ON THE EQUIVALENCE OF ALL MODELS FOR $(\infty,2)$-CATEGORIES

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Abstract

The goal of this paper is to provide the last equivalence needed in order to identify all known models for $(\infty,2)$-categories. We do this by showing that Verity's model of saturated 2-trivial complicial sets is equivalent to Lurie's model of ∞-bicategories, which, in turn, has been shown to be equivalent to all other known models for $(\infty,2)$-categories. A key technical input is given by identifying the notion of ∞-bicategories with that of *weak* ∞-bicategories, a step which allows us to understand Lurie's model structure in terms of Cisinski–Olschok's theory. Several of our arguments use tools coming from a new theory of *outer (co)cartesian fibrations*, further developed in a companion paper. In the last part of the paper we construct a homotopically fully faithful scaled simplicial nerve functor for 2-categories, give two equivalent descriptions of it, and show that the homotopy 2-category of an ∞-bicategory retains enough information to detect thin 2-simplices.

Contents

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Introduction

Nowadays, ∞-categories are widely recognized as an extremely important tool to develop homotopy coherent mathematics. These were first introduced by Boardman and Vogt [6], and later developed by Joyal and Lurie, in terms of simplicial sets admitting fillers for inner horns [12, 17]. Just as in order to develop category theory for ordinary categories one cannot avoid the framework of 2-categories (in particular, one has to consider the 2-category **Cat**), $(\infty, 1)$-categories are best understood when their theory is framed inside an $(\infty, 2)$-category.

There are currently many models for $(\infty, 2)$-categories, and almost all of them have been proven to be equivalent to one another in the works of Lurie, Rezk–Bergner, Ara, Barwick–Schommer-Pries and others ([16], [4], [5], [1], [3]). Yet, an important one is still left out: the complicial model. This has been developed by Verity, initially in its strict version motivated by proving the Street–Robert's conjecture [22], and later on weakened in [23], where a model structure for *weak complicial sets* is introduced. These are *stratified sets*, *i.e.*, simplicial sets bearing some extra structure in the form of a marking on n-simplices for $n > 0$, which satisfy an extension property with respect to a set of anodyne morphisms (the analogues of inner horns for ∞-categories). When we specialize to those which are *saturated*, *i.e.*, those in which the marked n-simplices satisfy the 2-out-of-6 property, and 2*-trivial*, *i.e.*, every n-simplex is marked when $n > 2$, we get a model for $(\infty, 2)$-categories. In [19], a model structure whose fibrant objects are the saturated n-trivial complicial sets is constructed, based on a general principle established in [23]. In this paper we will only consider the case where $n = 2$.

So far, the complicial model has not been proven to be equivalent to any other known one. This has resulted in an undesirable gap between the theory as developed by Verity and collaborators and the theory developed in other settings. In this paper we close this gap by providing a Quillen equivalence between the later and Lurie's bicategorical model structure on scaled simplicial sets [16]. We do so in two steps: first, we show that ∞-bicategories coincide with *weak* ∞-bicategories, where the latter are defined as those scaled simplicial sets with the extension property with respect to a certain set of anodyne morphisms. Second, using the identification above we *redefine* Lurie's model structure on scaled simplicial sets by invoking the combinatorial machinery of Cisinski–Olschok. The equivalence between the two definitions allows us to prove the result we previously mentioned: a Quillen equivalence with the model structure for 2-trivial saturated complicial sets. We may then conclude that the model of complicial sets is equivalent to all other known models. A quite comprehensive and hopefully self-explanatory diagram of the relationships between the known models for $(\infty, 2)$-categories (with arrows indicating right Quillen equivalences) is depicted in Figure 1 below. Our contribution then corresponds to the two dotted arrows.

In the final parts of the paper we construct a homotopy fully faithful nerve functor for 2-categories, which embeds them (in the ∞-categorical sense) in the category of scaled simplicial sets as a particular class of ∞-bicategories. We then show that the homotopy 2-category of an ∞-bicategory has a conservativity property: a 2-simplex in X is thin if and only if it represents an invertible 2-cell in $\mathrm{ho}_2(X)$.

Several of our arguments make use of a new theory of *outer (co)cartesian fibrations*. This notion, which we introduce in this paper, will be developed in greater

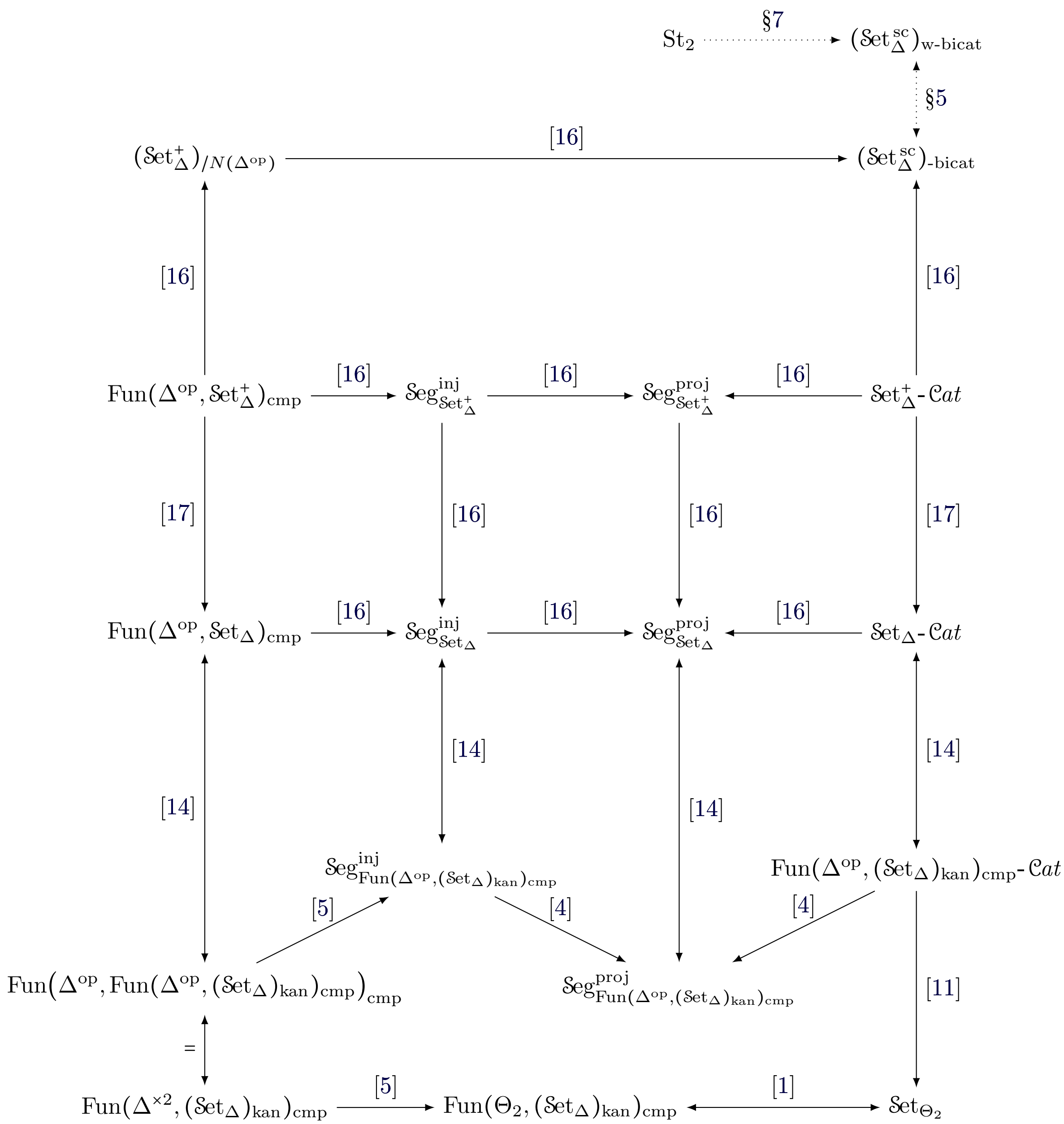

FIGURE 1. Models for $(\infty, 2)$-categories.

detail in the companion paper [9], where we also cover topics related to *Gray products* and *lax limits*.

This paper is organized as follows. In §1 we review all the necessary definitions and preliminary results concerning the model structures involved in what follows, including the one on scaled simplicial sets and that of n-trivial saturated complicial sets.

In §2 we introduce the notion of outer (co)cartesian fibrations and develop some of its basic properties. We then generalize the join and slice constructions to the setting of scaled simplicial sets, yielding in particular a model for the hom-∞-categories of an ∞-bicategory.

In §3 and §4 we do some preparatory work for the subsequent proofs by developing a few technical tools and proving some key results concerning thin triangles in weak ∞-bicategories.

In §5 we prove one of the main results of this paper, namely that weak ∞-bicategories and ∞-bicategories coincide. In particular, this implies that the fibrant objects in Lurie's model structure can be detected by means of a generating set of anodyne morphisms. This part uses the technology of outer cartesian fibrations.

In §6 we define the model structure for weak ∞-bicategories using the machinery developed in [7] and [18] and recorded in the appendix. Using the results of §5 we then prove that the fibrant objects are exactly Lurie's ∞-bicategories.

In §7 we construct a Quillen pair between the model structure for ∞-bicategories and that of 2-trivial saturated complicial sets, and we show in Theorem 7.7 that it is a Quillen equivalence. This is achieved by producing an explicit model for a fibrant replacement of the image of a weak ∞-bicategory under the left adjoint.

Finally, in Section §8 we construct a scaled simplicial nerve for 2-categories in two different ways, and prove they coincide in Proposition 8.2. Then, in Proposition 8.3, we show that this nerve is homotopy fully faithful. The left adjoint of this scaled 2-nerve gives a construction of the *homotopy* 2*-category* associated to an ∞-bicategory. We then prove a conservativity property by showing that the homotopy 2-category detects thin triangles.

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Notation

We will denote by $\mathcal{S}\mathrm{et}_\Delta$ the category of simplicial sets. We will employ the standard notation $\Delta^n \in \mathcal{S}\mathrm{et}_\Delta$ for the n-simplex, and for $\varnothing \neq S \subseteq [n]$ we write $\Delta^S \subseteq \Delta^n$ the $(|S|-1)$-dimensional face of Δ^n whose set of vertices is S. For $0 \leq i \leq n$ we will denote by $\Lambda^n_i \subseteq \Delta^n$ the i'th horn in Δ^n, that is, the subsimplicial set spanned by all $(n-1)$-dimensional faces containing the i'th vertex. By an ∞-category we will always mean a *quasi-category*, *i.e.*, a simplicial set X which admits extensions for all inclusions $\Lambda^n_i \to \Delta^n$, for all $n > 1$ and all $0 < i < n$ (known as *inner* horns). Given an ∞-category X, we will denote its homotopy category by $\mathrm{ho}(X)$. This is the ordinary category having as objects the 0-simplices of X, and as morphisms $x \to y$ the set of equivalence classes of 1-simplices $f\colon x \to y$ of X under the equivalence relation generated by identifying f and f' if there is a 2-simplex H in X with $H|_{\Delta^{\{1,2\}}} = f$, $H|_{\Delta^{\{0,2\}}} = f'$ and $H|_{\Delta^{\{0,1\}}}$ degenerate on x. We recall that the functor $\mathrm{ho}\colon \infty\text{-}\mathcal{C}at \to 1\text{-}\mathcal{C}at$ is left adjoint to the ordinary nerve functor.

1. Preliminaries

In this section we will review the main definitions and basic results concerning marked simplicial sets, marked-simplicial categories, scaled simplicial sets and stratified sets. In particular, we will recall the relevant model category structures and some of the Quillen adjunctions relating them.

1.1. Marked simplicial sets and marked-simplicial categories.

Definition 1.1. A *marked simplicial set* is a pair (X,E) where X is simplicial set and M is a subset of the set of 1-simplices of X, called *marked* simplices, such that it contains the degenerate ones. A map of marked simplicial sets $f\colon (X,E_X) \to (Y,E_Y)$ is a map of simplicial sets $f\colon X \to Y$ satisfying $f(E_X) \subseteq E_Y$.

The category of marked simplicial sets will be denoted by $\mathcal{S}\mathrm{et}_\Delta^+$.

Notation 1.2. For simplicity, we will often speak only of the non-degenerate marked edges when considering a marked simplicial set. For example, if X is a simplicial set and E is any set of edges in X then we will denote by (X,E) the marked simplicial set whose underlying simplicial set is X and whose marked edges are E together with the degenerate edges. In addition, when there is no risk of ambiguity we will omit the set of marked 1-simplices and just denote (X,E) by X.

Remark 1.3. The category $\mathcal{S}\mathrm{et}_\Delta^+$ of marked simplicial sets admits an alternative description, as the category of models of a limit sketch. In particular, it is a reflective localization of a presheaf category and it is a cartesian closed category.

Definition 1.4. Given a marked simplicial set (X,E) we define its *marked core* as the sub-simplicial set of X spanned by those n-simplices whose 1-dimensional faces are marked, *i.e.*, they belong to E. We will denote by $\kappa(X,E)$ the marked core of (X,E).

Theorem 1.5 ([17]). *There exists a model category structure on the category $\mathcal{S}et_\Delta^+$ of marked simplicial sets in which cofibrations are exactly the monomorphisms and the fibrant objects are marked simplicial sets (X,E) in which X is an ∞-category and E is the set of equivalences of X,* i.e.*, 1-simplices $f\colon\Delta^1 \to X$ which are invertible in* $\mathrm{ho}(X)$.

This is a special case of Proposition 3.1.3.7 in [17], when $S = \Delta^0$. We will refer to the model structure of Theorem 1.5 as the *marked categorical model structure*, and its weak equivalences as *marked categorical equivalences*.

Remark 1.6. Marked simplicial sets are a model for $(\infty,1)$-categories.

Recall that a *relative category* is a pair $(\mathcal{C},\mathcal{W})$, where $\mathcal{C}$ is a category and $\mathcal{W}$ is a subcategory of $\mathcal{C}$, called the subcategory of *weak equivalences in* $\mathcal{C}$, containing all the objects of $\mathcal{C}$. We denote by $\mathcal{RC}at$ the category of small relative categories having as morphisms the functors which preserve weak equivalences.

Definition 1.7. We define the *marked nerve*

$$\mathrm{N}^+\colon \mathcal{RC}at \to \mathcal{S}\mathrm{et}_\Delta^+\,,$$

to be the functor which sends a relative category $(\mathcal{C},\mathcal{W})$ to the marked simplicial set $(\mathrm{N}(\mathcal{C}),\mathrm{Arr}(\mathcal{W}))$, where $\mathrm{N}(\mathcal{C})$ is the standard nerve of the small category $\mathcal{C}$ and the marking $\mathrm{Arr}(\mathcal{W})$ consists of those edges of $\mathrm{N}(\mathcal{C})$ which are contained in $\mathrm{N}(\mathcal{W})$. The marked nerve functor admits a left adjoint

$$\overline{\mathrm{ho}}\colon \mathcal{S}\mathrm{et}_{\Delta}^{+} \to \mathcal{R}\mathcal{C}at$$

which can be explicitly described as follows: a marked simplicial set (X, E) is mapped to the relative category $(\mathrm{ho}(X), \mathrm{hIm}(X, E))$, where $\mathrm{ho}\colon \mathcal{S}\mathrm{et}_{\Delta} \to \mathcal{C}at$ is the standard left adjoint to the nerve functor and $\mathrm{hIm}\colon \mathcal{S}\mathrm{et}_{\Delta}^{+} \to \mathcal{C}at$ is defined by mapping (X, E) to the smallest subcategory containing the image of the functor $\mathrm{ho}\,\kappa(X, E) \to \mathrm{ho}(X)$.

Remark 1.8. For any two marked simplicial sets (X, E_X) and (Y, E_Y) we have

$$\begin{aligned}\overline{\mathrm{ho}}\big((X, E_X) \times (Y, E_Y)\big) &\cong \overline{\mathrm{ho}}(X \times Y, E_X \times E_Y)\\ &\cong \big(\mathrm{ho}(X \times Y), \mathrm{hIm}(X \times Y, E_X \times E_Y)\big)\\ &\cong \big(\mathrm{ho}(X) \times \mathrm{ho}(Y), \mathrm{hIm}(X, E_X) \times \mathrm{hIm}(Y, E_Y)\big)\\ &\cong \big(\mathrm{ho}(X), \mathrm{hIm}(X, E_X)\big) \times \big(\mathrm{ho}(Y), \mathrm{hIm}(Y, E_Y)\big)\\ &\cong \overline{\mathrm{ho}}(X, E_X) \times \overline{\mathrm{ho}}(Y, E_Y)\,;\end{aligned}$$

hence, the functor $\overline{\mathrm{ho}}\colon \mathcal{S}\mathrm{et}_{\Delta}^{+} \to \mathcal{R}\mathcal{C}at$ preserves finite products.

The canonical functor $\iota\colon \mathcal{C}at \to \mathcal{R}\mathcal{C}at$, mapping a small category $\mathcal{C}$ to the relative category $(\mathcal{C}, \mathrm{Iso}(\mathcal{C}))$ having as weak equivalences the set of isomorphisms of $\mathcal{C}$, has a left adjoint $L\colon \mathcal{R}\mathcal{C}at \to \mathcal{C}at$ which is the localization functor mapping a small relative category $(\mathcal{C}, \mathcal{W})$ to the small category $\mathcal{C}[W^{-1}]$, that is to say, the category in which we have formally inverted the arrows of W. It is well-known that the functor L preserves finite products. Composing the adjunctions $L \dashv \iota$ and $\overline{\mathrm{ho}} \dashv \mathrm{N}^{+}$ we now obtain an adjunction

$$\mathcal{S}\mathrm{et}_{\Delta}^{+} \underset{\mathrm{N}^{+}\iota}{\overset{L\overline{\mathrm{ho}}}{\rightleftarrows}} \mathcal{C}at \tag{1.1}$$

in which the left adjoint $L\overline{\mathrm{ho}}$ preserves finite products.

Recall that the category $\mathcal{C}at$ carries the *canonical model structure* in which the weak equivalences are the categorical equivalences, the fibrations are the isofibrations and the cofibrations are the functors which are injective on objects.

