which is the complement to $X(k)$. Following Poonen we consider families of curves parameterised by a curve with exactly one k-point. The new idea is to make the unique k-fibre a singular curve, geometrically a union of projective lines, and then use properties of rational and adelic points on singular curves.

The structure of the Picard group of a singular projective curve is well known, see [2, Section 9.2] or [9, Section 7.5]. In Section 1 we give a formula for the Brauer group of a reduced projective curve, see Theorem 1.3. A singular curve over k can have surprising properties that no smooth curve can ever have: it can contain infinitely many adelic points, only finitely many k-points or none at all, and yet have the trivial Brauer group. See Corollary 3.2 for a singular, geometrically connected, projective curve over an arbitrary number field k that is a counterexample to the Hasse principle not explained by the Brauer–Manin obstruction. In his forthcoming paper [6] the first author proves that every counterexample to the Hasse principle on a curve which geometrically is a union of projective lines, can be explained by finite descent (and hence by the étale Brauer–Manin obstruction). Here we note that geometrically connected and simply connected projective curves over number fields satisfy the Hasse principle, a statement that does not generalise to higher dimension, see Proposition 2.1 and the remark after it.

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1. The Brauer group of singular curves

Let k be a field of characteristic 0 with an algebraic closure $\bar{k}$ and the Galois group $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$. For a scheme X over k we write $\overline{X} = X \times_k \bar{k}$. All cohomology groups in this paper are Galois or étale cohomology groups. Let C be a *reduced, geometrically connected, projective* curve over k. We define the *normalisation* $\widetilde{C}$ as the disjoint union of normalisations of the irreducible components of C. The normalisation morphism $\nu : \widetilde{C} \to C$ factors as

$$\widetilde{C} \xrightarrow{\nu'} C' \xrightarrow{\nu''} C,$$

where C' is a maximal intermediate curve universally homeomorphic to C, see [2, Section 9.2, p. 247] or [9, Section 7.5, p. 308]. The curve C' is obtained from $\widetilde{C}$ by identifying the points which have the same image in C. In particular, there is a canonical bijection $\nu'' : C'(K) \tilde{\longrightarrow} C(K)$ for any field extension K/k. The curve C' has mildest possible singularities: for each singular point $s \in C'(\bar{k})$ the branches of $\overline{C}'$ through s intersect like n coordinate axes at $0 \in \mathbb{A}^n_{\bar{k}}$.

Let us define the following reduced 0-dimensional schemes:

$$\Lambda = \mathrm{Spec}(H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}})), \quad \Pi = C_{\mathrm{sing}}, \quad \Psi = \big(\Pi \times_C \widetilde{C}\big)_{\mathrm{red}}. \tag{1}$$

Here Λ is the k-scheme of geometric irreducible components of C (or the geometric connected components of $\widetilde{C}$); it is the disjoint union of closed points $\lambda = \mathrm{Spec}(k(\lambda))$ such that $k(\lambda)$ is the algebraic closure of k in the function field of the corresponding irreducible component $k(C_\lambda) = k(\widetilde{C}_\lambda)$. Next, Π is the union of singular points of C, and Ψ is the union of fibres of $\nu : \widetilde{C} \to C$ over the singular points of C with their reduced subscheme structure. The morphism ν'' induces an isomorphism $\big(\Pi \times_C C'\big)_{\mathrm{red}} \tilde{\longrightarrow} \Pi$, so we can identify these schemes. Let $i : \Pi \to C$, $i' : \Pi \to C$ and $j : \Psi \to \widetilde{C}$ be the natural closed immersions. We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\nu'} & C' & \xrightarrow{\nu''} & C \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle i'} & \nearrow{\scriptstyle i} & \\ \Psi & \xrightarrow{\nu'} & \Pi & & \end{array}$$

The restriction of ν to the smooth locus of C induces isomorphisms

$$\widetilde{C}\setminus j(\Psi)\tilde{\longrightarrow}C'\setminus i'(\Pi)\tilde{\longrightarrow}C\setminus i(\Pi).$$

An algebraic group over Π is a product $G=\prod_\pi i_{\pi*}(G_\pi)$, where π ranges over the irreducible components of Π, $i_\pi:\mathrm{Spec}(k(\pi))\to\Pi$ is the natural closed immersion, and G_π is an algebraic group over the field $k(\pi)$.

Proposition 1.1. *(i) The canonical maps* $\mathbb{G}_{m,C'}\to\nu'_*\mathbb{G}_{m,\widetilde{C}}$ *and* $\mathbb{G}_{m,C'}\to i'_*\mathbb{G}_{m,\Pi}$ *give rise to the exact sequence of étale sheaves on* C'

$$0\to\mathbb{G}_{m,C'}\to\nu'_*\mathbb{G}_{m,\widetilde{C}}\oplus i'_*\mathbb{G}_{m,\Pi}\to i'_*\nu'_*\mathbb{G}_{m,\Psi}\to 0, \tag{2}$$

where $\nu'_*\mathbb{G}_{m,\Psi}$ *is an algebraic torus over* Π.

(ii) The canonical map $\mathbb{G}_{m,C}\to\nu''_*\mathbb{G}_{m,C'}$ *gives rise to the exact sequence of étale sheaves on* C*:*

$$0\to\mathbb{G}_{m,C}\to\nu''_*\mathbb{G}_{m,C'}\to i_*\mathcal{U}\to 0, \tag{3}$$

where $\mathcal{U}$ *is a commutative unipotent group over* Π.

