Differential Geometry - Tirgul 1

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In this tirgul we will develop the classical proof of Gauss's Theorema Egregium, which states that the curvature of a surface

$$f : M \hookrightarrow \mathbb{R}^3$$

is determined by the metric g induced from f. This means that the curvature can be determined using only measurements inside the surface.

A common poetic description of this idea is to imagine a civilization living on a two dimensional surface, unable to sense the third dimension. For example, these people have no concept of the normal vector to the surface, which is the data we used in order to define the curvature. It is thus surprising that they can measure the curvature nonetheless.

In contrast, consider the analogous situation for curves in $\mathbb{R}^3$. A civilization living on a 1-dimensional curve and (could only measure distances inside the curve) would have no way to measure the curvature. The concept becomes meaningless without the embedding.

If t he curve is a closed curve then the only meaningful number that they can measure is the total length of their world. This statement can be made precise further on in the course - two riemannian structures on S^1 are isometric if and only if the total length is equal.

Now recall that the curvature can be expressed using the Weingarten map $L : T_pM \longrightarrow T_p$ which is defined by

$$L(X) = -X(n)$$

where $n(p)$ is a continuous choice of normal vector and $-X(n)$ means the derivative of n in the direction given by $X \in T_pM$. The curvature $K(p)$ at p can then be expressed as

$$K(p) = \det(L)$$

We want to show directly from this expression that $K(p)$ depends only on g. Choose local coordinates u^1, u^2 in some neighborhood of $p \in U \subseteq M$ and write g in these coordinates:

$$g = \sum_{i,j} g_{i,j} du^1 dv^2$$

Now the vectors

$$X_1 = \frac{\partial}{\partial u^1}(p), X_2 = \frac{\partial}{\partial u^1}(p)$$

form a basis for T_pM and by definition

$$g(X_i, X_j) = g_{i,j}(p)$$

Let us assume for simplicity that u^1, u^2 are chosen in such a way that X_1, X_2 form an orthonormal basis at p (if they are not we can do a linear change of coordinates to make them so). Note that $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ might not be orthonormal at other points in U.

Now since X_1, X_2 are orthonormal the matrix representing T in the basis X_1, X_2 is

$$\begin{pmatrix} g(L(X_1), X_1) & g(L(X_2), X_1) \\ g(L(X_1), X_2) & g(L(X_2), X_2) \end{pmatrix}$$

and

$$K(p) = g(L(X_1), X_1)g(L(X_2), X_2) - g(L(X_1), X_2)g(L(X_2), X_1)$$

The form $X, Y \longrightarrow g(L(X), Y)$ is known as the second fundamental form. It is sometimes custom to denote

$$l = g\left(L\left(\frac{\partial}{\partial u^1}\right), \frac{\partial}{\partial u^1}\right)$$

$$m = g\left(L\left(\frac{\partial}{\partial u^1}\right), \frac{\partial}{\partial u^2}\right)$$

$$n = g\left(L\left(\frac{\partial}{\partial u^2}\right), \frac{\partial}{\partial u^2}\right)$$

which are called the coefficients of the second fundamental form in the coordinates u^1, u^2.

Now since $g(n, X_i) = 0$ is constant it follows from Leibnitz formula that

$$g(L(X_i), X_j) = \left\langle -\frac{\partial n}{\partial u^i}, X_j \right\rangle = \left\langle n, \frac{\partial X_j}{\partial u^i} \right\rangle \overset{def}{=} \langle n, X_{i,j} \rangle$$

where $<,>$ is the standard scalar product in $\mathbb{R}^3$. This is also called the second fundamental form (evaluated on the vectors X_1, X_2). Note that $X_{i,j} = X_{j,i}$ so we can write

$$K(p) =< n, X_{1,1} >< n, X_{2,2} > - < n, X_{1,2} >^2$$

Now since X_1, X_2, n are an orthonormal basis for $\mathbb{R}^3$ it follows that

$$X_{i,j} =< X_{i,j}, X_i > X_i + < X_{i,j}, X_j > X_j + < X_{i,j}, n > n$$

Let us denote

$$a_{i,j,k} =< X_{i,j}, X_k >$$

Note that from Leibnitz formula

$$X_i(g_{j,k}) = X_i(< X_j, X_k >) = 2a_{i,j,k}$$

so all the terms of the form $a_{i,j,k}$ are determined by the metric coefficients.

Substituting in the expression above one gets:

$$K(p) = \langle X_{1,1} - a_{1,1,1}X_1 - a_{1,1,2}X_2,\ X_{2,2} - a_{2,2,1}X_1 - a_{2,2,2}X_2\rangle -$$

$$|X_{1,2} - a_{1,2,1}X_1 - a_{1,2,2}X_2|^2 =$$

$< X_{1,1}, X_{2,2} > - < X_{1,2}, X_{1,2} > +$ terms depending on $a_{i,j,k}$

Hence it remains to show that $< X_{1,1}, X_{2,2} > - < X_{1,2}, X_{1,2} >$ depends only on the metric coefficients. For that we use the Leibnitz formula one more time:

$$X_2(a_{1,1,2}) - X_1(a_{1,2,2}) = \langle X_2(X_{1,1}), X_2\rangle - \langle X_{1,1}, X_{2,2}\rangle -$$

$$\langle X_1(X_{1,2}), X_2\rangle + \langle X_{1,2}, X_{1,2}\rangle$$

and since partial derivatives commute in $\mathbb{R}^3$ we see that $X_1(X_{1,2}) = X_2(X_{1,1})$ so

$$X_2(a_{1,1,2}) - X_1(a_{1,2,2}) =< X_{1,1}, X_{2,2} > - < X_{1,2}, X_{1,2} >$$

and so $K(p)$ depends only on the coefficients of the metric (and their derivatives). This means that the metric is indeed intrinsic, as god intended.

Differential Geometry - Tirgul 2

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1 The Real Projective Space

The real projective space is an n-dimensional manifold which is a good example because it is constructed without using any concrete embedding in some ambient $\mathbb{R}^N$.

In order to define a smooth manifold (notation: smooth = differentiable in Do Carmo's book) we need to specify the set of points and an atlas $\{U_\alpha, x_\alpha\}$. For the real projective space $\mathbb{R}P^n$ the set of points is the set of lines in $\mathbb{R}^{n+1}$ which contain the origin, i.e. a point $L \in \mathbb{R}P^n$ is a subset of $\mathbb{R}^{n+1}$ of the form

$$L = \{\lambda v | \lambda \in \mathbb{R}\}$$

for some $0 \neq v \in \mathbb{R}^{n+1}$. Note that there exists a natural map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}P^n$ which sends $0 \neq v \in \mathbb{R}^{n+1}$ to the line $\{\lambda v | \lambda \in \mathbb{R}\}$. This map is surjective but not injective: for each $0 \neq v \in \mathbb{R}^{n+1}$ and $0 \neq \lambda \in \mathbb{R}$ we have $\pi(\lambda v) = \pi(v)$.

We will now put a smooth structure on $\mathbb{R}P^n$ in which the map π will be smooth. Let $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ be the map

$$f_i(x_1, ..., x_n) = (x_1, ..., x_i, 1, x_{i+1}, ..., x_n)$$

Then we can compose π and get a map $\phi_i = \pi \circ f_i : \mathbb{R}^n \longrightarrow \mathbb{R}P^n$. We claim that the set $\{(\mathbb{R}^n, \phi)\}$ is a smooth atlas (but not maximal). First of all we need to show that the ϕ_i's are injective. Suppose that $(x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{R}^n$ satisfy

$$\phi_i(x_1, ..., x_n) = \phi_i(y_1, ..., y_n)$$

Then the following lines in $\mathbb{R}^{n+1}$ are equal

$$\{\lambda(x_1, ..., 1, ..., x_n) | \lambda \in \mathbb{R}^{n+1}\} = \{\lambda(y_1, ..., 1, ..., y_n) | \lambda \in \mathbb{R}^{n+1}\}$$

This means that there exists a $\lambda \in \mathbb{R}^{n+1}$ such that

$$(y_1, ..., 1, ..., y_n) = \lambda(x_1, ..., 1, ..., x_n)$$

by considering the i'th coordinate we see that λ must be 1, which means that

$$(x_1, ..., x_n) = (y_1, ..., y_n)$$

Now we want to show that

$$\cap_i \phi_i(\mathbb{R}^n) = \mathbb{R}P^n$$

Let $L \in \mathbb{R}P^n$. Then there exists a $0 \neq v \in \mathbb{R}^{n+1}$ such that $\pi(v) = L$. Write $v = (x_1, ..., x_n)$. Since $v \neq 0$ there exists an i such that $x_i \neq 0$. Then we can divide v by x_i and get a vector

$$u = \left(\frac{x_1}{x_i}, ..., 1, ..., \frac{x_n}{x_i}\right)$$

such that $\pi(u) = L$ as well. But

$$u = f_i\left(\frac{x_1}{x_i}, ..., ..., \frac{x_n}{x_i}\right)$$

which means that

$$L = \phi_i\left(\frac{x_1}{x_i}, ..., ..., \frac{x_n}{x_i}\right)$$

so

$$L \in \cap_i \phi_i(\mathbb{R}^n) = \mathbb{R}P^n$$

It is left to show that the transition maps are smooth. Let $i > j \in \{1, ..., n+1\}$ and consider

$$W_{i,j} = \phi_i(\mathbb{R}^n) \cap \phi_j(\mathbb{R}^n)$$

Note that

$$\phi_i^{-1}(W_{i,j}) = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_j \neq 0\}$$
$$\phi_j^{-1}(W_{i,j}) = \{(y_1, ..., y_n) \in \mathbb{R}^n | y_{i-1} \neq 0\}$$

are both open. It is left to compute the map

$$\phi_j^{-1} \circ \phi_i : \phi_i^{-1}(W_{i,j}) \longrightarrow \phi_j^{-1}(W_{i,j})$$

and show that it is smooth. Let $(x_1, ..., x_n) \in \phi_i^{-1}(W_{i,j})$. Then

$$\phi_j^{-1}(\phi_i(x_1, ..., x_n)) = \phi_j^{-1}(\pi(x_1, ..., x_{i-1}, 1, x_i, ..., x_n)) =$$

$$\phi_j^{-1}\left(\pi\left(\frac{x_1}{x_j}, ..., \frac{x_{j-1}}{x_j}, 1, \frac{x_{j+1}}{x_j}, ..., \frac{x_{i-1}}{x_j}, \frac{1}{x_j}\frac{x_i}{x_j}, ..., \frac{x_n}{x_j}\right)\right)$$

$$\left(\frac{x_1}{x_j}, ..., \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, ..., \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, ..., \frac{x_n}{x_j}\right)$$

Clearly $\phi_j^{-1} \circ \phi_i$ is smooth and its inverse $\phi_i^{-1} \circ \phi_j$ is smooth as well. This means that $\{(\mathbb{R}^n, \phi_i)\}$ is a smooth atlas. Now we can complete it to a maximal atlas and get a smooth structure on $\mathbb{R}P^n$.