Lemma 1.9. *The adjunction (1.1) is a Quillen adjunction. In particular, the functor $L\overline{\mathrm{ho}}$ preserves weak equivalences (since all objects in $\mathcal{S}\mathrm{et}_{\Delta}^{+}$ are cofibrant).*

Proof. Since the objects of $L\overline{\mathrm{ho}}(X, E_X)$ are the vertices of X the functor $L\overline{\mathrm{ho}}$ clearly preserves cofibrations. Now let $f\colon (X, E_X) \to (Y, E_Y)$ be a trivial cofibration of marked simplicial sets, so that $f_*\colon L\overline{\mathrm{ho}}(X, E_X) \to L\overline{\mathrm{ho}}(Y, E_Y)$ is a cofibration. In order to prove that f_* is also an equivalence of categories it will suffice to show that for every category $\mathcal{C}$ the induced map $\mathrm{Fun}(L\overline{\mathrm{ho}}(Y, E_Y), \mathcal{C}) \to \mathrm{Fun}(L\overline{\mathrm{ho}}(X, E_X), \mathcal{C})$ is trivial fibration of categories. Replacing f by its pushout-products with $\partial\Delta^1 \to \Delta^1$ and $\partial\Delta^2 \to \Delta^2$ and using Remark 1.8 we may reduce to showing that the induced map $\mathrm{Fun}(L\overline{\mathrm{ho}}(Y, E_Y), \mathcal{C}) \to \mathrm{Fun}(L\overline{\mathrm{ho}}(X, E_X), \mathcal{C})$ is surjective on objects, that is, every functor $L\overline{\mathrm{ho}}(X, E_X) \to \mathcal{C}$ extends to $L\overline{\mathrm{ho}}(Y, E_Y)$. Finally, since f is a trivial cofibration it will suffice to check that $\mathrm{N}^{+}(\iota\mathcal{C}) = (\mathrm{N}(\mathcal{C}), \mathrm{Iso}(\mathcal{C}))$ is fibrant. Indeed, $\mathrm{N}(\mathcal{C})$ is an ∞-category whose equivalences are exactly $\mathrm{Iso}(\mathcal{C})$. □

Since $\mathcal{S}\mathrm{et}_\Delta^+$ is a model for ∞-categories, using enrichment in marked simplicial sets one can form a model for the theory of $(\infty,2)$-categories.

Definition 1.10. We let $\mathcal{C}at_\Delta^+$ denote the category of categories enriched over marked simplicial sets. We will refer to these as *marked-simplicial categories.*

By virtue of Proposition A.3.2.4 and Theorem A.3.2.24 of [17], the category $\mathcal{C}at_\Delta^+$ is endowed with a model category structure in which the weak equivalences are the *Dwyer–Kan equivalences.* More explicitly, these are the maps $f\colon \mathcal{C} \to \mathcal{D}$ which are

- *fully-faithful:* in the sense that the maps $f_*\colon \mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{D}}(f(x),f(y))$ are marked categorical equivalences;
- *essentially surjective:* in the sense that the functor of ordinary categories given by $f_*\colon \mathbf{ho}(\mathcal{C}) \to \mathbf{ho}(\mathcal{D})$ is essentially surjective, where for a marked-simplicial category $\mathcal{E}$ we denote by $\mathbf{ho}(\mathcal{E})$ the category whose objects are the objects of $\mathcal{E}$ and such that $\mathrm{Hom}_{\mathbf{ho}(\mathcal{E})}(x,y) := [\Delta^0, \mathrm{Map}_{\mathcal{C}}(x,y)]$ is the set of homotopy classes of maps from Δ^0 to $\mathrm{Map}_{\mathcal{C}}(x,y)$ with respect to the marked categorical model structure.

We also note that the trivial fibrations in $\mathcal{C}at_\Delta^+$ are the maps $f\colon \mathcal{C} \to \mathcal{D}$ which are surjective on objects and such that $f_*\colon \mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{D}}(f(x),f(y))$ is a trivial fibration of marked simplicial sets for every pair of objects (x,y) in $\mathcal{C}$.

Now since the left adjoint in the adjunction (1.1) above preserves products it induces in particular an adjunction on the level of enriched categories, which we will denote by

$$\mathcal{C}at_\Delta^+ \underset{\mathrm{N}_*}{\overset{\mathrm{ho}_*}{\rightleftarrows}} 2\text{-}\mathcal{C}at \quad . \tag{1.2}$$

In [15] Lack constructs a model structure on the category 2-$\mathcal{C}at$ of 2-categories, in which the weak equivalences are the 2-categories equivalences and the fibrations are the 2-isofibrations (that is, functors which are isofibrations on each mapping category and admit lifts for invertible 1-morphisms in the base). Furthermore, the trivial fibrations in the Lack model structure are the functors which are surjective on objects and induce trivial fibrations on the level of mapping categories. We note that 2-categorical equivalences can be described in a way analogous to Dwyer–Kan equivalences. In particular, a 2-functor $f\colon \mathcal{C} \to \mathcal{D}$ between 2-categories is a 2-categorical equivalence if and only if it is

- *fully-faithful:* in the sense that the maps $f_*\colon \mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{D}}(f(x),f(y))$ are categorical equivalences;
- *essentially surjective:* in the sense that the functor of ordinary categories given by $f_*\colon \mathbf{ho}(\mathcal{C}) \to \mathbf{ho}(\mathcal{D})$ is essentially surjective, where for a 2-category $\mathcal{E}$ we denote by $\mathbf{ho}(\mathcal{E})$ the category whose objects are the objects of $\mathcal{E}$ and such that $\mathrm{Hom}_{\mathbf{ho}(\mathcal{E})}(x,y) := [*, \mathrm{Map}_{\mathcal{C}}(x,y)]$ is the set of homotopy classes of maps from $*$ to $\mathrm{Map}_{\mathcal{C}}(x,y)$ with respect to the canonical model structure on $\mathcal{C}at$.

Proposition 1.11. *The adjunction* (1.2) *is a Quillen adjunction. Furthermore, the functor* ho_* *preserves weak equivalences.*

Proof. We first note that by Lemma 1.9 and the analogous description of trivial fibrations on both side we have that N_* preserves trivial fibrations, and so ho_* preserves cofibrations. We will now show that ho_* preserves weak equivalences (and hence in particular trivial cofibrations). First since $L\overline{\mathrm{ho}}$ preserves weak equivalences (Lemma 1.9) we have that ho_* preserves fully-faithful functors. To finish the proof it will hence suffice to show that ho_* preserves essentially surjective functors. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map such that $f_*\colon \mathbf{ho}(\mathcal{C}) \to \mathbf{ho}(\mathcal{D})$ is essentially surjective. Then the maps $[\Delta^0, X] \to [*, L\overline{\mathrm{ho}}(X)]$ induced by the left Quillen functor $L\overline{\mathrm{ho}}\colon \mathcal{S}\mathrm{et}_\Delta^+ \to \mathcal{C}at$ determine a commutative square of ordinary categories

$$\begin{array}{ccc} \mathbf{ho}(\mathcal{C}) & \longrightarrow & \mathbf{ho}(\mathrm{ho}_*(\mathcal{C})) \\ \downarrow & & \downarrow \\ \mathbf{ho}(\mathcal{D}) & \longrightarrow & \mathbf{ho}(\mathrm{ho}_*(\mathcal{D})) \end{array}$$

in which the horizontal maps are bijective on vertices. Given an object x in $\mathbf{ho}(\mathrm{ho}_*(\mathcal{D}))$, we may then lift it to an object $x' \in \mathbf{ho}(\mathcal{D})$. Since the left vertical map is assumed essentially surjective there exists a $y' \in \mathbf{ho}(\mathcal{C})$ and an isomorphism $\alpha\colon f(y') \xrightarrow{\cong} x$ in $\mathbf{ho}(\mathcal{D})$. The images of y' and α on the right hand side then show that x is in the essential image of the right vertical map, as desired. □

1.2. **Scaled simplicial sets and ∞-bicategories.** We now introduce scaled simplicial sets, which form another model for the theory of (∞,2)-categories.

Definition 1.12 ([16])**.** A *scaled simplicial set* is a pair (X,T) where X is simplicial set and T is a subset of the set of 2-simplices of X, called the subset of *thin* simplices, containing the degenerate ones. A map of scaled simplicial sets $f\colon (X,T_X) \to (Y,T_Y)$ is a map of simplicial sets $f\colon X \to Y$ satisfying $f(T_X) \subset T_Y$.

The category of scaled simplicial sets will be denoted by $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$.

Definition 1.13. Given a simplicial set X we will denote by $X_\flat = (X, \deg_2(X))$ the scaled simplicial consisting of X with only degenerate triangles as thin 2-simplices, and by $X_\sharp = (X, X_2)$ the scaled simplicial set consisting of X with all triangles thin.

Remark 1.14. The category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ admits an alternative description, as the category of models of a limit sketch. In particular, it is a reflective localization of a presheaf category. In fact, we can define a category Δ_{sc} having as set of objects the set $\{[k]\}_{k\geq 0} \cup \{[2]_t\}$, obtained from Δ by adding an extra object and maps $[2] \to [2]_t$, $\sigma_t^i\colon [2]_t \to [1]$ for $i = 0,1$ satisfying the obvious relations. The category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ is then the reflective localization of the category of presheaves $\mathrm{PSh}(\Delta_{\mathrm{sc}})$ (of sets) at the arrow $[2]_t \coprod_{[2]} [2]_t \to [2]_t$, where we have identified an object of Δ_{sc} with its corresponding representable presheaf. Equivalently, the local objects are those presheaves $X\colon \Delta_{\mathrm{sc}}^{\mathrm{op}} \to \mathbf{Set}$ for which $X([2]_t) \to X([2])$ is a monomorphism. In particular, the category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ is cartesian closed and it is easy to check that the reflector functor $\mathrm{PSh}(\Delta_{\mathrm{sc}}) \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ preserves monomorphisms and commutes with finite products.

Notation 1.15. For simplicity, we will often speak only of the non-degenerate thin 2-simplices when considering a scaled simplicial set. For example, if X is a simplicial set and T is any set of triangles in X then we will denote by (X,T) the scaled simplicial set whose underlying simplicial set is X and whose thin triangles are T together with the degenerate triangles. If $L \subseteq K$ is a subsimplicial set then we use $T|_L \coloneqq T \cap L_2$ to denote the set of triangles in L whose image in K is contained in T.

Definition 1.16. Let $\mathbf{S}$ be the set of maps of scaled simplicial sets consisting of:

(1) inner horns inclusions $\left(\Lambda^n_i, \{\Delta^{\{i-1,i,i+1\}}\}|_{\Lambda^n_i}\right) \subseteq \left(\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\}\right)$ for $n \geq 2$ and $0 < i < n$.

(2) the map $(\Delta^4, T) \to \left(\Delta^4, T \cup \{\Delta^{\{0,3,4\}},\ \Delta^{\{0,1,4\}}\}\right)$, where

$$T := \left\{\Delta^{\{0,2,4\}},\ \Delta^{\{1,2,3\}},\ \Delta^{\{0,1,3\}},\ \Delta^{\{1,3,4\}},\ \Delta^{\{0,1,2\}}\right\};$$

(3) the set of maps $\left(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}|_{\Lambda^n_0}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right)$ for $n \geq 2$.

We call $\mathbf{S}$ the set of *generating anodyne morphisms*, and its saturation is the class of *(scaled) anodyne maps*.

Remark 1.17. As observed in Remark 3.1.4 of [16], the inclusions of scaled simplicial sets $j_i\colon (\Delta^3, T_i) \to \Delta^3_\sharp$, for $i = 1,2$, where T_i is the collection of 2-simplices of Δ^3 containing the i'th vertex, are both anodyne.

Definition 1.18. We will say that a map of scaled simplicial sets $X \to Y$ is a *scaled fibration* if it has the right lifting property with respect to scaled anodyne maps.

The scaled fibrations whose codomain is a point are of special interest:

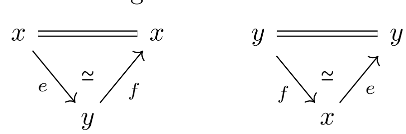
Definition 1.19 ([16])**.** A *weak ∞-bicategory* is a scaled simplicial set $\mathcal{C}$ which admits extensions along all scaled anodyne maps.

We observe that the map in the second point of Definition 1.16 ensures that marked 2-simplices of weak ∞-bicategories satisfy a *saturation* property, while the first set guarantees, among other things, that the subobject of a weak ∞-bicategory X spanned by those n-simplices whose 2-dimensional faces are thin is an ∞-category, which we call the *core* ∞-category of X. This will be denoted by X^{th}.

Definition 1.20. Let $\mathcal{C}$ be a weak ∞-bicategory. We will say that an edge $e\colon x \to y$ in $\mathcal{C}$ is *invertible* if it is invertible when considered in the core ∞-category $\mathcal{C}^{\mathrm{th}}$ of $\mathcal{C}$, that is, if the corresponding arrow in the homotopy category $\mathrm{ho}(\mathcal{C}^{\mathrm{th}})$ is an isomorphism. In this case we will also say that $e\colon x \to y$ is *en equivalence* in $\mathcal{C}$.

If X is an arbitrary scaled simplicial set then we say that an edge in X is invertible if its image in $\mathcal{C}$ is invertible for any scaled anodyne $X \hookrightarrow \mathcal{C}$ with $\mathcal{C}$ a weak ∞-bicategory (this does not depend on the choice of such a $\mathcal{C}$).

Remark 1.21. More explicitly, if $\mathcal{C}$ is a weak ∞-bicategory then $e\colon x \to y$ is invertible in $\mathcal{C}$ if and only if there exist triangles of the form



Indeed, this condition clearly implies that e corresponds to an isomorphism in $\mathrm{ho}(\mathcal{C}^{\mathrm{th}})$, and the implication in the other direction follows by applying Joyal's special outer horn theorem ([12, Theorem 1.3], see also [17, Proposition 1.2.4.3]) to $\mathcal{C}^{\mathrm{th}}$.

Definition 1.22. The *scaled coherent nerve* functor $\mathrm{N}^{\mathrm{sc}}\colon \mathcal{C}at^+_\Delta \to \mathcal{S}et^{\mathrm{sc}}_\Delta$ is defined by letting the underlying simplicial set of $\mathrm{N}^{\mathrm{sc}}(\mathcal{C})$ be the coherent nerve of the simplicially enriched category $\mathcal{C}$ (as in Definition 1.1.5.5 of [17]), and setting its thin

2-simplices to be those maps $f: \mathfrak{C}(\Delta^2) \to \mathcal{C}$ that send the unique non-degenerate 1-simplex of $\mathfrak{C}(\Delta^2)(0,2)$ to a marked 1-simplex in $\mathcal{C}(f(0), f(2))$.

The functor N^{sc} admits a left adjoint $\mathfrak{C}^{\mathrm{sc}}: \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}} \to \mathcal{C}at_{\Delta}^{+}$, whose explicit description can be found in Definition 3.1.30 of [16].

Theorem 1.23 ([16, Theorem 4.2.7])**.** *There exists a model structure on the category $\mathcal{S}et_{\Delta}^{\mathrm{sc}}$ of scaled simplicial sets, characterized as follows:*

- *a map $f: X \to Y$ is a cofibration if and only if it is a monomorphism;*
- *a map $f: X \to Y$ is a weak equivalence if and only $\mathfrak{C}^{\mathrm{sc}}(f): \mathfrak{C}^{\mathrm{sc}}(X) \to \mathfrak{C}^{\mathrm{sc}}(Y)$ is a weak equivalence in $\mathcal{C}at_{\Delta}^{+}$.*

Moreover, the adjoint pair

$$\mathcal{S}et_{\Delta}^{\mathrm{sc}} \underset{\mathrm{N}^{\mathrm{sc}}}{\overset{\mathfrak{C}^{\mathrm{sc}}}{\rightleftarrows}} \mathcal{C}at_{\Delta}^{+} \quad (\perp)$$

is a Quillen equivalence with respect to the model structures defined above.

We will refer to the model structure of Theorem 1.23 as the *bicategorical model structure*. Following [16] we will refer to the fibrant objects of this model category as ∞*-bicategories*, to its weak equivalences as *bicategorical equivalences*, and to its fibrations as *bicategorical fibrations*.

Proposition 1.24 ([16, 3.1.13])**.** *If f belongs to $\mathbf{S}$ then $\mathfrak{C}^{\mathrm{sc}}(f)$ is a trivial cofibration of $\mathcal{S}\mathrm{et}_{\Delta}^{+}$-categories. Therefore, every ∞-bicategory is a weak ∞-bicategory. Similarly, every bicategorical fibration is a scaled fibration.*

We will prove a converse to Proposition 1.24 in §5 below (see Theorem 5.1).

1.3. **Stratified sets.** In this subsection we introduce the notion of stratified sets, and define the model category for complicial sets and other variations such as n-trivial complicial sets.

Definition 1.25 ([23])**.** A *stratified set* is a pair (X, M) where X is a simplicial sets and M is a collection of its n-simplices for $n > 0$ that contains the degenerate ones. A map $f: (X, M_X) \to (Y, M_Y)$ of stratified sets is a map of simplicial sets $f: X \to Y$ such that $f(M_X) \subset M_Y$. We denote the category of stratified sets by St.

Remark 1.26. As with the case of scaled simplicial sets, the category of stratified sets admits a description in terms of a reflective localization of a category of presheaves.

Notation 1.27. If X is a simplicial set and M is any set of simplices in X then we will denote by (X, M) the stratified set whose underlying simplicial set is X and whose marked simplices are $M \cup \deg(X)$. If $L \subseteq K$ is a subsimplicial set then we use $M|_L$ to denote the set of simplices in L whose image in K is contained in M.

Warning 1.28. Comparing Notation 1.27 and Notation 1.15 reveals a certain abuse of notation: given a simplicial set K and a collection of triangles T, the symbol (K, T) can either mean the scaled simplicial set whose thin triangles are T plus the degenerate ones or the stratified set on K whose marked simplices are T plus the degenerate ones. A similar ambiguity exists between Notation 1.27 and 1.2. We believe however that in the way these terms are used in this paper there will be no place where an actual confusion can arise. We also note that the category $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}}$ of scaled simplicial set can be identified with the full subcategory of St consisting of those stratified sets for which all non-degenerate marked simplices are of dimension 2. Under this identification the ambiguity above disappears. This full inclusion $\iota\colon \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \to \mathrm{St}$ is also the left counterpart of the Quillen adjunction we consider in §7.

Remark 1.29. In the specific case of Δ^0 we will simplify notation and denote $(\Delta^0, \varnothing)$ by Δ^0, since here there is really no possibility for confusion.

Definition 1.30. Given an integer k we will denote by $\mathrm{th}_k(K)$ the stratified set whose underlying simplicial set is K and whose marked simplices are the degenerate ones and all those in dimension $> k$. In the case $k = 0$ we will simply write $\mathrm{th}(K)$ to $\mathrm{th}_0(K)$. If (X, M) is a stratified set then we will denote by $\mathbf{th}_k(X, M)$ the stratified set whose marked simplices consist of M together with all simplices of dimension $> k$.

Definition 1.31 ([23])**.** Given two stratified sets (X, M_X) and (Y, M_Y), we define their *join* $X \ast Y$ to be the stratified set whose underlying simplicial set consists of Joyal's join of simplicial sets $X \ast Y$, and such that an n-simplex $(x, y) \in (X \ast Y)_n$ with $x \in X_k$ and $y \in Y_{n-k-1}$ for $-1 \le k \le n$ is marked if and only if either x is marked in X *or* y is marked in Y. Here we employ the convention that X_{-1} and Y_{-1} contain a single element which is *not* considered as marked for the purpose of this definition.