Proof. This is essentially well known, see [2], the proofs of Propositions 9.2.9 and 9.2.10, or [9, Lemma 7.5.12]. By [11, Thm. II.2.15 (b), (c)] it is enough to check the exactness of (2) at each geometric point $\bar{x}$ of C'. If $\bar{x}\notin i'(\Pi)$, this is obvious since locally at $\bar{x}$ the morphism ν' is an isomorphism, and the stalks $(i'_*\mathbb{G}_{m,\Pi})_{\bar{x}}$ and $(i'_*\nu'_*\mathbb{G}_{m,\Psi})_{\bar{x}}$ are zero. Now let $\bar{x}\in i'(\Pi)$, and let $\mathcal{O}_{\bar{x}}$ be the strict henselisation of the local ring of $\bar{x}$ in C'. Each geometric point $\bar{y}$ of $\widetilde{C}$ belongs to exactly one geometric connected component of $\widetilde{C}$, and we denote by $\mathcal{O}_{\bar{y}}$ the strict henselisation of the local ring of $\bar{y}$ in its geometric connected component. By the construction of C' we have an exact sequence

$$0\longrightarrow\mathcal{O}_{\bar{x}}\longrightarrow k(\bar{x})\times\prod_{\nu'(\bar{y})=\bar{x}}\mathcal{O}_{\bar{y}}\longrightarrow\prod_{\nu'(\bar{y})=\bar{x}}k(\bar{y})\longrightarrow 0,$$

where $\mathcal{O}_{\bar{y}}\to k(\bar{y})$ is the reduction modulo the maximal ideal of $\mathcal{O}_{\bar{y}}$, and $k(\bar{x})\to k(\bar{y})$ is the multiplication by -1. We obtain an exact sequence of abelian groups

$$1\longrightarrow\mathcal{O}^*_{\bar{x}}\longrightarrow k(\bar{x})^*\times\prod_{\nu'(\bar{y})=\bar{x}}\mathcal{O}^*_{\bar{y}}\longrightarrow\prod_{\nu'(\bar{y})=\bar{x}}k(\bar{y})^*\longrightarrow 1.$$

Using [11, Cor. II.3.5 (a), (c)] one sees that this is the sequence of stalks of (2) at $\bar{x}$, so that (i) is proved.

To prove (ii) consider the exact sequence

$$0\to\mathbb{G}_{m,C}\to\nu''_*\mathbb{G}_{m,C'}\to\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C}\to 0.$$

Since ν'' is an isomorphism away from $i(\Pi)$, the restriction of the sheaf $\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C}$ to $C\setminus i(\Pi)$ is zero, hence $\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C} = i_*\mathcal{U}$ for some sheaf $\mathcal{U}$ on Π. To see that $\mathcal{U}$ is a unipotent group scheme it is enough to check the stalks at geometric points. Let $\bar{x}$ be a geometric point of $i(\Pi)$, and let $\bar{y}$ be the unique geometric point of C' such that $\nu''(\bar{y}) = \bar{x}$. Let $\mathcal{O}_{\bar{x}}$ and $\mathcal{O}_{\bar{y}}$ be the corresponding strictly henselian local rings. The stalk $(\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C})_{\bar{x}}$ is $\mathcal{O}^*_{\bar{y}}/\mathcal{O}^*_{\bar{x}}$, and according to [9, Lemma 7.5.12 (c)], this is a unipotent group over the field $k(\bar{x})$. This finishes the proof. □

Remark 1.2. The first part of Proposition 1.1 admits an alternative argument which is easier to generalize to higher dimensions. Let X be a projective k-variety with normalization map $\nu : \widetilde{X} \longrightarrow X$. Assume that $\widetilde{X}, X_{\text{sing}}$ and $\widetilde{X}_{\text{crit}}$ are smooth where $X_{\text{sing}} \subseteq X$ is the singular locus of X and $\widetilde{X}_{\text{crit}} = \nu^{-1}(X_{\text{sing}}) \subseteq \widetilde{X}$ is the critical locus of ν. These assumptions are automatically satisfied when X is a curve. The analogue of C' is then the K-variety X' given by the pushout in the square

$$\begin{array}{ccc} \widetilde{X}_{\text{crit}} & \xrightarrow{i'} & \widetilde{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle j} \\ X_{\text{sing}} & \xrightarrow{\nu'} & X' \end{array}$$

Such a pushout always exists in the category of K-varieties since i' is a closed embedding and g is an affine map between smooth projective varieties (see [8, Theorem 5.4]). One can then prove that the analogous sequence of sheaves

$$0 \longrightarrow \mathbb{G}_{m,X'} \longrightarrow \nu'_*\mathbb{G}_{m,\widetilde{X}} \oplus i'_*\mathbb{G}_{m,X_{\text{sing}}} \longrightarrow \nu'_*j_*\mathbb{G}_{m,\widetilde{X}_{\text{crit}}} \longrightarrow 0$$

is exact as follows: from the definition of X' we get that the square

$$\begin{array}{ccc} \mathcal{O}_{X'} & \longrightarrow & \nu'_*\mathcal{O}_{\widetilde{X}} \\ \downarrow & & \downarrow \\ i'_*\mathcal{O}_{X_{\text{sing}}} & \longrightarrow & \nu'_*j_*\mathcal{O}_{\widetilde{X}_{\text{crit}}} \end{array}$$

is a Cartesian diagram of sheaves of rings on X' (where Cartesian for sheaves means Cartesian on the stalks). Now the functor $\mathbb{G}_m$: Ring $\longrightarrow$ Ab which takes a ring to its group of units commutes with limits and filtered colimits (e.g. taking stalks). This means that the diagram

$$\begin{array}{ccc} \mathbb{G}_{m,X'} & \longrightarrow & \nu'_*\mathbb{G}_{m,\widetilde{X}} \\ \downarrow & & \downarrow \\ i'_*\mathbb{G}_{m,X_{\rm sing}} & \longrightarrow & \nu'_*j_*\mathbb{G}_{m,\widetilde{X}_{\rm crit}} \end{array}$$

is Cartesian as well. Hence we get that the sequence

$$0 \longrightarrow \mathbb{G}_{m,X'} \longrightarrow \nu'_*\mathbb{G}_{m,\widetilde{X}} \oplus i'_*\mathbb{G}_{m,X_{\rm sing}} \longrightarrow \nu'_*j_*\mathbb{G}_{m,\widetilde{X}_{\rm crit}}$$

is exact. To know that the last map is surjective it is enough to note that $j : \widetilde{X}_{\rm crit} \hookrightarrow \widetilde{X}$ is a closed embedding (hence $\mathbb{G}_{m,\widetilde{X}} \longrightarrow j_*\mathbb{G}_{m,\widetilde{X}_{\rm crit}}$ is surjective) and that ν'_* is exact by Milne, Corollary II 3.6 (since ν' is a finite map).

