Elliptic Regularity - Appendix B.4

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Let M be a smooth manifold of dimension $2n$ with a smooth almost complex structure $J : TM \longrightarrow TM$. Let $L \subseteq M$ be a closed sub-manifold of dimension n which is totally real with respect to J, i.e.

$$J_p(T_pL) \cap T_pL = \{0\}$$

for every $p \in L$ (in many interesting examples M will carry a J-compatible simplectic structure and L will be a lagrangian sub-manifold but we don't need to assume this here). Let Σ be a Riemann surface with boundary $\partial\Sigma$. Note that the boundary of a Riemann surface is always totally real with respect to its complex structure.

We are interested in J-holomorphic maps $u : \Sigma \longrightarrow M$ which send $\partial\Sigma$ to L. These are maps which solves the first order non-linear boundary value problem

$$(1)\;\; du_p(iv) = J_{u(p)}(du_p(v)), \quad u(\partial\Sigma) \subseteq V$$

where $p \in \Sigma, v \in T_p\Sigma$.

Note that one can also consider **weak** solutions to (1). These are functions $u \in W^{k,p}(\Sigma, M)$ (with $k \geq 1$) which satisfy $(*)$ almost everywhere. In this lecture we will show that weak solutions to (1) are actually smooth. This is the content of Corollary 0.5 which will be our final destination.

Let us begin by localizing the problem so we can write it with coordinates. Let $U \subseteq \Sigma$ be a small subset which is bi-holomorphic to an an open subset $\Omega \subseteq \mathbb{H} \cup \mathbb{R}$ where $\mathbb{H}$ is the upper half plane (open subsets of $\mathbb{H} \cup \mathbb{R}$ are the local models of Reimann surfaces with boundary. In particular this identifies $U \cap \partial\Sigma$ with $\Omega \cap \mathbb{R}$). Let $z = s + it$ be the holomorphic coordinate on Ω.

Assume that U is small enough so that $u(U)$ is contained in some $V \subseteq M$ which is diffeomorphic to $\mathbb{R}^{2n}$. We can further assume that this diffeomorphism identifies $V \cap L$ with $\mathbb{R}^n \times \{0\}$. The restriction of J to V can then be identified with a function $J : \mathbb{R}^{2n} \longrightarrow M_{n\times n}$ such that $(J_x)^2 = Id$. Then we are looking for functions $u : \Omega \longrightarrow \mathbb{R}^{2n}$ which satisfy the boundary value problem

$$(2)\;\; \partial_s u + (J \circ u)(\partial_t u) = 0, \quad u(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}^n \times \{0\}$$

This boundary problem is non-linear because $J \circ u$ depends non-linearly on u. In order to simplify the problem we will **linearize** it: let u be a solution of the above problem and consider the map $\mathcal{J} = J \circ u : \Omega \longrightarrow M_{n\times n}$. Then we see that u solves **linear** boundary problem:

$$(3)\ \ \partial_s u + \mathcal{J}\partial_t u = 0, \quad u(\Omega \cap \mathbb{R})$$

Note that here we think of $\mathcal{J}$ as a function on Ω and "forget" that we constructed it using u itself. Such equations will be called (linear) Cauchi-Riemann equations. Now for reasons that will be clear later and will be useful for us to consider the more general case of an non-homogenous Cauchi-Riemann equation, in which we add an non-homogenous term:

$$(4)\ \ \partial_s u + \mathcal{J}\partial_t u = \eta, \quad u(\Omega \cap \mathbb{R})$$

Now note the following fun property: if u is a solution of homogenous Cauchi-Riemann equation, then the derivative $v = \partial_s u$ satisfies the non-homogenous equation (4) with $\eta = -\partial_s J \partial_t u$. This property will be useful later.