We let Δ^3_{eq} be the stratified set whose underlying simplicial set is Δ^3, whose marked 1-simplices are $\Delta^{\{0,2\}}$ and $\Delta^{\{1,3\}}$ and whose n-simplices for n strictly greater than 1 are all marked.

Definition 1.32 ([20], [23])**.** The class of *complicial horns* is the saturation of the set of inclusions

$$(\Lambda^n_i, M_i|_{\Lambda^n_i}) \to (\Delta^n, M_i)$$

for $n \ge 1$ and $0 \le i \le n$, where M_i consists of all the degenerate simplices and all the simplices which contain the vertices in the set

$$\{i-1, i, i+1\} \cap \{0, \dots, n\} = \{j \in [n] : |j-i| \le 1\},$$

whose size may be either 2 or 3 depending on i and n.

The class of *thinness extensions* is the saturation of the set of inclusions of the form

$$(\Delta^n, M_i') \to (\Delta^n, M_i'')$$

with $n \ge 2$ where M_i' contains M_i as above as well as the two $(n-1)$-faces opposite to the vertices $i-1$ and $i+1$, while M_i'' contains M_i and as well as all $(n-1)$-faces.

The class of *k-trivializing morphisms* is the saturation of the set of inclusions

$$(\Delta^n, \varnothing) \to (\Delta^n, \{\Delta^n\})$$

for $n > k$. We consider the class of ∞-trivializing morphisms to be the empty class.

Finally, the class of *saturation morphisms* is the saturation of the set of inclusions of the form $\Delta^3_{\mathrm{eq}} \ast (\Delta^n, \varnothing) \to \mathrm{th}(\Delta^3) \ast (\Delta^n, \varnothing)$ for $n \ge -1$, where $\Delta^{-1} = \varnothing$ by convention.

The following definition isolates the stratified sets of interest.

Definition 1.33 ([20], [23])**.** A stratified set (X, M) is a *complicial set* if it has the right lifting property with respect to complicial horns and thinness extensions.

A stratified set (X, M) is *k-trivial* if it has the right lifting property with respect to k-trivializing morphisms.

A stratified set (X, M) is *saturated* if it has the right lifting property with respect to saturation morphisms.

The following result establishes the existence of a model category structure for (saturated, n-trivial) complicial sets.

Theorem 1.34 ([19], [20], [23])**.** *For every $0 \leq n \leq \infty$ there exists a model category structure on the category* St *of stratified sets characterized by the following properties:*

- *a map $f: X \to Y$ in* St *is a cofibration if and only if it is a monomorphism;*
- *a stratified set (X, M) is fibrant if and only if it is an n-trivial saturated complicial set.*

We denote this model category structure by St_n.

Remark 1.35. The thinness extensions $(\Delta^n, M_i') \to (\Delta^n, M_i'')$ for $n \geq 4$ are already contained in the weakly saturated closure of the 2-trivializing morphisms. When working in the model category St_2 one can hence restrict attention to the thinness extensions of dimensions 2 and 3. Concretely, these consists of the following maps:

- for $i = 0, 1, 2$ the map $(\Delta^2, M_{(2,i)}) \to \mathrm{th}(\Delta^2)$ when $M_{(2,i)}$ consists of all faces of Δ^2 which contain the vertex i;
- for $i = 0, 1, 2, 3$ the map $(\Delta^3, M_{(3,i)}) \to \mathrm{th}(\Delta^3)$ when $M_{(3,i)}$ consists of all faces of dimension ≥ 2 which contain the vertex i as well as the edge $\{i-1, i, i+1\} \cap \{0, 1, 2, 3\}$ in the case $i = 0, 3$.

Remark 1.36. The model category St_2 is cartesian closed. This has in particular the following implication on the collection of marked simplices in a given fibrant stratified set (X, M_X): if $h: \mathrm{th}(\Delta^1) \times (\Delta^n, \varnothing) \to (X, M_X)$ is a map, which we consider as encoding a natural equivalence from $\sigma_0 := h|_{\{0\} \times \Delta^n}$ to $\sigma_1 := h|_{\{1\} \times \Delta^n}$, then σ_0 is marked if and only if σ_1 is marked.

2. Outer cartesian fibrations

In the $(\infty, 2)$-categorical setting one often encounters $(\infty, 2)$-categories which are *fibered* into $(\infty, 1)$-categories over a given base. To describe such situations effectively one requires a robust theory which encompasses the four types of *variance* such a fibration may encode. One such type, which we will call *inner cocartesian fibration*, was studied in [16]. It corresponds to the situation where the fiber $\mathcal{D}_x$ depends *covariantly* on both 1-morphisms and 2-morphisms. If $\pi: \mathcal{C} \to \mathcal{D}$ is an inner cocartesian fibration then $\pi^{\mathrm{op}}: \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is an *inner cartesian fibration* (whose fibers are the opposites of the fibers of π). These correspond to the situation where $\mathcal{D}_x$ depends contravariantly on both 1-morphisms and 2-morphisms. The reason we use the term "inner" to describe these two types of fibrations is that they are inner fibrations in the sense of Lurie–Joyal on the level of the underlying simplicial sets. By contrast, the fibrations which describe a dependence which is covariant in 1-morphisms and contravariant in 2-morphisms (or the other way around) are generally not inner fibrations. Instead, we call them *outer (co)cartesian fibrations*. To our knowledge these type of fibrations have not yet appeared in the literature.

In this section we introduce the notion of outer cartesian fibration and investigate its basic properties. We will begin in §2.1 by presenting the main definitions.

The prototypical example we have in mind is that of the outer cartesian fibration represented by a given object. We will construct this fibration using the slice construction in §2.2 and show that its fibers model the corresponding mapping ∞-categories in §2.3. Finally, in §2.4 we will show that outer cartesian fibrations satisfy a lifting property for natural transformations — a key feature which we will exploit in later parts of the paper.

2.1. **Outer fibrations and cartesian edges.** We begin by introducing the basic definitions.

Definition 2.1. We will say that a map of scaled simplicial sets $X \to Y$ is a *weak fibration* if it has the right lifting property with respect to the following types of maps:

(1) All scaled inner horn inclusions of the form
$$(\Lambda^n_i, \{\Delta^{\{i,i-1,i\}}\}|_{\Lambda^n_i}) \subseteq (\Delta^n, \{\Delta^{\{i,i-1,i\}}\})$$
for $n \geq 2$ and $0 < i < n$.

(2) The scaled horn inclusions of the form:
$$\Big(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}|_{\Lambda^n_0}\Big) \subseteq \Big(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\Big)$$
for $n \geq 2$.

(3) The scaled horn inclusions of the form:
$$\Big(\Lambda^n_n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0, \{\Delta^{\{0,n-1,n\}}\}|_{\Lambda^n_n}\Big) \subseteq \Big(\Delta^n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0, \{\Delta^{\{0,n-1,n\}}\}\Big)$$
for $n \geq 2$.

Remark 2.2. The maps of type (1)-(3) in Definition 2.1 are trivial cofibrations with respect to the bicategorical model structure: indeed, the first two are scaled anodyne and the third is the opposite of a scaled anodyne map. It follows that every bicategorical fibration is a weak fibration.

Definition 2.3. Let $p\colon X \to Y$ be a weak fibration. We will say that an edge $e\colon \Delta^1 \to X$ is *p-cartesian* if the dotted lift exists in any diagram of the form

$$\begin{array}{ccc} (\Lambda^n_n, \{\Delta^{\{0,n-1,n\}}\}|_{\Lambda^n_n}) & \xrightarrow{\sigma} & (X, T_X) \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ (\Delta^n, \{\Delta^{\{0,n-1,n\}}\}) & \longrightarrow & (Y, T_Y) \end{array}$$

with $n \geq 2$ and $\sigma|_{\Delta^{n-1,n}} = e$.

Definition 2.4. Let $p\colon X \to Y$ be a weak fibration. We will say that p is an *outer fibration* if

(1) p detects thin simplices, that is, a triangle in X is thin if and only if its image in Y is thin;

(2) the map of simplicial sets underlying $X \to Y$ satisfies the right lifting property with respect to the inclusions
$$\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \subseteq \Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0 \quad \text{and} \quad \Lambda^n_n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0 \subseteq \Delta^n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0$$
for $n \geq 2$.

Remark 2.5. If $p: X \to Y$ is a weak fibration and detects thin triangles then p is a scaled fibration (Definition 1.18). In particular, every outer fibration is a scaled fibration.

Definition 2.6. Let $p: X \to Y$ be a map scaled simplicial sets. We will say that p is an *outer cartesian fibration* if the following conditions hold:

(1) The map p is an outer fibration.
(2) For every edge $e: y \to y'$ in Y and every $x' \in X$ such that $p(x') = y'$ there exists a p-cartesian edge $f: x \to x'$ such that $p(f) = e$.

Dually, we will say that $p: X \to Y$ is an *outer cocartesian fibration* if $p^{\mathrm{op}}: X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is an outer cartesian fibration.

Remark 2.7. The classes of weak fibrations, outer fibrations and outer (co)cartesian fibrations are all closed under base change.

Remark 2.8. It follows from Remarks 2.5 and 2.7 that if $X \to Y$ is an outer fibration then for every $y \in Y$ the fiber X_y is a weak ∞-bicategories in which every triangle is thin. Forgetting the scaling, we may simply consider these fibers as ∞-categories.

Remark 2.9. Let $p: X \to Y$ be an outer cartesian fibration and assume in addition that Y is a weak ∞-bicategory. Then X is a weak ∞-bicategory as well by Remark 2.5. The condition that p detects thin triangles then implies that $X^{\mathrm{th}} = X \times_S Y^{\mathrm{th}}$, and so the induced map $p^{\mathrm{th}}: X^{\mathrm{th}} \to Y^{\mathrm{th}}$ is a cartesian fibration of ∞-categories. In particular, p^{th} is a categorical fibration (see [17]) and so an isofibration. We may hence conclude that every outer cartesian fibration is an isofibration, that is, admits lifts for equivalences.

Remark 2.10. Let $p: X \to Y$ be a weak fibration between weak ∞-bicategories. If $e: x \to y$ is a p-cartesian edge of X then it is also cartesian with respect to the inner fibration of ∞-categories $X^{\mathrm{th}} \to Y^{\mathrm{th}}$. This implies, in particular, that any p-cartesian edge which lies above an equivalence in Y is necessarily an equivalence in X.

Our approach is that outer cartesian fibrations encode the data of functors from Y to $\mathcal{C}at_\infty$ which are contravariant on the level of 1-morphisms but covariant on the level of 2-morphisms. Similarly, outer cocartesian fibration encode functors which are covariant on the level of 1-morphisms but contravariant on the level of 2-morphisms. For example, we will see below how one can encode in this manner representable functors using a suitable slice construction (see §2.3). A more comprehensive treatment and justification of this approach will appear in [9].

2.2. **The join and slice constructions.** Let $\mathcal{C}$ be a weak ∞-bicategory and let $y \in \mathcal{C}$ be a vertex. Define a scaled simplicial set $\overline{\mathcal{C}}_{/y}$ as follows. The n-simplices of $\overline{\mathcal{C}}_{/y}$ are given by $(n+1)$-simplices $\Delta^{n+1}_\flat \to \mathcal{C}$ of $\mathcal{C}$ which send $\Delta^{\{n+1\}}$ to y, and a triangle in $\overline{\mathcal{C}}_{/y}$, corresponding to a 3-simplex $\sigma: \Delta^3_\flat \to \mathcal{C}$, is declared to be thin $\overline{\mathcal{C}}_{/y}$ if $\sigma|_{\Delta^{\{0,1,2\}}}$ is thin in $\mathcal{C}$. The scaled simplicial set $\overline{\mathcal{C}}_{/y}$ admits a natural map $\overline{\mathcal{C}}_{/y} \to \mathcal{C}$ sending $\sigma: \Delta^{n+1}_\flat \to \mathcal{C}$ to $\sigma|_{\Delta^{\{0,\dots,n\}}}$. If all the triangles in $\mathcal{C}$ are thin (that is, if $\mathcal{C}$ is actually an ∞-category) then the same holds for $\overline{\mathcal{C}}_{/y}$ and the projection $\overline{\mathcal{C}}_{/y} \to \mathcal{C}$ is a right fibration which classifies the presheaf on $\mathcal{C}$ represented by y. In this section we will show that for a general weak ∞-bicategory $\mathcal{C}$ the projection $\overline{\mathcal{C}}_{/y} \to \mathcal{C}$ is an *outer cartesian fibration.* In §2.3 we will show that the fibers of these fibrations are models for the mapping ∞-categories in $\mathcal{C}$ with target y.

In what follows it will be useful to work in a setting where both marking and scaling are allowed.

Definition 2.11. A *marked-scaled* simplicial set is a triple (X,E,T) where X is a simplicial set, $E \subseteq X_1$ is a collection of edges containing all the degenerate edges and $T \subseteq X_2$ is collection of 2-simplices containing all the degenerate 2-simplices. In particular, if (X,E,T) is a marked-scaled simplicial set then (X,E) is a marked simplicial set and (X,T) is a scaled simplicial set. A map of marked-scaled simplicial sets $f\colon(X,E_X,T_X) \to (Y,E_Y,T_Y)$ is a map of simplicial sets $f\colon X \to Y$ satisfying $f(E_X) \subseteq E_Y$ and $f(T_X) \subseteq T_Y$.

The collection of scaled simplicial sets and their morphisms forms a category which will be denoted as in [16] by $\mathrm{Set}_{\Delta}^{+,\mathrm{sc}}$. It is fairly standard to check that the category $\mathrm{Set}_{\Delta}^{+,\mathrm{sc}}$ has all small limits and colimits.

Notation 2.12. Given a simplicial set K, a set E of edges in K and a set T of triangles in K we will denote by (K,E,T) the marked-scaled simplicial set whose underlying simplicial set is K, whose marked edges are the degenerate ones and those contained in E and whose thin triangles are the degenerate ones and those contained in T. As in Notation 1.15, if $L \subseteq K$ is a subsimplicial set then we use $E|_L$ and $T|_L$ to denote the set of edges and triangles respectively in L whose image in K is contained in E and T respectively.

Given a scaled simplicial set S we will denote by $(\mathrm{Set}_{\Delta}^{+,\mathrm{sc}})_{/S}$ the category of marked-scaled simplicial sets (X,E_X,T_X) equipped with a map of scaled simplicial sets $(X,T_X) \to S$.

Definition 2.13. Let S be a scaled simplicial set. We will denote by $\mathcal{A}_S$ the smallest weakly saturated class of maps in the category $(\mathrm{Set}_{\Delta}^{+,\mathrm{sc}})_{/S}$ containing the following maps:

(1) The inclusion $(\Lambda^n_i,\varnothing,\{\Delta^{\{i-1,i,i+1\}}\}|_{\Lambda^n_i}) \subseteq (\Delta^n,\varnothing,\{\Delta^{\{i-1,i,i+1\}}\})$ for $0<i<n$ and every map $(\Delta^n,\{\Delta^{\{i-1,i,i+1\}}\}) \to S$.
(2) The inclusion $(\Lambda^n_n,\{\Delta^{\{n-1,n\}}\}|_{\Lambda^n_n},\varnothing) \subseteq (\Delta^n,\{\Delta^{\{n-1,n\}}\},\varnothing)$ for every $n \geq 1$ and every map $\Delta^n_\flat \to S$.
(3) The inclusion $(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0,\varnothing,\varnothing) \subseteq (\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0,\varnothing,\varnothing)$ for every $n \geq 2$ and every map $\Delta^n_\flat \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to S$.
(4) The inclusion $\Delta^2 \subseteq (\Delta^2,\varnothing,\{\Delta^2\})$ for every $\Delta^2_\sharp \to S$.

Proposition 2.14. *Let S be a scaled simplicial set, (X,E_X,T_X) a marked-scaled simplicial set and $p\colon(X,T_X) \to S$ a map of scaled simplicial sets. If the object of $(\mathrm{Set}^+_\Delta)_{/S}$ determines by p has the right lifting property with respect to the set $\mathcal{A}_S$ of Definition 2.13 then p is an outer cartesian fibration and every marked edge is p-cartesian.*

Proof. Since every degenerate edge in X belongs to E_X the right lifting property with respect to the generating maps of type (1),(2), (3) and (6) implies that p is an outer fibration. The lifting property against maps of type (2) further implies that any marked edge in X is in particular p-cartesian, and the case $n=1$ of maps of type (2) implies that for every arrow $f\colon x \to y$ in S and for every $y' \in X$ such that $p(y')=y$ there exists a marked edge $e'\colon x' \to y'$ in X such that $p(e')=e$. We may hence conclude that $p\colon(X,T_X) \to S$ is an outer cartesian fibration, and that all the marked edges are p-cartesian. □

Definition 2.15. Let (Z, E_Z, T_Z) be a marked-scaled simplicial set and (K, T_K) a scaled simplicial set. We define the *join* $(Z * K, T_{Z*K})$ to be the *scaled simplicial set* whose underlying simplicial set is the ordinary join of simplicial sets $Z * K$ and whose thin triangles T_{Z*K} are given by the subset

$$T_Z \coprod [E_Z \times K_0] \coprod \varnothing \coprod T_K$$

of the set

$$Z_2 \coprod [Z_1 \times K_0] \coprod [Z_0 \times K_1] \coprod K_2 = (Z * K)_2.$$

For a fixed scaled simplicial set (K, T_K) we may consider the association given by $(Z, E_Z, T_Z) \mapsto (Z * K, T_{Z*K})$ as a functor $\mathrm{Set}_\Delta^{+,\mathrm{sc}} \to \mathrm{Set}^{\mathrm{sc}}_{(K,T_K)/}$. As such, it becomes a colimit preserving functor which admits a right adjoint $\mathrm{Set}^{\mathrm{sc}}_{(K,T_K)/} \to \mathrm{Set}_\Delta^{+,\mathrm{sc}}$ by the adjoint functor theorem.

Definition 2.16. Given a map of scaled simplicial sets $f\colon (K, T_K) \to X$, considered as an object of $\mathrm{Set}^{\mathrm{sc}}_{(K,T_K)/}$, we will denote by $X_{/f}$ the marked-scaled simplicial set obtained by applying the above mentioned right adjoint. In particular, the marked-scaled simplicial set $X_{/f}$ is characterized by a mapping property of the form

$$\mathrm{Hom}((Z, E_Z, T_Z), X_{/f}) = \mathrm{Hom}((Z * K, T_{Z*K}), X).$$

We will denote by $\overline{X}_{/f}$ the scaled simplicial set underlying $X_{/f}$.