For fields $k_1, \ldots, k_n$, we have $\mathrm{Br}\left(\coprod_{i=1}^n \mathrm{Spec}(k_i)\right) = \oplus_{i=1}^n \mathrm{Br}(k_i)$.

Theorem 1.3. *Let C be a reduced, geometrically connected, projective curve, and let Λ, Π and Ψ be the schemes defined in (1). Let $\Lambda = \coprod_\lambda \mathrm{Spec}(k(\lambda))$ be the decomposition into the disjoint union of connected components, so that $\widetilde{C} = \coprod_\lambda \widetilde{C}_\lambda$, where $\widetilde{C}_\lambda$ is a geometrically integral, smooth, projective curve over the field $k(\lambda)$. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(C) \longrightarrow \mathrm{Br}(\Pi) \oplus \bigoplus_{\lambda\in\Lambda} \mathrm{Br}(\widetilde{C}_\lambda) \longrightarrow \mathrm{Br}(\Psi), \qquad (4)$$

where the maps are the composition of canonical maps

$$\mathrm{Br}(\widetilde{C}_\lambda) \to \mathrm{Br}(\widetilde{C}_\lambda \cap \Psi) \to \mathrm{Br}(\Psi),$$

and the opposite of the restriction map $\mathrm{Br}(\Pi) \to \mathrm{Br}(\Psi)$.

Proof. Let π range over the irreducible components of Π, so that $\mathcal{U} = \prod_\pi i_{\pi*}(U_\pi)$, where U_π is a commutative unipotent group over the field $k(\pi)$. Since i_* is an exact functor [11, Cor. II.3.6], we have $H^n(C, i_*\mathcal{U}) = H^n(\Pi, \mathcal{U}) = \prod_\pi H^n(k(\pi), U_\pi)$. The field k has characteristic 0, and it is well known that this implies that any commutative unipotent group has zero cohomology in degree $n > 0$. (Such a group has a composition series with factors $\mathbb{G}_a$, and $H^n(k, \mathbb{G}_a) = 0$ for any $n > 0$, see [15, X, Prop. 1].) Thus the long exact sequence of cohomology groups associated to (3) gives rise to an isomorphism $\mathrm{Br}(C) = H^2(C, \mathbb{G}_{m,C}) \tilde{\longrightarrow} H^2(C, \nu''_*\mathbb{G}_{m,C'})$. Since ν'' is finite, the functor ν''_* is exact [11, Cor. II.3.6], so we obtain an isomorphism $\mathrm{Br}(C) \tilde{\longrightarrow} \mathrm{Br}(C')$. We now apply similar arguments to (2). Hilbert's theorem 90 gives $H^1(\Pi, \nu'_*\mathbb{G}_{m,\Psi}) = H^1(\Psi, \mathbb{G}_{m,\Psi}) = 0$, so we obtain the exact sequence (4). □

Recall that a reduced, geometrically connected, projective curve S over a field k is called *semi-stable* if all the singular points of S are ordinary double points [2, Def. 9.2.6].

Definition 1.4. A semi-stable curve is called *bipartite* if it is a union of two smooth curves without common irreducible components.

Corollary 1.5. *Let $S = S^+ \cup S^-$ be a bipartite curve, where S^+ and S^- are smooth curves such that $S^+ \cap S^-$ is finite. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(S) \longrightarrow \mathrm{Br}(S^+) \oplus \mathrm{Br}(S^-) \longrightarrow \mathrm{Br}(S^+ \cap S^-), \qquad (5)$$

where $\mathrm{Br}(S) \to \mathrm{Br}(S^+) \oplus \mathrm{Br}(S^-)$ is the natural map, and $\mathrm{Br}(S^\pm) \to \mathrm{Br}(S^+ \cap S^-)$ is the restriction map multiplied by ± 1.

Proof. In our previous notation we have $\widetilde{S} = S^+ \coprod S^-$, $\Pi = S^+ \cap S^-$, and Ψ is the disjoint union of two copies of Π, namely $\Psi^+ = \Psi \cap S^+$ and $\Psi^- = \Psi \cap S^-$. In particular, the restriction map $\mathrm{Br}(\Pi) \to \mathrm{Br}(\Psi)$ is injective. Thus taking the quotients by $\mathrm{Br}(\Pi)$ in the middle and last terms of (4), we obtain (5). □

The constructions in Sections 2 and 3 use singular curves of the following special kind.

Definition 1.6. A reduced, geometrically connected, projective curve C over k is called *conical* if every irreducible component of $\overline{C}$ is rational.

For *bipartite conical* curves the calculation of the Brauer group can be carried out using only the Brauer groups of fields.

Corollary 1.7. *Let $C = C^+ \cup C^-$ be a bipartite conical curve, and let $\Lambda = \Lambda^+ \coprod \Lambda^-$ be the corresponding decomposition of Λ. Then $\mathrm{Br}(C)$ is the kernel of the map*

$$\bigoplus_{\lambda \in \Lambda^+} \mathrm{Br}(k(\lambda))/[\widetilde{C}_\lambda] \ \oplus \ \bigoplus_{\lambda \in \Lambda^-} \mathrm{Br}(k(\lambda))/[\widetilde{C}_\lambda] \ \longrightarrow \mathrm{Br}(C^+ \cap C^-),$$

where $[\widetilde{C}_\lambda] \in \mathrm{Br}(k(\lambda))$ is the class of the conic $\widetilde{C}_\lambda$ over the field $k(\lambda)$, and the map $\mathrm{Br}(k(\lambda))/[\widetilde{C}_\lambda] \to \mathrm{Br}(C^+ \cap C^-)$ is the restriction followed by multiplication by ± 1 when $\lambda \in \Lambda^\pm$.