Note that for each i, if we restrict π to $\pi^{-1}(\phi_i(\mathbb{R}^n))$ then

$$\phi_i^{-1}(\pi(x_1, ..., x_{n+1})) = \left(\frac{x_1}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right)$$

which is a smooth map. Hence π is smooth.

2 Orientability

Definition 2.1. Let M be a smooth manifold with atlas $\mathcal{A}$. An **orientable** on M is a sub-atlas $\{(U_\alpha, \phi_\alpha)\} \subseteq \mathcal{S}$ which covers M such that all transition functions within it have differentials with positive determinant. We say that M is orientable if is has an orientation.

Two orientations $\{(U_\alpha, \phi_\alpha)\}, \{(U_\beta, \phi_\beta)\}$ are called **equivalent** if their union is an orientation.

It is not hard to show that if M is orientable and connected then it has exactly two orientations up to equivalence.

Examples:

1. The sphere $S^n \subseteq \mathbb{R}^{n+1}$ has an atlas composed of two maps $\{(\mathbb{R}^n, \phi_1), (\mathbb{R}^n, \phi_2)\}$ such that the image of ϕ_1 is $S^n - \{(0, ..., 0, 1)\}$ and the image of ϕ_2 is $S^n - \{(0, ..., 0, -1)\}$.

   The intersection of these two charts is connected, and so the differential between them is either always positive or always negative. If it is always negative reverse the chart by composing a self diffeomorphism $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ of negative differential (like $(x_1, ..., x_n) \mapsto (-x_1, ..., x_n)$). Then the two charts will have a positive differential transition function so they will form an orientation. Hence S^n is orientable.

2. The real projective space $\mathbb{R}P^n$ is orientable iff n is odd. The reason for that is the following. Consider the embedding

   $$f : S^n \longrightarrow \mathbb{R}^{n+1}$$

   and compose on it the map $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}P^n$. We obtain a map $p = \pi \circ f : S^n \longrightarrow \mathbb{R}^{n+1}$ which is surjective but not injective. In fact every $L \in \mathbb{R}P^{n+1}$ has exactly two pre-images, which are the two points of norm 1 in L.

   Now consider the map $\sigma : S^n \longrightarrow S^n$ defined by $\sigma(x_1, ..., x_n) = -(x_1, ..., x_n)$. The clearly $p \circ \sigma = p$. Further more σ is a diffeomorphism so it sends an orientation $\{(U_\alpha, \phi_\alpha)\}$ to an orientation $\{(U_\alpha, \sigma \circ \phi_\alpha)\}$. Since there are exactly two orientations up two equivalence we see that σ either preserves them both or switches between them. It is an exercise to show that when n is odd then σ preserves each orientation and when n is even it switches them.

   The map p is actually very nice: for each $L \in \mathbb{R}P^{n+1}$ there exists a neighborhood $L \in V$ such that

   $$g^{-1}(V) = U \cup \sigma(U)$$

   with $U \cap \sigma(U) = \emptyset$ and such that p induces diffeomorphisms

   $$U \xrightarrow{\simeq} V, \sigma(U) \xrightarrow{\simeq} V$$

Now if σ preserves the orientation (i.e. n is odd) then we can use p to induce an orientation on $\mathbb{R}P^n$ by choosing an orientation $\{(U_\alpha, \phi_\alpha)\}$ on S^n which is fine enough so that p induces diffeomorphisms

$$\phi_\alpha(U_\alpha) \xrightarrow{\simeq} p(\phi_\alpha(U_\alpha))$$

We claim that in that case $\{(U_\alpha, p \circ \phi_\alpha)\}$ is an orientation of $\mathbb{R}P^n$. For suppose that $p(\phi_\alpha(U_\alpha)) \cap p(\phi_\beta(U_\beta)) \neq \emptyset$. If $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$ then the transition function has positive differential. If $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = \emptyset$ then $\phi_\alpha(U_\alpha) \cap \sigma(\phi_\beta(U_\beta)) \neq \emptyset$. But σ preserves orientation so the transition function is has again positive differential.

Now suppose that σ reverses orientation (i.e. n is even). Then we claim that $\mathbb{R}P^n$ is not orientable. The argument is similar but the other way around: suppose $\mathbb{R}P^n$ was orientable and choose an atlas $\{(V_\alpha, \psi_\alpha)\}$ for it which is fine enough so that

$$p^{-1}(\psi_\alpha(V_\alpha)) = W_\alpha \cup \sigma(W_\alpha)$$

such that $W_\alpha \cap \sigma(W_\alpha) = \emptyset$ and such that p induces diffeomorphisms

$$W_\alpha \xrightarrow{\simeq} \psi_\alpha(V_\alpha), \sigma(W_\alpha) \xrightarrow{\simeq} \psi_\alpha(V_\alpha)$$

Then it is a similar check to verify that $\{(V_\alpha, q \circ \psi_\alpha), (V_\alpha, \sigma \circ q \circ \psi_\alpha))\}$ would constitute an orientation for S^n where $q : V_\alpha \longrightarrow W_\alpha$ is the inverse diffeomorphism to p. But then σ would preserve this orientation and this is a contradiction.

We finish this tirgul with a theorem which says that this situation is actually quite general:

Theorem 2.2. *Let M be a smooth manifold of dimension n. Then there exists an orientable smooth manifold $\widetilde{M}$, a map $\widetilde{M} \xrightarrow{p} M$ and a self diffeomorphism $\sigma : \widetilde{M} \longrightarrow \widetilde{M}$ such that $\sigma \circ p = p$ and for each $x \in M$ there exists a neighborhood V such that*

$$p^{-1}(V) = U \cup \sigma(U)$$

with $U \cap \sigma(U) = \emptyset$ and p induces a diffeomorphism $U \xrightarrow{\simeq} V, \sigma(U) \xrightarrow{\simeq} V$.

Proof. Let V be an n-dimensional vector space over $\mathbb{R}$. We call two basis B_1, B_2 equivalent if the basis transition matrix has positive determinant. This is an equivalence relation on the set of basis of V. The equivalence classes of this relations are called **orientations** for V. It is clear that there are exactly two orientations for every V. If O is an orientation then we denote by $\overline{O}$ the other orientation.

Let us now construct $\widetilde{M}$. The points of M would be pairs (x, O) where $x \in M$ and O is an orientation for T_xM. Let $\{(U_\alpha, \phi_\alpha)\}$ be a maximal atlas for M. Since $U_\alpha \subseteq \mathbb{R}^n$ we can identify the tangent space T_vU_α with $\mathbb{R}^n$ for all

$v \in U_\alpha$. $\mathbb{R}^n$ has an orientation given by the standard basis $e_1, ..., e_n$ so we get an orientation O_α for $T_v U_\alpha$. Now consider the two maps $\widetilde{\phi}_\alpha : U_\alpha \longrightarrow \widetilde{M}$ given by

$$\widetilde{\phi}_\alpha(v) = (\phi_\alpha(v), \phi_\alpha(O_\alpha))$$

Then the set $\{(U_\alpha, \widetilde{\phi}_\alpha)\}$ is an atlas for $\widetilde{M}$. It is a simple exercise to show that this atlas is actually an orientation, so $\widetilde{M}$ is orientable.

The map p is defined by $p(x, O) = x$ and the map σ is defined by $\sigma(x, O) = (x, \overline{O})$. It is again a simple exercise to show that they satisfy the conditions in the theorem. □

Differential Geometry - Tirgul 2 Extra

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0.1 $\mathbf{SL}_n(\mathbb{R})$ and $\mathbf{O}_n(\mathbb{R})$

Let $M_n(\mathbb{R})$ denote the space of $n \times n$ matrices. This space can of course be identified with $\mathbb{R}^{n^2}$ and we can give it the standard smooth structure using the atlas $\{\mathbb{R}^{n^2} \xrightarrow{\simeq} M_n(\mathbb{R})\}$.

In class you've learned that if we have a smooth map $f : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ and if $x \in \mathbb{R}^k$ is a regular value then $M = f^{-1}(x) \subseteq \mathbb{R}^N$ admits a natural smooth structure under which the inclusion $M \hookrightarrow \mathbb{R}^N$ is an embedding.

The tangent space T_pM then embeds in $T_p\mathbb{R}^N = \mathbb{R}^N$ and can be identified with the kernel of the differential $df_p : \mathbb{R}^N \longrightarrow \mathbb{R}^k$. We will now explore the two examples $\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}(\mathbb{R}) \subseteq M_n(\mathbb{R})$.

Let subset $\mathrm{SL}_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$ is defined by the equation $\det(A) = 1$. Hence our $f = \det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ and we are taking the pre-image of $1 \in \mathbb{R}$. Let us show that 1 is a regular value for f. For that we need to compute the differential.

Let $A, B \in M_n(\mathbb{R})$ and think of B as a vector in $T_AM_n(\mathbb{R}) = M_n(\mathbb{R})$. We will compute $df_A(B)$ in the case $f(A) = 1$ and show that it is surjective. This would imply that $1 \neq x \in \mathbb{R}$ is a regular value (in fact all non-zero values are regular).

Consider the linear curve $\gamma(t) = A + Bt \in M_n(\mathbb{R})$. The tangent vector to γ at time 0 is

$$\left.\frac{d\gamma(t)}{dt}\right|_{t=0} = B$$

Hence the tangent vector to $f(\gamma(t))$ at $t = 0$ is by definition $df_A(B)$. Let us compute it:

$$\left.\frac{d}{dt}f(\gamma(t))\right|_{t=0} = \lim_{t\to 0}\left[\frac{f(\gamma(t)) - f(\gamma(0))}{t}\right] =$$

$$\lim_{t\to 0}\left[\frac{\det(A + tB) - 1}{t}\right] = \lim_{t\to 0}\left[\frac{\det(I + tA^{-1}B) - 1}{t}\right] =$$

$$\lim_{t\to 0}\left[\frac{t^n P_{A^{-1}B}(t^{-1}) - 1}{t}\right] = \mathrm{Tr}(A^{-1}B)$$

Where $P_{A^{-1}B}$ is the characteristic polynomial of $A^{-1}B$. This means that

$$df_A(B) = \mathrm{Tr}(A^{-1}B)$$

This is a linear into a 1-dimensional space so in order to show that it is surjective we just need to produce one B such that $\mathrm{Tr}(A^{-1}B) \neq 0$. But this is easy - just take $B = A$.