Lemma 2.17. *Let $f\colon (X, E_X, T_X) \to (Y, E_Y, T_Y)$ be a map of marked-scaled simplicial sets and $g\colon (A, T_A) \to (B, T_B)$ a map of scaled simplicial sets. If f belongs to $\mathcal{A}_S$ and g is a monomorpohism then the map of scaled simplicial sets*

$$(X * B, T_{X*B}) \coprod_{(X*A, T_{X*A})} (Y * A, T_{Y*A}) \to (Y * B, T_{Y*B}) \tag{2.1}$$

is in the weakly saturated closure of maps of type (1) and (3) in Definition 1.16. In particular, it is scaled anodyne.

Proof. It suffices to check the claim on generators, and so we may assume that g is either the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$, the inclusion $\Delta^2 \hookrightarrow \Delta^2_\sharp$, and f is one of the generating maps appearing in Definition 2.13. We first note that when g is the map $\Delta^2 \hookrightarrow \Delta^2_\sharp$ then (2.1) is an isomorphism. We may hence assume that g is the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ for some $n \geq 0$. Let us now consider the various possibilities for f case by case.

(1) When f is the inclusion $(\Lambda^m_i, \varnothing, \{\Delta^{\{i-1,i,i+1\}}\}|_{\Lambda^m_i}) \subseteq (\Delta^m, \varnothing, \{\Delta^{\{i-1,i,i+1\}}\})$ for $0 < i < m$ the map (2.1) is isomorphic to the map

$$\left(\Lambda^{[m]*[n]}_i, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^{[m]*[n]}, \{\Delta^{\{i-1,i,i+1\}}\}\right)$$

which is a map of type (1) in Definition 1.16.

(2) When f is the inclusion $(\Lambda^m_m, \{\Delta^{\{m-1,m\}}\}|_{\Lambda^m_m}, \varnothing) \subseteq (\Delta^m, \{\Delta^{\{m-1,m\}}\}, \varnothing)$ for $m \geq 1$ the map (2.1) is isomorphic to the map

$$\left(\Lambda^{[m]*[n]}_k, \{\Delta^{\{m-1,m,m+1\}}\}|_{\Lambda^{[m]*[n]}}\right) \to \left(\Delta^{[m]*[n]}, \{\Delta^{\{m-1,m,m+1\}}\}\right)$$

which is a map of type (1) in Definition 1.16.

(3) When f is the inclusion $(\Lambda^m_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \varnothing, \varnothing) \subseteq (\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^m, \varnothing, \varnothing)$ for $m \geq 2$ the map (2.1) is isomorphic to the map

$$\left(\Lambda^{[m]*[n]}_0 \coprod_{\Delta^{\{0,1\}*[n]}} \Delta^{[0]*[n]}, \varnothing\right) \to \left(\Delta^{[m]*[n]} \coprod_{\Delta^{\{0,1\}}*[n]} \Delta^{[0]*[n]}, \varnothing\right)$$

which is a pushout of a map type (3) in Definition 1.16.

(4) When f is the inclusion $\Delta^2 \subseteq (\Delta^2, \varnothing, \{\Delta^2\})$, then the map (2.1) is an isomorphism. □

Corollary 2.18. *Let X be a scaled simplicial set which satisfies the extension property with respect to maps of type (1) and (3) in Definition 1.16. Given a map $f\colon (K, T_K) \to X$ of scaled simplicial sets, the map $q\colon \overline{X}_{/f} \to X$ is an outer cartesian fibration and every edges which is marked in $X_{/f}$ is q-cartesian.*

Proof. Combine Proposition 2.14 and Lemma 2.17. □

In the situation of Corollary 2.18, if K is a point and the image of f is the vertex $y \in X$ then we will denote $X_{/f}$ and $\overline{X}_{/f}$ simply by $X_{/y}$ and $\overline{X}_{/y}$, respectively. In particular, the outer cartesian fibration $q\colon \overline{X}_{/y} \to X$ is the one we alluded to in the beginning of this section. We consider q as the outer cartesian fibration *represented* by y. A partial justification to this point of view will be offered in Proposition 2.24 below.

Remark 2.19. Let X be a scaled simplicial set and let $y \in X$ a vertex. Then for $i = 1, 2$, if the scaled simplicial set X satisfies the extension property with respect to the inclusion $(\Delta^3, T_i) \to \Delta^3_\sharp$ (where T_i denotes all triangles which contain the vertex i) then the marked-scaled simplicial set $X_{/y}$ satisfies the extension property with respect to the inclusions $(\Delta^2, E_i, \{\Delta^2\}) \to (\Delta^2, \Delta^2, \{\Delta^2\})$ (where E_i denotes all edges containing i). In particular, if X is a weak ∞-bicategory and $\sigma\colon \Delta^2 \to X_{/y}$ is a thin triangle such that $\sigma|_{\Delta^{\{1,2\}}}$ is marked then $\sigma|_{\Delta^{\{0,1\}}}$ is marked if and only if $\sigma|_{\Delta^{\{0,2\}}}$ is marked (see Remark 1.17).

Corollary 2.18 admits the following generalization, which can be deduced from Proposition 2.14 and Lemma 2.17 in the same manner:

Corollary 2.20. *Let $p\colon X \to S$ be a map of scaled simplicial sets which satisfies the right lifting property with respect to maps of type (1) and (3) in Definition 1.16. Let $f\colon (K, T_K) \to X$ be a map of scaled simplicial sets and let $i\colon (L, T_L) \hookrightarrow (K, T_K)$ be an inclusion of scaled simplicial sets. Then the map*

$$q\colon \overline{X}_{/f} \to \overline{X}_{/fi} \times_{\overline{S}_{/pfi}} \overline{S}_{/pf}$$

is an outer fibration, the marked edges of $X_{/f}$ are q-cartesian, and the base change of q to the marked core of $X_{/fi} \times_{S_{/pfi}} S_{/pf}$ is an outer cartesian fibration.

2.3. **Mapping categories.** Let $\mathcal{C}$ be a weak ∞-bicategory with underlying simplicial set $\overline{\mathcal{C}}$ and let $x, y \in \overline{\mathcal{C}}$ be two vertices. Recall the following explicit model for the mapping ∞-category from x to y in $\mathcal{C}$ constructed in [16, §4.2]). Let $\mathrm{Hom}_{\mathcal{C}}(x, y)$ be the marked simplicial set whose n-simplices are given by maps $f\colon \Delta^1 \times \Delta^n \to \overline{\mathcal{C}}$ such that $f|_{\{0\}\times\Delta^n}$ is constant on x, $f|_{\{1\}\times\Delta^n}$ is constant on y, and the triangle $f|_{\Delta^{\{(0,i),(1,i),(1,j)\}}}$ is thin for every $0 \le i \le j \le n$. The marked edges $\mathrm{Hom}_{\mathcal{C}}(x, y)$ are the edges corresponding to those maps $\Delta^1 \times \Delta^1 \to \overline{\mathcal{C}}$ which send both triangles of $\Delta^1 \times \Delta^1$ to thin triangles. As shown in [16, §4.2], the assumption that $\mathcal{C}$ is a weak ∞-bicategory implies that the marked simplicial set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is *fibrant* in the marked categorical model structure, that is, it is an ∞-category whose marked edges are exactly the equivalences.

This construction can also be understood in terms of the *Gray product* of scaled simplicial sets, which we will further study in [9]. For our purpose here let us consider the following limited variant:

Definition 2.21. Let K and L be two simplicial sets. The *Gray product* $K \times_{\mathrm{gr}} L$ is the *scaled* simplicial set whose underlying simplicial set is the cartesian product $K \times L$ and such that a 2-simplex $\sigma\colon \Delta^2 \to K \times L$ is thin if and only if the following conditions hold:

(1) The image of σ in both K and L is degenerate.
(2) Either $\sigma|_{\Delta^{\{1,2\}}}$ maps to a degenerate edge in K or $\sigma|_{\Delta^{\{0,1\}}}$ maps to a degenerate edge in L.

In terms of this Gray product we can describe the n-simplices in $\mathrm{Hom}_{\mathcal{C}}(x,y)$ as maps $f\colon \Delta^1 \times_{\mathrm{gr}} \Delta^n \to \mathcal{C}$ such that $f|_{\{0\}\times\Delta^n}$ is constant on x and $f|_{\{1\}\times\Delta^n}$ is constant on y. An edge $\Delta^1 \times_{\mathrm{gr}} \Delta^1 \to \mathcal{C}$ is marked exactly when it factors through the map $\Delta^1 \times_{\mathrm{gr}} \Delta^1 \to \Delta^1_\flat \times \Delta^1_\flat$.

Now recall from §2.2 that for $y \in \mathcal{C}$, the map of scaled simplicial sets $p\colon \overline{\mathcal{C}}_{/y} \to \mathcal{C}$ is an outer cartesian fibration (Corollary 2.18) and all the marked edges of $\mathcal{C}_{/y}$ are p-cartesian. If $x \in \mathcal{C}$ is then another vertex then by Remark 2.8 the fiber $(\overline{\mathcal{C}}_{/y})_x$ is a weak ∞-bicategory in which all triangles are thin, and every marked edge in $(\overline{\mathcal{C}}_{/y})_x$ is an equivalence. Forgetting the scaling, we will denote by $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$ the underlying *marked* simplicial set of $(\mathcal{C}_{/y})_x$. In particular, $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$ is an ∞-category endowed with a marking, and all marked edges are equivalences.

Construction 2.22. We construct a natural map of marked simplicial sets

$$i\colon \mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}}(x,y). \tag{2.2}$$

Explicitly the n-simplices of $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$ are given by maps

$$g\colon (\Delta^n * \Delta^0)_\flat = \Delta^{n+1}_\flat \to \mathcal{C}$$

which send $\Delta^{\{0,\ldots,n\}}$ to x and $\Delta^{\{n+1\}}$ to y (where an edge is marked exactly when it corresponds to a thin triangle), while the n-simplices of $\mathrm{Hom}_{\mathcal{C}}(x,y)$ correspond to maps $\Delta^1 \times_{\mathrm{gr}} \Delta^n \to \mathcal{C}$ (where an edge $\Delta^1 \times_{\mathrm{gr}} \Delta^1 \to \mathcal{C}$ is marked exactly with it factors through $\Delta^1_\flat \times \Delta^1_\flat$). The map 2.2 is then obtained by pulling back along the unique map $\Delta^1 \times_{\mathrm{gr}} \Delta^n \to (\Delta^n * \Delta^0)_\flat = \Delta^{n+1}_\flat$ which on vertices sends $(i,0)$ to i and $(i,1)$ to $n+1$ (this map indeed sends the thin triangles $\Delta^{\{(0,i),(1,i),(1,j)\}}$ of $\Delta^1 \times_{\mathrm{gr}} \Delta^n$ to degenerate triangles). It is straightforward to verify that this association is compatible with face and degeneracy maps, and with the markings on both sides.

Remark 2.23. Inspecting Construction 2.22 we see that the map (2.2) detects marked edges. Since the marked edges in $\mathrm{Hom}_{\mathcal{C}}(x,y)$ are exactly the equivalences it follows that every equivalence in $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$ is marked. In particular, the marked edges in $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$ are exactly the equivalence and $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$ is fibrant with respect to the marked categorical model structure.

Proposition 2.24. *Let $\mathcal{C}$ be a weak ∞-bicategory. Then the map*

$$i\colon \mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}}(x,y)$$

of Construction 2.22 is a marked categorical equivalence of fibrant marked simplicial sets for every $x, y \in \mathcal{C}$.

The proof of Proposition 2.24 will require the following lemma:

Lemma 2.25. *Let $\mathcal{C}$ be a weak ∞-bicategory and for $m \geq 2$ suppose given a diagram of marked simplicial sets*

$$(2.3)\qquad \begin{array}{ccc} (\partial\Delta^{\{1,\dots,m\}}, \varnothing) & \xrightarrow{f_0} & \mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y) \\ \big\downarrow & & \big\downarrow \\ (\Lambda^m_0, \{\Delta^{\{0,1\}}\}) & \xrightarrow{h_0} & \mathrm{Hom}_{\mathcal{C}}(x,y) \end{array}$$

where $\partial\Delta^{\{1,\dots,m\}}$ and Λ^m_0 are considered as sub simplicial sets of $\Delta^m = \Delta^{\{0,\dots,m\}}$. Then there exists an extension of (2.3) to a diagram the form

$$(2.4)\qquad \begin{array}{ccccc} (\partial\Delta^{\{1,\dots,m\}}, \varnothing) & \hookrightarrow & (\Delta^{\{1,\dots,m\}}, \varnothing) & \xrightarrow{f} & \mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ (\Lambda^m_0, \{\Delta^{\{0,1\}}\}) & \hookrightarrow & (\Delta^m, \{\Delta^{\{0,1\}}\}) & \xrightarrow{h} & \mathrm{Hom}_{\mathcal{C}}(x,y) \end{array}$$

Proof. Let

$$Z_0 := \Delta^1 \times_{\mathrm{gr}} \Lambda^m_0 \coprod_{\partial\Delta^1 \times_{\mathrm{gr}} \Lambda^m_0} \partial\Delta^1 \times_{\mathrm{gr}} \Delta^m \subseteq \Delta^1 \times_{\mathrm{gr}} \Delta^m.$$

Unwinding the definitions, what we need to prove amounts to solving an extension problem of the form

$$\begin{array}{ccc} Z_0 & \xrightarrow{g_0} & \mathcal{C} \\ \big\downarrow & \nearrow_{g} & \\ \Delta^1 \times_{\mathrm{gr}} \Delta^m & & \end{array}$$

under the assumptions that

(i) the map g_0 sends $\Delta^{\{0\}} \times_{\mathrm{gr}} \Delta^m$ to the vertex x and $\Delta^{\{1\}} \times_{\mathrm{gr}} \Delta^m$ to the vertex y;
(ii) the restriction of g_0 to $\Delta^1 \times_{\mathrm{gr}} \partial\Delta^{\{1,\dots,m\}}$ factors through $\partial\Delta^{\{1,\dots,m\}} * \Delta^0$; and
(iii) the map g_0 sends $\Delta^{\{(0,0),(0,1),(1,1)\}}$ to a thin triangle in $\mathcal{C}$;

and with the additional constraint that

(*) the restriction of g to $\Delta^1 \times_{\mathrm{gr}} \Delta^{\{1,\dots,m\}}$ factors through $\Delta^{\{1,\dots,m\}} * \Delta^0$.

Let $\tau_i\colon \Delta^m \to \Delta^1 \times \Delta^{\{1,\dots,m\}}$ for $i = 0, \dots, m-1$ be the m-simplex given on vertices by the formula:

$$\tau_i(j) = \begin{cases} (0, j+1) & j \leq i \\ (1, j) & j > i \end{cases}.$$

In particular, for $i = 0, \dots, m-2$ our constraint (*) above requires that the desired lift g sends σ_i to the degenerate m-simplex in $\mathcal{C}$ whose image is the $(i+1)$-simplex $(g_0 \circ \sigma_i)|_{\Delta^{\{0,\dots,i+1\}}}$. Let $Z_1 \subseteq \Delta^1 \times_{\mathrm{gr}} \Delta^{\{1,\dots,m\}}$ be the union of Z_0 and the simplices σ_i for $i \leq m-2$. We may then define $g_1\colon Z_1 \to \mathcal{C}$ to be the unique map whose restriction to Z_0 is g_0 and which such that $g_1 \circ \sigma_i$ is the composition $\Delta^{\{0,\dots,m\}} \to \Delta^{\{0,\dots,i+1\}} \to \mathcal{C}$, where the first map collapses $i+1, \dots, m$ to $i+1$ and the second is given by the restriction of $g_0 \circ \sigma_i$. This is indeed well-defined since g_0 is assumed to satisfy (ii).

We are hence left with solving the unconstrained extension problem

$$\begin{array}{ccc} Z_1 & \xrightarrow{g_1} & \mathcal{C} \\ \downarrow & \nearrow_{g} & \\ \Delta^1 \times_{\mathrm{gr}} \Delta^m & & \end{array}$$

For $i = 1, ..., m+1$ let $\sigma_i \colon \Delta^{m+1} \to \Delta^1 \times \Delta^m$ be the $(m+1)$-simplex given on vertices by the formula:

$$\sigma_i(j) = \begin{cases} (0,j) & j < i \\ (1,j+1) & j \geq i \end{cases}.$$

For $j = 2, ..., m+2$ let $Z_j \subseteq \Delta^1 \times_{\mathrm{gr}} \Delta^m$ be the union of Z_1 and the simplices σ_i for $i < j$. We then observe that $Z_{m+2} = \Delta^1 \times_{\mathrm{gr}} \Delta^m$ and that for $j = 1, ..., m$ we have a pushout square

$$\begin{array}{ccc} (\Lambda^{m+1}_j, \{\Delta^{\{j-1,j,j+1\}}\}) & \longrightarrow & Z_j \\ \downarrow & & \downarrow \\ (\Delta^{m+1}, \{\Delta^{\{j-1,j,j+1\}}\}) & \longrightarrow & Z_{j+1} \end{array}$$

In particular $Z_j \to Z_{j+1}$ is scaled anodyne for $j = 1, ..., m$ and since $\mathcal{C}$ is a weak ∞-bicategory we may extend g_1 to a map $g_{m+1} \colon Z_{m+1} \to \mathcal{C}$. Finally, we observe that we have a pushout square

$$\begin{array}{ccc} (\Lambda^{m+1}_0)_\flat & \longrightarrow & Z_{m+1} \\ \downarrow & & \downarrow \\ \Delta^{m+1}_\flat & \longrightarrow & Z_{m+2} \end{array}$$

and by (i) above $g_{m+1} \circ \sigma_{m+1} \colon \Lambda^{m+1}_0 \to \mathcal{C}$ sends $\Delta^{\{0,1\}}$ to degenerate edge. Now the vertical maps in the above square are not scaled anodyne. Nonetheless, since the map $(\Lambda^{m+1}_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,m+1\}}\}) \to (\Delta^{m+1} \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,m+1\}}\})$ is scaled anodyne we could finish the proof if we showed that $g_{m+1} \circ \sigma_{m+1} \colon \Lambda^{m+1}_0 \to \mathcal{C}$ sends $\Delta^{\{0,1,m+1\}}$ to a thin triangle, or, equivalently, that g_{m+1} sends $\Delta^{\{(0,0),(0,1),(1,m)\}}$ to a thin triangle. Indeed, since the map g_{m+1} sends the triangle $\Delta^{\{(0,0),(0,1),(1,1)\}}$ to a thin triangle by (iii), sends every triangle of the form $\Delta^{\{(0,i),(1,i),(1,j)\}}$ to a thin triangle by the definition of the Gray product and sends $\{1\} \times \Delta^m$ to a point it suffices to apply Remark 1.17 to the 3-simplex $g_{m+1}(\Delta^{\{(0,0),(1,0),(1,1),(1,m)\}})$ to deduce that g_{m+1} sends $\Delta^{\{(0,0),(1,1),(1,m)\}}$ to a thin triangle and then to apply Remark 1.17 to the 3-simplex $g_{m+1}(\Delta^{\{(0,0),(0,1),(1,1),(1,m)\}})$ to deduce that g_{m+1} sends $\Delta^{\{(0,0),(0,1),(1,m)\}}$ to a thin triangle, as desired. □

Proof of Proposition 2.24. We first note that the map $i \colon \mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}}(x,y)$ is bijective on vertices. It will hence suffice to show that it is fully-faithful.

