Homotopy Obstructions to Sections

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1 Introduction

The question of rational points has a natural generalization. Let S be a base scheme and $p : X \longrightarrow S$ a scheme over S. One then naturally wishes to know whether there exist a section for p, i.e. a map $s : S \longrightarrow X$ such that $p \circ s = Id$. In the case $S = \mathrm{spec}(K)$ for a field K these sections correspond to K-rational points, and if $S = \mathrm{spec}(O_K)$ for some integer ring O_K then these sections correspond to K-integral points. In general one can just think of them as S-points.

In this lecture we wish to suggest a relative version of the étale homotopy type which we believe can be useful to study S-points. In particular one can obtain obstructions to the existence of S-points. In the case of $S = \mathrm{spec}(O_K)$ there are known obstructions to the existence of integral points. We currently do not know if one can construct in this way an obstruction which is stronger than all known obstructions.

This lecture is based on work in progress with Tomer Schlank.

1.1 Sheaves of Sets and Simplicial Sheaves

Let $\mathcal{C}$ be a Grothendieck site. The example to keep in mind here is the étale site of a scheme S, and coverings are simply given by surjective maps $U \longrightarrow V$ over S.

A **presheaf of sets** is simply a functor form $\mathcal{C}^{op}$ to the category of sets, which we will denote by $\mathcal{S}ets$. A presheaf $F : \mathcal{C}^{op} \longrightarrow \mathcal{S}ets$ is called a **sheaf of sets** if for every $U \in \mathcal{C}$ and a covering $U \longrightarrow X$ in $\mathcal{C}$ the following diagram

$$F(X) \longrightarrow F(U) \rightrightarrows F(U \times_X U)$$

(where the two arrows come from the two natural maps $V \times_U V \longrightarrow V$) is an equalizer diagram. We denote by $Sh(C)$ the category of sheaves of sets on C.

Examples: Assume that $\mathcal{C}$ is the étale site of a scheme X. Here are two familiar examples:

1. If X is a point over an algebraically closed field then a sheaf F of sets is freely determined by specifying the value $F(X)$. In other words, the category $Sh(X)$ is then equivalent to the category of sets.

2. If X is a point over an general field K with absolution Galois group Γ then the category of sheaves of sets is equivalent to the category of sets A with an action of Γ such that each $a \in A$ has a finite orbit. This is very similar to the situation with sheaves of abelian groups.

Now a sheaf of sets is something that can be though of a family of sets parameterized by the points of X. For example, in étale site of a scheme S the representable functors $F_U(V) \mapsto \mathrm{Hom}_{\mathcal{C}}(V, U)$ are actually sheaves of sets, and the corresponding family of sets are the fibers of the map $U \longrightarrow X$.

We now wish to generalize this to describe families of **simplicial sets** parameterized by points in X. If we had such an object we would get in particular families of sets in each dimension n. The simplest way to capture such information is through the concept of a simplicial sheaf, i.e. a functor from Δ^{op} to $Sh(X)$.

Now if we have two simplicial sheaves we want to understand what is a good notion of maps between them. It turns out that actual maps of simplicial objects is too strict a notion. We need to find some sort of derived notion. In order to make this more clear lets recall a more familiar case.

Suppose we were working with complexes of sheaves. Then we would have the notion of the derived mapping complex. The cohomologies of this complex would give the familiar Ext functors. In particular the zero'th cohomology would give the set of derived mappings up to homotopy. How would we describe these objects? one way is get into homological algebra and consider injective resolutions and so on. But even if this is too complicated, one can always know that has encountered a derived mapping if he found a mapping between two complexes which are quasi-isomorphic to the original complexes. In fact Ext_0 can be described as the morphism set obtained when one takes the category of complexes and inverts quasi-isomorphisms.