This proves that 1 is a regular value and so $\mathrm{SL}_n(\mathbb{R})$ admits a smooth structure making it into a smooth sub-manifold of $M_n(\mathbb{R})$ od dimension $n^2 - 1$. If $A \in \mathrm{SL}_n(\mathbb{R})$ then $T_A\mathrm{SL}_n(\mathbb{R}) \subseteq T_A M_n(\mathbb{R})$ can be identified with the kernel of df_A. In particular

$$T_A\mathrm{SL}_n(\mathbb{R}) = \{AC | \mathrm{Tr}(C) = 0\}$$

The case of $\mathrm{O}_n(\mathbb{R})$ is similar. Let $\mathrm{Sym}_n(\mathbb{R})$ denote the space of symmetric $n \times n$ matrices. This is a vector space of dimension $\binom{n+1}{2}$ so it can be given a 1-map smooth atlas

$$\left\{ \mathbb{R}^{\binom{n+1}{2}} \right\} \xrightarrow{\simeq} \mathrm{Sym}_n(\mathbb{R})$$

Our map now will be $g : M_n(\mathbb{R}) \longrightarrow \mathrm{Sym}_n(\mathbb{R})$ defined by

$$g(Q) = Q^T Q$$

We want to show that $I \in \mathrm{Sym}_n(\mathbb{R})$ is a regular value so we will take an $Q \in M_n(\mathbb{R})$ such that $g(Q) = I$ and compute the differential dg_Q. As before let $B \in T_Q M_n(\mathbb{R}) = M_n(\mathbb{R})$ and $\gamma(t) = Q + tB$. We want to compute

$$\frac{d}{dt} g(\gamma(t))\Big|_{t=0} = \lim_{t \to 0} \left[\frac{g(\gamma(t)) - g(\gamma(0))}{t} \right] =$$

$$\lim_{t \to 0} \left[\frac{(Q + tB)^T (Q + tB) - I}{t} \right] = \lim_{t \to 0} \left[\frac{I + (Q^T B + B^T Q)t + t^2 B^T B - I}{t} \right] = Q^T B + B^T Q$$

Hence our differential is

$$dg_Q(B) = Q^T B + B^T Q \in T_Q\mathrm{Sym}_n(\mathbb{R}) = \mathrm{Sym}_n(\mathbb{R})$$

This is a map from $M_n(\mathbb{R})$ to $\mathrm{Sym}_n(\mathbb{R})$ and we need to show that it surjective. Let $C \in \mathrm{Sym}_n(\mathbb{R})$ be a matrix. Then

$$dg_Q \left(\frac{QC}{2} \right) = \frac{Q^T Q C}{2} + \frac{C^T Q^T Q}{2} = C$$

and so dg_Q is surjective.

This proves that I is a regular value for g and so $\mathrm{O}_n(\mathbb{R})$ admits a smooth structure making it into a smooth sub-manifold of $M_n(\mathbb{R})$ of dimension

$$n^2 - \binom{n+1}{2} = \binom{n}{2}$$

If $Q \in \mathrm{SO}_n(\mathbb{R})$ then $T_Q\mathrm{SO}_n(\mathbb{R}) \subseteq T_Q M_n(\mathbb{R})$ can be identified with the kernel of dg_Q. In particular

$$T_Q\mathrm{SO}_n(\mathbb{R}) = \{QD | D^T = -D\}$$

Note that each $Q \in \mathrm{O}_n(\mathbb{R})$ satisfies

$$\det(Q)^2 = \det(Q^T Q) = \det(I) = 1$$

so $\det(Q) = \pm 1$. It is easy that both signs can be obtained, so $\mathrm{O}_n(\mathbb{R})$ has two connected components. It is clear that each connected component inherits a smooth structure from $\mathrm{O}_n(\mathbb{R})$. In particular the connected component with $\det = 1$ is denoted by $\mathrm{SO}_n(\mathbb{R})$.

Let us note a nice fact which occurs in dimension $n = 3$. Note that if $Q \in \mathrm{SO}_3(\mathbb{R})$ and if we denote by $\widehat{x}, \widehat{y}, \widehat{z}$ the columns of Q then $\widehat{x}, \widehat{y}, \widehat{z}$ form an orthogonal basis (this is in all dimensions) and satisfy

$$\widehat{x} \times \widehat{y} = \widehat{z}$$

$$\widehat{y} \times \widehat{z} = \widehat{x}$$

$$\widehat{z} \times \widehat{x} = \widehat{y}$$

Now it is a direct computation to verify that if

$$D = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Is a general anti-symmetric matrix then the columns of QD are $v \times \widehat{x}, v \times \widehat{y}, v \times \widehat{z}$ for

$$v = a\widehat{x} + b\widehat{y} + c\widehat{z}$$

This proves question 4 in exercise 1, namely for any smooth path $\Phi(s) \in \mathrm{SO}_3(\mathbb{R})$ there exists a function $v(s) \in \mathbb{R}^3$ such that for every vector $u_0 \in \mathbb{R}^3$ the curve $u(s) = \Phi(s)u_0 \in \mathbb{R}^3$ satisfies

$$x'(s) = u(s) \times x(s)$$

Differential Geometry - Tirgul 3

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1 The Tangent Bundle

Let M be a smooth manifold of dimension n. In this section we will discuss how to construct an atlas on the tangent bundle TM giving it the structure of a smooth manifold of dimension $2n$.

As a set, the tangent bundle is defined to be

$$TM = \{(p, v) | v \in T_p M\}$$

We now wish to put an (oriented) atlas on TM. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M. Since $U_\alpha \subseteq \mathbb{R}^n$ we can identify for all $x \in U_\alpha$ the tangent space $T_x U_\alpha$ with $\mathbb{R}^n$ in a canonical way (the vector in $T_x U_\alpha$ which corresponds to $v \in \mathbb{R}^n$ is the equivalence class of the path $x + tv$ at $t = 0$). This means that we can consider the differential $(d\phi_\alpha)_x$ as map from $\mathbb{R}^n$ to $T_{\phi_\alpha(x)} M$. Consider the map

$$\widetilde{\phi}_\alpha : U_\alpha \times \mathbb{R}^n \longrightarrow TM$$

defined by

$$\widetilde{\phi}_\alpha(x, v) = (\phi(x), (d\phi_\alpha)_x(v))$$

We claim that $\{(U_\alpha \times \mathbb{R}^n, \widetilde{\phi}_\alpha)\}$ is an oriented atlas for TM ($U_\alpha \times \mathbb{R}^n$ is considered as an open subset of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. Hence this will be a $2n$-dimensional structure).

First we need to verify that all the maps are injective. This clear because ϕ_α is injective and each $(d\phi_\alpha)_x$ is an isomorphism and so injective. A similar argument shows that the images of $\widetilde{\phi}_\alpha$ cover M: each $p \in M$ is in the image of some U_α and since $(d\phi_\alpha)_x$ is an isomorphism it is surjective so all tangent vectors to x are covered.

We now want to show that the transition functions are smooth and with positive differential. Define

$$W_{\alpha,\beta} = \phi_\alpha^{-1}(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta))$$

and

$$\widetilde{W}_{\alpha,\beta} = \widetilde{\phi}_\alpha^{-1}(\widetilde{\phi}_\alpha(U_\alpha \times \mathbb{R}^n) \cap \widetilde{\phi}_\beta(U_\beta \times \mathbb{R}^n))$$

Now suppose that the image of $\widetilde{\phi}_\alpha$ intersects the image of $\widetilde{\phi}_\beta$. Then clearly from the definition (and the fact that each $(d\phi_\alpha)_x$ is an isomorphism) we get

$$\widetilde{\phi}_\alpha(U_\alpha \times \mathbb{R}^n) \cap \widetilde{\phi}_\beta(U_\beta \times \mathbb{R}^n) = T\phi_\alpha(U_\alpha) \cap T\phi_\beta(U_\beta) =$$

$$T(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta))$$

and so

$$\widetilde{W}_{\alpha,\beta} = \widetilde{\phi}_\alpha^{-1}(\widetilde{\phi}_\alpha(U_\alpha \times \mathbb{R}^n) \cap \widetilde{\phi}_\beta(U_\beta \times \mathbb{R}^n)) = \phi_\alpha^{-1}(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta)) \times \mathbb{R}^n = W_{\alpha,\beta} \times \mathbb{R}^n$$

Further more from the definition of $\widetilde{\phi}_\alpha$ we see that

$$\widetilde{\phi}_\beta^{-1} \circ \widetilde{\phi}_\alpha : \widetilde{W}_{\alpha,\beta} \longrightarrow \widetilde{W}_{\beta,\alpha}$$

is just

$$(\phi_\beta^{-1} \circ \phi_\alpha, d\phi_\beta^{-1} \circ d\phi_\alpha) : W_{\alpha,\beta} \times \mathbb{R}^n \longrightarrow W_{\beta,\alpha} \times \mathbb{R}^n$$

The first coordinate is a smooth function because we started with a smooth atlas for M. The second coordinate depends smoothly ont the first coordinate (because it is a differential of a smooth function) and is **linear** in the second coordinate, i.e. it is smooth. This means that we have a smooth atlas for TM.

We claim that this atlas is oriented, i.e. that these transition functions have a positive differential. This is because the differential of a linear map is it self. Further more, since in the map $\widetilde{\phi}_\beta^{-1} \circ \widetilde{\phi}_\alpha$ the last n coordinates depend only on the last n coordinates, its differential at any point takes the form

$$\begin{pmatrix} d\phi_\beta^{-1} \circ d\phi_\alpha & 0 \\ * & d\phi_\beta^{-1} \circ d\phi_\alpha \end{pmatrix}$$

whose determinant is

$$\det(d\phi_\beta^{-1} \circ d\phi_\alpha)^2 > 0$$

Hence TM is an orientable manifold of dimension $2n$.

There exists a natural map $p_M : TM \longrightarrow M$ which sends (x, v) to x. Note that for any α if we look at the maps ϕ_α and $\widetilde{\phi}_\alpha$ the map p_M looks like a simple projection on the first n coordinates. Hence p_M is a smooth function between these manifolds.

2 The Lie Bracket

Recall that we associate to a tangent vector $v \in T_xM$ and a smooth function $f \in C^\infty(U)$ (for some neighborhood U of x) a real number $v(f)$ which is the derivative of f at x in direction v. We have also seen in class that all maps from $d_x : C^\infty(U) \longrightarrow \mathbb{R}$ satisfy the local Leibnitz rule at x:

$$d_x(fg) = f(x)d_x(g) + d_x(f)g(x)$$

can be realized as $d_x(f) = v(f)$ for some $v \in T_xM$.

Having defined the tangent bundle TM as a smooth manifold we can naturally define now what is a **vector field**: it is a smooth map $X : M \longrightarrow TM$ such that $p_M \circ X = Id$. Now given a vector field X and a smooth function $f \in C^\infty(M)$ we can create a new function $X(f)$ by setting

$$X(f)(p) = X_p(f)$$

It is not hard to show that (check it in every map as in the previous section) that $X(f)$ is also a smooth map. We also get a global Leibnitz rule

$$X(fg) = X(f)g + fX(g)$$

Similarly to the case of a single tangent vector, it is quite straight forward to show that every map $C^\infty \longrightarrow C^\infty$ which satisfies the global Leibnitz rule is of the form $f \mapsto X(f)$ for some smooth vector field X (since clearly it satisfies the local Leibnitz rule at every point we get a tangent vector $X_p \in T_pM$ at every p. It is then not hard to check in coordinates that the resulting map $X : M \longrightarrow TM$ is smooth).

