$\mathrm{THH}(A)$ AS A CYCLOTOMIC SPECTRUM

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Contents

In this note we attempt to summarize the content of [NS, §3] concerning the construction of the topological Hochshild homology spectrum $\mathrm{THH}(A)$ of an associative ring spectrum A as a cyclotomic spectrum.

1. Recollections on the cyclic and paracyclic categories

Definition 1. Let Λ_∞ be the full subcategory of $\mathbb{Z}-\mathrm{PoSet}$ consisting of the sets $[n]_{\Lambda_\infty} := \frac{1}{n}\mathbb{Z} \subseteq \mathbb{Q}$ equipped with the partial order induced from the usual one on $\mathbb{Q}$ and with the $\mathbb{Z}$-action given by $(n, x) \mapsto x + n$.

Let $\mathbf{B}\mathbb{Z}$ denote the groupoid with one object whose endomorphism group is $\mathbb{Z}$. Since $\mathbb{Z}$ is an abelian group we have a canonical symmetric monoidal structure on $\mathbf{B}\mathbb{Z}$, and the category $\mathbb{Z} - \mathrm{PoSet} = \mathrm{Fun}(\mathbf{B}\mathbb{Z}, \mathrm{PoSet})$ inherits a canonical action of the $\mathbf{B}\mathbb{Z}$ induced by its action on itself. Unwinding the definition, such an action amounts to a natural transformation from the identity functor of $\mathbb{Z} - \mathrm{PoSet}$ to itself, i.e., a canonical self equivalence of every $\mathbb{Z}$-poset. This self equivalence is given by the action of $1 \in \mathbb{Z}$ (which is a map of $\mathbb{Z}$-posets since $\mathbb{Z}$ is abelian). This explicit formulation also makes it clear that the full subcategory $\Lambda_\infty \subseteq \mathbb{Z} - \mathrm{PoSet}$ inherits this action. In particular, we have an induced action of $\mathbb{Z}$ on every Hom set $\mathrm{Hom}_{\Lambda_\infty}([n]_{\lambda_\infty}, [m]_{\lambda_\infty})$ in Λ_∞ (where the $k \in \mathbb{Z}$ acts by post-composing with the action of k on $[m]_{\lambda_\infty}$).

Definition 2. For a positive integer $q \geq 1$ let Λ_q be the quotient category $\Lambda_\infty/\mathbf{B}(q\mathbb{Z})$. Explicitly, we may identify Λ_q as the category whose objects are the same as those of Λ_∞ (where the object corresponding to $[n]_{\lambda_\infty}$ will now be denoted $[n]_{\Lambda_q}$ to avoid abuse) and such that $\mathrm{Hom}_{\Lambda_q}([n]_{\Lambda_q}, [m]_{\Lambda_q}) = \mathrm{Hom}_{\Lambda_\infty}([n]_{\Lambda_\infty}, [m]_{\Lambda_\infty})/q\mathbb{Z}$. When $q = 1$ we will denote Λ_1 simply by Λ. We note that Λ_q inherits a remaining $\mathbf{B}(\mathbb{Z}/q)$-action.

Remark 3. The category Λ_∞ admits a self duality $\mathrm{D} : \Lambda_\infty \xrightarrow{\simeq} \Lambda_\infty^{\mathrm{op}}$ which sends every object to itself and sends a map $f : [n]_{\Lambda_\infty} \longrightarrow [m]_{\Lambda_\infty}$ to the map $\mathrm{D}(f) : [m]_{\Lambda_\infty} \longrightarrow [n]_{\Lambda_\infty}$ given by $\mathrm{D}(f)(x) = \min(\{y|f(y) \geq x\})$. Furthermore, the duality D intertwines the $\mathbf{B}\mathbb{Z}$-action on Λ_∞ with the $\mathbf{B}\mathbb{Z}$ action on Λ_q^{op} via the isomorphism $\mathbb{Z} \xrightarrow{[-1]} \mathbb{Z}$, and hence descends to a self duality $\mathrm{D}_q : \Lambda_q \xrightarrow{\simeq} \Lambda_q^{\mathrm{op}}$ for every $q \in \mathbb{N}$.

Using the fact that the hom sets $\mathrm{Hom}_{\Lambda_\infty}([n]_{\Lambda_\infty}, [m]_{\Lambda_\infty})$ are all free as $\mathbb{Z}$-sets one can show that the quotient $\Lambda_q = \Lambda_\infty/\mathbf{B}(q\mathbb{Z})$ coincides with the corresponding **homotopy quotient** in Cat_∞. This means, in particular, that the classifying space $|\Lambda_q|$ is the homotopy quotient of $|\Lambda_\infty|$ by the induced action of $|\mathbf{B}(q\mathbb{Z})| \simeq S^1$. One can then show that the category $\Lambda_\infty^{\mathrm{op}}$ is sifted, and hence $|\Lambda_\infty| \simeq |\Lambda_\infty^{\mathrm{op}}| \simeq *$. It then follows that $|\Lambda_q| \simeq |\Lambda_q^{\mathrm{op}}| \simeq \mathbf{B}(\mathbf{B}(q, \mathbb{Z}))$ is a classifying space for circle actions. Furthermore, one can show that the resulting square of ∞-categories

$$\begin{tikzcd}
\Lambda_\infty^{\mathrm{op}} \arrow[r] \arrow[d] & {|\Lambda_\infty^{\mathrm{op}}|} \arrow[d] & \simeq & * \\
\Lambda_q^{\mathrm{op}} \arrow[r] & {|\Lambda_q^{\mathrm{op}}|} & \simeq \mathbf{B}(\mathbf{B}(q\mathbb{Z}, 2)) &
\end{tikzcd}$$

is Cartesian, i.e., $\Lambda_\infty^{\mathrm{op}}$ is the homotopy fiber of the map $\Lambda_q^{\mathrm{op}} \longrightarrow |\Lambda_q^{\mathrm{op}}|$ (over any object in $|\Lambda_q^{\mathrm{op}}|$: they are all equivalent).

This property of the categories Λ_q^{op} has the following important consequence: if $\mathcal{C}$ is an ∞-category which admits colimits and $\mathcal{F} : \Lambda_q^{\mathrm{op}} \longrightarrow \mathcal{C}$ is a functor then by the Beck-Chevalley property for left Kan extensions we may consider the diagram $\mathrm{Lan}(\mathcal{F}) : |\Lambda_q^{\mathrm{op}}| \longrightarrow \mathcal{C}$ as encoding a $\mathbf{B}(q\mathbb{Z})$-action on $\mathrm{colim}(\mathcal{F}|_{\Lambda_\infty^{\mathrm{op}}})$. We also note that this $\mathbf{B}(q\mathbb{Z})$-action is compatible with the action of $\mathbf{B}(\mathbb{Z}/q)$ on Λ_q^{op} in the sense that the resulting functor $\mathrm{Fun}(\Lambda_q^{\mathrm{op}}, \mathcal{C}) \longrightarrow \mathcal{C}^{\mathbf{B}(\mathbf{B}(q\mathbb{Z}))}$ is $\mathbf{B}(\mathbb{Z}/q)$-equivariant. Explicitly, this means that for each $[m]_{\Lambda_q} \in \Lambda_q$ the natural map

$$\mathcal{F}([m]_{\Lambda_q^{\mathrm{op}}}) \longrightarrow \mathrm{colim}(\mathcal{F}|_{\Lambda_\infty^{\mathrm{op}}})$$

intertwines the associated $\mathbb{Z}/q$-action on $\mathcal{F}([m]_{\Lambda_q^{\mathrm{op}}})$ with the $\mathbf{B}(q\mathbb{Z})$ action on $\mathrm{colim}(\mathcal{F}|_{\Lambda_\infty^{\mathrm{op}}})$ with respect to the map $\mathbb{Z}/q\mathbb{Z} \longrightarrow \mathbf{B}(q\mathbb{Z})$ (which is equivalent to the inclusion of the q-torsion subgroup of $\mathbf{B}(q\mathbb{Z}) \simeq S^1$).

2. The cyclic bar construction and THH

Now suppose that $\mathcal{C}$ is a symmetric monoidal ∞-category admitting colimits and let A be an associative algebra in $\mathcal{C}$ equipped with an action of $\mathbb{Z}/q$ (via associative algebra maps). We note that the ∞-operad which controls such an algebraic structure is the ∞-operad

$$\mathrm{Ass}_q^\otimes := \mathbf{B}(\mathbb{Z}/q) \otimes \mathrm{Ass}^\otimes \simeq \mathbf{B}(\mathbb{Z}/q)^{\amalg} \times_{\mathrm{Fin}_*} \mathrm{Ass}^\otimes$$

where $\otimes$ denotes the ∞-operadic Boardman-Vogt tensor product and $\mathbf{B}(\mathbb{Z}/q)^{\amalg} \longrightarrow \mathrm{Fin}_*$ is the coCartesian operad of $\mathbf{B}(\mathbb{Z}/q)$. In particular, if $\mathcal{C}^\otimes \longrightarrow \mathrm{Fin}_*$ is the coCartesian fibration encoding the symmetric monoidal structure on $\mathcal{C}$ then an associative algebra object in $\mathcal{C}$ equipped with a $\mathbb{Z}/q$-action is encoded by a map

$$A : \mathbf{B}(\mathbb{Z}/q)^{\amalg} \times_{\mathrm{Fin}_*} \mathrm{Ass}^\otimes \longrightarrow \mathcal{C}^\otimes$$