Let us now return to simplicial sheaves. In this case instead of a complex of derived mappings we are goings to have a simplicial set of derived mappings. We would refer to it as the derived mapping space. The homotopy groups of this simplicial set would be analogous to the Ext functors and in particular π_0 of this simplicial set will be the set of derived mappings up to homotopy. Now we will not give an explicit description of this derived mapping space, but we will explain what is analogous to quasi-isomorphism. If we ignore for a moments the problem of base point we can take a simplicial sheaf and generated from it a sheaf groups by applying the π_n functor and then sheafifying. We call these the associated sheaves of homotopy groups. We now define a weak equivalence of simplicial sheaves to be a map which induces an isomorphism on the sheaves of homotopy groups. The reader should rest assured that the base point issue can be properly overcome.

The most important examples of weak equivalences are obtained from hypercoverings. If $U_\bullet \longrightarrow X$ is a hypercovering then corresponding map of simplicial sheaves is a weak equivalence. In fact, one can show that under certain conditions on a simplicial sheaf F, all derived maps from the constant simplicial sheaf X to F are given by maps $U_\bullet \longrightarrow F$ where $U_\bullet \longrightarrow X$ is a hypercovering.

2 The Relative Étale Homotopy Type

Let S be a base scheme and $p : X \longrightarrow S$ a scheme over S. The map p induces a functor from the étale site of S to the étale site of X given by $U \mapsto U \times_S X$. We denote this functor by p^*. Now in fairly general circumstances there exists a left adjoint p^*, denoted by $f_!$. For example if $S = \mathrm{spec}(K)$ for a field K then $p_! U$ is the 0-dimensional scheme of connected components of U.

Note that the étale site is a full subcategory of the category of sheaves (embedding given by representable sheaves) and in this situations the functors p^* and $p_!$ are actually defined for all sheaves.

We now come to the definition of the relative étale homotopy type. Assume as above that $p_!$ exists. For each hypercovering $U_\bullet \longrightarrow X$, we apply the functor $p_!$ level wise and get a simplicial object in the étale site of S, or a simplicial sheaf. We denote this simplicial sheaf by Π_U. Let I be the homotopy category of hypercoverings of X. The association $U \mapsto \Pi_U$ gives a functor from I to the homotopy category simplicial sheaves (this times actual maps and actual simplicial homotopies). The pro-object $\{\Pi_U\}$ is the relative étale homotopy type.

3 Obstruction Theory

Suppose now that we have a section $s : S \longrightarrow X$. From the adjunction of p^* and $p_!$ we get for every hypercovering $U_\bullet \longrightarrow X$ a natural map

$$U_\bullet \longrightarrow p^* p_! U_\bullet$$

Composing this map with s^* (pulling back via s) we get a natural map of simplicial sheaves over S:

$$s^* U_\bullet \longrightarrow s^* p^* p_! U_\bullet$$

But $s^* p^* = Id$ because s is a section so this gives a natural map

$$s^* U_\bullet \longrightarrow p_! U_\bullet = \Pi_U$$

It is not hard to verify to the pullback of a hypercovering is hypercovering so we got a map from a hypercovering of S to Π_U. This is in particular a derived map between the constant simplicial sheaf on S to Π_U, or what can also be called a **derived section**. Hence if one shows that Π_U does not have any derived sections then there couldn't be an S-point.

One can construct obstructions to the existence of a derived section in the following way. Let F be a simplicial sheaf on S and $V_\bullet$ a hypercovering of S. Thinking of F as a presheaf of simplicial sets we can compose it with V, considered as a functor from Δ^{op} to $S_{\acute{e}t}$. We get a functor from Δ to simplicial sets:

$$F(V_0) \rightrightarrows F(V_1) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} F(V_2) \cdots$$

It can be shown that if there is map of simplicial sheaves from V to F if and only if there exist a map of cosimplicial simplicial sets from $\Delta^\bullet$ to $F(V_\bullet)$. But for this there is an obstruction theory by interpreting this is the topological notion of a **homotopy limit**. There obstructions leave in

$$\check{H}^{n+1}(V_\bullet, \pi_n(F))$$

Since we just want there to be a map from **some** hypercovering we take the direct limit of these obstructions and get an obstruction in

$$H^{n+1}(S, \pi_n(F))$$

where $\pi_n(F)$ is the sheaf of homotopy groups associated with F.