Given vertices $\alpha, \beta \in \mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)$, let $X_{\alpha,\beta} = ([\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,y)]_{/\alpha})_\beta$ and similarly $Y_{\alpha,\beta} = (\mathrm{Hom}_{\mathcal{C}}(x,y)_{/\alpha})_\beta$. As established in [17] these are Kan complexes which model the correponding mapping spaces on both sides. We hence need to show that the map

$$X_{\alpha,\beta} \to Y_{\alpha,\beta}$$

is a homotopy equivalence of Kan complexes. Applying lemma 2.25 for diagrams of the form (2.3) for which f_0 sends $\Delta^{\{1,\dots,m-1\}}$ to α and $\Delta^{\{m\}}$ to β and h_0 sends $\Delta^{\{0,\dots,m-1\}}$ to α we may conclude that for $m \geq 2$ every diagram of the form

$$\begin{array}{ccc} \partial\Delta^{\{1,\dots,m-1\}} & \xrightarrow{f_0} & X_{\alpha,\beta} \\ \downarrow & & \downarrow \\ \Lambda^{m-1}_0 & \xrightarrow{h_0} & Y_{\alpha,\beta} \end{array}$$

extends to a diagram of the form

$$\begin{array}{ccccc} \partial\Delta^{\{1,\dots,m-1\}} & \hookrightarrow & \Delta^{\{1,\dots,m-1\}} & \xrightarrow{f} & X_{\alpha,\beta} \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda^{m-1}_0 & \hookrightarrow & \Delta^{m-1} & \xrightarrow{h} & Y_{\alpha,\beta} \end{array}$$

It then follows that all the homotopy fibers of $X_{\alpha,\beta} \to Y_{\alpha,\beta}$ are contractible, and so the desired result follows. □

2.4. **Cartesian lifts of natural transformations.** Our goal in this subsection is to prove the following:

Proposition 2.26 (Lifting natural transformations)**.** *Let $p\colon X \to Y$ be a weak fibration of scaled simplicial sets and let $A \subseteq B$ an inclusion of simplicial sets. Consider a lifting problem of the form*

$$\begin{array}{ccc} \Delta^{\{1\}} \times B_\flat \coprod_{\Delta^{\{1\}}\times A_\flat} \Delta^1_\flat \times A_\flat & \xrightarrow{f} & X \\ \downarrow & \overset{\widetilde{H}}{\nearrow} & \downarrow p \\ \Delta^1_\flat \times B_\flat & \xrightarrow{H} & Y \end{array}$$

such that f sends every edge of the form $\Delta^1 \times \{a\}$ (for $a \in A$) to a p-cartesian edge. Suppose that for every $b \in B$ there exists a p-cartesian edge with target $f(\Delta^{\{1\}} \times \{b\})$ which lifts $H(\Delta^1 \times \{b\})$. Then the dotted lift $\widetilde{H}\colon \Delta^1_\flat \times B \to X$ exists. Furthermore, $\widetilde{H}$ can be chosen so that the edges $\widetilde{H}(\Delta^1 \times \{b\})$ are any prescribed collection of p-cartesian lifts.

In the proof of Proposition 2.26 we will make use of the following type of filtration, which will also employ in several other proofs later on:

Construction 2.27. For $i = 0, \dots, n$, let T_i be the collection of all triangles in Δ^{n+1} which are either degenerate or contain the edge $\Delta^{\{i,i+1\}}$, and let

$$\tau_i\colon (\Delta^{n+1}, T_i) \to \Delta^1_\flat \times \Delta^n_\flat$$

be the map given on vertices by the formula

$$\tau_i(m) = \begin{cases} (0,m) & m \leq i \\ (1,m-1) & m > i \end{cases}$$

For $k = 0, \dots, n+1$ let $Z^k \subseteq \Delta^1_\flat \times \Delta^n_\flat$ be the union of $[\Delta^1_\flat \times \partial\Delta^n_\flat] \coprod_{\Delta^{\{0\}}\times\partial\Delta^n_\flat} [\Delta^{\{0\}} \times \Delta^n_\flat]$ and the simplices τ_i for $i \geq k$. We then have an ascending filtration of scaled simplicial sets

$$[\Delta^1_\flat \times \partial\Delta^n_\flat] \coprod_{\Delta^{\{0\}}\times\partial\Delta^n_\flat} [\Delta^{\{0\}} \times \Delta^n_\flat] = Z^{n+1} \subseteq Z^n \subseteq \ldots \subseteq Z^0 = \Delta^1_\flat \times \Delta^n_\flat \tag{2.5}$$

and for each $k = 0, \ldots, n$ we have a pushout square of scaled simplicial sets

$$\begin{array}{ccc} (\Lambda^{n+1}_k, T_k|_{\Lambda^{n+1}_k}) & \longrightarrow & Z^{k+1} \\ \downarrow & & \downarrow \\ (\Delta^{n+1}, T_k) & \longrightarrow & Z^k \end{array}$$

where the composed map $(\Delta^{n+1}, T_k) \to Z^k \to \Delta^1_\flat \times \Delta^n_\flat$ is the simplex τ_k.

Dually, for $k = 0, \ldots, n+1$ let $Z_k \subseteq \Delta^n_\flat \times \Delta^1_\flat$ be the union of

$$[\Delta^1_\flat \times \partial\Delta^n_\flat] \coprod_{\Delta^{\{1\}}\times\partial\Delta^n_\flat} [\Delta^{\{1\}} \times \Delta^n_\flat]$$

and the simplices τ_i, for $i < k$. We then have an ascending filtration of scaled simplicial sets

$$[\Delta^1_\flat \times \partial\Delta^n_\flat] \coprod_{\Delta^{\{1\}}\times\partial\Delta^n_\flat} [\Delta^{\{1\}} \times \Delta^n_\flat] = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_{n+1} = \Delta^1_\flat \times \Delta^n_\flat \tag{2.6}$$

and for each $k = 0, \ldots, n$ we have a pushout square of scaled simplicial sets

$$\begin{array}{ccc} (\Lambda^{n+1}_{k+1}, T_k|_{\Lambda^{n+1}_{k+1}}) & \longrightarrow & Z_k \\ \downarrow & & \downarrow \\ (\Delta^{n+1}, T_k) & \longrightarrow & Z_{k+1} \end{array}$$

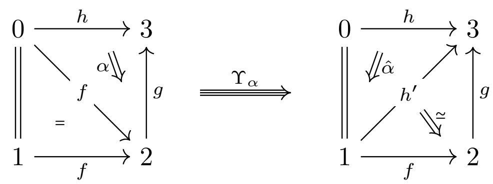
where the composed map $(\Delta^{n+1}, T_k) \to Z_k \to \Delta^1_\flat \times \Delta^n_\flat$ is the simplex τ_k.

Proof of Proposition 2.26. Arguing simplex by simplex it will suffice to prove the claim for $L \subseteq K$ being the inclusion $\partial\Delta^n \subseteq \Delta^n$. In the case $n = 0$ the claim is tautological, since we assume the existence of cartesian lifts. In the case $n \geq 1$ the map $\partial\Delta^n \subseteq \Delta^n$ is bijective on vertices and so we just need to construct a lift without the additional constraints on the edges. In this case we use the filtration (2.6) of Construction 2.27 to construct the lift step by step, where in the last step of the filtration we use the assumption that every edge $\Delta^1 \times \Delta^{\{i\}}$ is mapped to a p-cartesian edge of X. □

3. Thin triangles in weak ∞-bicategories

In this section we will establish some useful properties of thin triangles which we will need in the subsequent sections. We begin with the following lemma which turns an arbitrary 2-simplex into a 2-morphism:

Lemma 3.1. *Given a 2-simplex α in a weak ∞-bicategory $\mathcal{C}$, there exists a 3-simplex of the form:*



in $\mathcal{C}$.

Proof. We first construct a thin triangle $\Delta^{\{1,2,3\}}_{\sharp} \to \mathcal{C}$ by extending the 2-dimensional horn $(f,g)\colon \Lambda^2_1 \to \mathcal{C}$ along the scaled anodyne map $\Lambda^2_1 \subseteq \Delta^2_{\sharp}$, thus obtaining the bottom right triangle in the right square above. This triangle together with α and the degenerate triangle in the left square determine a map $h\colon (\Lambda^3_2, \{\Delta^{\{1,2,3\}}\}) \to \mathcal{C}$. Extending along the scaled anodyne inclusion

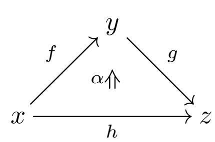
$$(\Lambda^3_2, \{\Delta^{\{1,2,3\}}\}) \subseteq (\Delta^3, \{\Delta^{\{1,2,3\}}\})$$

we get the desired 3-simplex Υ_α. □

Remark 3.2. We identify $\hat{\alpha}$ with an edge $[\hat{\alpha}]$ in the marked simplicial set $(\mathcal{C}_{/\alpha(2)})_{\alpha(0)}$, which in turn is weakly equivalent to the mapping ∞-category $\mathrm{Hom}_{\mathcal{C}}(\alpha(0), \alpha(2))$ by Proposition 2.24. It follows from Remark 1.17 that α is thin if and only if $\hat{\alpha}$ is thin, that is, if and only if the edge $[\hat{\alpha}]$ is marked.

Proposition 3.3. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a bicategorical equivalence of weak ∞-bicategories and let $\sigma\colon \Delta^2 \to \mathcal{C}$ be a triangle. Then σ is thin in $\mathcal{C}$ if and only if $f\sigma$ is thin in $\mathcal{D}$.*

Proof. Let us depict σ as a diagram



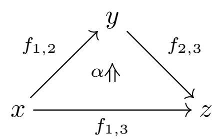
which is commutative up to a (not-necessarily-invertible 2-cell) $\alpha\colon h \Rightarrow fg$. Applying Lemma 3.1 and Remark 3.2 we may reduce to the case where f is degenerate. In this case we may consider α is encoding an edge in $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,z)$, which is marked if and only if α is thin. Since f is a bicategorical equivalence it is in particular fully-faithful, and hence Proposition 2.24 implies that the map

$$\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,z) \to \mathrm{Hom}^{\triangleright}_{\mathcal{D}}(f(x), f(z)) \tag{3.1}$$

is a marked categorical equivalence. Since $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(x,z)$ and $\mathrm{Hom}^{\triangleright}_{\mathcal{D}}(f(x), f(z))$ are fibrant marked simplicial sets (see Remark 2.23) their marked edges are exactly the respective equivalences, and hence (3.1) detects marked edges. We may then conclude that α is thin in $\mathcal{C}$ if and only if $f\alpha$ is thin in $\mathcal{D}$, as desired. □

Proposition 3.4. *Let $\mathcal{C}$ be a weak ∞-bicategory and $\sigma\colon (\Delta^3, T) \to \mathcal{C}$ a map of scaled simplicial sets, where $T = \{\Delta^{\{0,1,3\}}, \Delta^{\{0,2,3\}}\}$. If $\sigma(\Delta^{\{0,1\}})$ is an equivalence and $\sigma(\Delta^{\{0,1,2\}})$ is thin then $\sigma(\Delta^{\{1,2,3\}})$ is thin. Dually, if $\sigma(\Delta^{\{2,3\}})$ is an equivalence and $\sigma(\Delta^{\{1,2,3\}})$ is thin in $\mathcal{C}$ then $\sigma(\Delta^{\{0,1,2\}})$ is thin.*

Proof. We prove the first claim. The proof of the second is completely analogous. Since $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}^{\mathrm{sc}}$ is a Quillen equivalence there exists a fibrant Set^+_Δ-enriched category $\mathcal{E}$ equipped with a bicategorical equivalence $f\colon \mathcal{C} \to \mathrm{N}^{\mathrm{sc}}(\mathcal{E})$. By Proposition 3.3 we have that $\sigma|_{\Delta^{\{1,2,3\}}}$ is thin in $\mathcal{C}$ if and only if $f\sigma|_{\Delta^{\{1,2,3\}}}$ is thin in $\mathrm{N}^{\mathrm{sc}}(\mathcal{E})$. We may hence reduce to the case where $\mathcal{C} = \mathrm{N}^{\mathrm{sc}}(\mathcal{E})$. Let $\sigma^{\mathrm{ad}}\colon \mathfrak{C}^{\mathrm{sc}}(\Delta^3, T) \to \mathcal{E}$ be the adjoint map. Then the restriction of σ^{ad} to $\mathfrak{C}^{\mathrm{sc}}(\Delta_\flat^{\{1,2,3\}})$ determines a diagram in $\mathcal{E}$ of the form



where the α corresponds to an edge in the marked simplicial set $\mathrm{Map}_{\mathcal{E}}(x,z)$ going from $f_{1,3}$ to $f_{2,3} \circ f_{1,2}$. To show that $\sigma|_{\Delta^{\{1,2,3\}}}$ is thin in $\mathrm{N}^{\mathrm{sc}}(\mathcal{E})$ we need to show that α is a marked edge of $\mathrm{Map}_{\mathcal{E}}(x,z)$. Now since $\sigma|_{\Delta^{\{0,1\}}}$ is an equivalence in $\mathrm{N}^{\mathrm{sc}}(\mathcal{E})$ we have that the edge $f_{0,1}\colon x' \to x$ in $\mathcal{E}$ determined by $\sigma^{\mathrm{ad}}|_{\mathfrak{C}(\Delta^{\{0,1\}})}$ is an equivalence in $\mathcal{E}$, i.e., admits an inverse up to homotopy. This implies that the pre-composition map

$$\mathrm{Map}_{\mathcal{E}}(x,z) \xrightarrow{f_{0,1}^*} \mathrm{Map}_{\mathcal{E}}(x',z)$$

is a categorical equivalence of (fibrant) marked simplicial sets. Such an equivalence detects marked edges (which coincide in this case with the collection of equivalences), and hence it will suffice to show that the edge $f_{0,1}^*\alpha\colon \Delta^1 \to \mathrm{Map}_{\mathcal{E}}(x',z)$ is marked. Now the map $\sigma^{\mathrm{ad}}_*\colon \mathrm{Map}_{\mathfrak{C}^{\mathrm{sc}}(\Delta^3,T)}(0,3) \to \mathrm{Map}_{\mathcal{E}}(x',z)$ determines a commutative square of the form

$$\begin{array}{ccc} f_{0,3} & \longrightarrow & f_{1,3} \circ f_{0,1} \\ \downarrow & & \downarrow{\scriptstyle f_{0,1}^*\alpha} \\ f_{2,3} \circ f_{0,2} & \longrightarrow & f_{2,3} \circ f_{1,2} \circ f_{0,1} \end{array} \tag{3.2}$$

in the marked simplicial set $\mathrm{Map}_{\mathcal{E}}(x',z)$. Since σ sends the triangles $\Delta^{\{0,1,3\}}, \Delta^{\{0,2,3\}}$ and $\Delta^{\{0,1,2\}}$ to thin triangles in $\mathrm{N}^{\mathrm{sc}}(\mathcal{E})$ it follows that the two horizontal arrows and the left vertical arrow in (3.2) are marked. Since $\mathrm{Map}_{\mathcal{E}}(x',z)$ is fibrant it follows that $f_{0,1}^*\alpha$ is marked as well, and so the proof is complete. □

Corollary 3.5. *Let $\mathcal{C}$ be a weak ∞-bicategory and let $\rho_0, \rho_1\colon \Delta^2 \to \mathcal{C}$ be two triangles. Let $h\colon \rho_0 \to \rho_1$ be a natural transformation from ρ_0 to ρ_1, i.e., a map $h\colon \Delta^1_\flat \times \Delta^2_\flat \to \mathcal{C}$ such that $h|_{\Delta^{\{i\}} \times \Delta^2} = \rho_i$. Then the following holds:*

(1) If $h|_{\Delta^1 \times \{0\}}$ is an equivalence in $\mathcal{C}$ and ρ_0 is thin then ρ_1 is thin.
(2) If $h|_{\Delta^1 \times \{2\}}$ is an equivalence in $\mathcal{C}$ and ρ_1 is thin then ρ_0 is thin.

Proof. For $i = 0,1,2,3$ let $\alpha_i\colon \Delta^2 \to \Delta^1 \times \Delta^2$ be the 2-simplex given by

$$\alpha_i(j) = \begin{cases} (0,j) & j < i \\ (1,j) & j \geq i \end{cases}$$

By the definition of the cartesian product of scaled simplicial sets all the triangles in $\Delta^1_\flat \times \Delta^2_\flat$ are thin except the α_i's. To prove (1), assume that $\rho_0 = h|_{\alpha_3}$ is thin in X. Applying Remark 1.17 to the 3-simplex $\Delta^{\{(0,0),(0,1),(0,2),(1,2)\}}$ we get that $h|_{\alpha_2}$ is thin. Applying Remark 1.17 to the 3-simplex $\Delta^{\{(0,0),(0,1),(1,1),(1,2)\}}$ we get that

$h|_{\alpha_1}$ is thin. Finally, applying Proposition 3.4 to the 3-simplex $\Delta^{\{(0,0),(1,0),(1,1),(1,2)\}}$ we get that $\rho_1 = h|_{\alpha_0}$ is thin. To prove (2), assume that $\rho_1 = h|_{\alpha_0}$ is thin in X. Applying Remark 1.17 to the 3-simplex $\Delta^{\{(0,0),(1,0),(1,1),(1,2)\}}$ we get that $h|_{\alpha_1}$ is thin. Applying Remark 1.17 to the 3-simplex $\Delta^{\{(0,0),(0,1),(1,1),(1,2)\}}$ we get that $h|_{\alpha_2}$ is thin. Finally, applying Proposition 3.4 to the 3-simplex $\Delta^{\{(0,0),(0,1),(0,2),(1,2)\}}$ we get that $\rho_0 = h|_{\alpha_3}$ is thin. □

4. The moving lemma

Throughout this section let us fix a surjective map of simplices $\rho\colon \Delta^n \to \Delta^m$ and a section $\sigma\colon \Delta^m \to \Delta^n$ such that $\sigma(i)$ is minimal in $\rho^{-1}(i)$ for every $i \in [m]$. We will denote such a pair by $\sigma \dashv \rho$, since we can think of it as an adjunction between the posets $[n]$ and $[m]$.

Definition 4.1. Let $A \subseteq \Delta^n$ be a subsimplicial set. We will say that A is $(\sigma \dashv \rho)$-*admissible* if it satisfies the following two properties:

- The image $\sigma\rho(A)$ is contained in A.
- If A contains a vertex $i \in [n]$ which is not in the image of σ then A contains every face of Δ^n which has i as a terminal vertex.