over Fin_* which carry inert maps to inert maps. The operad $\mathbf{B}(\mathbb{Z}/q)^{\coprod}$ can also be understood using the "geometric" model $\mathrm{Tor}_{\mathbb{Z}/q} \simeq \mathbf{B}(\mathbb{Z}/p)$ given by the ∞-category of $\mathbb{Z}/q$-torsors (indeed, up to isomorphism there is a unique $\mathbb{Z}/p$-torsor and its automorphism group is $\mathbb{Z}/p$). Using this model one can show that the base change $\mathbf{B}(\mathbb{Z}/q)^{\coprod}_{\mathrm{act}} \longrightarrow \mathrm{Fin}$ (restricted along the functor $(-)\coprod\{*\} : \mathrm{Fin} \longrightarrow \mathrm{Fin}_*$) can equivalently be described as the quotient functor

$$(-)_{\mathbb{Z}/q} : \mathrm{Free}_{\mathbb{Z}/q} \longrightarrow \mathrm{Fin} \tag{1}$$

where $\mathrm{Free}_{\mathbb{Z}/q}$ is the category of finite free $\mathbb{Z}/q$-sets and $S_{\mathbb{Z}/q} := S/(\mathbb{Z}/q) \in \mathrm{Fin}$ is the associated quotient. We have a natural functor

$$U_q : \Lambda_q \longrightarrow \mathrm{Free}_{\mathbb{Z}/q} \tag{2}$$

which sends $[n]_{\Lambda_q}$ to the set $\frac{1}{n}\mathbb{Z}/q$ equipped with its natural $\mathbb{Z}/q$-action by translations. Furthermore, the composition $\Lambda_q \longrightarrow \mathrm{Free}_{\mathbb{Z}/q} \longrightarrow \mathrm{Fin}$ of (1) and (2), which sends $[n]_{\Lambda_q}$ to the finite set $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ admits a natural lift along $\mathrm{Ass}^{\otimes} \longrightarrow \mathrm{Fin}$. To prove this it will suffice to show that the map $\Lambda_\infty \longrightarrow \mathrm{Fin}$ sending $[n]_{\Lambda_\infty}$ to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ admits a $\mathbf{B}(q\mathbb{Z})$-equivariant lift to $\mathrm{Ass}^{\otimes}$. We now observe that if $f : \frac{1}{n}\mathbb{Z} \longrightarrow \frac{1}{m}\mathbb{Z}$ is a $\mathbb{Z}$-equivariant order preserving map then for every $x \in \frac{1}{m}\mathbb{Z}$ the fiber $f^{-1}(x) \subseteq \frac{1}{n}\mathbb{Z}$ acquires a natural linear ordering (which is invariant under replacing f with $f+1$), and that the square of sets

$$\begin{array}{ccc} \frac{1}{n}\mathbb{Z} & \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \\ \downarrow f & & \downarrow f_* \\ \frac{1}{m}\mathbb{Z} & \longrightarrow & \frac{1}{m}\mathbb{Z}/\mathbb{Z} \end{array}$$

is Cartesian. We may now conclude that $U_q : \Lambda_q \longrightarrow \mathrm{Free}_{\mathbb{Z}/q}$ lifts to a map

$$V_q : \Lambda_q \longrightarrow \mathrm{Free}_{\mathbb{Z}/q} \times_{\mathrm{Fin}} \mathrm{Ass}^{\otimes}_{\mathrm{act}} \qquad V_q([n]_{\Lambda_q}) = \left(\frac{1}{n}\mathbb{Z}/q\mathbb{Z}, \frac{1}{n}\mathbb{Z}/\mathbb{Z}\right). \tag{3}$$

We will denote similarly denote by

$$V_q^{\mathrm{D}} : \Lambda_q^{\mathrm{op}} \longrightarrow \mathrm{Free}_{\mathbb{Z}/q} \times_{\mathrm{Fin}} \mathrm{Ass}^{\otimes}_{\mathrm{act}}$$

the composition of V_q with the self duality functor $\Lambda_q^{\mathrm{op}} \xrightarrow{\simeq} \Lambda_q$ of Remark 3.

Definition 4. Let $\mathcal{C}$ be a symmetric monoidal ∞-category that admits colimits and let

$$A : \mathbf{B}(\mathbb{Z}/q)^{\coprod} \times_{\mathrm{Fin}_*} \mathrm{Ass}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$$

be a $\mathbb{Z}/q$-equivariant associative algebra object in $\mathcal{C}$. We define the **q-cyclic Bar construction** of A to be the composed map

$$\mathrm{Bar}_q(A) : \Lambda_q^{\mathrm{op}} \xrightarrow{V_q^{\mathrm{D}}} \mathrm{Free}_{\mathbb{Z}/q} \times_{\mathrm{Fin}} \mathrm{Ass}^{\otimes}_{\mathrm{act}} \xrightarrow{A_{\mathrm{act}}} \mathcal{C}^{\otimes}_{\mathrm{act}} \longrightarrow \mathcal{C}$$

where the last functor is the canoical symmetric monoidal projection $\mathcal{C}^{\otimes}_{\mathrm{act}} \longrightarrow \mathcal{C}$ sending $(X_1, ..., X_n)$ to $X_1 \otimes ... \otimes X_n$. We then define the **topological q-Hochshild homology** of A to be

$$\mathrm{THH}_q(A) := \mathrm{colim}(\mathrm{Bar}_q\,|\Lambda_\infty) \in \mathcal{C}^{\mathbf{B}(q\mathbb{Z})}.$$

When $q = 1$ we will denote $\mathrm{Bar}_q(A)$ and $\mathrm{THH}_q(A)$ simply by $\mathrm{Bar}(A)$ and $\mathrm{THH}(A)$.

Before we proceed further let us record the following colimit-like property of THH_q:

Lemma 5. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a lax symmetric monoidal functor between symmetric monoidal ∞-categories with colimits. Then there is a canonical natural transformation*

$$\tau_F : \mathrm{THH}_q(\mathcal{F}(-)) \longrightarrow \mathcal{F}(\mathrm{THH}_q(-))$$

of functors $\mathrm{Alg}_{\mathrm{Ass}_q}(\mathcal{C}) \longrightarrow \mathcal{D}^{\mathbf{B}(q\mathbb{Z})}$. Furthermore, if $\mathcal{F}$ is symmetric monoidal and preserves colimits then τ_F is an equivalence.

Proof. Let $\mathcal{F}^{\otimes} : \mathcal{C}^{\otimes} \longrightarrow \mathcal{D}^{\otimes}$ the ∞-operad map encoding the lax symmetric monoidal structure of $\mathcal{F}$. By the universal property of $\mathcal{C}^{\otimes}_{\mathrm{act}}$ ([Lur, Proposition 2.2.4.9]) there is a canonical natural transformation from the composed symmetric monoidal functor

$$\mathcal{C}^{\otimes}_{\mathrm{act}} \xrightarrow{\mathcal{F}^{\otimes}_{\mathrm{act}}} \mathcal{D}^{\otimes}_{\mathrm{act}} \longrightarrow \mathcal{D} \qquad (X_1, ..., X_n) \mapsto \mathcal{F}(X_1) \otimes ... \otimes \mathcal{F}(X_n)$$

to the composed lax symmetric monoidal functor

$$\mathcal{C}^{\otimes}_{\mathrm{act}} \longrightarrow \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D} \qquad (X_1, ..., X_n) \mapsto \mathcal{F}(X_1 \otimes ... \otimes X_n)$$

which is an equivalence if $\mathcal{F}$ is symmetric monoidal. In particular, we have a natural transformation $\mathrm{Bar}_q(\mathcal{F}(A)) \longrightarrow \mathcal{F}(\mathrm{Bar}_q(A))$ for $A \in \mathrm{Alg}_{\mathrm{Ass}_q}(\mathcal{C})$, which is an equivalence if $\mathcal{F}$ is symmetric monoidal. We then obtain τ_F by composing this natural transformation with the natural transformation

$$\mathrm{Lan}(\mathcal{F}\,\mathrm{Bar}_q(A)) \Rightarrow \mathcal{F}(\mathrm{Lan}\,\mathrm{Bar}_q(A))$$

where $\mathrm{Lan} : \mathrm{Fun}(\Lambda_q, \mathcal{C}) \longrightarrow \mathrm{Fun}(|\Lambda_q|, \mathcal{C})$ is the left Kan extension functor, which is an equivalence when $\mathcal{F}$ preserves colimits, as desired. □

We also have the following compatibility properties between THH_q and THH. We note that we have two natural functors $\mathrm{Alg}_{\mathrm{Ass}}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathrm{Ass}_q(\mathcal{C})}$. The first functor takes an associative algebra object A and considers is as being $\mathbb{Z}/q$-equivariant with constant action. Let us denote the resulting $\mathbb{Z}/q$-equivariant associative algebra by $\mathrm{Inf}_q(A)$. The second functor takes A and associates to it the $\mathbb{Z}/q$-equivariant algebra object $\otimes_{x \in \mathbb{Z}/q} A$ obtained by the q-fold tensor product of A with itself, where the action of $\mathbb{Z}/q$ is by permuting the factors. To describe this functor formally, let us invoke some machinery.