Now suppose we have two vector fields X, Y. Consider the map $Z : C^\infty(M) \longrightarrow C^\infty(M)$ defined by

$$Z(f) = X(Y(f)) - Y(X(f))$$

We will show that this map satisfies the global Leibnitz rule:

$$X(Y(fg)) - Y(X(fg)) = X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) =$$

$$X(Y(f))g+Y(f)X(g)+X(f)Y(g)+fX(Y(g))-Y(X(f))g-X(f)Y(g)-Y(f)X(g)-fY(X(g)) =$$

$$[X(Y(f)) - Y(X(f))]\, g + f\, [X(Y(f)) - Y(X(f))] = Z(f)g + fZ(g)$$

Hence there exist a vector field, which we will denote by $[X, Y]$, such that

$$[X, Y](f) = Z(f)$$

This vector field is called the **Lie Bracket** of X and Y. Comment: Note that we can't take the Lie bracket of two tangent vectors, only of two vector fields.

Let us calculate the Lie bracket in coordinates. For this it is enough to calculate the Lie bracket of two vector fields on $\mathbb{R}^n$ with coordinates $u^1, ..., u^n$. Let

$$X = \sum_i X_i \frac{\partial}{\partial u^i}$$

$$Y = \sum_i Y_i \frac{\partial}{\partial u^i}$$

be two vector fields on $\mathbb{R}^n$, where each X_i, Y_i is a function from $\mathbb{R}^n$ to $\mathbb{R}$. Then if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a smooth function then

$$X(f) = \sum_i X_i \frac{\partial f}{\partial u^i}$$

$$Y(f) = \sum_i Y_i \frac{\partial f}{\partial u^i}$$

and

$$X(Y(f)) - Y(X(f)) = \sum_{i,j} X_i \frac{\partial}{\partial u^i}\left[Y_j \frac{\partial f}{\partial u^j}\right] - \sum_{i,j} Y_i \frac{\partial}{\partial u^i}\left[X_j \frac{\partial f}{\partial u^j}\right] =$$

$$\sum_{i,j} X_i \frac{\partial Y_j}{\partial u^i}\frac{\partial f}{\partial u^j} - Y_i \frac{\partial X_j}{\partial u^i}\frac{\partial f}{\partial u^j}$$

which means that

$$[X, Y] = \sum_j \left[\sum_i \left(X_i \frac{\partial Y_j}{\partial u^i} - Y_i \frac{\partial X_j}{\partial u^i}\right)\right] \frac{\partial}{\partial u^j}$$

The Lie Bracket satisfies these three basic properties

1. $[-,-]$ is $\mathbb{R}$-bilinear.
2. $$[X, Y] = -[Y, X]$$
3. Jacobi's identity:
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

The first two properties are obvious. The third one can be proved by a direct calculation. Consider 3 vectors fields X_1, X_2, X_3. Denote for shortage

$$\{i, j, k\} = X_i(X_j(X_k(f)))$$

Then

$$[X_1, [X_2, X_3]]f = X_1([X_2, X_3]f) - [X_2, X_3](X_1(f)) = X_1(X_2(X_3(f))) - X_1(X_3(X_2(f))) - X_2(X_3(X_1(f))) + X_3(X$$

which means that

$$[X_1, [X_2, X_3]]f = (\{1, 2, 3\} - \{2, 3, 1\}) + (\{3, 2, 1\} - \{1, 3, 2\})$$

$$[X_2, [X_3, X_1]]f = (\{2, 3, 1\} - \{3, 1, 2\}) + (\{1, 3, 2\} - \{2, 1, 3\})$$

$$[X_3, [X_1, X_2]]f = (\{3, 1, 2\} - \{1, 2, 3\}) + (\{2, 1, 3\} - \{3, 2, 1\})$$

and so the sum of these three terms vanishes.

Differential Geometry - Tirgul 4

Yonatan Harpaz

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1 Flows and Local Behavior at Critical Points

Let M be a smooth compact manifold and X a vector field defined on all of M. We now that in that case the flow $\Phi_t : M \longrightarrow M$ is defined for all $t \in \mathbb{R}$ with $\Phi_0 = Id$. Now let $p \in M$ be a point such that $X_p = 0$. In that case the constant curve $\gamma(t) = p$ is an integral curve of X which means that

$$\Phi_t(p) = p$$

for all t. We say that p is a **fixed point** of the flow. We wish to describe the local behavior of the flow around p. For example, a natural question to ask is about **stability**: suppose we pick a point which is very close to p, would is flow towards p ($\lim_{t\to\infty} \Phi_t(p') = p$), stay in some neighborhood of p, or maybe flow away (at least in the nearby time)?

In order to answer this question, let us start by making the following assumption: there exists a chart around p with coordinates $x^1, ..., x^n$ such that X in these coordinates is **linear**, i.e. there exists constants $a_{i,j} \in \mathbb{R}$ such that

$$X(x^1, ..., x^n) = \sum_{i,j=1}^{n} a_{i,j} x^j \frac{\partial}{\partial x^i}$$

In particular

$$X_p = X(0, ..., 0) = 0$$

Let $A = [a_{i,j}]$. Now if $\gamma(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}$ is in an integral curve of this linear vector field then it satisfies the linear ODE

$$\frac{d}{dt} \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix} = A \cdot \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}$$

From the classical theory of linear ODE's we know that in order to analyze this situation we need first find the eigenvalues and eigenvectors of A. Assume for simplicity that A is non-singular and diagonalizable over $\mathbb{C}$. Let $\lambda_1, ..., \lambda_k$ be the real eigenvalues of A and $\alpha_1 \pm \beta_1 i, ..., \alpha_r \pm \beta_r i$, the non-real eigenvalues (note that $k + 2r = n$). Similarly let $v_1, ..., v_k$ be the corresponding real eigenvectors and $u_1 \pm iw_1, ..., u_r \pm iw_r$ the corresponding non-real eigenvectors.

From the classical theory we know that if γ is an integral curve of X in the coordinate chart such that

$$\gamma(0) = \sum_{i=1}^{k} A_i v_i + \sum_{j=1}^{r} [B_j u_j + C_j w_j] =$$

$$\sum_{i=1}^{k} A_i v_i + \frac{1}{2} \sum_{j=1}^{r} [(B_j - iC_j)(u_j + iw_j) + (B_j + iC_j)(u_j - iw_j)]$$

with $A_i, B_j, C_j \in \mathbb{R}$ then

$$\gamma(t) = \sum_{i=1}^{k} e^{\lambda_i t} A_i v_i + \frac{1}{2} \sum_{j=1}^{r} \left[e^{(\alpha_j + \beta_j i)t} (B_j - iC_j)(u_j + iw_j) + e^{(\alpha_j - \beta_j i)t} (B_j + iC_j)(u_j - iw_j) \right] =$$

$$\sum_{i=1}^{k} e^{\lambda_i t} A_i v_i + \sum_{j=1}^{r} e^{\alpha_j t} [\cos(\beta_j t)(B_j u_j + C_j w_j) + \sin(\beta_j t)(C_j u_j - B_j w_j)]$$

This formula allows us to analyze what happens if we move away from $p = (0, ..., 0)$ in some direction which is an eigenvector of A. If we move a bit in the direction of v_i (respectively u_j), then we would flow back to p if $\lambda_i < 0$ (respectively $\alpha_i < 0$) and away form p if $\lambda_i > 0$ (respectively $\alpha_i > 0$).

If at least one of the λ_i, α_i's is > 0 then in most directions (all the directions which have a non-zero coefficient in the corresponding eigenvector) the point will flow away from p. We then say that the fixed point is **unstable**. If all the λ_i, α_i's are < 0 then in every direction we go we will flow back to p. In that case we say that the fixed point is **stable**.

If $\alpha_i = 0$ (this is possible without making A singular, as apposed to $\lambda_i = 0$) then if we move away from p in a direction spanned by u_j, w_j then we would enter a periodic orbit with period of time $\frac{2\pi}{\beta_j}$. If we move away in a direction which is a span of $u_1, w_1, ..., u_r, w_r$ then we would enter a bounded orbit (i.e. stay in the proximity of p) but without an orbit (unless the β_j's are linearly dependent over $\mathbb{Q}$).

Now suppose that X wasn't linear but a general vector field

$$X = \sum_i f_i \frac{\partial}{\partial x^i}$$

satisfying $f_i(0, ..., 0) = 0$. Then we could approximate the f_i's (in the standard Taylor expansion way) by the linear maps

$$f_i \simeq \sum_j \frac{\partial f_i}{\partial x^j} x^j$$

and do the same analysis only with the matrix $A = \left[\frac{\partial f_i}{\partial x_i}\right]$. This would give us at least a good approximation of the dynamics in a small neighborhood of $(0, ..., 0)$, and will answer the basic stability questions.

Now return to the case of a general vector field X on a manifold M such that $X_p = 0$. We can put coordinates near p and analyze the matrix A given above to understand the local behavior around p in terms of stability, periodic orbits etc. But what if we now change the coordinate system? we would get a different matrix, although clearly the dynamical behavior is (at least qualitatively) not coordinate dependent.

To explain this phenomenon we shall show that the matrix A is coordinate independent up to conjugation (i.e., if we change to a different coordinate system A would change by conjugation). This is because A represents a map $T_pM \longrightarrow T_pM$ defined internally by X at every point p where $X_p = 0$.

To see this map think of X as a smooth map $X : M \longrightarrow TM$. Recall that we have two familiar maps connecting M and TM. One is the projection $\pi : TM \longrightarrow M$ which takes (p, v) to p and the other is the zero-section $z : M \longrightarrow TM$ which sends p to $(p, 0)$.

Clearly from the definition $\pi \circ z = Id$ which means that

$$d\pi_{(p,0)} \circ dz_p = Id$$

in particular this implies that dz_p is injective, $d\pi_{(p,0)}$ is surjective and that we have a direct sum decomposition

$$T_{(p,0)}TM = \ker(d\pi_{(p,0)}) \oplus \mathrm{Im}\,(dz_p)$$

This direct sum decomposition induces a projection ρ from $T_{(p,0)}TM$ to $\ker(d\pi_{p,0})$ which in turn can be naturally identified with the tangent space to the submanifold $\pi^{-1}(p)$ at $(p, 0)$. Since $\pi^{-1}(p) = T_pM$ is a vector space we can identify its tangent space at every point with T_pM itself.

Now since $X_p = 0$ we get a differential

$$dX_p : T_pM \longrightarrow T_{(p,0)}TM$$

composing this with the projection ρ we get a well defined map

$$\rho \circ dX_p : T_pM \longrightarrow T_pM$$

It can be shown that if we work in local coordinates where

$$X = \sum_i f_i \frac{\partial}{\partial x^i}$$

then this linear transformation is represented by the matrix $A = \left[\frac{\partial f_i}{\partial x_i}\right]$. Hence this matrix is intrinsic up to conjugation, and in particular all its eigenvalues (and their dynamical consequences) are well defined invariants of X.

Differential Geometry - Tirgul 6

Yonatan Harpaz

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1 Some calculations of Geodesics

1.1 The Sphere

Let us solve question 2 from HW 5:

1. Let M be a surface and $V \subseteq \mathbb{R}^3$ an affine plane such that M is locally (around some intersection point $p \in M \cap V$) symmetric with respect to reflection by V. By maybe replacing M by some open subset of it we can assume WLOG that M it self is symmetric with respect to V.