Proof. This follows directly from Proposition 1.5 and the well known fact that the Brauer group of a conic over k is the quotient of $\mathrm{Br}(k)$ by the cyclic subgroup generated by the class of this conic. □

In some cases one can compute $\mathrm{Br}(C)$ using the Hochschild–Serre spectral sequence $H^p(k, H^q(\overline{C}, \mathbb{G}_m)) \Rightarrow H^{p+q}(C, \mathbb{G}_m)$. In [7, III, Cor. 1.2] Grothendieck proved that $\mathrm{Br}(\overline{C}) = 0$ for any curve C. Hence the spectral sequence identifies the cokernel of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$ with a subgroup of $H^1(k, \mathrm{Pic}(\overline{C}))$.

The structure of Γ_k-module $\mathrm{Pic}(\overline{C})$ is well known, at least up to its maximal unipotent subgroup. It is convenient to describe this structure in combinatorial terms. Recall that $\overline{\Lambda}$, $\overline{\Pi}$, $\overline{\Psi}$ are the $\bar{k}$-schemes obtained from Λ, Π, Ψ by extending the ground field to $\bar{k}$. We associate to C the *incidence graph* $X(C)$ defined as the directed graph whose vertices are $X(C)_0 = \overline{\Lambda} \cup \overline{\Pi}$ and the edges are $X(C)_1 = \overline{\Psi}$. The edge $Q \in \overline{\Psi}$ goes from $L \in \overline{\Lambda}$ to $P \in \overline{\Pi}$ when $\nu(Q) = P$ and Q is contained in the irreducible component L of $\widetilde{C}$. The source and target maps $X(C)_1 \to X(C)_0$ can be described as a morphism of k-schemes

$$(s,t) : \Psi \longrightarrow \Lambda \coprod \Pi,$$

where $t : \Psi \to \Pi$ is induced by ν', and s is the composition of the closed immersion $j : \Psi \to \widetilde{C}$ and the canonical morphism $\widetilde{C} \to \Lambda$. By construction $X(C)$ is a connected bipartite graph with a natural action of the Galois group Γ_k.

For a reduced 0-dimensional k-scheme $p : \Sigma \to \mathrm{Spec}(k)$ of finite type, the k-group scheme $p_*\mathbb{G}_m$ is an algebraic torus over k. If we write $\Sigma = \coprod_{i=1}^n \mathrm{Spec}(k_i)$, where $k_1, \ldots, k_n$ are finite field extensions of k, then $p_*\mathbb{G}_m$ is the product of Weil restrictions $\prod_{i=1}^n R_{k_i/k}(\mathbb{G}_m)$. For a reduced 0-dimensional scheme $p' : \Sigma' \to \mathrm{Spec}(k)$ of finite type a morphism of k-schemes $f : \Sigma' \to \Sigma$ gives rise to a canonical morphism $\mathbb{G}_{m,\Sigma} \to f_*\mathbb{G}_{m,\Sigma'}$, and hence to a canonical homomorphism of k-tori $p_*\mathbb{G}_m \to p'_*\mathbb{G}_m$, which we denote by f^*. Let us denote the structure morphism $\Lambda \to \mathrm{Spec}(k)$ by p_Λ, and use the same convention for Ψ and Π. Let T be the algebraic k-torus defined by the exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow p_{\Lambda *}\mathbb{G}_m \oplus p_{\Pi *}\mathbb{G}_m \longrightarrow p_{\Psi *}\mathbb{G}_m \longrightarrow T \longrightarrow 1,$$

where the middle arrow is $s^*(t^*)^{-1}$.

Remark. The character group $\hat{T}$ with its natural action of the Galois group Γ_k, is canonically isomorphic to $H_1(X(C), \mathbb{Z})$, the *first homology group* of the graph $X(C)$. Since $X(C)$ is connected, we have $T = \{1\}$ if and only if $X(C)$ is a tree.

Proposition 1.8. *Let C be a reduced, geometrically connected, projective curve over k. We have the following exact sequences of Γ_k-modules:*

$$0 \longrightarrow T(\bar{k}) \longrightarrow \mathrm{Pic}(\overline{C'}) \longrightarrow \mathrm{Pic}(\widetilde{C} \times_k \bar{k}) \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow U(\bar{k}) \longrightarrow \mathrm{Pic}(\overline{C}) \longrightarrow \mathrm{Pic}(\overline{C'}) \longrightarrow 0, \tag{7}$$

where U is a commutative unipotent algebraic group over k.

Proof. To obtain (6) we apply the direct image functor with respect to the structure morphism $C' \to \mathrm{Spec}(k)$ to the exact sequence (2). The sequence (7) is obtained from (3) in a similar way. □

Corollary 1.9. *If C is a conical curve such that $X(C)$ is a tree, then $H^1(k, \mathrm{Pic}(\overline{C})) = 0$, so that the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$ is surjective.*

Proof. Since k has characteristic 0, we have $H^n(k, U) = 0$ for $n > 0$. Thus (7) gives an isomorphism $H^1(k, \mathrm{Pic}(\overline{C})) = H^1(k, \mathrm{Pic}(\overline{C'}))$. We have $T = \{1\}$, hence $\mathrm{Pic}(\overline{C'}) = \mathrm{Pic}(\widetilde{C} \times_k \bar{k})$. The curve $\widetilde{C}$ is the disjoint union of conics defined over finite extensions of k, thus the free abelian group $\mathrm{Pic}(\widetilde{C} \times_k \bar{k})$ has a natural Γ_k-stable $\mathbb{Z}$-basis. Hence $H^1(k, \mathrm{Pic}(\widetilde{C} \times_k \bar{k})) = 0$. □

2. Weak approximation

Let k be a number field. Recall that the étale Brauer–Manin set $X(\mathbb{A}_k)^{\text{ét,Br}}$ is the set of adelic points $(M_v) \in X(\mathbb{A}_k)$ satisfying the following property: for any torsor $f : Y \to X$ of a finite k-group scheme G there exists a k-torsor Z of G such that (M_v) is the image of an adelic point in the Brauer–Manin set of $(Y \times_k Z)/G$. Here G acts simultaneously on both factors, and the morphism $(Y \times_k Z)/G \to X$ is induced by $Y \to X$. It is clear that the étale Brauer–Manin obstruction is a functor from the category of varieties over k to the category of sets. Note that $(Y \times_k Z)/G \to X$ is a torsor of an inner form of G, called the twist of Y/X by Z, see [17, pp. 20–21] for details.