Now let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and consider the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. This ∞-category admits a symmetric monoidal functor given by **Day convolution**, whose algebra objects can be identified with **lax monoidal functors** $\mathcal{C} \longrightarrow \mathcal{C}$. In particular, the ∞-category $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C})$ inherits the day convolution monoidal structure. Since Fin is the free symmetric monoidal ∞-category generated by an commutative algebra object $* \in \mathrm{Fin}$, and since the identify functor $\mathrm{Id} : \mathcal{C} \longrightarrow \mathcal{C}$ admits a canonical lax monoidal structure it follows that there is an essentially unique symmetric monoidal functor

$$\mathrm{Fin} \longrightarrow \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C}) \tag{4}$$

which sends the singleton $* \in \mathrm{Fin}$ to identity functor $\mathrm{Id} \in \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C})$ (considered as a lax monoidal functor). Unwinding the definitions we see that (4) corresponds to the functor $\mathrm{Fin} \times \mathcal{C} \longrightarrow \mathcal{C}$ which sends (S, X) to $\otimes_{s \in S} X$. Now consider the functor $\mathrm{Tor}_{\mathbb{Z}/q} \times \mathrm{Tor}_{\mathbb{Z}/q} \longrightarrow \mathrm{Fin}$ which sends a pair (S, S') of $\mathbb{Z}/q$-torsors to the contracted product $S \times^{\mathbb{Z}/q} S'$. Composing with (4) we now obtain a functor

$$\mathrm{Tor}_{\mathbb{Z}/q} \times \mathrm{Tor}_{\mathbb{Z}/q} \longrightarrow \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C}) \tag{5}$$

Unwinding the definitions we see that (5) corresponds to the functor

$$\mathrm{N}^{\mathrm{naive}}_{(-)} : \mathrm{Tor}_{\mathbb{Z}/q} \longrightarrow \mathrm{Fun}(\mathrm{Tor}_{\mathbb{Z}/q}, \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C},\mathcal{C})) \simeq \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C}^{\mathbf{B}(\mathbb{Z}/q)}) \tag{6}$$

which sends (S, X) to the (naively) $\mathbb{Z}/q$-equivariant spectrum $\mathrm{N}^{\mathrm{naive}}_S(X) := \otimes_{s\in S} X$. For a fixed S the functor $\mathrm{N}^{\mathrm{naive}}_S(-)$ is lax monoidal, and hence if A is an associative algebra object in $\mathcal{C}$ then $\mathrm{N}^{\mathrm{naive}}_S(A)$ is an associative algebra object in $\mathcal{C}^{\mathbf{B}(\mathbb{Z}/q)}$. In fact, it will be convenient to use the (self-commuting) $\mathbb{Z}/q$-action twice to treat $\mathrm{N}^{\mathrm{naive}}_S(A)$ as an Ass_q-algebra object in $\mathcal{C}^{\mathbf{B}(\mathbb{Z}/q)}$. For this, we note that since the functor $\mathrm{N}^{\mathrm{naive}}_{(-)}(-) : \mathrm{Tor}_{\mathbb{Z}/q} \times \mathcal{C} \longrightarrow \mathcal{C}^{\mathbf{B}(\mathbb{Z}/q)}$ is lax monoidal in its second argument it extends to a map of ∞-operads

$$\mathrm{N}^{\mathrm{naive},\otimes} : \mathrm{Tor}^{\amalg}_{\mathbb{Z}/q} \times_{\mathrm{Fin}} \mathcal{C}^{\otimes} \longrightarrow (\mathcal{C}^{\mathbf{B}(\mathbb{Z}/q)})^{\otimes}.$$

In particular, if A is given by a map $\mathrm{Ass}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ over Fin_*, then we will denote by $\mathrm{N}^{\mathrm{naive}}_q(A)$ the Ass_q-algebra object in $\mathcal{C}^{\mathbf{B}(\mathbb{Z}/q)}$ encoded by the map

$$\mathrm{Tor}^{\amalg}_{\mathbb{Z}/p} \times_{\mathrm{Fin}} \mathrm{Ass}^{\otimes} \xrightarrow{A} \mathrm{Tor}^{\amalg}_{\mathbb{Z}/p} \times_{\mathrm{Fin}} \mathcal{C}^{\otimes} \xrightarrow{\mathrm{N}^{\mathrm{naive},\otimes}} (\mathcal{C}^{\mathbf{B}(\mathbb{Z}/q)})^{\otimes}.$$

We then have the following compatibility statement:

Lemma 6.

(1) There is a natural $\mathbf{B}(q\mathbb{Z})$-equivariant equivalence

$$\mathrm{THH}_q(\mathrm{Inf}_q(A)) \simeq \mathrm{res}^{\mathbf{B}\mathbb{Z}}_{\mathbf{B}(q\mathbb{Z})} \mathrm{THH}(A).$$

where $\mathrm{res}^{\mathbf{B}\mathbb{Z}}_{\mathbf{B}(q\mathbb{Z})} : \mathcal{C}^{\mathbf{B}\mathbb{Z}} \longrightarrow \mathcal{C}^{\mathbf{B}(q\mathbb{Z})}$ is the restriction along the q-fold covering map $\mathbf{B}(q\mathbb{Z}) \longrightarrow \mathbf{B}\mathbb{Z}$.

(2) There is a natural $\mathbf{B}\mathbb{Z}$-equivariant equivalence

$$\mathrm{THH}_q(\mathrm{N}^{\mathrm{naive}}_q(A)) \simeq \mathrm{THH}(A).$$

where the $\mathbf{B}\mathbb{Z}$-action on the left hand side is obtained via the equivalence $\mathbf{B}(q\mathbb{Z}) \xrightarrow{\simeq} \mathbf{B}\mathbb{Z}$ which sends the generator $q \in q\mathbb{Z}$ to the generator $1 \in \mathbb{Z}$.

Proof. To prove (1), observe that we have a natural equivalence

$$\mathrm{Bar}_q(\mathrm{Inf}_q(A)) \simeq \mathrm{Bar}(A)|_{\Lambda_q}$$

and so the result follows from the Beck-Chevalley property for left Kan extension applied to the Cartesian square

$$\begin{array}{ccccc} \Lambda^{\mathrm{op}}_q & \longrightarrow & |\Lambda^{\mathrm{op}}_q| & \simeq & K(q\mathbb{Z}, 2) \\ \downarrow & & \downarrow & & \\ \Lambda^{\mathrm{op}} & \longrightarrow & |\Lambda^{\mathrm{op}}| & \simeq & K(\mathbb{Z}, 2) \end{array}$$

To prove (2), we note that this time we have a natural equivalence

$$\mathrm{Bar}_q(\mathrm{N}^{\mathrm{naive}}_q(A)) \simeq \mathrm{sd}^*_q \mathrm{Bar}(A)$$

where $\mathrm{sd}_q : \Lambda^{\mathrm{op}}_q \longrightarrow \Lambda^{\mathrm{op}}$ is the subdivision functor which sends $[n]_{\Lambda_q}$ to $[qn]_{\Lambda_q}$ and sends the equivalence class of an order preserving map $f : \frac{1}{n}\mathbb{Z} \longrightarrow \frac{1}{m}\mathbb{Z}$ to the equivalence class of the order preserving map $\frac{1}{q}f(q(-)) : \frac{1}{qn}\mathbb{Z} \longrightarrow \frac{1}{qm}\mathbb{Z}$. The one can check that $\mathrm{sd}_q : \Lambda_q^{\mathrm{op}} \longrightarrow \Lambda^{\mathrm{op}}$ is cofinal and in particular induces an equivalence on classifying spaces $\mathbf{B}(q\mathbb{Z}) \simeq |\Lambda_q^{\mathrm{op}}| \xrightarrow{\simeq} |\Lambda^{\mathrm{op}}| \simeq \mathbf{B}\mathbb{Z}$ (sending the generator q of $q\mathbb{Z}$ to the generator 1 of $\mathbb{Z}$) and a compatible equivalence between the $\mathrm{Lan}(\mathrm{Bar}_q(\mathrm{N}_q^{\mathrm{naive}}(A))) : |\Lambda_q^{\mathrm{op}}| \longrightarrow \mathcal{C}$ and $\mathrm{Lan}(\mathrm{Bar}(A)) : |\Lambda^{\mathrm{op}}| \longrightarrow \mathcal{C}$, as desired. □

3. The Tate diagonal and the cyclotomic structure on THH

Let us now specialize to the case $\mathcal{C} = \mathrm{Sp}$. Given an associative ring spectrum A, we wish to promote $\mathrm{THH}(A)$ to a **cyclotomic spectrum** in the sense of [NS]. For this, we need to equip $\mathrm{THH}(A)$ with a collection of $\mathbf{B}\mathbb{Z}$-equivariant **Frobenius maps** $\varphi_p : \mathrm{THH}(A) \longrightarrow \mathrm{THH}(A)^{t(\mathbb{Z}/p)}$. To define those, we will need to recall the definition of the **Tate diagonal**. Let us now fix a positive prime number p.

Definition 7. We will denote by

$$T_{(-)} : \mathrm{Tor}_{\mathbb{Z}/p} \xrightarrow{\mathrm{N}_{(-)}^{\mathrm{naive}}} \mathrm{Fun}_{\mathrm{lax}}(\mathrm{Sp}, \mathrm{Sp}^{\mathbf{B}(\mathbb{Z}/p)}) \xrightarrow{(-)_*^{t(\mathbb{Z}/p)}} \mathrm{Fun}_{\mathrm{lax}}(\mathrm{Sp}, \mathrm{Sp})$$

the composition of (6) and the functor induced by post-composing with the (lax monoidal) Tate functor $(-)^{t(\mathbb{Z}/p)} : \mathrm{Sp}^{\mathbf{B}(\mathbb{Z}/p)} \longrightarrow \mathrm{Sp}$. Unwinding the definitions, we see that $T_{(-)}$ corresponds to the functor

$$T_{(-)}(-) : \mathrm{Tor}_{\mathbb{Z}/p} \times \mathrm{Sp} \longrightarrow \mathrm{Sp} \qquad (S, X) \mapsto (\otimes_{s \in S} X)^{t(\mathbb{Z}/p)}$$

Lemma 8. *For any fixed $S \in \mathrm{Tor}_{\mathbb{Z}/p}$ the functor*

$$T_S(-) : \mathrm{Sp} \longrightarrow \mathrm{Sp}$$

is exact.