   Note that V will never be tangent to M (because then since M is symmetric to reflection it will not look locally like $\mathbb{R}^2$ at around that point). Hence by the implicit function theorem the intersection id a curve γ. We want to show that γ is a geodesic.

   From the existence and uniqueness theorem for geodesics we know that there exists a unique geodesic γ_g such that

   $$\gamma_g(0) = \gamma(0)$$

   $$\gamma_g'(0) = \gamma'(0)$$

   Let $\sigma : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ denote the reflection by V. Then σ induces an isometry form M to itself and so $\sigma(\gamma_g(t))$ is also a geodesic. But

   $$(\sigma(\gamma_g(t)))' = d\sigma(\gamma_g'(t)) = d\sigma(\gamma'(r)) = \gamma'(t)$$

   because $\sigma(\gamma(t)) = \gamma(t)$. Hence $\gamma_g(t)$ and $\sigma(\gamma_g(t))$ are two geodesics which agree on the initial point and initial velocity. From uniqueness we get that

   $$\sigma(\gamma_g(t)) = \gamma_g(t)$$

   which means that $\gamma_g(t)$ is contained in V. Since it is also contained in M and since the intersection of V and M is a curve we see that $\gamma_g(t) = \gamma(t)$. Hence $\gamma(t)$ is a geodesic.

2. The great circles on the sphere are intersections of the sphere with a plane which contains the center of the sphere. Clearly the sphere is symmetric to reflection by such a plane which means by section 1 that great circles are geodesics.

3. Let $\gamma(t)$ be a geodesic on S^2 with $T(t), n(t)$ the tangent and normal respectively. Let V be the plane spanned by $T(0), n(0)$ and consider the intersection of $\gamma(0) + V$ with S^2.

   Since $\gamma(0) + V$ would contain the center of the sphere we see that this intersection is actually a great circle, hence defining a curve γ_M which is a geodesic. We can choose the direction and speed of γ_M so that $\gamma_M'(0) = T(0)$. Then from uniqueness of geodesics we get that $\gamma(t) = \gamma_M(t)$.

2 The Poincare Upper Half Plane

Recall the Poincare upper half place $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ with the metric $g = \frac{dx^2+dy^2}{y}$ (i.e. in the x, y coordinates $g_{1,1} = g_{2,2} = \frac{1}{y}$ and $g_{1,2} = g_{2,1} = 0$). As in the last tirgul we will use the notation $\begin{pmatrix} a \\ b \end{pmatrix}$, where a, b are smooth functions from $\mathbb{R}^2_+$ to $\mathbb{R}$, to denote the vector field

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$$

We saw in the previous tirgul that the Levi-Civita connection on $\mathbb{R}^2_+$ is given by

$$\nabla_Y \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} Y(a) \\ Y(b) \end{pmatrix} + \Gamma(Y) \begin{pmatrix} a \\ b \end{pmatrix}$$

where

$$\Gamma \begin{pmatrix} c \\ d \end{pmatrix} = -\frac{1}{y} \begin{pmatrix} d & c \\ -c & d \end{pmatrix}$$

We want find all the geodesics on $\mathbb{R}^2_+$. Let us first find all the geodesics $\gamma(t)$ which whose tangent $\gamma'(t)$ is parallel to the y-axis. The geodesic equation is

$$\nabla_{\gamma'(t)}\gamma'(t) = 0$$

which explicitly gives

$$\gamma''(t) + \Gamma(\gamma'(y))\gamma'(t) = 0$$

Since $\gamma'(0)$ has only a $\frac{\partial}{\partial y}$ component we see that $\Gamma(\gamma'(0))$ is a multiple of the identity matrix. Hence $\gamma''(0)$ is also parallel to the y-axis. This hints to us that we should look for a solution which is contained in a line parallel to the y-axis. Let $\gamma(0) = (x_0, y_0)$ and look for a solution of the form $\gamma(t) = (x_0, c(t))$. Then the geodesic equation becomes

$$c''(t) + \frac{c'(t)^2}{c(t)} = 0$$

Note that all the functions of the form $c(t) = Ae^{at}$ solve this equation. From our initial conditions we get that if $\gamma'(0) = a\frac{\partial}{\partial y}$ then

$$\gamma(t) = (x_0, y_0 e^{at})$$

How will find all the other geodesics? For this we will use Mobius transformations. Identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way and think of the upper half plane as complex numbers with positive imaginary part. A **Mobius** transformation is a map of the form

$$\rho(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ab - cd \neq 0$. Note that $\rho\left(-\frac{d}{c}\right)$ is apparently ill defined because the denominator vanishes. The standard way to fix this (see any basic course on complex functions) is to add a point to $\mathbb{C}$, called ∞, and put a topology on $\widehat{\mathbb{C}} = \mathbb{C} \cap \{\infty\}$ which makes it homeomorphic to S^2. It is easy to show that then ρ becomes a continuous map $\widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ which is actually homeomorphism. Its inverse is

$$\rho^{-1}(z) = \frac{dz-b}{-cz+a}$$

Now suppose that $a, b, c, d \in \mathbb{R}$. Then $\rho(\mathbb{R} \cap \{\infty\}) = \mathbb{R} \cap \{\infty\}$ and since ρ is a homeomorphism it maps $X = \widehat{\mathbb{C}} \setminus (\mathbb{R} \cap \{\infty\})$ to itself (homeomorphically). But X has two connected components - the upper and lower half planes. It is an easy exercise to show that the condition $ad - bc > 0$ (this expression is called the **determinant** of ρ) is equivalent to the fact that ρ maps the upper half plane to itself.

Now to top all that, we shall now show that ρ preserves the metric g. Let us write the differential of ρ in the x, y coordinates (i.e. we use the vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ as a basis for each tangent space). Since ρ is a holomorphic, we know that if we identify the tangent spaces of each point of the upper half plane with $\mathbb{C}$ in the canonical way, then $d\rho : T_z\mathbb{R}^2_+ \longrightarrow T_{\rho(z)}\mathbb{R}^2_+$ would be multiplication by the complex number $\rho'(z)$. Such map are always a composite of a rotation by $\arg(\rho'(z))$ and expansion by $|\rho'(z)|$.

Since the metric is $\frac{dx^2+dy^2}{y^2}$ we see that in order to preserve the metric the expansion part $|\rho'(z)|$ has to be equal to

$$\frac{\operatorname{Im}(\rho(z))}{\operatorname{Im}(z)} = \frac{\rho(z) - \rho(\overline{z})}{z - \overline{z}} = \frac{\frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}}{z - \overline{z}} =$$

$$\frac{(c\overline{z}+d)(az+b) - (cz+d)(a\overline{z}+b)}{|cz+d|^2(z-\overline{z})} = \frac{ad-bc}{|cz+d|^2}$$

and indeed:

$$|\rho'(z)| = \left|\frac{ad-bc}{(cz+d)^2}\right| = \frac{ad-bc}{|cz+d|^2}$$

Now this means that real Mobius maps with positive determinant are self isometries of the upper half plane. Hence they must take geodesics to geodesics. Now we claim that for each $z_1, z_2 \in \mathbb{R}^2_+$ there exists a mobius map sending z_1 to z_2. The proof is simple: since z_1 is not real we get that z_1 and 1 are linearly independent over $\mathbb{R}$. Hence there exists $a, b \in \mathbb{R}$ such that

$$az_1 + b = z_2$$

with $a > 0$ (becomes both z_1, z_2 have positive imaginary part). Hence this is a real Mobius map with positive determinant.

Now we claim that for every $v \in T_{z_1}\mathbb{R}^2_+, u \in T_{z_2}\mathbb{R}^2_+$ such that $g(v, v) = g(u, u)$ there exists a real positive Mobius map ρ such that

$$\rho(z_1) = z_2$$

$$d\rho(v) = u$$

Since we already know that we can move z_1 to z_2, it is enough to show this under the assumption that $z_1 = z_2$, and in fact we can even assume that $z_1 = z_2 = i$. Now the Mobius maps of the form

$$\rho(z) = \frac{\cos(\alpha)z + \sin(\alpha)}{-\sin(\alpha)z + \cos(\alpha)}$$

satisfy

$$\rho(i) = i$$

$$\det(\rho) = 1 > 0$$

and

$$\rho'(z) = \frac{1}{(-\sin(\alpha)i + \cos(\alpha))^2} = \cos(2\alpha) + i\sin(2\alpha)$$

which means that $\rho'(z)$ rotates $T_i\mathbb{R}^2_+$ in an angle of 2α. Since we can do this for every α we can send every vector in $T_i\mathbb{R}^2_+$ to any other vector of the same length.

Now from uniqueness of geodesics we know that they are determined by an initial point and an initial velocity. Hence from the above considerations we get that every geodesic is an image under some real positive ρ of the line vertical line $x = 0$. From complex functions we know that the image of a line is either a line or a circle, and that a holomorphic map preserves angles (in the usual sense) on $\mathbb{C}$. Hence we get that the image of the straight line $x = 0, y > 0$ is either another straight line of the form $x = b, y > 0$ or half a circle which is orthogonal to the line $y = 0$.

Differential Geometry - Tirgul 7

Yonatan Harpaz

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1 The Curvature Tensor and Parallel transport

In this tirgul we will see how the curvature tensor is connected to parallel transport. Let $p, q \in M$ be two points and γ a smooth path from p to q. Let $P_\gamma : T_pM \longrightarrow T_qM$ be the parallel transport map. The curvature tensor measures in some sense how much P_γ will change if we change γ smoothly to another curve γ' (without moving the end points).

The prototypical case here is the case $q = p$. Suppose I have a path which can be smoothly deformed (without moving the end points) to the constant path from p to p. How different will P_γ be from the identity, or in other words, what will P_γ be?

In this tirgul we shall prove a theorem for 2 dimensional manifolds (there is an analogous general statement but it intails some technical difficulties which we don't wish to enter).

Theorem 1.1. *Let M be a 2-dimensional Riemannian manifold with Levi-Civita connection ∇ and*

$$R_{X,Y}(Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

the associated curvature tensor.

Consider $\mathbb{R}^2$ with coordinates u^1, u^2 and $D \subseteq \mathbb{R}^2$ the open unit disc. Let γ be a smooth closed curve in M and $F : D \longrightarrow M$ such that

$$F|_{\partial D} = \gamma$$

(with respect to the counter clockwise direction).

We will write $\frac{\partial F}{\partial u^i}$ for $dF\left(\frac{\partial}{\partial u^i}\right)$. Let X, Y be some orthonormal frame defined on an open set containing $F(\overline{D})$ such that the determinant

$$\det(dF) = \begin{vmatrix} \frac{\partial F}{\partial u^1} & \frac{\partial F}{\partial u^1} \end{vmatrix} \geq 0$$

where the determinant is taken with respect to the orthonormal basis given by X, Y.