In this section we construct a simple example of a smooth projective surface X over k such that $X(\mathbb{A}_k)^{\text{ét,Br}}$ is infinite but $X(k)$ contains only one point. Thus $X(k)$ is far from dense in $X(\mathbb{A}_k)^{\text{ét,Br}}$; in fact, infinitely many points of $X(\mathbb{A}_k)^{\text{ét,Br}}$ are contained in the complement to $X(k)$ in X.

We start with the following statement which shows that on an everywhere locally soluble conical curve C such that $X(C)$ is a tree all the adelic points survive the étale Brauer–Manin obstruction.

Proposition 2.1. *Let k be a number field, and let C be a conical curve over k such that $X(C)$ is a tree and $C(\mathbb{A}_k) \neq \emptyset$. Then*

(1) *$C(k) \neq \emptyset$;*
(2) *the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$ is an isomorphism;*
(3) *$C(\mathbb{A}_k)^{\text{ét,Br}} = C(\mathbb{A}_k)$.*

Remark. Proposition 2.1 (1) implies that *geometrically connected and simply connected projective curves over number fields satisfy the Hasse* *principle*. It is easy to see that this statement does not generalise to higher dimension. Consider a conic $C \subset \mathbb{P}^2_k$ without a k-point, and choose a quadratic extension K/k so that all the places v with $C(k_v) = \emptyset$ are split in K. Then the union of two planes conjugate over K and intersecting at C is a geometrically connected and simply connected projective surface that is a counterexample to the Hasse principle.

Proof of Proposition 2.1. Let us prove (1). It is well known that any group acting on a finite connected tree fixes a vertex or an edge. (This is easily proved by induction on the diameter of a tree, that is, on the length of a longest path contained in it.) We apply this to the action of the Galois group Γ_k on $X(C)$. If Γ_k fixes a point of $\overline{\Pi}$ or $\overline{\Psi}$, then $C(k) \neq \emptyset$. If Γ_k fixes a point of $\overline{\Lambda}$, then C has an irreducible component C_0 which is a geometrically integral geometrically rational curve. Let $\widetilde{C}_0$ be the normalisation of C_0. Since $X(C)$ is a tree, the morphism $\widetilde{C}_0 \to C_0$ is a bijection on points. Thus if we can prove that $C_0(\mathbb{A}_k) \neq \emptyset$, then $\widetilde{C}_0(\mathbb{A}_k) \neq \emptyset$, and by the Hasse–Minkowski theorem $\widetilde{C}_0(k) \neq \emptyset$, so that finally $C_0(k) \neq \emptyset$. Since $X(C)$ is a connected tree, each geometric connected component of $\overline{C} \setminus \overline{C}_0$ meets $\overline{C}_0$ in exactly one point. Let k_v be a completion of k. Suppose that $C_0(k_v) = \emptyset$. Since $C(k_v) \neq \emptyset$, at least one of the geometric connected components of $\overline{C} \setminus \overline{C}_0$ is fixed by the Galois group Γ_{k_v}, and hence it intersects C_0 in a k_v-point. This contradiction proves (1).

By (1) the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$ has a retraction, and so is injective, but it is also surjective by Corollary 1.9. This proves (2).

Let us prove (3). Let G be a finite k-group scheme, and let $\mathcal{T} \to C$ be a torsor of G. Fix a k-point P in C. The fibre $\mathcal{T}_P$ is then a k-torsor. The twist of $\mathcal{T}$ by $\mathcal{T}_P$ is the quotient of $\mathcal{T} \times_k \mathcal{T}_P$ by the diagonal action of G. This is a C-torsor of an inner form of G such that the fibre at P has a k-point, namely the quotient by G of the diagonal in $\mathcal{T}_P \times_k \mathcal{T}_P$. Thus, twisting $\mathcal{T}$ by a k-torsor of G, and replacing G by an inner form we can assume that $\mathcal{T}_P$ contains a k-point Q. Since all geometric irreducible components of $\overline{C}$ are homeomorphic to $\mathbb{P}^1_{\bar{k}}$, the torsor $\overline{\mathcal{T}} \to \overline{C}$ trivialises over each component of $\overline{C}$. But $X(C)$ is a tree, and this implies that the torsor $\overline{\mathcal{T}} \to \overline{C}$ is trivial, that is, $\overline{\mathcal{T}} \simeq (C \times_k G) \times_k \bar{k}$. The geometric connected component of $\overline{\mathcal{T}}$ that contains Q is thus defined over k, and hence it gives a section s of $\mathcal{T} \to C$ such that $s(P) = Q$. Hence any adelic point on C lifts to an adelic point on $s(C) \subset \mathcal{T}$. Since $\mathrm{Br}(C) = \mathrm{Br}(k)$ we conclude that

$C(\mathbb{A}_k)$ is contained in, and hence is equal to the étale Brauer–Manin set $C(\mathbb{A}_k)^{\text{ét,Br}}$. □

Let k be a number field, and let $f(x,y)$ be a separable homogeneous polynomial such that its zero locus $Z^f \subseteq \mathbb{P}^1_k$ is a 0-dimensional scheme violating the Hasse principle. It is easy to see that such a polynomial exists for any number field k. For example, for $k=\mathbb{Q}$ one can take

$$f(x,y) = (x^2-2y^2)(x^2-17y^2)(x^2-34y^2). \tag{8}$$

For an arbitrary number field k the following polynomial will do:

$$f(x,y) = (x^2-ay^2)(x^2-by^2)(x^2-aby^2)(x^2-cy^2),$$

where $a,b,c \in k^* \setminus k^{*2}$ are such that $ab \notin k^{*2}$, whereas $c \in k_v^{*2}$ for all places v such that $\mathrm{val}_v(a) \neq 0$ or $\mathrm{val}_v(b) \neq 0$, and also for the archimedean places, and the places with residual characteristic 2. (For fixed a and b one finds c using weak approximation.) Let $d = \deg(f)$.