Proof. We follow the proof given in [NS]. We first prove that $T_S(-)$ preserves direct sums. Indeed:

$$T_S(X_0 \oplus X_1) \simeq \left(\bigoplus_{(i_s) \in \{0,1\}^S} \otimes_{s \in S} X_{i_s} \right)^{t(\mathbb{Z}/p)} \simeq$$

$$T_S(X_0) \oplus T_S(X_1) \oplus \left[\bigoplus_{(i_s) \in \{0,1\}^S \setminus \{(0,...,0),(1,...,1)\}} \otimes_{s \in S} X_{i_s} \right]^{t(\mathbb{Z}/p)} \simeq T_S(X_0) \oplus T_S(X_1).$$

where the last equivalence is due to the fact that the action of $\mathbb{Z}/p$ on $\{0,1\}^S \setminus \{(0,...,0),(1,...,1)\}$ is free, and the Tate functor vanishes on induced modules. It will now suffice to show that $T_S(-)$ preserves fiber sequences. Here the argument is similar: if $X_0 \longrightarrow \widetilde{X} \longrightarrow X_1$ is a fiber sequence then $\otimes_{s \in S} \widetilde{X}$ inherits a filtration whose successive quotients are either $\oplus_{s \in S} X_0$, $\oplus_{s \in S} X_1$, or one of the induced modules appearing above. The desired result follows from the fact that $(-)^{t(\mathbb{Z}/p)}$ exact and vanishes on induced modules. □

In light of Lemma 8 we may consider $T_{(-)}$ as a functor

$$T_{(-)} : \mathrm{Tor}_{\mathbb{Z}/p} \longrightarrow \mathrm{Fun}_{\mathrm{lax}}^{\mathrm{ex}}(\mathrm{Sp}, \mathrm{Sp}).$$

We now recall a previous result of Nikolaus ([Nik]) which states the following:

Proposition 9 (Nikolaus). *The identity functor is the initial object of* $\mathrm{Fun}_{\mathrm{lax}}^{\mathrm{ex}}(\mathrm{Sp}, \mathrm{Sp})$.

Corollary 10. *There exists a unique levelwise lax symmetric monoidal natural transformation*

$$\Delta_{(-)} : I_{(-)} \longrightarrow T_{(-)}$$

where $I_{(-)} : \mathrm{Tor}_{\mathbb{Z}/p} \longrightarrow \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{lax}}(\mathrm{Sp}, \mathrm{Sp})$ *is the constant functor taking the value* $\mathrm{Id} \in \mathrm{Fun}^{\mathrm{ex}}_{\mathrm{lax}}(\mathrm{Sp}, \mathrm{Sp})$.

Definition 11. Let S be a $\mathbb{Z}/p$-torsor and X a spectrum. Then the natural map

$$\Delta_S(X) : X \longrightarrow T_S(X) \simeq (\otimes_{s\in S} X)^{t(\mathbb{Z}/p)}$$

furnished by Corollary 10 is called the **Tate diagonal**.

Let us now fix a $\mathbb{Z}/p$-torsor S and an associative ring spectrum A. Since the functor $T_S(-)$ is lax monoidal we have an induced associative ring structure on $T_S(A)$ and since the natural transformation $\Delta_S(-)$ is lax monoidal the Tate diagonal map

$$\Delta_S(A) : A \longrightarrow T_S(A)$$

is naturally a map of associative ring spectra. As above, we can use the extra S-argument in order to promote $T_S(A)$ to an Ass_p-algebra object. This results in a homotopically trivial action, but it is still formally convenient to do so. In particular, since $T_{(-)}(-) : \mathrm{Tor}_{\mathbb{Z}/p} \times \mathrm{Sp} \longrightarrow \mathrm{Sp}$ is lax monoidal in the second coordinate it extends to a map of ∞-oprads

$$T_p^{\otimes} : \mathrm{Tor}^{\amalg}_{\mathbb{Z}/p} \times_{\mathrm{Fin}} \mathrm{Sp}^{\otimes} \longrightarrow \mathrm{Sp}^{\otimes} .$$

If A is given by a map $\mathrm{Ass}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ over Fin_*, then we will denote by $T_p(A)$ the Ass_q-algebra object in Sp encoded by the map

$$\mathrm{Tor}^{\amalg}_{\mathbb{Z}/p} \times_{\mathrm{Fin}} \mathrm{Ass}^{\otimes} \xrightarrow{A} \mathrm{Tor}^{\amalg}_{\mathbb{Z}/p} \times_{\mathrm{Fin}} \mathrm{Sp}^{\otimes} \xrightarrow{T_p^{\otimes}} \mathrm{Sp}^{\otimes} .$$

The natural transformation $\Delta_{(-)}$ of Corollary 10 then induces a map of Ass_p-ring spectra

$$\Delta_p(A) : \mathrm{Inf}_p(A) \longrightarrow T_p(A) \simeq \mathrm{N}^{\mathrm{naive}}_p(A)^{t(\mathbb{Z}/p)}$$

which can be considered as an $\mathbb{Z}/p$-equivariant version of the Tate diagonal. We may then consider the composed map

$$\mathrm{res}^{\mathbf{B}\mathbb{Z}}_{\mathbf{B}(q\mathbb{Z})} \mathrm{THH}(A) \xrightarrow{\simeq} \mathrm{THH}(\mathrm{Inf}_p A) \xrightarrow{(\Delta_p(A))_*} \mathrm{THH}_p\left(\mathrm{N}^{\mathrm{naive}}_p(A)^{t(\mathbb{Z}/p)}\right) \xrightarrow{\tau}$$
$$\mathrm{THH}_p(\mathrm{N}^{\mathrm{naive}}_p(A))^{t(\mathbb{Z}/p)} \xrightarrow{\simeq} \mathrm{THH}(A)^{t(\mathbb{Z}/p)} \tag{7}$$

where the first arrow is the equivalence of Lemma 6(1), τ is the natural transformation of Lemma 5 associated to the lax symmetric monoidal $(-)^{t(\mathbb{Z}/p\mathbb{Z})} : \mathrm{Sp}^{\mathbf{B}(\mathbb{Z}/p)} \longrightarrow \mathrm{Sp}$ and the last arrow is the equivalence of Lemma 6(2). Tracing the compatibility with the circle action specified in Lemma 6 we see that φ_p intertwines the $\mathbf{B}(p\mathbb{Z})$-action on $\mathrm{res}^{\mathbf{B}\mathbb{Z}}_{\mathbf{B}(p\mathbb{Z})} \mathrm{THH}(A)$ with the $\mathbf{B}(p\mathbb{Z})$-action on $\mathrm{THH}(A)^{t(\mathbb{Z}/p)}$ obtained by taking the $\mathbf{B}(p\mathbb{Z})$-action on $\mathrm{THH}(A)$ (via the identification $\mathbf{B}(p\mathbb{Z}) \simeq \mathbf{B}\mathbb{Z}$) and then passing to the induced action on $\mathrm{THH}(A)^{t(\mathbb{Z}/p)}$. In both these actions $\mathbf{B}(\mathbb{Z}/p)$ ends up acting trivially and hence φ_p descends to a $\mathbf{B}\mathbb{Z}$-equivariant map $\mathrm{THH}(A) \longrightarrow \mathrm{THH}(A)^{t(\mathbb{Z}/p)}$.

Definition 12. Let A be a ring spectrum. We define the **Frobenius map**

$$\varphi_p : \mathrm{THH}(A) \longrightarrow \mathrm{THH}(A^{t(\mathbb{Z}/p)})$$

to be the $\mathbf{B}\mathbb{Z}$-equivariant induced by (7) as explained above.

4. Construction of the Frobenius using genuine $\mathbb{Z}/p$-equivariant spectra

Let us now describe an alternative way of defining the cyclotomic structure on $\mathrm{THH}(A)$, this time using genuine $\mathbb{Z}/p$-equivariant spectra. The Hill-Hopkins-Ravanel norm functor refines

$$\mathrm{N}^{\mathrm{naive}}_{(-)} : \mathrm{Tor}_{\mathbb{Z}/p} \longrightarrow \mathrm{Fun}_{\mathrm{lax}}(\mathrm{Sp}, \mathrm{Sp}^{\mathbf{B}(\mathbb{Z}/p)}) \tag{8}$$

to a functor

$$\mathrm{N}^{\mathrm{gen}}_{(-)} : \mathrm{Tor}_{\mathbb{Z}/p} \longrightarrow \mathrm{Fun}_{\mathrm{lax}}(\mathrm{Sp}, (\mathbb{Z}/p)\,\mathrm{Sp}) \tag{9}$$

where $(\mathbb{Z}/q)\,\mathrm{Sp}$ denotes the ∞-category of genuine $\mathbb{Z}/q$-spectra. For a given $\mathbb{Z}/p$-torsor S the genuine $\mathbb{Z}/p$-equivariant structure $\mathrm{N}^{\mathrm{gen}}_S(X)$ enjoys the following two properties:

(1) There is a natural equivalence $\sigma : X \longrightarrow \Phi^{\mathbb{Z}/p}\mathrm{N}^{\mathrm{gen}}_S(X)$, where $\Phi^{\mathbb{Z}/p}$ is the geometric fixed point functor.
(2) The composed map $X \xrightarrow{\sigma} \Phi^{\mathbb{Z}/p}\mathrm{N}^{\mathrm{gen}}_S(X) \longrightarrow \mathrm{N}^{\mathrm{naive}}_S(X)^{t(\mathbb{Z}/p)}$ is naturally homotopic to the Tate diagonal.