Then the parallel transport P_γ along γ is a rotation by angle α (counter clockwise in the orientation given by X, Y) where

$$\alpha = \int_D g\left(R_{\frac{\partial F}{\partial u^1}, \frac{\partial F}{\partial u^2}}(X), Y\right) du^1 du^2 = \int_D \kappa(F(u^1, u^2)) \det(dF) du^1 du^2$$

Before we come to the proof we give a short lemma:

Lemma 1.2. *Let X_1, X_2 be an orthonormal frame defined on some open set $U \subseteq M$. Then for every vector field Z on U, then expression $g(\nabla_Z X_i, X_j)$ is anti-symmetric with respect to i, j. In particular*

$$\nabla_Z X_1 = g(\nabla_Z X_1, X_2) X_2$$

$$\nabla_Z X_2 = -g(\nabla_Z X_1, X_2) X_1$$

Proof. Since $g(X_i, X_j)$ is a constant function (its either constantly 0 or constantly 1) we get that

$$0 = Z(g(X_i, X_j)) = g(\nabla_Z X_i, X_j) + g(X_i, \nabla_Z X_j)$$

which means that

$$g(\nabla_Z X_i, X_j) = -g(X_i, \nabla_Z X_j)$$

this means in particular that $\nabla_Z X_1$ is a multiple of X_2, and so

$$\nabla_Z X_1 = g(\nabla_Z X_1, X_2) X_2$$

Similarly

$$\nabla_Z X_2 = g(\nabla_Z X_2, X_1) X_2 = -g(\nabla_Z X_1, X_2) X_1$$

□

We now come to the proof of theorem 1.1:

Proof. From question 1 in assigned problems 3 we know that parallel transport preserves the Riemannian inner product and orientation (here we have an orientation defined locally around the image of F by the basis X, Y). Hence the praralel transport must be a rotation by some angle. Note that this angle is determined by looking at the image of a single vector.

Let $V(t)$ be the parallel transport of the vector $X_{\gamma(0)}$ along γ. Note that $V(t)$ is always of norm 1 and so we can write

$$V(t) = \cos(\alpha(t))X(t) + \sin(\alpha(t))Y(t)$$

for some function $\alpha : [0,1] \longrightarrow \mathbb{R}$.

Then the parallel transport along γ results in a rotation by $\alpha(1)$, so this is the quantity we wish to compute. From the definition of parallel transport we get

$$0 = \frac{DV}{dt} = \nabla_{\gamma'(t)} V(t) =$$

$$\cos(\alpha(t))\nabla_{\gamma'(t)} X + \sin(\alpha(t))\nabla_{\gamma'(t)} Y - \alpha'(t)\sin(\alpha(t))X + \alpha'(t)\cos(\alpha(t))Y$$

Now using lemma 1 we can write this as

$$g\left(\nabla_{\gamma'(t)} X, Y\right)(\sin(\alpha(t))X - \cos(\alpha(t))Y) = \alpha'(t)\left(-\sin(\alpha(t))X + \cos(\alpha(t))Y\right)$$

which means that

$$\alpha'(t) = -g\left(\nabla_{\gamma'(t)}X, Y\right)$$

Hence we need to compute the integral

$$\alpha(1) = -\int_0^1 g(\nabla_{\gamma'(t)}X, Y)dt$$

Write

$$F_1 = g\left(\nabla_{\frac{dF}{du^1}}X, Y\right)$$

$$F_2 = g\left(\nabla_{\frac{dF}{du^2}}X, Y\right)$$

We will do it by using Stocks's theorem from advanced infi (or from electromagnetism to the Physicists among you):

$$\alpha(1) = -\int_0^1 g(\nabla_{\gamma'(t)}X, Y)dt = -\int_\gamma \left[F_1du^1 + F_2du^2\right] = \int_D \left[\frac{dF_1}{du^2} - \frac{dF_2}{du^1}\right]du^1du^2$$

From $(*)$ we know that

$$g\left(\nabla_{\frac{dF}{du^1}}X, \nabla_{\frac{dF}{du^2}}Y\right) = g\left(\nabla_{\frac{dF}{du^2}}X, \nabla_{\frac{dF}{du^1}}Y\right) = 0$$

hence we get

$$\alpha(1) = \int_D g\left(\nabla_{\frac{\partial F}{\partial u^2}}\nabla_{\frac{\partial F}{\partial u^1}}X - \nabla_{\frac{\partial F}{\partial u^2}}\nabla_{\frac{\partial F}{\partial u^1}}X, Y\right)du^1du^2 =$$

$$\int_D g\left(R_{\frac{\partial F}{\partial u^1},\frac{\partial F}{\partial u^2}}(X), Y\right)du^1du^2$$

because $\left[\frac{\partial F}{\partial u^1}, \frac{\partial F}{\partial u^2}\right] = 0$.

□

If you are not familiar with the equality

$$g\left(R_{\frac{\partial F}{\partial u^1},\frac{\partial F}{\partial u^2}}(X), Y\right) = \kappa(F(u^1, u^2))\det(dF)$$

Recall that in the last tirgul we saw that the Levi-Civita connection on a surface in $M \subseteq \mathbb{R}^3$ with the induced Riemannian metric is given by

$$\nabla_Z X = \overline{\nabla}_Z X - g(L(Z), X)n$$

where L is the Weingarten map. Hence in particular if Y is any vector field on M then

$$\langle\nabla_Z X, Y\rangle = \left\langle\overline{\nabla}_Z X - g(L(Z), X)n, Y\right\rangle = \left\langle\overline{\nabla}_Z X, Y\right\rangle$$

Hence

$$\left\langle R_{\frac{\partial F}{\partial u^1},\frac{\partial F}{\partial u^2}}(X), Y\right\rangle = \left\langle\nabla_{\frac{\partial F}{\partial u^2}}\nabla_{\frac{\partial F}{\partial u^1}}X - \nabla_{\frac{\partial F}{\partial u^1}}\nabla_{\frac{\partial F}{\partial u^2}}X, Y\right\rangle = \left\langle\overline{\nabla}_{\frac{\partial F}{\partial u^2}}\nabla_{\frac{\partial F}{\partial u^1}}X - \overline{\nabla}_{\frac{\partial F}{\partial u^1}}\nabla_{\frac{\partial F}{\partial u^2}}X, Y\right\rangle$$

Now since $\overline{\nabla}$ has no curvature we get that

$$\overline{\nabla}_{\frac{\partial F}{\partial u^2}} \overline{\nabla}_{\frac{\partial F}{\partial u^1}} = \overline{\nabla}_{\frac{\partial F}{\partial u^1}} \overline{\nabla}_{\frac{\partial F}{\partial u^2}}$$

and so the above term equals

$$\left\langle \overline{\nabla}_{\frac{\partial F}{\partial u^1}} g\left(L\left(\frac{\partial F}{\partial u^2}\right), X\right) n - \overline{\nabla}_{\frac{\partial F}{\partial u^2}} g\left(L\left(\frac{\partial F}{\partial u^1}\right), X\right) n, Y \right\rangle =$$

$$g\left(L\left(\frac{\partial F}{\partial u^1}\right), X\right) g\left(L\left(\frac{\partial F}{\partial u^2}\right), Y\right) - g\left(L\left(\frac{\partial F}{\partial u^1}\right), Y\right) g\left(L\left(\frac{\partial F}{\partial u^2}\right), X\right)$$

which is the determinant of the matrix representing L as a map from the basis $\frac{\partial F}{\partial u^1}, \frac{\partial F}{\partial u^2}$ to the basis X, Y. Hence

$$g\left(R_{\frac{\partial F}{\partial u^1}, \frac{\partial F}{\partial u^2}}(X), Y\right) = \kappa(F(u^1, u^2)) \det(dF)$$

since $\det(dF)$ can be viewed as the determinant of the base change matrix from the basis X, Y to the basis function $\frac{\partial F}{\partial u^1}, \frac{\partial F}{\partial u^2}$.

Corollary 1.3. *In the above situation, if the curvature tensor vanishes then $P_\gamma = Id$ whenever γ can be smoothly deformed to a constant curve without moving the end points. In fact - for those of you who took algebraic topology - it means that if the curvature tensor vanishes then for any γ, P_γ depends only on the (end-points preserving) homotopy type of γ.*

Examples:

1. Consider the unit sphere $S^2 \subseteq \mathbb{R}^3$. We know that its curvature κ is constantly 1. Further more every path in S^2 can be smoothly deformed to the constant path, and so we see that the parallel transport along a curve simply equals the spherical area bounded by the curve. In particular parallel transport along a geodesic should give the identity, because the area of a hemisphere is 2π.

   Now consider a latitude of hight z_0. Then it is an easy (and some what surprising) geometric exercise to see that the area in the cape above the latitude is exactly $2\pi(1 - z_0)$. Hence the total angle we get by doing parallel transport is $2\pi(1 - z_0) = -2\pi z_0 (\mod 2\pi)$. This is consistent with an exercise in assigned problems 3.

2. Consider a curve which is the boundary of the domain

   $$D = \{(x, y) | x^2 + y^2 < r, y > y_0\}$$

   this is a piece-wise smooth curve composed of the circular piece

   $$\{(x, y) | x^2 + y^2 = 1, y > y_0\}$$

and the linear piece

$$\{(x,y)|y=y_0, x^2+y^2<r\}$$

Let $X = y\frac{\partial}{\partial x}, Y = y\frac{\partial}{\partial y}$ be a global orthonormal frame.

The circular piece is a geodesic, so we know that the parallel transport along it (measured in the orthonormal frame X, Y) will be a rotation by the angle between $\gamma'(t)$ at its beginning and $\gamma'(t)$ at its end (measured in the same orthonormal basis). In particular here this angle is $2\theta_0$ where $0 \le \theta_0 \le \pi/2$ is the angle satisfying $\cos(\theta_0) = y_0$.

Also from question 8 in assigned problems 3 we know that the parallel transport along the curve $\gamma(t) = (t, y_0)$ from $t = -\sqrt{r^2 - y_0^2}$ to $t = \sqrt{r^2 - y_0^2}$ results in a rotation by an angle of $-\frac{2\sqrt{r^2-y_0^2}}{y_0} = -2\tan(\theta_0)$. Hence the total angle obtained by doing parallel transport along γ is

$$2\theta_0 - 2\tan\theta_0$$

Let us see that this agrees with our theorem. Now in the next tirgul we will show that the curvature of the Poincare upper half plane is constantly -1. Hence the integral in the theorem becomes

$$-Area(D) = -\int_{y_0}^{r} \frac{2\sqrt{r^2-y^2}}{y^2} dy$$

Substituting variables by

$$y = r\cos(\theta)$$

$$dy = -r\sin(\theta)d\theta$$

we get

$$-\int_0^{\theta_0} \frac{2\sin^2\theta}{\cos^2\theta} d\theta = -2\int_0^{\theta_0} \frac{d\theta}{\cos^2\theta} + 2\theta_0 = 2\theta_0 - 2\tan\theta_0$$

2 Applications

1. **Geodesic Triangles:**

   A geodesic triangle on Riemannian manifold is a triangle all of whose vertices are geodesics. Now it is simple to see that if we do parallel transport along a geodesic triangle with angles $\alpha_1, \alpha_2, \alpha_3$ we get total rotation

   $$\alpha = \sum_i (\alpha_i - \pi) = \sum_i \alpha_i - \pi (mod\ 2\pi)$$

   Hence we get from the theorem as a corollary a famous theorem of Gauss:

Corollary 2.1. *Let T be a geodesic triangle on a surface $M \subseteq \mathbb{R}^3$ with angles $\alpha_1, \alpha_2, \alpha_3$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \int_T \kappa \det(g)$$

In particular the usual triangles we know from Euclidean geometry whose angles sum up to π are explained by the fact that the curvature is 0. On a sphere, for example, the curvature is always 1 which means that the sum of degrees is always greater than π. In the upper half plane the curvature is always -1 which means that the sum of degrees is always smaller then π.