Let $C^f \subseteq \mathbb{P}^2_k$ be the curve with the equation $f(x,y)=0$. It is geometrically connected and has a unique singular point $P=(0:0:1) \in C^f \subset \mathbb{P}^2_k$ which is contained in all irreducible components of C^f. Since $Z^f(k)=\emptyset$ we see that P is the only k-point of C^f. The intersection of C^f with any line in $\mathbb{P}^2_k$ that does not pass through P is isomorphic to Z^f, hence *the smooth locus of* C^f *contains an infinite subset of* $C^f(\mathbb{A}_k) = C^f(\mathbb{A}_k)^{\text{ét,Br}}$, where the equality is due to Proposition 2.1 (3).

Now let $g(x,y,z)$ be a homogeneous polynomial over k of the same degree as $f(x,y)$ such that the subset of $\mathbb{P}^2_k$ given by $g(x,y,z) = f(x,y) = 0$ consists of d^2 distinct $\bar{k}$-points. Consider the projective surface $Y \subseteq \mathbb{P}^2_k \times \mathbb{P}^1_k$ given by the equation

$$\lambda f(x,y) + \mu g(x,y,z) = 0,$$

where $(\lambda:\mu)$ are homogeneous coordinates on $\mathbb{P}^1_k$. One immediately checks that Y is smooth. The fibre of the projection $Y \to \mathbb{P}^1_k$ over $\infty = (1:0)$ is C^f.

Let E be a smooth, geometrically integral, projective curve over k containing exactly one k-rational point M. By [13] such curves exist over any global field k. Choose a dominant morphism $\varphi : E \to \mathbb{P}^1_k$ such that $\varphi(M)=\infty$, and φ is not ramified over all the points of $\mathbb{P}^1_k$ where Y has a singular fibre (including ∞). Define

$$X = E \times_{\mathbb{P}^1_k} Y. \tag{9}$$

Proposition 2.2. *The surface X is smooth, projective and geometrically integral. The set $X(k)$ has exactly one point, whereas the set $X(\mathbb{A}_k)^{\text{ét,Br}}$ is infinite. Furthermore, an infinite subset of $X(\mathbb{A}_k)^{\text{ét,Br}}$ is contained in the Zariski open set $X \setminus X(k)$.*

Proof. The inverse image of the unique k-point of E in X is C^f, hence X has exactly one k-point. Since the étale Brauer–Manin obstruction is a functor from the category of k-varieties to the category of sets, the inclusion $\iota : C^f \hookrightarrow X$ induces an inclusion

$$\iota_* : C^f(\mathbb{A}_k) = C^f(\mathbb{A}_k)^{\text{ét,Br}} \hookrightarrow X(\mathbb{A}_k)^{\text{ét,Br}}.$$

We conclude that $X(\mathbb{A}_k)^{\text{ét,Br}}$ contains infinitely many points that belong to the Zariski open set $X \setminus X(k)$. □

Remark. Following a suggestion of Ambrus Pál made in response to the first version of this paper we now sketch a more elementary construction of a counterexample to weak approximation not explained by the étale Brauer–Manin obstruction. Consider any irreducible binary quadratic form $f(x, y)$ over k. Then our method produces a smooth, projective and geometrically integral surface X fibred into conics over E. The fibre over the unique k-point of E is the irreducible singular conic C^f, hence the set $X(k)$ consists of the singular point of C^f. Let D be the discriminant of $f(x, y)$. If v is a finite place of k that splits in $k(\sqrt{D})$, then $C^f(k_v)$ is the union of two projective lines over k_v. Thus $C^f(\mathbb{A}_k) = C^f(\mathbb{A}_k)^{\text{ét,Br}}$ is infinite, and hence so is $X(\mathbb{A}_k)^{\text{ét,Br}}$. This construction gives a surface of simpler geometric structure than the surface in Proposition 2.2, but it does not possess the stronger property of the previous example: here no element of $X(\mathbb{A}_k)^{\text{ét,Br}}$ is contained in the Zariski open set $X \setminus X(k)$. For this argument we assume that the Jacobian of E has rank 0 and a finite Shafarevich–Tate group, e.g. E is an elliptic curve over $\mathbb{Q}$ of analytic rank 0. Let (M_v) be an adelic point in $X(\mathbb{A}_k)^{\text{ét,Br}}$. Its image (N_v) in E is contained in $E(\mathbb{A}_k)^{\text{Br}}$, but this set is just the connected component of $0 = E(k)$ in $E(\mathbb{A}_k)$, see [17, Prop. 6.2.4]. Thus for all finite places v we have $N_v = 0$. For any place v that does not split in $k(\sqrt{D})$, this implies $M_v \in X(k)$.

3. The Hasse principle

In this section we construct a smooth projective surface X over an arbitrary number field k such that $X(\mathbb{A}_k)^{\text{ét,Br}}$ is infinite and $X(k)$ is empty. This means that X does not satisfy the Hasse principle and this failure is not explained by the étale Brauer–Manin obstruction.

Let $f(x, y)$ and Z^f be as in the previous section. The scheme Z^f is the disjoint union of $\text{Spec}(K_i)$ for $i = 1, \ldots, n$, where K_i is a finite extension of k. We assume $d = \deg(f) \geq 5$. (For $k = \mathbb{Q}$ one can take $f(x, y)$ of degree 6 as in (8), and in general degree 8 will suffice.) We choose field extensions L/k and F/k such that $L \otimes_k K_i$ and $F \otimes_k K_i$ are fields for all $i = 1, \ldots, n$, and, moreover, $[L : k] = d/2 - 1$, $[F : k] = d/2$

if d is even, and $[L:k]=(d-1)/2$, $[F:k]=(d+1)/2$ if d is odd. Fix an embedding

$$\mathrm{Spec}(L)\coprod\mathrm{Spec}(F)\hookrightarrow\mathbb{P}^1_k$$

and let D be the following curve in $\mathbb{P}^1_k\times\mathbb{P}^1_k$:

$$D=\big(Z^f\times\mathbb{P}^1_k\big)\cup\big(\mathbb{P}^1_k\times\mathrm{Spec}(L)\big)\cup\big(\mathbb{P}^1_k\times\mathrm{Spec}(F)\big).$$

This is a bipartite conical curve without k-points, see Definition 1.4. The class of D in $\mathrm{Pic}(\mathbb{P}^1_k\times\mathbb{P}^1_k)$ is $(d,d-1)$ or (d,d) depending on the parity of d.