Now since (4) is a linear equation we can now consider even weaker solutions, i.e. solutions which are in $L^p_{\text{loc}}(\Omega)$. Note that if u is a smooth solution of (4) and $\phi \in C_0^\infty(\Omega, \mathbb{R}^{2n})$ such that $\phi(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}^n \times \{0\}$ then

$$\int_\Omega \langle u, \partial_s \phi + \mathcal{J}^t \partial_t \phi \rangle = \int_\Omega \langle \eta + (\partial_t J)u, \phi \rangle$$

This equality makes since for a general $u \in L^p_{\text{loc}}(\Omega, \mathbb{R}^{2n})$ so we can use to define when a function in $L^p_{\text{loc}}(\Omega, \mathbb{R}^{2n})$ is a weak solution of (4).

We are interested in regularity results for Cauchi-Riemann equations of types $(1) - (4)$. Our first theorem is the opening step towards regularity for the non-homogenous Cauchi-Riemann equations (4).

Theorem 0.1. *Let $\Omega \subseteq \mathbb{H} \cup \mathbb{R}$ be an open subset and $2 < p \leq \infty$, $0 < q \leq \infty$ and $0 < r \leq \infty$ real numbers such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

Assume that $u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^{2n})$ is a weak solution of (4) for some $\mathcal{J} \in W^{1,p}_{\text{loc}}(\Omega, M_{n\times n})$ and $\eta \in L^r_{\text{loc}}(\Omega, \mathbb{R}^{2n})$. Then u is in $W^{1,r}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$.

Proof. Let $\psi \in C_0^\infty(\Omega, \mathbb{R}^{2n})$ be such that

$$\psi(s, 0) \in \mathbb{R}^n \times \{0\}$$

and

$$\partial_t \psi(s, 0) \in \{0\} \times \mathbb{R}^n$$

consider the function

$$\phi = \partial_s \psi - \mathcal{J}^T \partial_t \psi$$

is in $W^{1,p}(\Omega, \mathbb{R}^{2n})$ and satisfies the boundary condition $\phi(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}^n \times \{0\}$. Further more one has

$$\partial_s \phi + \mathcal{J}^T \partial_t \phi = \partial_s \left[\partial_s \psi - \mathcal{J}^T \partial_t \psi \right] + \mathcal{J}^T \partial_t \left[\partial_s \psi - \mathcal{J}^T \partial_t \psi \right] =$$

$$\partial_{s,s}\psi - \partial_s(\mathcal{J}^T\partial_t\psi) + \mathcal{J}^t\partial_{t,s}\psi - \mathcal{J}^T\partial_t(\mathcal{J}^T\partial_t\psi) =$$
$$\partial_{s,s}\psi - (\partial_s\mathcal{J}^T)\partial_t\psi - \mathcal{J}^T\partial_{s,t}\psi + \mathcal{J}^T\partial_{s,t}\psi - \mathcal{J}^T\partial_t(\mathcal{J}^T)\psi - \mathcal{J}^T\mathcal{J}^T\partial_{t,t}\psi =$$
$$\triangle\psi - (\partial_s\mathcal{J}^T)\partial_t\psi - \mathcal{J}^T\partial_t(\mathcal{J}^T)\partial_t\psi$$

so $\partial_t\mathcal{J}$ and $\mathcal{J}$ anti-commute at every point.

Now since u is a weak solution we have

$$\int_\Omega \left\langle \partial_s\phi + \mathcal{J}^T\partial_t\phi, u \right\rangle = -\int_\Omega \langle \phi, \eta + (\partial_t\mathcal{J})u\rangle = -\int_\Omega \left\langle \partial_s\psi - \mathcal{J}^T\partial_t\psi, \eta + (\partial_t\mathcal{J})u \right\rangle =$$
$$-\int_\Omega \langle \partial_s\psi, \eta + (\partial_t\mathcal{J})u\rangle + \int_\Omega \left\langle \mathcal{J}^T\partial_t\psi, \eta + (\partial_t\mathcal{J})u \right\rangle$$