We will refer to non-degenerate simplices in A via the corresponding subset $I \subseteq [n]$. Similarly, we will refer to the simplices in $\Delta^1 \times A$ via the corresponding subset $J \subseteq [1] \times [n]$. Given such a J we denote $J_0 := \{j \in [n] \mid (0,j) \in J\}$ and $J_1 := \{j \in [n] \mid (1,j) \in J\}$.

Definition 4.2. Let (X, T_X) be a scaled simplicial set and $A \subseteq \Delta^n$ a $(\sigma \dashv \rho)$-admissible subsimplicial set. By a $(\sigma \dashv \rho)$-*transformation* we will mean a map of simplicial sets $h\colon \Delta^1 \times A \to X$ which satisfies the following properties:

(1) Suppose that $J \subseteq [1] \times [n]$ is such that $J_0 \neq \varnothing$, $\Delta^{J_0} \subseteq A$, $i := \max(J_0)$ is in the image of σ and $J_1 = \{(1,i)\}$. Then $h|_{\Delta^J}$ factors through the retraction $\Delta^J \to \Delta^{J \smallsetminus \{(0,i)\}}$ which maps $(0,i)$ to $(1,i)$.
(2) Suppose that $J \subseteq [1] \times [n]$ is such that $J_0 \neq \varnothing$, $\Delta^{J_0} \subseteq A$, $i := \max(J_0)$ is not in the image of σ,J_0 contains $i-1$, and J_1 is either $\varnothing$ or $\{(1,i)\}$. Then $h|_{\Delta^J}$ factors through the retraction $\Delta^J \to \Delta^{J \smallsetminus \{(0,i)\}}$ which maps $(0,i)$ to $(0,i-1)$.
(3) For each edge $\Delta^{\{i,j\}} \subseteq A$ the triangle $h(\Delta^{\{(0,i),(1,i),(1,j)\}})$ is thin in $\mathcal{C}$.

Remark 4.3. Condition (1) implies in particular that the map h sends the edge $\Delta^1 \times \{\sigma(i)\} \subseteq \Delta^1 \times A$ to a degenerate edge in X for every $i \in [m]$.

Remark 4.4. Given an ∞-bicategory $\mathcal{C} := (\overline{\mathcal{C}}, T_{\mathcal{C}})$, the notion of a $(\sigma \dashv \rho)$-transformation $h\colon \Delta^1 \times A \to \overline{\mathcal{C}}$ is weaker than that of a natural transformation in $\mathcal{C}$, as h may fail to extend to a map of scaled simplicial sets $\Delta^1_\flat \times A_\flat \to \mathcal{C}$. Nonetheless, by Condition (3) h sends every triangle of the form $\Delta^{\{(0,i),(1,i),(1,j)\}}$ to a thin triangle in $\mathcal{C}$, and hence extends to a scaled map $\Delta^1 \times_{\mathrm{gr}} A \to \mathcal{C}$. We consequently consider it as a *lax natural transformations*, a concept we will investigate further in [9]. We note that this lax natural transformation is of a rather special kind, due to the degeneracy conditions imposed by Conditions (1) and (2). These conditions imply, in particular, that for an edge $\Delta^{\{i,j\}} \subseteq A$, if either j is in the image of σ or $j = i+1$ and $\rho(i) = \rho(j)$ then the triangle $h(\Delta^{\{(0,i),(0,j),(1,j)\}})$ is degenerate and hence thin as well, so that $h|_{\Delta^1 \times \Delta^{\{i,j\}}}$ extends to a natural transformation $\Delta^1_\flat \times \Delta^{\{i,j\}}_\flat \to \mathcal{C}$.

Notation 4.5. Given a $(\sigma \dashv \rho)$-admissible simplicial set $A \subseteq \Delta^n$, we shall denote by L_A the collection of triangles of $\Delta^1 \times A$ which are either of the form $\Delta^{\{(0,i),(1,i),(1,j)\}}$ for $i < j \in [n]$ or of the form $\Delta^{\{(0,i),(0,j),(1,j)\}}$ if either j is in the image of σ or $j = i+1$ and $\rho(i) = \rho(j)$. In particular, if (X, T_X) is a scaled simplicial set then by Remark 4.4 every $(\sigma \dashv \rho)$-transformation $h\colon \Delta^1 \times A \to X$ extends to a map of scaled simplicial sets $(\Delta^1 \times A, L_A) \to (X, T_X)$.

Remark 4.6. By induction, Condition (1) implies, for example, that for every $j \in [m]$ the simplex $g|_{\Delta^{\{0\}\times\rho^{-1}(j)}}$ degenerates to a point, provided $\max(\rho^{-1}(j)) \in A$. We warn the reader however that the map $g|_{\{0\}\times A}$ does not in general factor through the image of A in Δ^m.

Remark 4.7. Let $\sigma \dashv \rho$ be as above and assume that $n \geq 2$. Since $\Delta^n \subseteq \Delta^n$ is always $(\sigma \dashv \rho)$-admissible we may consider the set of triangles L_{Δ^n} of $\Delta^1 \times \Delta^n$ defined in Notation 4.5. Since $n \geq 2$ all these triangles are contained in $\Delta^1 \times \partial\Delta^n$, and so we also have $L_{\partial\Delta^n} = L_{\Delta^n}$. Consider the filtration

$$[(\Delta^1 \times \partial\Delta^n, L_{\Delta^n})] \coprod_{\Delta^{\{0\}}\times\partial\Delta^n} [\Delta^{\{0\}} \times \Delta^n_\flat] = Z^{n+1} \subseteq Z^n \subseteq ... \subseteq Z^0 = (\Delta^1 \times \Delta^n, L_{\Delta^n})$$

whose underlying filtration of simplicial sets is as in the filtration (2.5) of Construction 2.27, and the scaling on Z^k is given by L_{Δ^n}. Since L_{Δ^n} contains every triangle of the form $\Delta^{\{(0,i),(0,i+1),(1,i+1)\}}$ we have that the inclusion $Z^{k+1} \subseteq Z^k$ is an inner scaled anodyne map for $k \in \{1, ..., n\}$. In the last step of the filtration, since L_{Δ^n} contains all the triangles $\Delta^{\{(0,0),(1,0),(1,i)\}}$ we obtain a pushout square of the form

$$\begin{array}{ccc} (\Lambda^{n+1}_0, T_0) & \longrightarrow & Z^1 \\ \downarrow & & \downarrow \\ (\Delta^{n+1}_0, T_0) & \longrightarrow & Z^0 \end{array}$$

where T_0 is the set of all triangles which contain the edge $\Delta^{\{0,1\}}$. Since 0 is always in the image of σ it then follows, for example, that if $\mathcal{C} = (\overline{\mathcal{C}}, T_{\mathcal{C}})$ is a weak ∞-bicategory and $h\colon \Delta^1 \times \partial\Delta^n \to \overline{\mathcal{C}}$ is a $(\sigma \dashv \rho)$-transformation (so that the edge $h_{|\Delta^1\times\{0\}}$ is degenerate by Remark 4.3), then any extension of $h|_{\Delta^{\{0\}}\times\partial\Delta^n}$ to $\{0\} \times \Delta^n$ can be prolonged to an extension of h to $\Delta^1 \times \Delta^n$.

The following is the key lemma of this section. Unlike the situation in Remark 4.7, it allows us to extend an $(\sigma \dashv \rho)$-transformations given an extension of its value at 1 (as apposed to 0). The combination of Remark 4.7 and Lemma 4.8 is what gives the notion of an $(\sigma \dashv \rho)$-transformation its power in practice.

Lemma 4.8 (The moving lemma)**.** *Let $(\sigma \dashv \rho)$ be as above and let $A \subseteq B \subseteq \Delta^n$ be an inclusion of $(\sigma \dashv \rho)$-admissible subsimplicial sets of Δ^n. Let $\mathcal{C} = (\overline{\mathcal{C}}, T_{\mathcal{C}})$ be a scaled simplicial set which satisfies the right lifting property with respect to scaled inner horn inclusions (that is, the horn inclusions of type (1) in Definition 1.16). Suppose that we are given a map*

$$g\colon \Delta^1 \times A \coprod_{\{1\}\times A} \{1\} \times B \to \overline{\mathcal{C}}$$

whose restriction to $\Delta^1 \times A$ is a $(\sigma \dashv \rho)$-transformation. Then g extends to a $(\sigma \dashv \rho)$-transformation $h\colon \Delta^1 \times B \to \overline{\mathcal{C}}$.

Remark 4.9. Considering for simplicity the case where A is empty, Lemma 4.8 enables one to take a given map $B \to \overline{\mathcal{C}}$ and modify it in such a way that it becomes degenerate in certain specific ways (see, e.g., Remark 4.6). We will use this "moving trick" in the proofs of Proposition 5.3 and Lemma 7.5 below.

Remark 4.10. Though we formulate Lemma 4.8 in a rather generic manner, we will only apply it in cases where B is either $\Delta^n, \partial\Delta^n$ or Λ^n_0 and A is either $\varnothing, \partial\Delta^n$ or Λ^n_0. The reader who wishes to have a concrete picture in mind while reading the proof below is invited to take $A = \varnothing$ and $B = \Delta^n$.

Proof of Lemma 4.8. Let $\mathcal{B}$ be the collection of non-degenerate simplices of $\Delta^1 \times B$ which are not contained in $\Delta^1 \times A \coprod_{\{1\}\times A} \{1\} \times B$. Adopting the notation of Definition 4.1 we will refer these simplices simply by their subset of vertices $J \subseteq [1] \times [n]$ and will denote $J_0, J_1 \subseteq [n]$ as in that definition. We call $m(J) \coloneqq \max(J_0)$ the *index* of J. Let $\mathcal{B}_0 \subseteq \mathcal{B}$ be the subset of those simplices $J \subseteq [1] \times [n]$ belonging to one of the following mutually exclusive cases:

(i) The index $m(J)$ belongs to the image of σ and $J_1 = \{m(J)\}$.
(ii) The index $m(J)$ does not belong to the image of σ, J contains $(0, m(J) - 1)$ and $J_1 \subseteq \{m(J)\}$.
(iii) $|J_1| \geq 2$ and J contains the edge $\{(0, m(J)), (1, m(J))\}$.

Choose a total ordering on $\mathcal{B}_0$ such that $J' < J$ whenever the dimension of J' is smaller than that of J, or whenever they have the same dimension but the index of J' is smaller than that of J, or whenever they have the same dimension and the same index but $|J'_1| < |J_1|$. More precisely, we arbitrarily extend to a total order that set of relations, by choosing a total order on those simplices who have the same dimension, index and cardinality of vertices at height 1. For $J \in \mathcal{B}_0$ let $Z_{<J} \subseteq \Delta^1 \times B$ be the subsimplicial set given by the union of $\Delta^1 \times A \coprod_{\{1\}\times A} \{1\} \times B$ with all the simplices $J' \in \mathcal{B}_0$ such that $J' < J$, and similarly let $Z_{\leq J} \subseteq \Delta^1 \times B$ be the subsimplicial set given by the union of $\Delta^1 \times A \coprod_{\{1\}\times A} \{1\} \times B$ with all the simplices $J' \in \mathcal{B}_0$ such that $J' \leq J$

We note that out of the simplices in $\mathcal{B}$, Conditions (1)-(3) of Definition 4.2 only concern simplices which are in $\mathcal{B}_0$ (more specifically, Condition (1) concerns simplices of type (i), Condition (2) simplices of type (ii) and Condition (3) simplices of type (iii)). We now show by induction that for every $J \in \mathcal{B}_0$ the map g extends to a map $g_{\leq J}\colon Z_J \to \overline{\mathcal{C}}$ such that Conditions (1)-(3) hold for the simplices in $Z_{\leq J}$. We note that every simplex in $\mathcal{B}$ which is not in $\mathcal{B}_0$ is a face of a simplex in $\mathcal{B}_0$ and so $Z_{\leq J} = \Delta^1 \times B$ when J is the maximal element of $\mathcal{B}_0$.

Let now J be a simplex in $\mathcal{B}_0$ and assume we have already extended the map g to a map $g_{<J}\colon Z_{<J} \to \overline{\mathcal{C}}$ in a way that Conditions (1)-(3) of Definition 4.2 hold. To extend $g_{<J}$ to a map $g_{\leq j}\colon Z_{\leq J} \to \overline{\mathcal{C}}$ we deal with each of the cases (i)-(iii) separately:

(1) Suppose that J is a simplex of type (i) and index $i \coloneqq m(J)$. We first note that $Z_{<J}$ already contains all the maximal face of J except the face opposite to the vertex $(1, i)$. Indeed, the faces opposite to vertices of the form $(0, j)$ for $j < i$ are either in $\Delta^1 \times A$ or are simplices in $\mathcal{B}_0$ of type (i) of lower dimension. The face opposite to the vertex $(0, i)$ is either in $\{1\} \times B$ or is a maximal face of a simplex J' in $\mathcal{B}_0$ of type (iii) and of index $j = \max(J_0 \setminus \{i\}) < i$, and is hence contained in $Z_{<J}$. On the other hand, the face opposite $(1, i)$ cannot belong to $\Delta^1 \times A$ nor to $\{1\} \times B$, does not belong to $\mathcal{B}_0$, and is a maximal face of a unique simplex in $\mathcal{B}_0$, namely J. It is hence not contained in $Z_{<J}$.

To respect Condition (1) we now define $h_{\leq J}\colon Z_{\leq J} \to \overline{\mathcal{C}}$ by mapping Δ^J in a degenerate manner via the retraction $\Delta^J \to \Delta^{J\smallsetminus\{(0,i)\}}$ which maps $(0,i)$ to $(1,i)$ (so that $(h_{\leq J})|_{\Delta^J}$ is determines by $(h_{<J})|_{\Delta^{J\smallsetminus\{(0,i)\}}}$). We note that this definition is compatible with all the faces which are already in $Z_{<J}$: this is automatically true for the face opposite $(0,i)$, and holds for the other ones thanks to our assumption that $g_{<J}$ satisfies Condition (1).

(2) Suppose that J is a simplex of type (ii) and index $i \coloneqq m(J)$. Then we claim that $Z_{<J}$ already contains all the maximal face of J except the one opposite to the vertex $(0,i-1)$. Indeed, the faces opposite to vertices of the form $(0,j)$ for $j < i-1$ are either in $\Delta^1 \times A$ or are simplices in $\mathcal{B}_0$ of type (ii) of lower dimension. The face opposite $(1,i)$ (if $(1,i) \in J$) is itself a simplex of type (ii) and a lower dimension. For the face opposite the vertex $(0,i)$, if $J_1 = \{(1,i)\}$ then this face is also a maximal face of a simplex in $\mathcal{B}_0$ of type (iii) and of a index $i-1$. If $J_1 = \varnothing$ then we argue as follows: if $i-1$ is in the image of σ then we view this face as the one opposite to $(1,i-1)$ in $J \smallsetminus \{(0,i)\} \cup \{(1,i-1)\}$, which is of type (i) and smaller index than J. If $i-1$ is not in the image of σ and $i-2 \in J_0$, then we view this face as the one opposite the vertex $(1,i-1)$ of $J \smallsetminus \{(0,i)\} \cup \{(1,i-1)\}$, which is of type (ii) and smaller index than J. Finally, if $i-1$ is not in the image of σ and $i-2 \notin J_0$, then we view this face as the one opposite to the vertex $(0,i-2)$ in $J \smallsetminus \{(0,i)\} \cup \{(0,i-2)\}$, which is of type (ii) and a smaller index than J.

On the contrary, the face opposite to $(0,i-1)$ cannot belong to $\{1\} \times B$ nor to $\Delta^1 \times A$ (indeed, since A is $(\sigma \dashv \rho)$-admissible, if A contained the simplex spanned by $J_0 \smallsetminus \{i-1\}$ then it would contain J_0), and is not an element of $\mathcal{B}_0$. If we consider which simplices in $\mathcal{B}_0$ other than J contain this face as a maximal face we see that they must either be of type (iii) and the same index as J or of type (ii) and a bigger index. They are hence necessarily bigger than J in the total order we chose, and hence this face is not contained in $Z_{<J}$.

To respect Condition (2) we now define $h_{\leq J}\colon Z_{\leq J} \to \overline{\mathcal{C}}$ by mapping Δ^J in a degenerate manner via the retraction $\Delta^J \to \Delta^{J\smallsetminus\{(0,i)\}}$ which maps $(0,i)$ to $(0,i-1)$. We note that this definition is compatible with all the faces which are already in $Z_{<J}$: this is automatically true for the face opposite $(0,i)$, and holds for the other ones thanks to our assumption that $g_{<J}$ satisfies Condition (2).

(3) Suppose that J is a simplex of type (iii) and index $i \coloneqq m(J)$. We first note that $Z_{<J}$ already contains all the maximal face of J except the face opposite to the vertex $(1,i)$. Indeed, the faces opposite vertices of the form $(0,j)$ for $j < i$ are either in $\Delta^1 \times A$ or are simplices of type (iii) of a smaller dimension, and the same holds for faces opposite vertices of the form $(1,j)$ for $j > i$ such that $(1,j)$ is not maximal in J. The face opposite to the vertex $(0,i)$ is either in $\{1\} \times B$ or is a maximal face of a simplex in $\mathcal{B}_0$ of type (iii) and of index $j = \max(J_0 \smallsetminus \{i\}) < i$, and is hence contained in $Z_{<J}$. Finally, if $(1,j)$ is the maximal vertex in J, then the face opposite $(1,j)$, if it is not in $\Delta^1 \times A$ and not in $\mathcal{B}_0$ then it must be that the index i is not in the image of σ, J_0 does not contain $i-1$ and $J_1 = \{(1,i),(1,j)\}$. In this case, the face opposite $(1,j)$ is the maximal face of the simplex $(J \smallsetminus \{(1,j)\}) \cup \{(0,i-1)\}$, which belongs to $\mathcal{B}_0$ since B is $(\sigma \dashv \rho)$-admissible, is of the same index and dimension as J but is still smaller than J with respect to our linear order since its intersection with $\{1\} \times [n]$ has less elements than that of J. We may then conclude that

this face is also contained in $Z_{<J}$. On the other hand, the face opposite $(1,i)$ is not in $\mathcal{B}_0$ and is the maximal face of exactly two simplices in $\mathcal{B}_0$, the simplex J and another simplex of type (iii) whose index is $\min(J_1 \setminus i) > i$. This face is consequently not contained in $Z_{<J}$. We may thus conclude that we have a pushout square of simplicial sets of the form

$$\begin{array}{ccc} \Lambda^J_{(1,i)} & \longrightarrow & Z_{<J} \\ \downarrow & & \downarrow \\ \Delta^J & \longrightarrow & Z_{\leq J} \end{array}$$

where the vertical map is an inner horn inclusion since $(1,i)$ is not maximal nor minimal in J. Now if $|J| = 3$ then, by our assumption on $\mathcal{C}$, we can extend $g_{<J}$ to $g_{\leq J}$ in such a way that Δ^J is sent to a thin triangle, thus assuring that Condition (3) continues to hold for $g_{<J}$. On the other hand, if $|J| > 3$ then, since we assumed that $g_{<J}$ satisfies Condition (3), we have that $g_{<J}$ maps $\Delta^{\{(0,i),(1,i),(1,j)\}}$ to a thin triangle, where $j = \min(J_1 \setminus i)$. By our assumption on $\mathcal{C}$ we can then extend $g_{<J}$ to a map $g_{\leq J}\colon Z_{\leq J} \to \overline{\mathcal{C}}$. □

Corollary 4.11. *Let $(\sigma \dashv \rho)$ and $A \subseteq B \subseteq \Delta^n$ be as in the previous lemma and L_A, L_B as in Notation 4.5. Let $\mathcal{C} = (\overline{\mathcal{C}}, T_{\mathcal{C}})$ be a scaled simplicial set which satisfies the right lifting property with respect to the maps of type (1) in Definition 1.16. Suppose that we are given a map*

$$g\colon (\Delta^1 \times A, L_A) \coprod_{\{1\}\times A_\flat} \{1\} \times B_\flat \to \mathcal{C}$$

such that the underlying simplicial map of its restriction to $\Delta^1 \times A$ is a $(\sigma \dashv \rho)$-transformation. Then g extends to a map of scaled simplicial sets. $h\colon (\Delta^1 \times B, L_B) \to \mathcal{C}$ whose underlying simplicial map is a $(\sigma \dashv \rho)$-transformation.