Remark 13. The functors $\Phi^{\mathbb{Z}/p}$ and $\mathrm{N}^{\mathrm{gen}}_S(-)$ are symmetric monoidal and the natural equivalence σ can be promoted to a symmetric monoidal natural equivalence from the identity functor $\mathrm{Id} : \mathrm{Sp} \longrightarrow \mathrm{Sp}$ to the functor $\Phi^{\mathbb{Z}/p} \circ \mathrm{N}^{\mathrm{gen}}_S : \mathrm{Sp} \longrightarrow \mathrm{Sp}$. Since the natural transformation $\Phi^{\mathbb{Z}/p}\mathrm{N}^{\mathrm{gen}}_S(-) \longrightarrow \mathrm{N}^{\mathrm{naive}}_S(-)^{t(\mathbb{Z}/p)}$ is lax monoidal as well Proposition 9 now implies the existence of a unique homotopy as in (2).

In particular, if A is an associative ring spectrum then $\mathrm{N}^{\mathrm{gen}}_S(A)$ is naturally an Ass-algebra object in $(\mathbb{Z}/p)\,\mathrm{Sp}$. As above we can spend the extra dependence in S to obtain an Ass_p-algebra object in $(\mathbb{Z}/p)\,\mathrm{Sp}$, which we will denote by $\mathrm{N}^{\mathrm{gen}}_p(X)$. We note that in this case we may consider $\Phi^{\mathbb{Z}/p}\mathrm{N}^{\mathrm{gen}}_p(X)$ as an Ass_p-algebra object in Sp (with trivial action) and σ as a map $\sigma : \mathrm{Inf}_p(A) \longrightarrow \Phi^{\mathbb{Z}/p}\mathrm{N}^{\mathrm{gen}}_p(X)$ of Ass_p-algebras. The associated topological Hochschild homology $\mathrm{THH}_p(\mathrm{N}^{\mathrm{gen}}_p(A))$ is now a $\mathbf{B}(p\mathbb{Z})$-equivariant object in genuine $\mathbb{Z}/p$-spectra whose underlying spectrum is equivalent to $\mathrm{THH}(A)$ by Lemma 6(2). We may hence consider $\mathrm{THH}_p(\mathrm{N}^{\mathrm{gen}}_p(A))$ as promoting the $\mathbb{Z}/p$-action on $\mathrm{THH}(A)$ to a genuine one.

Applying now Lemma 5 to the symmetric monoidal colimit preserving functor $\Phi^{\mathbb{Z}/p}$ and using Lemma 6 we get a composed natural $\mathbf{B}\mathbb{Z}$-equivariant equivalence

$$\begin{aligned}\psi_p : U(\mathrm{THH}_p(\mathrm{N}^{\mathrm{gen}}_p(A))) \simeq \mathrm{THH}(A) \xrightarrow{\simeq} \mathrm{THH}_p(\mathrm{Inf}_p\, A) \xrightarrow[\sigma_*]{\simeq} \\ \mathrm{THH}_p(\Phi^{\mathbb{Z}/p}\mathrm{N}^{\mathrm{gen}}_p(A)) \xrightarrow{\simeq} \Phi^{\mathbb{Z}/p}\,\mathrm{THH}_p(\mathrm{N}^{\mathrm{gen}}_p(A))\end{aligned} \tag{10}$$

where $U : (\mathbb{Z}/p)\,\mathrm{Sp} \longrightarrow \mathrm{Sp}$ is the forgetful functor. Using the (lax monoidal) natural transformation

$$\Phi^{\mathbb{Z}/p}(-) \Rightarrow U(-)^{t(\mathbb{Z}/p)}$$

of functors $(\mathbb{Z}/p)\,\mathrm{Sp} \longrightarrow \mathrm{Sp}$ we now obtain another Forbenius map

$$\varphi_p : \mathrm{THH}(A) \longrightarrow \mathrm{THH}(A)^{t\mathbb{Z}/p}.$$

To see that this construction is compatible with the previous one we observe that Lemma 5 furnishes a commutative diagram

$$\begin{tikzcd}
\mathrm{THH}(A) \arrow[r, "\simeq"] & \mathrm{THH}_p(\Phi^{\mathbb{Z}/p}\mathrm{N}^{\mathrm{gen}}_p A) \arrow[r] \arrow[d] & \Phi^{\mathbb{Z}/p}\,\mathrm{THH}_p(\mathrm{N}^{\mathrm{gen}}_p A) \arrow[d] & \\
& \mathrm{THH}_p((\mathrm{N}^{\mathrm{naive}} A)^{t\mathbb{Z}/p}) \arrow[r] & \mathrm{THH}_p(\mathrm{N}^{\mathrm{naive}} A)^{t\mathbb{Z}/p} \arrow[r, "\simeq"] & \mathrm{THH}(A)^{t\mathbb{Z}/p}
\end{tikzcd}$$

in which the top path traces the Frobenius maps of the second construction and the bottom path traces the Frobenius maps of the first construction by Property 2 of the Hill-Hopkins-Ravanel norm.

5. An S^1-genuine structure on THH

We note that the Hill-Hopkins-Ravanel norm exists for every positive integer q. In particular, for every q, we may consider $\mathrm{THH}_q(\mathrm{N}^{\mathrm{gen}}_q A)$ as promoting $\mathrm{THH}(A)$ to a genuine $\mathbb{Z}/q\mathbb{Z}$-spectrum, in a way that is compatible with the circle action on $\mathrm{THH}(A)$. Furthermore, the $\mathbb{Z}/q\mathbb{Z}$-structures for various q fit together to endow $\mathrm{THH}(A)$ with the structure of a genuine S^1-spectrum (with respect to the family of finite subgroups of S^1). Let us informally survey how this goes, based on ideas of Denis Nardin. In what follows, all references to genuine S^1-structures will always mean genuine with respect to the collection of finite subgroups $H \subseteq S^1$.

Let $\mathcal{O}_{S^1} \subseteq \mathcal{S}^{\mathbf{B}S^1}$ be the full subcategory spanned by the S^1-spaces of the form S^1/H for all finite subgroups $H \subseteq S^1$. Recall that an S^1-category is simply a coCartesian fibration $\pi : \mathcal{C} \longrightarrow \mathcal{O}_{S^1}^{\mathrm{op}}$. Such a fibration encodes with the data of a functor $\mathcal{O}_{S^1}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ sending S^1/H to the fiber $\mathcal{C}_H := \mathcal{C}_{S^1/H}$. We can then think of $\mathcal{C}$ as consisting of an underlying category $\mathcal{C}_e$ equipped with an S^1-action and of each of the fibers $\mathcal{C}_H$ as specifying an ∞-category of "H-genuine objects in $\mathcal{C}$". We will then define the notion of a **genuine S^1-object** in $\mathcal{C}$ to be a section $\mathcal{O}_{S^1}^{\mathrm{op}} \longrightarrow \mathcal{C}$ of π.

Let us now discuss some examples of interest. For this let us note that the data of a functor $\mathcal{O}_{S^1}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ can informally be described as associating to each finite subgroup $H \subseteq S^1$ an ∞-category $\mathcal{C}_H$ equipped with an action of S^1/H and for each inclusion $H \subseteq H'$ an S^1/H-equivariant functor $\mathcal{C}_{H'} \longrightarrow \mathcal{C}_H$. Now for a finite subgroup $H \subseteq S^1$ let us denote by $\mathbb{Z}_H = p^{-1}(H) \subseteq \mathbb{R}$ the preimage of H under the universal covering $p : \mathbb{R} \longrightarrow S^1$. We note that $\mathbb{Z}_H$ is a group isomorphic to $\mathbb{Z}$. We may then identify $\mathbf{B}\mathbb{Z}_H \simeq \mathbb{R}/\mathbb{Z}_H \simeq S^1/H$. We will be interested in the following examples of S^1-categories:

(1) For $H \subseteq S^1$ let Sp^H denote the ∞-category of genuine H-spectra. Then Sp^H carries a canonical action of $\mathbf{B}(H)$ which we can inflate to a $\mathbf{B}\mathbb{Z}_H$-action via the natural quotient map $\mathbb{Z}_H \longrightarrow H$. For every subgroup inclusion $H \subseteq H'$ the forgetful functor $\mathrm{Sp}^{H'} \longrightarrow \mathrm{Sp}^H$ is then $\mathbf{B}\mathbb{Z}_H$-equivariant and so the association $H \mapsto \mathrm{Sp}^H$ determines an S^1-category
$$\mathrm{Sp}^{\mathrm{gen}} \longrightarrow \mathcal{O}_{S^1}^{\mathrm{op}}$$
whose underlying S^1-equivariant ∞-category is Sp with trivial S^1-action.