This means that

$$\int_\Omega \langle \triangle\psi, u\rangle =$$
$$-\int_\Omega \langle \partial_s\psi, \eta + (\partial_t\mathcal{J})u\rangle + \int_\Omega \left\langle \mathcal{J}^T\partial_t\psi, \eta + (\partial_t\mathcal{J})u \right\rangle + \int_\Omega \left\langle (\partial_s\mathcal{J}^T)\partial_t\psi, u \right\rangle + \int_\Omega \left\langle \mathcal{J}^T\partial_t(\mathcal{J}^T)\partial_t\psi, u \right\rangle =$$
$$-\int_\Omega \langle \partial_s\psi, \eta + (\partial_t\mathcal{J})u\rangle + \int_\Omega \langle \partial_t\psi, \mathcal{J}\eta + (\partial_s\mathcal{J})u + ((\partial_t\mathcal{J})\mathcal{J} + \mathcal{J}(\partial_t\mathcal{J}))u\rangle$$

Since $\mathcal{J}^2$ is a constant function we get that

$$(\partial_t\mathcal{J})\mathcal{J} + \mathcal{J}(\partial_t\mathcal{J}) = \partial_t(\mathcal{J}^2) = 0$$

and so

$$\int_\Omega \langle \triangle\psi, u\rangle = -\int_\Omega \langle \partial_s\psi, \eta + (\partial_t\mathcal{J})u\rangle + \int_\Omega \langle \partial_t\psi, \mathcal{J}\eta + (\partial_s\mathcal{J})u\rangle$$

Now by Holder's inequality we get that both $\eta + (\partial_t\mathcal{J})u$ and $\mathcal{J}\eta + (\partial_s\mathcal{J})u$ are in $L^r_{\mathrm{loc}}(\Omega)$. Now if $\Omega \cap \mathbb{R} = \emptyset$ then we can simply apply our regularity results for the Laplacian operator and get that $u \in W^{1,r}_{\mathrm{loc}}(\Omega)$. If $\Omega \cap \mathbb{R} \neq \emptyset$ then we need to deal with the boundary points. The way to do this is two replace Ω with

$$\Omega_0 = \{s + it \in \mathbb{C} | s + i|t|\} \in \Omega$$

and extend ψ and ϕ from Ω to Ω_0. We can then apply the same regularity result to Ω_0 instead of Ω. The details are left to the reader.

It is left to show that $u(\Omega \cap \mathbb{R}) \in \mathbb{R}^n \times \{0\}$. This is written on page 537 and we leave it for homework reading.

□

Our next theorem takes the regularity of (4) one step further:

Theorem 0.2. *Let $\Omega \subseteq \mathbb{H} \cup \mathbb{R}$ be an open subset, $2 < p \leq \infty$ a real number and $0 < l$, $0 \leq k \leq l$ integers. Assume that $u \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ is a weak solution of (4) for some $\mathcal{J} \in W^{l,p}_{\mathrm{loc}}(\Omega, M_{n\times n})$ and $\eta \in W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$. Then u is in $W^{k+1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$.*

Proof. We will give the proof assuming $\Omega \cap \mathbb{R} = \emptyset$. Before we go to the prove we are going to prove a quick lemma:

Lemma 0.3. *Let $\Omega \subseteq \mathbb{R}^2$ be an open subset. Let $p > 2$ and $1 < r \leq p$ be real numbers and $f \in W^{1,p}(\Omega), g \in W^{1,r}(\Omega)$ functions. Then the product fg is in $W^{1,r}_{\text{loc}}(\Omega)$.*

Proof. Note that $d(fg) = (df)g + f(dg)$ so it is enough to show that $(df)g, f(dg) \in L^r_{\text{loc}}(\Omega)$. Since $p > 2$ we get from Sobolev embedding theorems that f is continuous. Since $g \in W^{1,r}(\mathbb{R}^2)$ we get that $f(dg)$ is in $L^r_{\text{loc}}(\Omega)$.

For $(df)g$ we separate to the cases $r > 2$ and $r \leq 2$. If $r > 2$ then by Sobolev embedding g is continuous. Since $p \geq r$ and $df \in L^p(\Omega)$ we get that $df \in L^r_{\text{loc}}(\Omega)$ and so $(df)g \in L^r_{\text{loc}}(\Omega)$.

If $r \leq 2$ then r has to be strictly smaller then p. In this case define $q = \frac{pr}{p-r}$. We then get that $q < \frac{2r}{2-r}$ and so by Sobolev embedding $dg \in L^q_{\text{loc}}(\Omega)$. Then from Holder's inequality we get that $f(dg) \in L^r_{\text{loc}}(\Omega)$.