Proposition 3.1. *The natural map* $\mathrm{Br}(k)\to\mathrm{Br}(D)$ *is an isomorphism.*

Proof. Since $D(L)\neq\emptyset$, the natural map $\mathrm{Br}(L)\to\mathrm{Br}(D\times_k L)$ has a retraction, and so is injective. The composition of the restriction $\mathrm{Br}(k)\to\mathrm{Br}(L)$ and the corestriction $\mathrm{Br}(L)\to\mathrm{Br}(k)$ is the multiplication by $[L:k]$, hence for any x in the kernel of the natural map $\mathrm{Br}(k)\to\mathrm{Br}(D)$ we have $[L:k]x=0$. Similarly, $D(F)\neq\emptyset$ implies that $[F:k]x=0$. But $[F:k]=[L:k]+1$, therefore the natural map $\mathrm{Br}(k)\to\mathrm{Br}(D)$ is injective. By Corollary 1.7 we need to prove that $\mathrm{Br}(k)$ is the kernel of the map

$$\mathrm{Br}(L)\oplus\mathrm{Br}(F)\oplus\bigoplus_{i=1}^n\mathrm{Br}(K_i)\longrightarrow\bigoplus_{i=1}^n\mathrm{Br}(LK_i)\oplus\bigoplus_{i=1}^n\mathrm{Br}(FK_i).$$

Recall that the maps $\mathrm{Br}(L)\to\mathrm{Br}(LK_i)$ are the restriction maps, whereas $\mathrm{Br}(K_i)\to\mathrm{Br}(LK_i)$ are opposites of the restriction maps. The same convention applies with F in place of L.

Suppose that we have $\alpha_i\in\mathrm{Br}(K_i)$, $i=1,\dots,n$, $\beta\in\mathrm{Br}(L)$ and $\gamma\in\mathrm{Br}(F)$ such that (α_i,β,γ) goes to zero. Let v be a place of k, and let k_v be a completion of k at v. Since $Z^f(k_v)\neq\emptyset$, there is an index i such that v splits in K_i. Let w be a degree 1 place of K_i over v, so that the natural map $k_v\to K_{i,w}$ is an isomorphism. Let $a_v\in\mathrm{Br}(k_v)=\mathrm{Br}(K_{i,w})$ be the image of α_i under the restriction map $\mathrm{Br}(K_i)\to\mathrm{Br}(K_{i,w})$. This defines $a_v\in\mathrm{Br}(k_v)$ for any place v of k, moreover, we have $a_v=0$ for almost all v since each α_i is unramified away from a finite set of places (and the Brauer group of the ring of integers of k_ν is trivial). We have a commutative diagram of restriction maps

$$\begin{array}{ccccc}\mathrm{Br}(L)&\longrightarrow&\mathrm{Br}(L\otimes_k K_i)&\longleftarrow&\mathrm{Br}(K_i)\\\downarrow&&\downarrow&&\downarrow\\\mathrm{Br}(L\otimes_k k_v)&\xrightarrow{\sim}&\mathrm{Br}(L\otimes_k K_{i,w})&\longleftarrow&\mathrm{Br}(K_{i,w})\end{array}$$

Here for a family of fields $\{F_i\}$ we write $\mathrm{Br}(\oplus F_i) = \oplus \mathrm{Br}(F_i)$. Since $(\alpha_i, \beta, \gamma)$ goes to zero, the restrictions of α_i and β to $\mathrm{Br}(L \otimes_k K_i)$ coincide. Hence the image β_v of β in $\mathrm{Br}(L \otimes_k k_v)$ is the image of $a_v \in \mathrm{Br}(k_v)$. From the global reciprocity law applied to $\beta \in \mathrm{Br}(L)$ we deduce $[L:k]\sum_v \mathrm{inv}_v(a_v) = 0$. The same argument with γ instead of β gives $[F:k]\sum_v \mathrm{inv}_v(a_v) = 0$. Since $[L:k]$ and $[F:k]$ are coprime we obtain $\sum_v \mathrm{inv}_v(a_v) = 0$. By global class field theory there exists $\alpha \in \mathrm{Br}(k)$ such that a_v is the image of α in $\mathrm{Br}(k_v)$. Since the map $\mathrm{Br}(L) \to \oplus_v \mathrm{Br}(L \otimes_k k_v)$ is injective it follows that α goes to β under the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(L)$, and similarly for γ. Modifying α_i, β and γ by the image of α we can now assume that $\beta = 0$ and $\gamma = 0$. Since α_i goes to zero in $\mathrm{Br}(LK_i)$, a standard restriction-corestriction argument gives $[L:k]\alpha = 0$. Similarly, α_i goes to zero in $\mathrm{Br}(FK_i)$, and hence $[F:k]\alpha = 0$. Therefore, $\alpha = 0$. □

Corollary 3.2. *We have* $D(\mathbb{A}_k)^{\mathrm{Br}} = D(\mathbb{A}_k)$.