Proof. Combine the Lemma 4.8 and Remark 4.4. □

5. Weak ∞-bicategories are ∞-bicategories

Our goal in this section is to prove the following result:

Theorem 5.1. *Let $\mathcal{C}$ be a weak ∞-bicategory. Then $\mathcal{C}$ is an ∞-bicategory,* i.e., *$\mathcal{C}$ is fibrant in $\mathrm{Set}^{\mathrm{sc}}_{\Delta}$.*

We advance towards Theorem 5.1 in a sequence of lemmas.

Lemma 5.2 (Special outer horns)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be weak ∞-bicategories and let*

$$p\colon \mathcal{C} \to \mathcal{D}$$

be a map which satisfies the right lifting property with respect to maps of type (1) and (3) in Definition 1.16. Let T be the collection of all triangles in Δ^n which are either degenerate or contain the edge $\Delta^{\{0,1\}}$, and let $T_0 \subseteq T$ be the subset of those triangles which are contained in Λ^n_0. Then the dotted lift exists in any square of the *form*

$$(5.1)\qquad \begin{array}{ccc} (\Lambda^n_0, T_0) & \xrightarrow{f} & \mathcal{C} \\ \downarrow & \nearrow & \downarrow p \\ (\Delta^n, T) & \xrightarrow{g} & \mathcal{D} \end{array}$$

such that $f|_{\Delta^{\{0,1\}}}$ *is an equivalence in* $\mathcal{C}$*.*

Proof. Let $f_0 \coloneqq f|_{\partial\Delta^{\{2,\ldots,n\}}}$ and let $f_1 \coloneqq f|_{\Delta^{\{2,\ldots,n\}}}$. By Corollary 2.20 the projection

$$q\colon \overline{\mathcal{C}}_{/f_1} \to \overline{\mathcal{C}}_{/f_0} \times_{\overline{\mathcal{D}}_{/pf_0}} \overline{\mathcal{D}}_{/pf_1}$$

is an outer fibration whose base change to the marked core of $\mathcal{C}_{/f_0} \times_{\mathcal{D}_{/pf_0}} \mathcal{D}_{/pf_1}$ is an outer cartesian fibration. Unwinding the definitions, we see that finding a solution to the lifting problem (5.1) is equivalent to finding a lift in a diagram of marked-scaled simplicial sets of the form

$$(5.2)\qquad \begin{array}{ccc} \Delta^{\{0\}} & \longrightarrow & \mathcal{C}_{/f_1} \\ \downarrow & & \downarrow q \\ (\Delta^1, \{\Delta^1\}, \varnothing) & \xrightarrow{\eta} & \mathcal{C}_{/f_0} \times_{\mathcal{D}_{/pf_0}} \mathcal{D}_{/pf_1} \end{array}$$

where η is a marked edge of $\mathcal{C}_{/f_0} \times_{\mathcal{D}_{/pf_0}} \mathcal{D}_{/pf_1}$ whose image in $\mathcal{C}$ is an equivalence. By Corollary 2.18 the map

$$q_{\mathcal{C}}\colon \overline{\mathcal{C}}_{/f_0} \to \mathcal{C}$$

is an outer cartesian fibrations and the edge e_η of $\overline{\mathcal{C}}_{/f_0}$ determined by the corresponding component of η is $q_{\mathcal{C}}$-cocartesian (since it is marked in $\mathcal{C}_{/f_0}$). By Remark 2.5 we have that $\overline{\mathcal{C}}_{/f_0}$ is a weak ∞-bicategory and by Remark 2.10 the edge e_η is an equivalence. Similarly, the map $\overline{\mathcal{C}}_{/f_0} \times_{\overline{\mathcal{D}}_{/pf_0}} \overline{\mathcal{D}}_{/pf_1} \to \overline{\mathcal{C}}_{/f_0}$, which is a base change of $\overline{\mathcal{D}}_{/pf_1} \to \overline{\mathcal{D}}_{/pf_0}$, is an outer fibration such that the edges which are marked in $\mathcal{C}_{/f_0} \times_{\mathcal{D}_{/pf_0}} \mathcal{D}_{/pf_1}$ are cartesian. By Remark 2.5 we then have that $\overline{\mathcal{C}}_{/f_0} \times_{\overline{\mathcal{D}}_{/pf_0}} \overline{\mathcal{D}}_{/pf_1}$ is a weak ∞-bicategory and by Remark 2.10 we may conclude that the edge η is an equivalence.

Now by Remark 2.19 the marked edges in $\mathcal{C}_{/f_0} \times_{\mathcal{D}_{/pf_0}} \mathcal{D}_{/pf_1}$ are closed under composition, and so the marked core of $\mathcal{C}_{/f_0} \times_{\mathcal{D}_{/pf_0}} \mathcal{D}_{/pf_1}$ is also a weak ∞-bicategory. Since the base change of q to this marked core is an outer cartesian fibration this base change is also an isofibration by Remark 2.9. We may thus conclude that the dotted lift in (5.2) exists, as desired. □

Proposition 5.3. *Let* $p\colon \mathcal{C} \to \mathcal{D}$ *be a bicategorical equivalence of weak* ∞*-bicategories and suppose given a square of the form*

$$\begin{array}{ccc} \partial\Delta^n_\flat & \xrightarrow{f} & \mathcal{C} \\ \downarrow & & \downarrow p \\ \Delta^n_\flat & \xrightarrow{g} & \mathcal{D} \end{array}$$

with $n \geq 1$. Then there exists an extension $\overline{f}\colon \Delta^n_\flat \to \mathcal{C}$ and a natural transformation $T\colon \Delta^1_\flat \times \Delta^n_\flat \to \mathcal{D}$ from g to $p\overline{f}$ whose restriction to $\partial\Delta^n_\flat$ is the identity transformation on pf (in particular, T is a levelwise equivalence since $n \geq 1$).

Proof. If $n = 1$ the claim amounts to the induced maps of (naturally marked) ∞-categories $\mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(p(-),p(-))$ being essentially surjective. We may hence assume that $n \geq 2$.

Let $x = f(0) \in \mathcal{C}$ and $y = f(n) \in \mathcal{C}$. Our first step is to "move" f so that it sends all the vertices except the last one to x. Let $\overline{\mathcal{C}}$ be the underlying simplicial set of $\mathcal{C}$. Applying Corollary 4.11 with $A = \varnothing, B = \partial\Delta^n$ and $\rho\colon \Delta^n \to \Delta^1$ the map which sends $\{0,...,n-1\}$ to 0 and n to 1 we may find a map of scaled simplicial sets

$$h\colon (\Delta^1 \times \partial\Delta^n, L_{\partial\Delta^n}) \to \mathcal{C}, \tag{5.3}$$

whose underlying simplicial map $\Delta^1 \times \partial\Delta^n \to \overline{\mathcal{C}}$ is a $(\sigma \dashv \rho)$-transformation, and such that $h|_{\{1\}\times\partial\Delta^n_\flat} = f$ (here the scaling $L_{\partial\Delta^n}$ is as in Notation 4.5). In addition, by the definition of a $(\sigma \dashv \rho)$-transformation we have that $h|_{\Delta^1\times\{0\}}$ and $h|_{\Delta^1\times\{n\}}$ are degenerate (see Remark 4.3) and $h|_{\{0\}\times\Delta^{\{0,...,n-1\}}_\flat}$ is degenerate on x (see Remark 4.6). Let us set $f' := h|_{\{0\}\times\partial\Delta^n_\flat}$. The map f' then determines a commutative square of scaled simplicial sets

$$\begin{array}{ccc} \partial\Delta^{\{0,...,n-1\}}_\flat & \longrightarrow & \overline{\mathcal{C}}_{/y} \\ \downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{C}}} \\ \Delta^{\{0,...,n-1\}}_\flat & \longrightarrow & \mathcal{C} \end{array} \tag{5.4}$$

in which the bottom horizontal map is given by the restriction of f' to $\{0\}\times\Delta^{\{0,...,n-1\}}_\flat$, and is in particular constant with image the vertex x. The top horizontal map in (5.4) then determines a map $f'_x\colon \partial\Delta^{\{0,...,n-1\}}_\flat \to (\overline{\mathcal{C}}_{/y})_x$. Projecting down to $\mathcal{D}$ we may now consider the map

$$(\Delta^1 \times \partial\Delta^n, L_{\partial\Delta^n}) \coprod_{\Delta^{\{1\}}\times\partial\Delta^n_\flat} \Delta^{\{1\}} \times \Delta^n_\flat \to \mathcal{D} \tag{5.5}$$

determined by ph and g. Applying again Corollary 4.11 with respect to $A = \partial\Delta^n$ and $B = \Delta^n$ we may extend (5.5) to a map

$$H\colon (\Delta^1 \times \Delta^n, L_{\Delta^n}) \to \mathcal{D},$$

whose underlying simplicial map is a $(\sigma \dashv \rho)$-transformation, so that, in particular, H maps $\{0\} \times \Delta^{\{0,...,n-1\}}$ to $p(x)$. Let us denote by g' the restriction of H to $\{0\} \times \Delta^n$. The map g' now allows us to extend the map $f'_x\colon \partial\Delta^{\{0,...,n-1\}} \to (\overline{\mathcal{C}}_{/y})_x$ above to a commutative square

$$\begin{array}{ccc} \partial\Delta^{\{0,...,n-1\}}_\flat & \xrightarrow{f'_x} & (\overline{\mathcal{C}}_{/y})_x \\ \downarrow & & \downarrow{\scriptstyle p_*} \\ \Delta^{\{0,...,n-1\}}_\flat & \xrightarrow{g'_x} & (\overline{\mathcal{D}}_{/p(y)})_{p(x)} \end{array} \tag{5.6}$$

Since $p\colon \mathcal{C} \to \mathcal{D}$ is a bicategorical equivalence of weak ∞-bicategories Proposition 2.24 implies that the map $(\overline{\mathcal{C}}_{/y})_x \to (\overline{\mathcal{D}}_{/p(y)})_{p(x)}$ is a an equivalence of ∞-categories (or, more precisely, of ∞-bicategories in which every triangle is thin). It then follows that in the square of Kan complexes

$$\begin{array}{ccc} \mathrm{Fun}^{\simeq}(\Delta_\flat^{\{0,\ldots,n-1\}}, (\overline{\mathcal{C}}_{/y})_x) & \longrightarrow & \mathrm{Fun}^{\simeq}(\Delta_\flat^{\{0,\ldots,n-1\}}, (\overline{\mathcal{D}}_{/p(y)})_{p(x)}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\simeq}(\partial\Delta_\flat^{\{0,\ldots,n-1\}}, (\overline{\mathcal{C}}_{/y})_x) & \longrightarrow & \mathrm{Fun}^{\simeq}(\partial\Delta_\flat^{\{0,\ldots,n-1\}}, (\overline{\mathcal{D}}_{/p(y)})_{p(x)}) \end{array}$$

the horizontal maps are equivalences and the vertical maps are categorical fibrations, which implies that this square induces equivalences on vertical fibers. We may therefore conclude that the map $f'_x\colon \partial\Delta_\flat^{\{0,\ldots,n-1\}} \to (\overline{\mathcal{C}}_{/y})_x$ extends to a map

$$\overline{f}'_x\colon \Delta_\flat^{\{0,\ldots,n-1\}} \to (\overline{\mathcal{C}}_{/y})_x,$$

such that we have an invertible natural transformation $\eta_x\colon \Delta^1_\flat \times \Delta^{n-1}_\flat \to (\overline{\mathcal{D}}_{/p(y)})_{p(x)}$ from g'_x to $p_*\overline{f}'_x$, whose restriction to $\partial\Delta^{n-1}_\flat$ is the identity on $p_*f'_x$. The map $\overline{f}'_x$ then corresponds to a map $\overline{f}'\colon \Delta^n_\flat \to \mathcal{C}$ and the natural equivalence η_x determines a natural equivalence $\eta\colon \Delta^1_\flat \times \Delta^n_\flat \to \mathcal{D}$ from g' to $p\overline{f}'$, whose restriction to $\partial\Delta^n_\flat$ is the identity on pf'. Before continuing let us introduce the shorthand notation

$$X := \Delta^{\{0\}} \times \Delta^n_\flat \coprod_{\Delta^{\{0\}}\times\partial\Delta^n_\flat} (\Delta^1 \times \partial\Delta^n, L_{\partial\Delta^n}) \to (\Delta^1 \times \Delta^n, L_{\Delta^n}) := Y.$$

The map h of (5.3) together with $\overline{f}'$ above then determine a map $\varphi\colon X \to \mathcal{C}$ which sends the edge $\Delta^1 \times \{0\}$ to a degenerate edge (see Remark 4.3). Using the filtration of Remark 4.7 we now see that the map $X \coprod_{\Delta^1_\flat\times\{0\}} \Delta^0 \to Y \coprod_{\Delta^1_\flat\times\{0\}} \Delta^0$ is scaled anodyne and hence the map φ extends to a map $\overline{\varphi}\colon Y \to \mathcal{C}$. Projecting to $\mathcal{D}$ we now obtain two different maps $p\overline{\varphi}, H\colon Y \to \mathcal{D}$, whose restriction to X gives the two maps $p\varphi, H|_X\colon X \to \mathcal{D}$. By construction, the map $p\varphi$ is determined by the pair $(p\overline{f}', ph)$, while the map $H_{|X}$ is determined by the pair (g', ph). These two maps are related via a levelwise invertible natural transformation $\overline{\eta}\colon \Delta^1_\flat \times X \to \mathcal{D}$ whose value on $\Delta^{\{0\}} \times \Delta^n_\flat \subseteq X$ is η, and which is constant on $(\Delta^1 \times \partial\Delta^n_\flat, L_{\partial\Delta^n}) \subseteq X$. We may hence consider the resulting extension problem

$$\begin{array}{ccc} \partial\Delta^1 \times Y \coprod_{\partial\Delta^1\times X} \Delta^1_\flat \times X & \xrightarrow{\left((H,p\overline{\varphi}),\overline{\eta}\right)} & \mathcal{D} \\ \downarrow & \nearrow & \\ \Delta^1_\flat \times Y & & \end{array} \tag{5.7}$$

By the above, the top horizontal map sends $\Delta^1_\flat \times \Delta^1_\flat \times \{0\} \subseteq \Delta^1_\flat \times X$ to a point. Since the map $X \coprod_{\Delta^1_\flat\times\{0\}} \Delta^0 \to Y \coprod_{\Delta^1_\flat\times\{0\}} \Delta^0$ is scaled anodyne and scaled anodyne are closed under pushout-products with arbitrary inclusions [16, Prop.3.1.8] the extension problem (5.7) admits a solution $\psi\colon \Delta^1_\flat \times Y \to \mathcal{D}$. Restricting the map $\overline{\varphi}\colon Y \to \mathcal{C}$ and the natural transformation $\psi\colon \Delta^1_\flat \times Y \to \mathcal{D}$ to $\Delta^{\{1\}} \times \Delta^n_\flat \subseteq Y$ now yields an extension $\overline{f}\colon \Delta^n_\flat \to \mathcal{C}$ of the original map $f\colon \partial\Delta^n_\flat \to \mathcal{C}$ and a natural transformation from $g\colon \Delta^n_\flat \to \mathcal{D}$ to $p\overline{f}\colon \Delta^n_\flat \to \mathcal{D}$, whose restriction to $\partial\Delta^n_\flat$ is the identity on pf, as desired. $\square$

Proposition 5.4. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a bicategorical equivalence of weak ∞-bicategories and let $j\colon A \to B$ an injective map of scaled simplicial sets. Suppose that we are given a commutative diagram of the form*

$$\begin{array}{ccc} \Delta^{\{1\}} \times A & \xrightarrow{\;r_0\;} & \mathcal{C} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ \Delta^{\{0\}} \times B \coprod_{\Delta^{\{0\}}\times A} \Delta^1_\flat \times A & \xrightarrow{\;h_0\;} & \mathcal{D} \end{array} \tag{5.8}$$

such that $(h_0)|_{\Delta^1\times\{a\}}$ is an equivalence in $\mathcal{D}$ for every $a \in A$. Then (5.8) extends to a diagram of the form

$$\begin{array}{ccccc} \Delta^{\{1\}} \times A & \longrightarrow & \Delta^{\{1\}} \times B & \xrightarrow{\;r\;} & \mathcal{C} \\ \downarrow{\scriptstyle j} & & \downarrow & & \downarrow{\scriptstyle i} \\ \Delta^{\{0\}} \times B \coprod_{\Delta^{\{0\}}\times A} \Delta^1_\flat \times A & \longrightarrow & \Delta^1_\flat \times B & \xrightarrow{\;h\;} & \mathcal{D} \end{array} \tag{5.9}$$

whose external rectangle is (5.8) and such that $h|_{\Delta^1\times\{b\}}$ is an equivalence in $\mathcal{D}$ for every $b \in B$.

Proof. Working simplex by simplex, it will suffice to prove the case where the map $j\colon A \hookrightarrow B$ is one of the inclusions $j_n\colon \partial\Delta^n_\flat \subseteq \Delta^n_\flat$ or the inclusion $\Delta^2_\flat \subseteq \Delta^2_\sharp$. We then note that for $j_0\colon \varnothing \subseteq \Delta^0$ the desired statement is equivalent to p being essentially surjective. When j is the inclusion $\Delta^2_\flat \subseteq \Delta^2_\sharp$ the horizontal maps in the left square of (5.9) are both isomorphisms on the underline simplicial sets. In this case the result can be obtained by invoking the fact that p detects thin triangles (Proposition 3.3) and that the collection of thin triangles is closed under levelwise invertible natural transformations (Corollary 3.5). We may hence assume that $j = j_n$ for some $n \geq 1$.