Differential Geometry - Tirgul 8

Yonatan Harpaz

June 7, 2009

1 Completions from the last tirgul

1.1 The Curvature of the Poincare Upper Half plane

Recall the Poincare upper half plane $\mathbb{R}^2_+ = \{(x,y)|y > 0\}$ with the metric $\frac{dx^2+dy^2}{y^2}$. We claim that it has constant curvature -1. To see this recall that the Levi-Civita connection is given by

$$\nabla_X Z = \overline{\nabla}_X Z + \Gamma(X)Z$$

where $\overline{\nabla}$ is the standard connection on $\mathbb{R}^2$ (i.e. just derivation of the coefficients) and

$$\Gamma(Z) = -\frac{1}{y}\begin{pmatrix} b & a \\ -a & b \end{pmatrix}$$

where

$$Z = a\frac{\partial}{\partial x} + f\frac{\partial}{\partial y} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Now in order to compute the curvature let us pick an orthonormal basis

$$X = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

Now we calculate

$$\nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial x}}X = \nabla_{\frac{\partial}{\partial y}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{y}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{y^2}Y$$

and

$$\nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial y}}X = \nabla_{\frac{\partial}{\partial x}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = 0$$

Combined together we get a

$$g\left(\nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial x}}X - \nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial y}}X, Y\right) = -\frac{1}{y^2}$$

Since

$$\left|\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}\right| = \left|\tfrac{1}{y}X \quad \tfrac{1}{y}Y\right| = \frac{1}{y^2}$$

we get that the curvature is constantly -1.

1.2 Hyperbolic structures on compact surfaces

Consider the surface Σ with "two handles" (draw on board). We would like to put a Riemannian structure on Σ which has constant curvature -1. In order to do this we will cut Σ along 4 circles which meet at a unique point in the way showed on the board. We will obtain an open polygon P with 8 edges.

In order to put a riemannian metric on Σ we will put a Riemannian metric on this open polygon. We will then need to show that this metric extends to a metric o Σ. The Riemannian metric we will put P will be induced by an embedding $P \hookrightarrow \mathbb{R}^2_+$ in the Poincare upper half plane as a regular geodesic 8-gon (i.e. a polygon with 8 edges which are geodesics of equal length and equal angles between them).

Note that in Euclidean space, the some of angles in an 8-gon is 6π. From a similar argument to what we did for geodesic triangles, the sum of angles in a geodesic 8-gon P in a general Riemannian surface is

$$(*)\ 6\pi + \int_P \kappa \det(g)$$

Now in order for P to glue back on Σ (see drawing on board) we need the some of angles to be 2π. Hence we need the integral of the curvature to be -4π. This is why, for example, we can't put a metric on Σ with constant positive curvature.

We shall now want to convince the reader of the existence of regular polygons in $\mathbb{R}^2_+$ whose some of angles is 2π. We will do this heuristically as follows: since regular polygons posses a rotational symmetry we will wish to work with a model for $\mathbb{R}^2_+$ which has a more obvious rotational symmetry. Recall from the course complex functions that the mobius map

$$\rho(z) = \frac{z-i}{z+i}$$

maps the upper half plane diffeomorphically to the unit disc D. Hence we can use it to induce a Riemannian structure on D making ρ into an isometry (clearly there is a unique such Riemannian metric. In fact, one can think of $(\mathbb{R}^2_+, \phi)$ as a chart on D, making x, y coordinates on D with the Poincare metric coefficients).

Now recall that the Mobius maps with real coefficients were isometries of $\mathbb{R}^2_+$ and we saw that for each $p, q \in \mathbb{R}^2_+$ and $v \in T_p\mathbb{R}^2_+, u \in T_q\mathbb{R}^2_+$ of the same norm there exists a Mobius mao ρ such that $\rho(p) = q$ and $d\rho(v) = u$. From uniqueness geodesics we get that we can map any geodesic to any other geodesic using a mobius map, and similarly any geodesic segment to any other geodesic segment of the same length.

Note also that for any geodesic we have an orientation reversing isometry preserving that geodesic but switching both its sides (for the vertical geodesic $x = 0$ it is the map $\sigma(x, y) = (-x, y)$ and for any geodesic we can conjugate σ with a real Mobius map sending that geodesic to $x = 0$). Hence we can send any geodesic segment to any other geodesic segment of the same length such that a certain "side" of our geodesic segment is mapped to any of the two sides of the other segment. Since D is isometric to $\mathbb{R}^2_+$ it also posses these properties.

Now recall that every Mobius map preserves generalized circles and angles (in the usual sense on $\mathbb{C}$). Hence every circular arc in D that meets ∂D in right angles is mapped by ρ^{-1} to either a vertical line or a half circle that meet the real line in right angles. This means that every such an arc in D is a geodesic.

From all this we obtain that for each $0 < r < 1$ the sequence of points

$$p_n = r \operatorname{cis}\left(\frac{2\pi i n}{8}\right) \in D$$

form (in that order) the vertices of a unique regular geodesic 8-gon. It is clear that the sum of angles of this 8-gon tends to 0 when $r \longrightarrow 1$, and tends to 6π as r approaches 0 (this can be seen from formula $(*)$ as the area clearly approaches 0). Hence there exists an r such that this regular 8-gon has a sum of angles of exactly 2π. This is the polygon we will use.

In order to show that we can extend the metric to Σ let p be a point on one of circles we have removed (but not the point that is common to all the circles). There are exactly two points q_1, q_2 on the boundary of P mapping to p and they sit on different edges E_1, E_2. Let $U \subseteq \Sigma$ be a small neighborhood of p. Then U is partitioned by the removed circles into two halves. The closure of the first half forms a neighborhood U_1 of q_1 in $\overline{P}$ and the second a neighborhood U_2 of q_2 (see drawing).

Since all the edges are of the same length there exists an isometry τ of D (with respect to the poincare metric) which maps E_1 to E_2 such that the P-inward direction is mapped to the P-outward direction. Hence in particular $U_2 \cup \tau(U_1)$ forms an open neighborhood in D (!) of q_2. Hence we can use this to extend our Rimannian metric consistently to all of U. A similar argument (only a bit more involved) can be made for the point common to all the circles. We will of course have to use the fact that the sum of angles is exactly 2π.

1.3 Convex Neighborhoods and Applications

Let M be a riemannian manifold. We recall the definition of a strongly convex neighborhood:

Definition 1.1. A subset $U \subseteq M$ is called a **strongly convex neighborhood** if it is open and

1. For every $p, q \in \overline{U}$ there exists a unique minimizing geodesic γ connecting p, q.

2. The interior of this minimizing geodesic is contained in U.

Definition 1.2. A set $U \subseteq M$ is called **totally normal** if there exists a $\delta > 0$ such that for every $p \in U$, the exponential map exp restricted to the ball $B_\delta \subseteq T_pM$ is a diffeomorphism on its image $\exp(B_\delta)$ and $\exp(B_\delta)$ contains U.

We shall now show some applications of the following theorem:

Theorem 1.3. *Every point $p \in M$ has a neighborhood which is a strongly convex and totally normal.*

Applications

Definition 1.4. A topological space X is called contractible if there exists a continuous map $f : I \times X \longrightarrow X$ such that $f(1, x) = x$ and $f(0, x) = x_0$ does not depend on x.

As a first application we shall prove question 4 in assigned problems 4 - we shall show that every Riemannian manifold has a cover $M = \bigcup U_i$ such that any finite intersection of the U_i's is contractible.

By theorem 1.2 we can cover M by strongly convex totally normal open sets. We shall show that this covering satisfies the required property. This will follow from the following two lemmas:

Lemma 1.5. *A finite intersection of strongly convex totally normal neighborhoods is strongly convex and totally normal.*

Proof. The property of being totally normal is inherited by open subsets so its no problem. As for strongly convex, we shall prove it for the intersection of two strongly convex neighborhoods and the rest will follow by induction. Let U_1, U_2 be two strongly convex sets. Let

$$p, q \in \overline{U_1 \cap U_2} \subseteq \overline{U_1} \cap \overline{U_2}$$

then since U_1 is strongly convex there exists a unique minimizing geodesic from p to q. Since both U_1, U_2 are strongly convex the interior of this geodesic must lie in both U_1, U_2. Hence it lies in their intersection and we are done. □

Lemma 1.6. *Any non-empty strongly convex totally normal neighborhood is contractible.*

Proof. Let U be a non-empty strongly convex totally normal neighborhood and take a $p \in U$. Let $\delta > 0$ be such that the exponential map exp is a diffeomorphism when restricted to $B_\delta \subseteq T_pM$ and such that $U \subseteq \exp(B_\delta)$. Then $V = \exp^{-1}(U)$ is open in B_δ and exp induces by restriction a diffeomorphism from V to U.

Now let $q \in U$ be any point. Since $U \subseteq \exp(B_\delta)$ there exists a unique minimizing geodesic from p to q, and it is given by the image under exp of the linear segment in B_δ from 0 to $\exp^{-1}(q)$. Since U is strongly convex and $p, q \in U$ we get that this geodesic must lie inside U. Hence the linear segment from 0 to $\exp^{-1}(q)$ lies in V.

This means that for every $t \in [0, 1]$, the set

$$tV = \{tv | v \in V\}$$

is contained in V. Hence V is contractible by the function $H : I \times V \longrightarrow V$ given by $(t, v) \mapsto tv$. Since U is diffeomorphic to V it is contractible as well. $\square$

Differential Geometry - Tirgul 10

Yonatan Harpaz

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1 The Energy Functional and Curvature

Let M be a Riemannian manifold with Riemannian metric g. For $p, q \in M$ let $C(p,q)$ be the space of smooth maps $\gamma : [0,1] \longrightarrow M$ such that $\gamma(0) = p, \gamma(1) = q$. Define the **energy functional** to be the map $E : C(p,q) \longrightarrow \mathbb{R}$ given by

$$E(\gamma) = \int_0^1 g(\gamma'(t), \gamma'(t))dt$$

This notation comes of course from physics (where this expression will actually be called the **action** of the curve, and not the energy). A word of mathematical caution is in order here: the energy functional does depend on the specific parametrization of γ. Hence we will always work with parameterized curves.