□

We now prove the result in 3 steps:

1. **Step 1**: prove for $l = 1$ and $k = 0$.

   Let $a = \frac{p}{p-1}, b = \frac{2p}{p-2}$ and consider the function $h : (a, b) \longrightarrow (2, \infty)$ defined by

   $$h(q) = \frac{2pq}{2p + 2q - pq} = \frac{1}{\frac{1}{q} - \frac{1}{b}}$$

   It is easy to see that h is a monotonically increasing diffeomorphism satisfying $q < h(q)$. Now let $a < q_0 \leq p$ be any number. Then there has to be an m such that $h^m(q_0) > b$. We then set $q_i = h^i(q_0)$ for $i = 0, ..., m$ and $r_i = \frac{pq_i}{p+q_i}$.

   Now since $q_0 \leq p$ we get that $u \in L^{q_0}_{\text{loc}}(\Omega)$ and since each $r_i < p$ then $\eta \in L^{r_i}_{\text{loc}}(\Omega)$ for every i. Theorem 0.1 then tells us that

   $$u \in L^{q_i}_{\text{loc}}(\Omega) \Rightarrow u \in W^{1,r_i}_{\text{loc}}(\Omega)$$

   Now direct calculation verifies that $q_{i+1} = \frac{2r_i}{2-r_i}$ and so Sobolev's embedding theorems tells us that

   $$u \in W^{1,r_i}_{\text{loc}}(\Omega) \Rightarrow u \in L^{q_{i+1}}_{\text{loc}}(\Omega)$$

   Hence by induction we get that $u \in W^{1,r_m}$. Since $q_m > b = \frac{2p}{p-2}$ we get that $r_m > 2$ and so by u is continuous. Applying Theorem 0.1 again for $q = \infty$ and $r = p$ one gets that

   $$u \in W^{1,p}_{\text{loc}}(\Omega)$$

   This finishes the proof for $l = 1$ and $k = 0$.

2. **Step 2**: prove for $l = 1$ and $k = 1$. First of all from step 1 we get that $u \in W^{1,p}_{\text{loc}}(\Omega)$. We need to show that it is also in $W^{2,p}_{\text{loc}}(\Omega)$.

   Let $q_0, ..., q_m$ be as in step 1. Note that since $q_0 \leq p$ and $u \in W^{1,p}_{\text{loc}}(\Omega)$ we get that $u \in W^{1,q_0}_{\text{loc}}(\Omega)$. Now we will show that for each i one has

   $$u \in W^{1,q_i}_{\text{loc}}(\Omega) \Rightarrow u \in W^{2,r_i}_{\text{loc}}(\Omega)$$

   Consider the (distributional) derivative $u' = \partial_s u$. We claim that u' satisfies a certain non-homogenous Cauchi-Riemann equation (in the weak sense). To see this let $\phi \in C^\infty_0(\Omega, \mathbb{R}^{2n})$ be a test function such that $\phi(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}^n \times 0$. Then one has

   $$\int_\Omega \left\langle \partial_s u, \partial_s \phi + \mathcal{J}^t \partial_t \phi \right\rangle = \int_\Omega \left\langle u, \partial_s(\partial_s \phi) + \partial_s(\mathcal{J}^t \partial_t \phi) \right\rangle =$$

   $$\int_\Omega \left\langle u, \partial_s(\partial_s \phi) + \mathcal{J}^t \partial_t(\partial_s \phi) + \partial_s(\mathcal{J}^t)\partial_t \phi \right\rangle =$$

   $$-\int_\Omega \langle \eta + (\partial_t \mathcal{J})u, \partial_s \phi \rangle + \int_\Omega \langle \partial_s(\mathcal{J})u, \partial_t \phi \rangle =$$

   $$\int_\Omega \langle \partial_s(\eta + (\partial_t \mathcal{J})u) - \partial_t(\partial_s \mathcal{J})u, \phi \rangle =$$

   $$\int_\Omega \langle \partial_s \eta - (\partial_s \mathcal{J})\partial_t u + (\partial_t \mathcal{J})\partial_s u, \phi \rangle$$