We now construct a conical curve $C \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ with one singular point such that C and D are linearly equivalent. Let $P = (P_1, P_2)$ be a k-point in $\mathbb{P}^1_k \times \mathbb{P}^1_k$. In the tangent plane to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at P choose a line ℓ through P such that ℓ is not one of the two tangent directions. Assume first that d is odd, so that $\mathcal{O}(D) = \mathcal{O}(d,d)$. For $i = 1, \ldots, d$ let $C_i \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ be pairwise different geometrically irreducible curves through P tangent to ℓ such that $\mathcal{O}(C_i) = \mathcal{O}(1,1)$. (If one embeds $\mathbb{P}^1_k \times \mathbb{P}^1_k$ as a quadric $Q \subset \mathbb{P}^3_k$, then C_i are smooth conic sections of Q by pairwise different hyperplanes passing through ℓ.) Define the curve C as the union of the conics C_i, for $i = 1, \ldots, d$. Since $(C_i^2) = 2$ and all the curves C_i are tangent to each other, we have $C_i \cap C_j = P$ if $i \neq j$. Thus P is the unique singular point of C. If d is even, we define C as the union of C_i, for $i = 1, \ldots, d-1$, and $L = P_1 \times \mathbb{P}^1_k$. We have $L \cap C_i = P$ for $i = 1, \ldots, d-1$, so P is a unique singular point of C. Therefore, for any d the curve C is conical and $X(C)$ is a tree, and C and D have the same class in $\mathrm{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k)$, and so are linearly equivalent.

By choosing P outside D we can arrange that C does not meet D_{sing}. Then each $\bar{k}$-point s of $C \cap D$ belongs to exactly one geometric irreducible component of each C and D, and these components meet transversally at s.

Let $r(x,y;u,v)$ and $s(x,y;u,v)$ be the bi-homogeneous polynomials of bi-degree (d,d) if d is odd, and $(d,d-1)$ if d is even, such that their zero sets are the curves D and C, respectively. Let $Y \subset (\mathbb{P}^1_k)^3$ be the surface given by

$$\lambda r(x,y;u,v) + \mu s(x,y;u,v) = 0,$$

where $(\lambda : \mu)$ are homogeneous coordinates on the third copy of $\mathbb{P}^1_k$. It is easy to check that Y is smooth. The projection to the third factor $(\mathbb{P}^1_k)^3 \to \mathbb{P}^1_k$ defines a surjective morphism $Y \to \mathbb{P}^1_k$ with fibres $Y_0 = C$ and $Y_\infty = D$. The generic fibre of $Y \to \mathbb{P}^1_k$ is geometrically integral.

As in the previous section, we pick a smooth, geometrically integral, projective curve E with a unique k-point M, and a dominant morphism $\varphi : E \to \mathbb{P}^1_k$ such that $\varphi(M) = \infty$, and φ is not ramified over all points of $\mathbb{P}^1_k$ where Y has a singular fibre (including 0 and ∞). We define X by (9); this is smooth, geometrically integral and projective surface. Let $p : X \to E$ be the natural projection. Since $E(k) = \{M\}$ and $D = p^{-1}(M)$ has no k-points, we see that $X(k) = \emptyset$.

To study the étale Brauer–Manin set of X we need to understand X-torsors of an arbitrary finite k-group scheme G. In the following general proposition the word 'torsor' means 'torsor with structure group G'.

Proposition 3.3. *Let k be a field of characteristic zero, and let X and B be varieties over k. Let $p : X \to B$ be a proper morphism with geometrically connected fibres. Assume that p has a simply connected geometric fibre. Then for any torsor $f : X' \to X$ there exists a torsor $B' \to B$ such that torsors $X' \to X$ and $X \times_B B' \to X$ are isomorphic.*

Proof. Let $\delta = |G(\bar{k})|$. Let $X' \xrightarrow{g} B' \xrightarrow{h} B$ be the Stein factorisation of the composed morphism $X' \xrightarrow{f} X \xrightarrow{p} B$ (EGA III.4.3.1). Thus we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{h} & B \end{array}$$

where g is proper with geometrically connected fibres, and h is a finite morphism. (The variety B' can be defined as the relative spectrum of $(pf)_*\mathcal{O}_{X'}$.) The Stein factorisation is a functor from the category of proper schemes over B to the category of finite schemes over B. Thus we obtain an induced action of G on B' such that g is G-equivariant.

Let δ' be the degree of h. We claim that $\delta' \geq \delta$. Indeed, let $\bar{x}$ be a geometric point of B such that $p^{-1}(\bar{x})$ is simply connected. Thus $f^{-1}(p^{-1}(\bar{x}))$ is isomorphic to a disjoint union of δ copies of $p^{-1}(\bar{x})$. Hence $h^{-1}(\bar{x})$ has cardinality δ, and so $\delta' \geq \delta$.

The projection $\pi_X : X \times_B B' \to X$ is a finite morphism of degree δ'. The composition of the natural map $X' \to X \times_B B'$ with π_X is $f : X' \to X$, and this is a finite étale morphism of degree δ. Since $\delta' \geq \delta$ it follows that $X' \to X \times_B B'$ is finite and étale of degree 1, and hence is an isomorphism. We also see that $\delta' = \delta$ and that π_X is an étale morphism of degree δ. It follows that h is also étale of degree δ. Now the action of G equips B' with the structure of a B-torsor. $\square$

Finally, we can prove the main result of this section.

Theorem 3.4. *The set $X(k)$ is empty, whereas the set $X(\mathbb{A}_k)^{\text{ét,Br}}$ contains $D(\mathbb{A}_k)$ and so is infinite.*

Proof. Since $\overline{C}$ is a connected and simply connected variety over $\bar{k}$ which is a geometric fibre of $X \to E$, we can use Proposition 3.3. Thus any X-torsor of a finite k-group scheme has the form $\mathcal{T}_X = \mathcal{T} \times_E X$ for some torsor $\mathcal{T} \to E$. After twisting we can assume that the fibre of $\mathcal{T}$ over the unique k-point of E has a k-point. Thus the restriction of the torsor $\mathcal{T}_X \to X$ to the curve $D \subset X$ has a section σ. Therefore every adelic point on $D \subset X$ is the image of an adelic point on $\sigma(D) \subset \mathcal{T}_X$. By Corollary 3.2 and the functoriality of the Brauer–Manin set we have

$$\sigma(D)(\mathbb{A}_k) = \sigma(D)(\mathbb{A}_k)^{\text{Br}} \subset \mathcal{T}_X(\mathbb{A}_k)^{\text{Br}}.$$

Thus, by the definition of the étale Brauer–Manin set, $X(\mathbb{A}_k)^{\text{ét,Br}}$ contains the infinite set $D(\mathbb{A}_k)$, and hence is infinite. $\square$

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