We first wish to explain the connection between the energy functional and geodesics. The idea is that geodesics from p to q are precisely the critical points of E in $C(p,q)$.

A critical point of a function into $\mathbb{R}$ is a point where the differential vanishes. The space $C(p,q)$ is not a manifold, so apriori we don't know how to define this differential.

What we will do is we will consider small "smooth" paths $F : (-\varepsilon, \varepsilon) \longrightarrow C(p,q)$ such that $F(0) = \gamma$ is a fixed path. Then $E \circ F$ will be a function from $\mathbb{R}$ to $\mathbb{R}$ and we can ask if its derivative vanishes. If $C(p,q)$ was a regular manifold then this vanishing (for all small paths) would have been equivalent to the vanishing of the differential.

How will we define what it means for a function $F : (-\varepsilon, \varepsilon) \longrightarrow C(p,q)$ to be smooth? Note that any such function defines a function $\widetilde{F} : (-\varepsilon, \varepsilon) \times [0,1] \longrightarrow M$. We say that F is smooth if the corresponding $\widetilde{F}$ is smooth. Such an F satisfying $F(0) = \gamma$ is called a **variation** of γ. Note that since F is a map into $C(p,q)$ then $F(s,0) = p$ and $F(s,1) = q$ for all $s \in (-\varepsilon, \varepsilon)$.

Let us assume for simplicity that $\widetilde{F}$ is an embedding when restricted to $(-\varepsilon, \varepsilon) \times (0,1)$ and consider

$$T = d\widetilde{F}\left(\frac{\partial}{\partial t}\right)$$

$$S = d\widetilde{F}\left(\frac{\partial}{\partial s}\right)$$

as vector fields on the image $U = \widetilde{F}\left((-\varepsilon, \varepsilon) \times (0,1)\right)$, on which we use t, s as coordinates (this assumption is not necessary but it will simplify the analysis for us). Note that in this formalism T is not well defined at the limit points p and q (but is bounded there), and S converges to 0 there (because $\widetilde{F}(s,0)$ and $\widetilde{F}(s,1)$ are constant). This means in particular that for every bounded vector field V on U we have

$$\int_0^1 \frac{\partial}{\partial t} g\left(S(s,t), V(s,t)\right) dt = 0$$

which implies that (here and below we drop the explicit depends on s, t for convenience of writing):

$$(*) \quad \int_0^1 g\left(\nabla_T S, V\right) dt = -\int_0^1 g\left(S, \nabla_T V\right) dt$$

Now to the calculation. We want to differentiate the function

$$E(F(s)) = \int_0^1 g\left(T(s,t), T(s,t)\right) dt$$

with respect to s. This gives:

$$\frac{\partial(E \circ F)}{\partial s} = \int_0^1 \frac{\partial}{\partial s}(g\left(T, T\right))dt = 2\int_0^1 g\left(\nabla_S T, T\right) dt =$$

$$2\int_0^1 g\left(\nabla_T S, T\right) dt = -2\int_0^1 g\left(S, \nabla_T T\right) dt$$

where we have use the symmetry of ∇ and the $(*)$ principle. Setting $s = 0$ and denoting $V(t) = S(0,t)$ we get

$$\left.\frac{\partial(E \circ F)}{\partial s}\right|_{s=0} = -2\int_0^1 g\left(V(t), \nabla_{\gamma'(t)}\gamma'(t)\right) dt$$

Then vector field $V(t)$ is called the **variational field** of F. It is reasonable to consider the vector field $V(t)$ as the tangent vector to F in $C(p,q)$ and to think of the space of vector fields along γ vanishing at the end points as the tangent space to $C(p,q)$. This perception is strengthened by the fact that for every such vector field $V(t)$ there exists a smooth variation $F : (-\varepsilon, \varepsilon) \times [0,1] \longrightarrow M$ such that

$$d\widetilde{F}\left(\frac{\partial}{\partial s}\right)(0,t) = V(t)$$

The reason is quite simple: since the image of γ is a compact set there exists an $\varepsilon > 0$ such that $\exp_{\gamma(t)}$ is well defined on a ball of radius $\varepsilon|V(t)|$ for all $t \in [0,1]$. We can then define

$$F(s,t) = \exp_{\gamma(t)}(sV(t))$$

and we get

$$d\widetilde{F}\left(\frac{\partial}{\partial s}\right) = V(t)$$

We then conclude that $\frac{\partial(E\circ F)}{\partial s} = 0$ for all variations F if and only if

$$\int_0^1 g\left(V(t), \nabla_{\gamma'(t)}\gamma'(t)\right) dt = 0$$

for all vector fields $V(t)$ along γ vanishing at the end points. This is easily seen to be equivalent to the fact that $\nabla_{\gamma'(t)}\gamma'(t) = 0$ which is just the condition for γ to be a geodesic.

So now we know that the critical points of E are the geodesics. What about geodesic which are length minimizing?

Recall the Swartz inequality

$$\left|\int_0^1 fg dt\right| \leq \sqrt{\int_0^1 f^2 dt}\sqrt{\int_0^1 g^2 dt}$$

Substituting in $f(t) = |\gamma'(t)|$ and $g(t) = 1$ one gets

$$\int_0^1 |\gamma'(t)| dt \leq \sqrt{\int_0^1 |\gamma'(t)|^2 dt} = \sqrt{E(\gamma)}$$

which means that the length of a curve is bounded from above by the square root of the energy (note that this is true even though the energy depends on the parametrization and the length doesn't, as long as we parameterizing our curve along a unit time interval).

Now if $|\gamma'(t)|$ is constant, for example if γ is a geodesic, then clearly the length **equals** $\sqrt{E(\gamma)}$. Now suppose that γ is a length minimizing geodesic. Then any other curve from p to q has a longer length and so a bigger energy. Hence in that case γ is also a minima of E.

If γ is a critical point of E, them the property of being a local minima of E can be determined from the second derivative of E in various deformations, in particular, all deformations must have positive second derivative.

Let us now calculate the second derivative of E along a deformation of a **geodesic** γ.

$$\frac{\partial^2(E\circ F)}{\partial s^2} = -2\int_0^1 \frac{\partial}{\partial s} g\left(S, \nabla_T T\right) dt = -2\int_0^1 g\left(\nabla_S S, \nabla_T T\right) dt - 2\int_0^1 g\left(S, \nabla_S \nabla_T T\right) dt$$

Since γ is a geodesic the first term vanishes at $s = 0$. Let us then calculate only the second term. Note that T and S are coordinate vector field and so don't have Lie brackets. Hence

$$-2\int_0^1 g\left(S, \nabla_S \nabla_T T\right) dt = -2\int_0^1 g\left(S, R_{T,S} T\right) dt - 2\int_0^1 g\left(S, \nabla_T \nabla_S T\right) dt =$$

$$-2\int_0^1 g\left(S, R_{T,S}T\right)dt + 2\int_0^1 g\left(\nabla_T S, \nabla_S T\right)dt =$$

$$(**) \quad -2\int_0^1 g\left(S, R_{T,S}T\right)dt + 2\int_0^1 g\left(\nabla_T S, \nabla_T S\right)dt =$$

$$-2\int_0^1 g\left(S, R_{T,S}T\right)dt - 2\int_0^1 g\left(S, \nabla_T \nabla_T S\right)dt$$

Hence we get

$$\frac{\partial^2(E\circ F)}{\partial s^2} = -2\int_0^1 g\left(S, R_{T,S}T + \nabla_T\nabla_T S\right)dt$$

Note that this expression at $s = 0$ depends only on $V(t) = S(0,t)$ and can be written as

$$\frac{\partial^2(E\circ F)}{\partial s^2} = -2\int_0^1 g\left(V(t), R_{\gamma'(t),V(t)}\gamma'(t) + \nabla_{\gamma'(t)}\nabla_{\gamma'(t)}V(t)\right)dt$$

It is also referred to as **second order variation** of the energy along $V(t)$.

Now from line $(**)$ see that if the sectional curvature is always negative then the second variation is always positive, resulting in the fact that every geodesic is a local minima of the energy. In fact, it can be show that in negatively curved manifold every geodesic is length minimizing.

When $V(t)$ satisfies

$$R_{\gamma'(t),V(t)}\gamma'(t) + \nabla_{\gamma'(t)}\nabla_{\gamma'(t)}V(t) = 0$$

we call it a **Jacobi field**. Jacobi fields correspond to deformations which (up to second order approximation) don't change the length of the curve. In particular it can be shown that every Jacobi field is a variational field of a variation F for which $F(s)$ is a geodesic for every $s \in (-\varepsilon, \varepsilon)$.

Let us now use this formula in order to prove a very nice theorem relating the curvature of a Riemannian manifold to its "size":

Theorem 1.1. *Let M be a complete Riemannian manifold for which the sectional curvature is bounded below by some constant $\kappa > 0$. Then M is compact and its diameter (maximal distance between points) is bounded by $\frac{\pi}{\sqrt{\kappa}}$.*

Proof. Since M is complete every two points $p, q \in M$ can be joined by a length minimizing geodesic γ. Let l be the length of γ and assume that $l > \frac{\pi}{\sqrt{\kappa}}$. Suppose that γ is parameterized on $[0,1]$ and so $|\gamma'(t)| = l$ for all t.

Let $E(t)$ be a parallel field of unit length along γ which is orthogonal to $\gamma'(t)$ (this is possible since γ is a geodesic and so $\gamma'(t)$ is parallel). Define

$$V(t) = \sin(\pi t)E(t)$$

Put $T(t) = \frac{\gamma'(t)}{|\gamma(t)|}$. We then calculate the second order variation of the energy along $V(t)$:

$$-2\int_0^1 g\left(V(t), R_{\gamma'(t),V(t)}\gamma'(t) + \nabla_{\gamma'(t)}\nabla_{\gamma'(t)}V(t)\right) dt =$$

$$-2\int_0^1 l^2 \sin^2(\pi t) g(E(t), R_{T(t),E(t)}T(t))dt + 2\int_0^1 \pi^2 \sin^2(\pi t) g(E(t), E(t))dt$$

Since $|T(t)| = |E(t)| = 1$ and $E(t)$ is orthogonal to $T(t)$ we see that

$$g(E(t), R_{T(t),E(t)}T(t))$$

is the sectional curvature of the hyperplane spanned by $T(t), E(t)$, and so is bounded below by κ. Further more $g(E(t), E(t)) = 1$ and so the above term is bounded from above by

$$-2\int_0^1 l^2\kappa \sin^2(\pi t)dt + 2\int_0^1 \pi^2 \sin^2(\pi t)dt = -l^2\kappa + \pi^2 < -\pi^2 + \pi^2 = 0$$

Hence γ cannot be a local minima for the energy, in contrast to the fact that it is a length minimizing geodesic. Hence the distance between every two points is at most $\frac{\pi}{\sqrt{\kappa}}$. Now since M is complete the exponential maps are defined on all tangent vectors, and so we see that M is the image of $\exp_p$ when restricted to a ball of radius $\frac{\pi}{\sqrt{\kappa}}$. Hence M is compact and we are done. $\square$