(2) For $H \subseteq S^1$ let $\Lambda^H_\infty \subseteq \mathbb{Z}_H - \mathrm{PoSet}$ be the full subcategory spanned by the $\mathbb{Z}_H$-posets of the form $\mathbb{Z}_{H'}$ for $H' \supseteq H$. We note that each Λ^H_∞ is equivalent to Λ_∞ and carries a natural $\mathbf{B}\mathbb{Z}_H$-action, and that for each subgroup inclusion

$H \subseteq H'$ we have a $\mathbf{B}\mathbb{Z}_H$-equivariant forgetful functor $\Lambda_\infty^{H'} \longrightarrow \Lambda_\infty^H$ obtained by restricting the action from $\mathbb{Z}_{H'}$ to $\mathbb{Z}_H$. The association $H \mapsto \Lambda_\infty^H$ then determined an S^1-category

$$\Lambda_\infty^{\mathrm{gen}} \longrightarrow \mathcal{O}_{S^1}^{\mathrm{op}}$$

whose underlying S^1-equivariant category is Λ_∞. We note that the fiberwise dualities $D^H : (\Lambda_\infty^H)^{\mathrm{op}} \xrightarrow{\simeq} \Lambda_\infty^H$ don't respect the functorial dependence on H, but instead intertwine it with the functorial dependence obtained by pulling back along the automorphism $[-1] : \mathcal{O}_{S^1}^{\mathrm{op}} \longrightarrow \mathcal{O}_{S^1}^{\mathrm{op}}$ which is the identity on objects and acts by conjugation by $[-1] : S^1/H \longrightarrow S^1/H$ on all mapping spaces. Hence if we let $\Lambda_\infty^{\mathrm{op-gen}} \longrightarrow \mathcal{O}_{S^1}^{\mathrm{op}}$ be the coCartesian fibration classifying the functor

$$\mathcal{O}_{S^1}^{\mathrm{op}} \xrightarrow{[-1]} \mathcal{O}_{S^1}^{\mathrm{op}} \xrightarrow{(\Lambda_\infty^{(-)})^{\mathrm{op}}} \mathrm{Cat}_\infty$$

then the fiberwise dualities D^H assemble to a functor $D^{\mathrm{gen}} : \Lambda_\infty^{\mathrm{op-gen}} \longrightarrow \Lambda_\infty^{\mathrm{gen}}$ of S^1-categories.

(3) For $H \subseteq S^1$ let $\mathrm{Fin}_{S^1/H}$ denote the category of free $\mathbb{Z}_H$-sets with finitely many orbits (which is equivalent to the comma category of finite sets equipped with a map to S^1/H). Then $\mathrm{Fin}_{S^1/H}$ carries a canonical action of $\mathbf{B}\mathbb{Z}_H$ and for each subgroup inclusion $H \subseteq H'$ we have the $\mathbf{B}\mathbb{Z}_H$-equivariant forgetful functor $\mathrm{Fin}_{S^1/H'} \longrightarrow \mathrm{Fin}_{S^1/H}$. The association $H \mapsto \mathrm{Fin}_{S^1/H}$ then determines an S^1-category

$$\mathrm{Fin}_{S^1}^{\mathrm{gen}} \longrightarrow \mathcal{O}_{S^1}^{\mathrm{op}}$$

whose underlying S^1-equivariant category is Fin_{S^1}.

(4) For $H \subseteq S^1$ let $\mathrm{Ass}_{S^1/H} := \mathrm{Fin}_{S^1/H} \times_{\mathrm{Fin}} \mathrm{Ass}_{\mathrm{act}}^{\otimes}$, where the map $\mathrm{Fin}_{S^1/H} \longrightarrow \mathrm{Fin}$ is the quotient by $\mathbb{Z}_H$ (alternatively, it's the functor that forgets the map to S^1/H). Then $\mathrm{Ass}_{S^1/H}$ carries a canonical action of $\mathbf{B}\mathbb{Z}_H$ and the association $H \mapsto \mathrm{Ass}_{S^1/H}$ then determines an S^1-category

$$\mathrm{Ass}_{S^1}^{\mathrm{gen}} \longrightarrow \mathcal{O}_{S^1}^{\mathrm{op}}$$

whose underlying S^1-equivariant category is Ass_{S^1}.

Let us now recall that an **S^1-symmetric monoidal structure** on an S^1-category $\mathcal{C} \longrightarrow \mathcal{O}_{S^1}^{\mathrm{op}}$ is an extension of the functor $S^1/H \mapsto \mathcal{C}_H$ to a product-preserving functor $A^{\mathrm{aff}}(S^1) \longrightarrow \mathrm{Cat}_\infty$, where $A^{\mathrm{aff}}(S^1)$ is the effective Burnside category of S^1 given by the span ∞-category of the coproduct completion of $\mathcal{O}_{S^1}$. Informally speaking, we may describe an S^1-symmetric monoidal structure on $\mathcal{C}$ as follows: for every subgroup inclusion $H \subseteq H'$, every finite free H'-set I and every H'/H-equivariant map $X : I/H \longrightarrow \mathcal{C}_H$ we have associated a new object $\otimes_{I/H} X_i \in \mathcal{C}_{H'}$, subject to a variety of compatibility constraints.

Out of the four examples above, the first, third and forth carry S^1-symmetric monoidal structures. In terms of the informal description above involving a finite free H'-set and an H'/H-equivariant map $I \longrightarrow \mathcal{C}_H$, these are given by a suitable form of the norm construction in the case of $\mathrm{Sp}^{\mathrm{gen}}$, and by the formula

$$\otimes_{I/H} X_i := \coprod_{i \in I/H} X_i \in \mathrm{Fin}_{S^1/H'}$$

in the case of $\mathrm{Fin}_{S^1}^{\mathrm{gen}}$, where the right hand side is endowed with a suitable $\mathbb{Z}_{H'}$-action. The latter also induces an S^1-symmetric monoidal structure on $\mathrm{Ass}_{S^1}^{\mathrm{gen}}$.

Furthermore, one can show that $\mathrm{Fin}^{\mathrm{gen}}_{S^1}$ and $\mathrm{Ass}^{\mathrm{gen}}_{S^1}$ enjoy certain universal properties. More precisely, $\mathrm{Fin}^{\mathrm{gen}}_{S^1}$ is the free S^1-symmetric monoidal ∞-category containing a commutative algebra object lying over S^1/e, while $\mathrm{Ass}^{\mathrm{gen}}_{S^1}$ is the free S^1-symmetric monoidal ∞-category containing an associative algebra object lying above S^1/e. We note that the underlying symmetric monoidal ∞-category Ass_{S^1} is also known in the context of factorization homology as the symmetric monoidal ∞-category of S^1-framed 1-disks.

By the above universal properties it follows that if $A \in \mathrm{Sp} = \mathrm{Sp}^{\mathrm{gen}}_e$ is an associative algebra object then A determines an essentially unique S^1-symmetric monoidal functor

$$A : \mathrm{Ass}^{\mathrm{gen}}_{S^1} \longrightarrow \mathrm{Sp}^{\mathrm{gen}} .$$

On the other hand, there is a natural S^1-functor

$$V : \Lambda^{\mathrm{gen}}_{\infty} \longrightarrow \mathrm{Ass}^{\mathrm{gen}}_{S^1}$$

induced by the functor $V_H(\mathbb{Z}_{H'}) = (\mathbb{Z}_{H'}, \mathbb{Z}_{H'}/\mathbb{Z}_H) \in \mathrm{Fin}_{S^1/H} \times_{\mathrm{Fin}} \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ as in (3). Similarly we have a map $V^D : \Lambda^{\mathrm{op-gen,op}}_{\infty} \longrightarrow \mathrm{Ass}^{\mathrm{gen}}_{S^1}$ obtained by composing with the duality $D^{\mathrm{gen}} : \Lambda^{\mathrm{op-gen}}_{\infty} \xrightarrow{\simeq} \Lambda^{\mathrm{gen}}_{\infty}$. Since $\mathrm{Sp}^{\mathrm{gen}} \longrightarrow \mathcal{O}^{\mathrm{op}}_{S^1}$ has relative colimits then we can define

$$\mathrm{THH}^{\mathrm{gen}}(A) : \mathcal{O}^{\mathrm{op}}_{S^1} \longrightarrow \mathrm{Sp}^{\mathrm{gen}}$$

to be section obtained by relatively left Kan extending V^D along $\Lambda^{\mathrm{op-gen,op}}_{\infty} \longrightarrow \mathcal{O}^{\mathrm{op}}_{S^1}$. The section $\mathrm{THH}^{\mathrm{gen}}(A)$ is then by definition a genuine S^1-spectrum whose underlying S^1-equivariant spectrum is $\mathrm{THH}(A)$.

6. Comparison with the point-set model construction of [ABGHLM]

Recall that the symmetric monoidal ∞-category Sp of spectra can be presented by the symmetric monoidal topological model category Sp^{O} of **orthogonal spectra**. Here, we will identify the underlying category of Sp^{O} with the category of Top_*-enriched functors $\mathrm{O} \longrightarrow \mathrm{Top}_*$ where

(1) Top_* is the category of pointed compactly generated weak Hausdorff spaces endowed with the (pointed) Serre model structure in which cofibrations are retracts of pointed relative cell complexes, fibrations are Serre fibrations and weak equivalences are the weak homotopy equivalences.
(2) O is the Top_*-enriched category whose objects are the finite dimensional inner product spaces V and whose mapping spaces are given by $\mathrm{Map}_{\mathrm{O}}(V, W) = \mathrm{Emb}(V, U)^{\tau}$ where $\mathrm{Emb}(V, U)$ is the space of isometric emebddings $V \hookrightarrow U$, $\tau : E \longrightarrow \mathrm{Emb}(V, U)$ is the vector bundle whose fiber over $\iota : V \hookrightarrow U$ is the orthogonal complement of $\iota(V)$ in U, and $(-)^{\tau}$ denotes the corresponding Thom space.