   which means that v satisfies the Cauchi-Riemann equation with non-homogenous term

   $$\eta' = \partial_s \eta - (\partial_s \mathcal{J})\partial_t u$$

   Now $\partial_s \eta$ and $\partial_s \mathcal{J}$ are in L^p_{loc} and so if $u \in W^{1,q_i}_{\text{loc}}(\Omega)$ then by Holder $\eta' \in L^{r_i}_{\text{loc}}(\Omega)$. Hence by Theorem 0.1 we get that $\partial_s u \in W^{1,r_i}_{\text{loc}}$. Note that since each $r_i \leq p$ we get that $\eta \in W^{1,r_i}_{\text{loc}}(\Omega)$ and so by Lemma 0.3 we get that

   $$\partial_t u = J(\partial_s u - \eta)$$

   is in W^{1,r_i} as well. This means that $u \in W^{2,r_i}$.

   We now get by induction that $u \in W^{2,r_m}$. Since $r_m > 2$ we get that u is continuously differentiable so v is continuous. Applying the argument above again for $q = \infty$ and $r = p$ yields in

   $$u \in W^{2,r_i}$$

   and we are done.

3. **The general case**. After the first two steps the general case is easier. Note that if the theorem is true for a pair (k, l) then it is true for $(k, l+1)$. Hence it is enough to prove that if it is true for $(k, k+1)$ then it is true for $(k+1, k+1)$. Since we've done the pairs $(0, 1)$ and $(1, 1)$ we can assume $k \geq 1$.

   Now from the induction hypothesis we already know that $u \in W^{k+1,p}_{\text{loc}}(\Omega)$ and we want to show that it is also in $W^{k+2,p}_{\text{loc}}(\Omega)$. Let $u' = \partial_s u$ and

   $$\eta' = \partial_s \eta - (\partial_s \mathcal{J})\partial_t u$$

   be like in step 2. Then $\partial_s \eta, \partial_t u$ and $\partial_s \mathcal{J}$ are all in $W^{k,p}_{\text{loc}}(\Omega)$. Since $k \geq 1$ we can use Lemma 0.3 to conclude that $\eta' \in W^{k,p}(\Omega)$ as well. Applying the induction hypothesis again to u', η' we get that $u' \in W^{k+1,p}_{\text{loc}}(\Omega)$ and a similar argument to Lemma 0.3 shows that

   $$\partial_t u = J(\partial_s u - \eta)$$

   is also in $W^{k+1,p}_{\text{loc}}(\Omega)$.

□

We now apply the regularity results for the linear Cauchi-Riemann equations to the non-linear case:

Theorem 0.4. *Let $\Omega \subseteq \mathbb{H} \cup \mathbb{R}$ be an open subset, $2 < p \leq \infty$ a real number and $0 < l$ an integer. Assume that $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$ is a weak solution of* (2) *for some $J \in C^l(\Omega, M_{n \times n})$. Then u is in $W^{l,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$.*

Proof. The theorem is trivial if $l = 1$ so we assume $l > 1$. Let $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$ be a solution of

$$\partial_s u + (J \circ u)(\partial_t u) = 0$$

Since J is of class C^l the composition $\mathcal{J} = J \circ u$ is of class $W^{1,p}_{\text{loc}}$. By Theorem 0.2 (with $\eta = 0$) we get that $u \in W^{2,p}_{\text{loc}}(\Omega)$. If $l = 2$ we are done. If $l > 2$ then now $\mathcal{J} = J \circ u$ turns out to be of class $W^{2,p}_{\text{loc}}$ so we can apply 0.2 again to get that $u \in W^{3,p}_{\text{loc}}(\Omega)$. We continue like this until we get that $u \in W^{l,p}_{\text{loc}}(\Omega)$. □

Corollary 0.5. *Let M be a smooth $2n$-dimensional manifold with a smooth almost complex structure J and a totally real sub-manifold $L \subseteq M$. Let*

$$u : \Sigma \longrightarrow M$$

be a $W^{1,p}$-class function which is a weak solution of (1). *Then u is smooth.*