The category Sp^{O} can then be endowed with the **stable model structure** in which the cofibrations are the projective cofibrations in $\mathrm{Fun}(\mathrm{O}, \mathrm{Top}_*)$ and whose weak equivalences are the stable equivalences. The model category Sp^{O} is then symmetric monoidal (with cofibrant unit) and tensored over Top_*. Using the left Quillen functor $(-)_+ : \mathrm{Top} \longrightarrow \mathrm{Top}_*$ (where Top is endowed with the unpointed Serre model structure) we may also consider Sp^{O} as tensored over Top.

Given an orthogonal ring spectrum A, i.e., an associative algebra object in Sp^{O}, we may define its cyclic bar construction $\mathrm{Bar}(A) : \Lambda^{\mathrm{op}} \longrightarrow \mathrm{Sp}^{\mathrm{O}}$ as above using the point-set smash product. To make sure that this has the right type we will only consider the case where the underlying object of A is cofibrant. In fact, given that the unit S^0 of Sp^{O} is cofibrant it will be convenient to require instead the slightly stronger condition that the unit map $S^0 \longrightarrow A$ is a cofibration. We note that this holds, for example, for every cofibrant algebra with respect to the transferred model structure on associative algebras.

As discussed above, we know that the homotopy colimit $\operatorname{hocolim} \mathrm{Bar}(A)|_{\Lambda_\infty^{\mathrm{op}}}$ should carry a circle action on the level of the underlying ∞-category of spectra. Given that Sp^{O} is tensored over Top (i.e., it is a **topological** model category), it makes since to talk about objects in Sp^{O} equipped with a point-set action of the topological group S^1 itself. We would hence like to realize the homotopy colimit $\operatorname{hocolim} \mathrm{Bar}(A)|_{\Lambda_\infty^{\mathrm{op}}}$ using an explicit point set model which carries a natural circle action. For this we may consider the composed functor

$$Q : \Lambda \longrightarrow |\Lambda| \simeq \mathbf{B}S^1 \subseteq \mathcal{S}^{\mathbf{B}S^1}$$

where the last inclusion is the Yoneda embedding. We then observe that if $x \in \Lambda$ is an object and $R_x := \mathrm{Hom}(-, x) : \Lambda^{\mathrm{op}} \longrightarrow \mathrm{Set}$ its associated representable functor then $\operatorname{colim} R_x|_{\Lambda_\infty^{\mathrm{op}}}$ is naturally equivalent to $Q_\infty(x)$ as an S^1-space. As a result, if $\mathcal{C}$ is an ∞-category with colimits $\mathcal{F} : \Lambda^{\mathrm{op}} \longrightarrow \mathcal{C}$ is a functor, then the S^1-object $\operatorname{colim}(\mathcal{F}|_{\Lambda_\infty^{\mathrm{op}}})$ is naturally equivalent to the coend $\int_\Lambda \mathcal{F} \otimes Q$. If $\mathcal{M}$ is now a topological model category, then to get an explicit point set model for the circle action on homotopy colimits of the form $\operatorname{hocolim} \mathcal{F}|_{\Lambda^{\mathrm{op}}}$ for functors $\Lambda^{\mathrm{op}} \longrightarrow \mathcal{M}$ it will suffice to fix a point-set model $Q : \Lambda \longrightarrow \mathrm{Top}^{S^1}$ for the above ∞-functor Q. For such an $\mathcal{M}$ there is an associated Quillen bifunctor

$$\mathcal{M} \times \mathrm{Top}^{S^1} \longrightarrow \mathcal{M}^{S^1}$$

where Top^{S^1} and $\mathcal{M}^{S^1}$ are equipped with the respective projective model structures. Given a projectively cofibrant functor $\mathcal{F} : \Lambda^{\mathrm{op}} \longrightarrow \mathcal{M}$, the coend $\|\mathcal{F}\|_Q := \int_\Lambda \mathcal{F} \otimes Q \in \mathcal{M}^{S^1}$ is then a model for $\operatorname{hocolim} \mathcal{F}|_{\Lambda_\infty^{\mathrm{op}}}$ equipped with a point-set S^1-action. There is, in fact, a natural choice for such a Q, which furthermore posses several good formal properties. In fact, this choice can be done in a compatible way for all the Λ_q's, including $q = \infty$. More precisely, let $Q_\infty : \Lambda_\infty \longrightarrow \mathrm{Top}^{\mathbb{R}}$ be the composition of the duality $D : \Lambda_\infty \xrightarrow{\cong} \Lambda_\infty^{\mathrm{op}}$ and the functor $\Lambda_\infty^{\mathrm{op}} \longrightarrow \mathrm{Top}^{\mathbb{R}}$ which sends the $\mathbb{Z}$-poset $\frac{1}{n}\mathbb{Z}$ to the space of ordering preserving $\mathbb{Z}$-equivariant maps $\mathrm{Map}(\frac{1}{n}\mathbb{Z}, \mathbb{R})$ (topologized in the natural way). For every positive integer q the composed functor $\Lambda_\infty \longrightarrow \mathrm{Top}^{\mathbb{R}} \xrightarrow{(-)_{q\mathbb{Z}}} \mathrm{Top}^{\mathbb{R}/q\mathbb{Z}}$ factors uniquely through a functor $Q_q : \Lambda_q \longrightarrow \mathrm{Top}^{\mathbb{R}/q\mathbb{Z}}$. For $q \leq \infty$ and a functor $\mathcal{F} : \Lambda_q^{\mathrm{op}} \longrightarrow \mathcal{M}$ we then set

$$\|\mathcal{F}\|_q := \int_{\Lambda_q} \mathcal{F} \otimes Q_q \in \mathcal{M}^{\mathbb{R}/q\mathbb{Z}}. \tag{11}$$

This definition enjoys the following compatibility properties:

Lemma 14.

(1) *For $q | q' \leq \infty$ and $\mathcal{F} : \Lambda_q^{\mathrm{op}} \longrightarrow \mathcal{M}$ there is a natural $\mathbb{R}/q'\mathbb{Z}$-equivariant isomorphism*

$$\|\mathcal{F}|_{\Lambda_q'}\|_{q'} \cong \mathrm{res}^{\mathbb{R}/q\mathbb{Z}}_{\mathbb{R}/q'\mathbb{Z}} \|\mathcal{F}\|_q.$$

(2) For $q|q' < \infty$ and $\mathcal{F} : \Lambda_q^{\mathrm{op}} \longrightarrow \mathcal{M}$ there is a natural $\mathbb{R}/q'\mathbb{Z}$-equivariant isomorphism

$$\|(\mathrm{sd}^q_{q'})^*\mathcal{F}\|_{q'} \cong \|\mathcal{F}\|_q.$$

where the $\mathbb{R}/q'\mathbb{Z}$-action on the right hand side is obtained via the multiplication by $\frac{q'}{q}$ isomorphism $\mathbb{R}/q'\mathbb{Z} \cong \mathbb{R}/q\mathbb{Z}$.

(3) For $\mathcal{F} : \Lambda_\infty^{\mathrm{op}} \longrightarrow \mathcal{M}$ there is a natural isomorphism in $\mathcal{M}$

$$\|\mathcal{F}|_{\Delta^{\mathrm{op}}}\| \cong \|\mathcal{F}\|_\infty$$

after forgetting the $\mathbb{R}$-action on the right hand side, where $\| - \|$ denotes the standard geometric realization of simplicial objects.

Remark 15. Lemma 6 implies that if $\mathcal{F} : \Lambda_\infty^{\mathrm{op}} \longrightarrow \mathcal{M}$ is a functor then $\|\mathcal{F}\|_\infty$ is a model for the homotopy colimit of $\mathcal{F}$ as soon as $\mathcal{F}|_{\Delta^{\mathrm{op}}}$ is Reedy cofibrant, i.e., as soon as the latching maps are cofibrations in $\mathcal{M}$.

Let us now assume that $\mathcal{M}$ is a topological model category which carries a compatible symmetric monoidal structure (with cofibrant unit). We will say that an associative algebra object A in $\mathcal{M}$ is **weakly cofibrant** if the unit map $1_\mathcal{M} \longrightarrow A$ is a cofibration. Given a weakly cofibrant associative algebra object $A \in \mathcal{M}$ with a $\mathbb{Z}/q\mathbb{Z}$-action we now define

$$\mathrm{THH}_q(A) := \|\,\mathrm{Bar}_q(A)\|_q \in \mathcal{M}^{\mathbb{R}/q\mathbb{Z}}.$$

Remark 16. Let A_∞ denote the image of A in the underlying ∞-category $\mathcal{M}_\infty$. Then there is a natural comparison map

$$\mathrm{THH}_q(A_\infty) \longrightarrow \mathrm{THH}_q(A) \tag{12}$$

which comes from comparing the coend operation 11 with its derived counterpart. By Lemma 14(3) the map (12) is an equivalence whenever the simplicial object $\mathrm{Bar}(A)|_{\Delta^{\mathrm{op}}}$ is Reedy cofibrant. This holds in our setting for every weakly cofibrant algebra object.

We have the following analogue of Lemma 17:

Lemma 17. *Let $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ be a lax symmetric monoidal topological functor between topological symmetric monoidal model categories. Then there is a canonical natural transformation*

$$\tau_F : \mathrm{THH}_q(\mathcal{F}(-)) \longrightarrow \mathcal{F}(\mathrm{THH}_q(-))$$

of functors $\mathrm{Alg}_{\mathrm{Ass}_q}(\mathcal{M}) \longrightarrow \mathcal{N}^{\mathbb{R}/q\mathbb{Z}}$. Furthermore, if $\mathcal{F}$ is symmetric monoidal and preserves geometric realizations then τ_F is an isomorphism.

Let us now return to the case where $\mathcal{M} = \mathrm{Sp}^{\mathrm{O}}$. We wish to define point-set models for the maps ψ_p of (10) using the above point-set model for THH(A). For this we will follow the approach of [ABGHLM]. We note that the first construction of the cyclotomic structure on THH is due to Bökstedt-Hsiang-Madsen ([BHM]), and it is the latter construction to which Nikolaus-Scholze compare their construction. The construction of [BHM] and the construction of [ABGHLM] are in turn shown to be equivalent in [DMPSW].

Given a finite group G let $\mathrm{Sp}^G := \mathrm{Fun}(\mathbf{B}G, \mathrm{Sp}^{\mathrm{O}}) = \mathrm{Fun}(\mathrm{O}, \mathrm{Top}_*^G)$ denote the category of G-objects in Sp^{O}. It can then be shown that Sp^G is equivalent (as an ordinary category) to the category of Top_*^G-enriched functors $\mathrm{Rep}(G) \longrightarrow \mathrm{Top}_*^G$

where $\mathrm{Rep}(G)$ is the Top_*^G-category whose objects are the finite dimensional orthogonal representations V of G and whose mapping spaces are given by the mapping spaces in O between the underlying quadratic spaces equipped with the corresponding conjugation G-action. In particular, the functor $I_G(X) : \mathrm{Rep}(G) \longrightarrow \mathrm{Top}_*^G$ associated to an orthogonal spectrum X by the above equivalence is given by $I_G(X)(V) = \mathrm{Iso}(\mathbb{R}^n, V) \times^{\mathrm{O}_n} X_n$ equipped with the diagonal G-action.

Using the identification $\mathrm{Sp}^G \cong \mathrm{Fun}(\mathrm{Rep}(G), \mathrm{Top}_*^G)$ we may endow Sp^G with a model structure presenting the ∞-category of genuine G-spectra. Recall first that the category Top_*^G of pointed G-spaces can be equipped with a model structure in which a map $f : X \longrightarrow Y$ is a fibration (resp. weak equivalence) if and only if for every subgroup $H \subseteq G$ the induced map $f^H : X^H \longrightarrow Y^H$ is a Serre fibration (resp. weak homotopy equivalence). This model category is cofibrantly generated with generating cofibrations of the form $(G/H)_+ \wedge S^n_+ \hookrightarrow (G/H)_+ \wedge D^n_+$ and constitutes a model categorical presentation of the ∞-category of genuine G-spaces. We may then consider the projective model structure on the enriched functor category $\mathrm{Fun}(\mathrm{Rep}(G), \mathrm{Top}_*^G)$ in which the cofibrations are generated by maps of the form $S^{-V} \wedge G/H_+ \wedge S^n_+ \hookrightarrow S^{-V} \wedge G/H_+ \wedge D^n_+$, where V is a G-representation and $S^{-V} : \mathrm{Rep}(G) \longrightarrow \mathrm{Top}_*^G$ is the functor corepresented by V, and then localize it so that the new fibrant objects are the orthogonal G-Ω-spectra (i.e., those for which for each representation embedding $V \hookrightarrow U$ with complement $V^\perp$ the induced map $X(V) \longrightarrow \mathrm{Map}_*(S^{V^\perp}, X(U))$ is a weak equivalence in Top_*^G). This yields a model categorical presentation of the ∞-category of genuine G-spectra.

Given a normal subgroup $H \triangleleft G$ we will denote by $\mathrm{Rep}(G)^H$ the Top_*^H-enriched category obtained by applying the fixed point functor $(-)^H$ to each mapping space in $\mathrm{Rep}(G)$. In particular, given an orthogonal spectrum we obtain a functor $I_G^H : \mathrm{Rep}(G)^H \longrightarrow \mathrm{Top}_*^H$ given by $V \mapsto (I_G(X)(V))^H = (\mathrm{Iso}(\mathbb{R}^n, V) \times^{\mathrm{O}_n} X_n)^H$. We now define the geomertric fixed point functor

$$\Phi^H : \mathrm{Sp}^G \longrightarrow \mathrm{Sp}^{G/H}$$

by the formula

$$\Phi^G(X) = \mathrm{Lan}(I_G^H(X))$$

where $\mathrm{Lan} : \mathrm{Fun}(\mathrm{Rep}^H(G), \mathrm{Top}_*^H) \longrightarrow \mathrm{Fun}(\mathrm{Rep}(G/H), \mathrm{Top}_*^H) \cong \mathrm{Sp}^H$ is the left Kan extension along the map $\mathrm{Rep}^H(G) \longrightarrow \mathrm{Rep}(G/H)$ given by $V \mapsto V^H$. We note that Φ^H is a point-set model for the geometric fixed points functor discussed in §4. Then functor Φ^H enjoys the following properties (see [HHR, Proposition 2.53], [ABGHLM, Theorem 2.31, Lemma 4.5]):

(1) Φ^H preserves cofibrations and trivial cofibrations, and so in particular weak equivalences between cofibrant objects.
(2) Φ^H carries a lax monoidal structure

$$\rho_{X,Y} : \Phi^H(X) \wedge \Phi^H(Y) \longrightarrow \Phi^H(X \wedge Y)$$

and $\rho_{X,Y}$ is an isomorphism whenever X, Y are cofibrant.
(3) For a G-representation V and a pointed G-space K there is a canonical isomorphism

$$\Phi^H(S^{-V} \wedge K) \cong S^{-V^H} \wedge K^H.$$

(4) Φ^H sends pushout squares in Sp^G

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & Z \end{tikzcd}$$

in which f is a cofibration to a pushout square in $\mathrm{Sp}^{G/H}$. In particular Φ^H preserves geometric realizations of Reedy cofibrant objects.

(5) Let $Z(G) \subseteq G$ be the center of G and $Z(G/H)$ the center of G/H. Then Φ^H intertwines the $\mathbf{B}Z(G)$-action on Sp^G with the $\mathbf{B}Z(G/H)$-action on $\mathrm{Sp}^{G/H}$ via the natural map $\mathbf{B}Z(G) \longrightarrow \mathbf{B}Z(G/H)$. In particular, if X is an orthogonal G-spectrum $z \in Z(G)$ is an element and $z_* : X \longrightarrow X$ is the associated G-equivariant map then the induced map $\Phi^G(z_*) : \Phi^G(X) \longrightarrow \Phi^G(X)$ is the identity.

In order to construct a point-set model for ψ_p we will also need to make use of Hill-Hopkins-Ravanel norm functor, whose ∞-categorical version was described in §4. In the model of orthogonal spectra the norm functor is given explicitly as as

$$\mathrm{N}^{C_q} : \mathrm{Sp} \longrightarrow \mathrm{Sp}^{C_q} \qquad X \mapsto \bigwedge_{i \in C_q} X$$

and enjoys the following properties:

(1) N^{C_q} preserves cofibrations and trivial cofibrations, and so in particular weak equivalences between cofibrant objects.
(2) N^{C_q} is symmetric monoidal.
(3) There is a natural symmetric monoidal transformation

$$\sigma_q : X \longrightarrow \Phi^{C_q}\mathrm{N}^{C_q}(X)$$

which is an isomorphism when X is cofibrant.

Let now A be a weakly cofibrant orthogonal ring spectrum. Then for every prime p the norm $\mathrm{N}^p(A)$ is a weakly cofibrant ring spectrum and $\Phi^{C_p}(\mathrm{N}^p(A))$ is a weakly cofibrant ring spectrum (isomorphic to A by property (3) above). Denoting by $U : \mathrm{Sp}^{C_p} \longrightarrow \mathrm{Sp}^{\mathrm{O}}$ the forgetful functor we may now apply Lemma 17 and Lemma 14 we get a composed natural $\mathbb{R}/\mathbb{Z}$-equivariant equivalence

$$\psi_p : U(\mathrm{THH}_p(\mathrm{N}^{C_p}(A))) \simeq \mathrm{THH}(A) \xrightarrow{\simeq} \mathrm{THH}_p(\mathrm{Inf}_p A) \xrightarrow[(\sigma_p)_*]{\simeq}$$

$$\mathrm{THH}_p(\Phi^{C_p}\mathrm{N}^{C_p}(A)) \xrightarrow{\simeq} \Phi^{C_p}\,\mathrm{THH}_p(\mathrm{N}^{C_p}(A))$$

where the last map is an isomorphism when A is weakly cofibrant since Φ^G is symmetric monoidal on cofibrant objects and commutes with geometric realizations of Reedy cofibrant objects. To show that ψ_p is a point-set model for the map ψ_p defined in §4 it will suffice to check that $\mathrm{N}^{C_p}, \Phi^{C_p}$ and σ_p are point-set models for their ∞-categorical counterparts. This is essentially true by definition.

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