Let

$$[\Delta^1_\flat \times \partial\Delta^n_\flat] \coprod_{\Delta^{\{0\}}\times\partial\Delta^n_\flat} [\Delta^{\{0\}} \times \Delta^n_\flat] = Z^{n+1} \subseteq Z^n \subseteq \ldots \subseteq Z^0 = \Delta^1_\flat \times \Delta^n_\flat \tag{5.10}$$

be the filtration (2.5) of Construction 2.27. Then the inclusions $Z^{k+1} \subseteq Z^k$ are scaled anodyne for $k \geq 1$ and so we may extend the map h_0 along the filtration (5.10) all the way to a map $h'\colon Z^1 \to \mathcal{D}$. In the last filtration step we have a pushout diagram of the form

$$\begin{array}{ccc} (\Lambda^{n+1}_0, T_0|_{\Lambda^{n+1}_0}) & \longrightarrow & Z^1 \\ \downarrow & & \downarrow \\ (\Delta^{n+1}, T_0) & \longrightarrow & Z^0 \end{array}$$

where the composed map $\Delta^{n+1} \to Z_0 \to \Delta^n_\flat \times \Delta^1_\flat$ is the simplex τ_0. Our assumption that $h_0|_{\{0\}\times\Delta^1}$ is an equivalence in $\mathcal{D}$ implies that $h' \circ \tau_0\colon \Lambda^{n+1}_0$ sends $\Delta^{\{0,1\}}$ to an equivalence and hence by Lemma 5.2 we may extend $h'\colon Z^1 \to \mathcal{D}$ to a map $h\colon Z^0 = \Delta^1_\flat \times \Delta^n_\flat \to \mathcal{D}$. We have thus constructed a natural transformation from $h_0|_{\{0\}\times\Delta^n_\flat}$ to some n-simplex $\sigma\colon \Delta^n_\flat \to \mathcal{D}$, extending the given natural transformation on $\partial\Delta^n_\flat \subseteq \Delta^n_\flat$. In particular, $\sigma|_{\partial\Delta^n_\flat}$ lifts to $\mathcal{C}$. Applying Proposition 5.3 we may conclude that σ admits a natural transformation (constant on $\partial\Delta^n$ and in particular levelwise invertible) to an n-simplex which lifts to $\mathcal{C}$. Composing this natural transformation with h we may as well assume that σ itself lifts to $\mathcal{C}$, as desired. □

Proof of Theorem 5.1. Let $\mathcal{C}$ be a weak ∞-bicategory. Choose a trivial cofibration $p\colon \mathcal{C} \hookrightarrow \mathcal{D}$ where $\mathcal{D}$ is fibrant. Then in particular $\mathcal{D}$ is a weak ∞-bicategory. Applying Proposition 5.4 to the diagram

$$\begin{array}{ccc} \mathcal{C} \times \Delta^{\{1\}} & \xrightarrow{\cong} & \mathcal{C} \\ \downarrow & & \downarrow p \\ \mathcal{D} \coprod_{\mathcal{C}\times\Delta^{\{0\}}} \Delta^1 \times \mathcal{C} & \longrightarrow & \mathcal{D} \end{array}$$

in which the bottom map restricts to the identity on $\mathcal{D}$ and to the identify transformation from p to itself on $\Delta^1 \times \mathcal{C}$, we may conclude that $\mathcal{C}$ is a deformation retract of $\mathcal{D}$ and hence in particular fibrant. □

Since the notion of a bicategorical equivalence is invariant under the duality operator $(X, T_X) \mapsto (X^{\mathrm{op}}, T_X)$, Theorem 5.1 implies that the notion of a weak ∞-bicategory is self-dual. We may summarize the situation as follows:

Corollary 5.5. *Let (X, T_X) be a scaled simplicial set. Then the following conditions are equivalent:*

(1) (X, T_X) is a weak ∞-bicategory.
(2) (X^{op}, T_X) is a weak ∞-bicategory.
(3) (X, T_X) is fibrant in Set^+_Δ.

In addition, when these equivalent conditions hold the map $(X, T_X) \to \Delta^0$ is a weak fibration.

6. The Cisinski model structure for ∞-bicategories

In this section, we give a different construction of Lurie's model structure for ∞-bicategories on the category $\mathcal{S}\mathrm{et}^{\mathrm{sc}}_\Delta$ of scaled simplicial sets using the machinery of Cisinski–Olschok recalled in the appendix. We choose as our subset of monomorphism the set $\mathbf{S}$ the generating anodyne maps of Definition 1.16. As interval, we choose $J_\sharp$, whose underlying simplicial set is the 1-categorical nerve of the free living groupoid on an invertible arrow (alternatively, it can be described as $J = \mathrm{Cosk}_0(\{0,1\})$), *i.e.*, the 0-coskeleton of the set with two elements, whose two non-degenerate 2-simplices are marked. The pair $(J_\sharp, \{0,1\} \to J_\sharp)$ is then indeed a cylinder object in the sense of Definition A.3, see also Remark A.4.

Definition 6.1. We will call the *Cisinski model structure* on $\mathcal{S}\mathrm{et}^{\mathrm{sc}}_\Delta$ the model structure of Theorem A.6 associated to the set of maps $\mathbf{S}$ and the interval object $J_\sharp$. Note that Assumption A.1 holds for $\mathcal{S}\mathrm{et}^{\mathrm{sc}}_\Delta$ thanks to Remark 1.14 and Remark A.2.

By definition, the cofibrations of the Cisinski model structure on $\mathrm{Set}^{\mathrm{sc}}_\Delta$ are the monomorphisms, and the fibrant objects are the scaled simplicial sets X which admit extensions for the generating anodyne maps of Notation A.5. The following result justifies our choice of an interval object:

Proposition 6.2. *The inclusions $i_0\colon \{0\} \to J_\sharp$ and $i_1\colon \{1\} \to J_\sharp$ are ∞-bicategorical equivalences of ∞-bicategories.*

Proof. It will suffice to show that the terminal map $J_\sharp \to \Delta^0$ is a trivial fibration, i.e., satisfies the right lifting property with respect to all inclusions of scaled simplicial sets. Since every triangle in $J_\sharp$ is thin it will suffice to check that the underlying simplicial set J satisfies the right lifting property with respect to all inclusions of simplicial sets. Better yet, since J is defined to be the 0-coskeleton of $\{0,1\}$ it will suffice to check that $\{0,1\}$ has the right lifting property with respect to all inclusions of sets. Indeed, every set has this property. □

Corollary 6.3. *The class of fibrant objects in the Cisinski model structure contains all ∞-bicategories and is contained in the class of all weak ∞-bicategories. It thus coincides with the class of ∞-bicategories by Theorem 5.1.*

Proof. Since $\mathbf{S}$ generates the class of scaled anodyne maps it follows that every Cisinski-fibrant object is a weak ∞-bicategory. On the other hand, combining the previous Proposition with the fact that Lurie's model structure on scaled simplicial sets is cartesian (see [16, Proposition 3.1.8] and [16, Lemma 4.2.6]) we may conclude that every generating anodyne in the Cisinski model structure is a trivial cofibration in the bicategorical model structure, and hence every ∞-bicategory is Cisinski-fibrant.

□

Since model structures are determined by the class of cofibrations and fibrant objects ([13], Proposition E.1.10) we may conclude that:

Corollary 6.4. *The Cisinski model structure on the category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ of scaled simplicial sets with generating set $\mathbf{S}$ and interval given by $J_\sharp$ coincides with Lurie's model structure for ∞-bicategories.*

7. The main equivalence

In this section we define an adjunction of the form:

$$\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \underset{U}{\overset{\iota}{\rightleftarrows}} \mathrm{St}_2 \qquad (\iota \dashv U)$$

which we show to be a Quillen one. After having established this, we prove it is a Quillen equivalence by making use, among other things, of an explicit fibrant replacement for ιX, when X is an ∞-bicategory.

Definition 7.1. Define the functor $\iota\colon \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \to \mathrm{St}$ by sending a scaled simplicial set (X, T_X) to the stratified set $\iota(X, T_X) \coloneqq (X, T_X \cup \deg(X))$. By definition, this is a stratified set whose only non-degenerate marked n-simplices have $n = 2$.

We observe that this functor can be equivalently described as the following left Kan extension:

$$\begin{array}{ccc} \Delta_{\mathrm{sc}} & \xrightarrow{\;i\;} & \mathrm{St}_2 \\ {\scriptstyle y}\downarrow & \nearrow_{\mathbf{Lan}_y(i) \cong \iota} & \\ \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} & & \end{array}$$

where $i: \Delta_{\mathrm{sc}} \to \mathrm{St}_2$ is defined by setting:

$$\begin{cases} i([n]) \coloneqq (\Delta^n, \varnothing) \\ i([2]_t) \coloneqq (\Delta^2, \{\Delta^2\}) \end{cases}$$

It thus follows that ι admits a right adjoint $U: \mathrm{St}_2 \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$. Explicitly, U forgets all the marking except in dimension 2, and preserves the underlying simplicial set. We will call $U(X, M)$ the *underlying scaled simplicial set* of the stratified set (X, M).

Lemma 7.2. *The functor $\iota: \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \to \mathrm{St}_2$ is fully faithful, preserves monomorphisms and preserves pushouts. In addition, ι preserves products and pushout-products up to homotopy.*

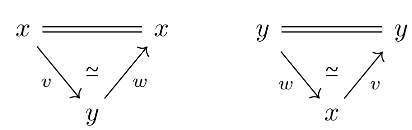
By "preserves products up to homotopy" we mean that the natural map

$$\iota(X \times Y) \to \iota X \times \iota Y$$

is an anodyne extension in St_2, and similarly for pushout-products.

Proof. We first note that monomorphisms of stratified sets are detected on the level of the underlying simplicial sets, and hence also on the level of the underlying scaled simplicial sets. The first part is then a consequence of the fact that ι is a left adjoint functor and the associated unit map $X \to U(\iota(X))$ is an isomorphism. Since U preserves products this also shows that $\iota(X \times Y)$ and $\iota X \times \iota Y$ have the same underlying scaled simplicial set, so that the marking of $\iota(X \times Y)$ and $\iota X \times \iota Y$ only differs in dimension strictly greater than 2. The comparison map (both for the case of products and pushout-products) is therefore in the saturation of the set $\{(\Delta^n, \varnothing) \to (\Delta^n, \{\Delta^n\})\}_{n>2}$, and is hence an anodyne extension. □

The previous lemma tells us that ι preserves cofibrations, since they coincide with monomorphisms for both model structures involved. Hence, this proves half of the Proposition 7.4. Before completing its proof, we need a preliminary lemma. Given a stratified set $X \in \mathrm{St}$ denote by eq_X the set of equivalences of X, *i.e.*, 1-simplices $v: x \to y$ that admit extensions to $\mathrm{th}_1 E_2$, as displayed below.



Here, E_2 denotes the 2-skeleton of the simplicial set $J = \mathrm{Cosk}_0(\{0,1\})$ (where we adopt the notation of [23]). Clearly, the set of equivalences of a stratified set only depends on the underlying scaled simplicial set.

Lemma 7.3. *Given an ∞-bicategory (X, T_X) in $\mathcal{S}et_\Delta^{\mathrm{sc}}$ we have that*

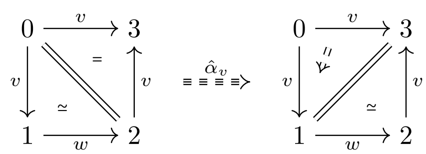
$$\tilde{X} \coloneqq \mathbf{th}_2(X, T_X \cup eq_X)$$

is a 2-trivial saturated complicial set and the map $\iota(X, T_X) \to \tilde{X}$ is a trivial cofibration of stratified sets. In particular, $\tilde{X}$ is a fibrant replacement of $\iota(X, T_X)$.

Proof. We begin by showing that the inclusion $\iota(X, T_X) \to \tilde{X}$ is an anodyne extension. We can factor it as the composite of the obvious maps

$$\iota(X, T_X) \to \mathbf{th}_2 \iota(X, T_X) \to \tilde{X},$$

the first one clearly being an anodyne morphism. Turning to the second one, we will show that it belongs to the saturation of the set of maps given by $\{\Delta^3_{\mathrm{eq}} \to \mathrm{th}(\Delta^3)\}$. For every equivalence $v \in X_1$ pick a witness $\alpha_v\colon \mathrm{th}_1(E_2) \to \mathbf{th}_2\iota(X, T_X)$, and consider the inner horn $\Lambda^3_2 \to X$ given by:

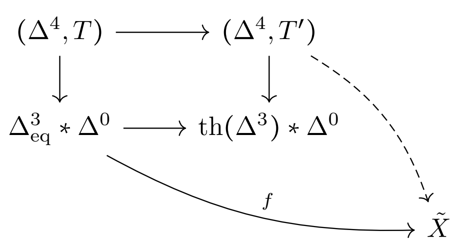


where the 2-simplices denoted by "≃" are those that constitute α_v. These data admit an extension to a 3-simplex $\hat{\alpha}_v$, since (X, T_X) is an ∞-bicategory, and the resulting 2-simplex $\hat{\alpha}_v|_{\Delta^{\{0,1,3\}}}$ has to be marked, thanks to Remark 1.17. We thus get a map $\alpha_v^\sharp\colon \Delta^3_{\mathrm{eq}} \to \mathbf{th}_2\iota(X, T_X)$. We now observe that the inclusion $\mathbf{th}_2\iota(X, T_X) \to \tilde{X}$ is obtained as a transfinite composite of pushouts of the maps $\alpha_v^\sharp$ along the map $\Delta^3_{\mathrm{eq}} \to \mathrm{th}(\Delta^3)$, as v varies through all the equivalences of (X, T_X), and is thus an anodyne map.

Let us now show that $\tilde{X}$ is fibrant, i.e., admits all lifts against generating anodyne morphisms (Definition 1.32). Since $\tilde{X}$ is 2-trivial by construction and is marked by its equivalences, [23, Theorem 56] tells us that it is enough to check that X admits lifts against inner complicial horn inclusions, inner thinness extensions of dimension ≥ 3 and saturation maps. Now for complicial inner horns $(\Lambda^n, M_i|_{\Lambda^n_i}) \to (\Delta^n, M_i)$, these lifts already exist at the level of the underlying scaled simplicial set $(X, T_X) = U(\tilde{X})$ since it is assumed to be an ∞-bicategory.

Turning to inner thinness extensions in dimension ≥ 3, we only have to check it for $(\Delta^3, M_i') \to (\Delta^3, M_i'')$ since $\tilde{X}$ is 2-trivial. In this case, the existence of a lift follows from Remark 1.17, which ensures it already exists at the level of the underlying scaled simplicial set, as above.

Let us now show $\tilde{X}$ is saturated. Since all the simplices of dimension ≥ 3 in $\tilde{X}$ are marked it will suffice to show that $\tilde{X}$ admits extensions against the maps $\Delta^3_{\mathrm{eq}} \to \mathrm{th}(\Delta^3)$ and $\Delta^3_{\mathrm{eq}} \star \Delta^0 \to \mathrm{th}(\Delta^3) \star \Delta^0$; indeed, for any $m > 0$, a marked 2-simplex of $\Delta^3_{\mathrm{eq}} \star \Delta^m$ (resp. $\mathrm{th}(\Delta^3) \star \Delta^m$) lives inside a stratified sub-simplicial set of the form $\Delta^3_{\mathrm{eq}} \star \Delta^{\{k\}}$ (resp. $\mathrm{th}(\Delta^3) \star \Delta^{\{k\}}$), for some $k \in \{0, 1, ..., m\}$. In the former case, we observe that, since both Δ^3_{eq} and $\mathrm{th}(\Delta^3)$ are 1-trivial by definition, any map $\Delta^3_{\mathrm{eq}} \to \tilde{X}$ factors through its core ∞-category. Therefore, a lift exists by the 2-out-of-6 property of equivalences in an ∞-category. For $\Delta^3_{\mathrm{eq}} \star \Delta^0 \to \mathrm{th}(\Delta^3) \star \Delta^0$, consider the following diagram of stratified sets (omitting the functor ι for sake of simplicity):



where

$$T := \{\Delta^{\{0,2,4\}},\ \Delta^{\{1,2,3\}},\ \Delta^{\{0,1,3\}},\ \Delta^{\{1,3,4\}},\ \Delta^{\{0,1,2\}}\}$$

and
$$T' := T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\}.$$
The precomposition of f with $(\Delta^4, T) \to \Delta^3_{\mathrm{eq}} * \Delta^0$ admits the dotted extension to (Δ^4, T'), since (X, T_X) is an ∞-bicategory and $(\Delta^4, T) \to (\Delta^4, T')$ is a generating anodyne map for the model structure for ∞-bicategories. We hence obtain an extension of f to a map $g\colon W \to \tilde{X}$, where $W := (\Delta^4, T') \coprod_{(\Delta^4, T)} \Delta^3_{\mathrm{eq}} * \Delta^0$. Since $\tilde{X}$ admits extensions against $\Delta^3_{\mathrm{eq}} \subseteq \mathrm{th}(\Delta^3)$ as we saw above we may further extend g to a map $g'\colon W' \to \tilde{X}$, where $W' := W \coprod_{\Delta^3_{\mathrm{eq}}} \mathrm{th}(\Delta^3)$.

We now claim that the map g' extends to all of $\mathrm{th}(\Delta^3) * \Delta^0$. To see this, note that W' and $\mathrm{th}(\Delta^3) * \Delta^0$ have the same underlying simplicial set Δ^4 and the same marked edges, and that all the triangles contained in $\Delta^{\{0,1,2,3\}}$ are marked in W'. Out of the six triangles in Δ^4 which contain the vertex 4 we have that exactly four are marked in W': $\Delta^{\{0,2,4\}}, \Delta^{\{1,3,4\}}, \Delta^{\{0,1,4\}}$ and $\Delta^{\{0,3,4\}}$, where as in $\mathrm{th}(\Delta^3) * \Delta^0$ all six are marked. Now since the marked edges in $\tilde{X}$ are exactly those which are equivalences in the ∞-bicategory (X, T_X) we get that g' sends the edges $\Delta^{\{0,1\}}$ and $\Delta^{\{1,2\}}$ to equivalence. Applying Proposition 3.4 to the 3-simplex $g'|_{\Delta^{\{0,1,2,4\}}}$ we may now conclude that $g'(\Delta^{\{1,2,4\}})$ is in T_X, and the same lemma applied to the 3-simplex $g'|_{\Delta^{\{1,2,3,4\}}}$ shows that $g'(\Delta^{\{2,3,4\}})$ is in T_X. We may hence conclude that g' extends to $\mathrm{th}(\Delta^3) * \Delta^0$, as desired. □

Proposition 7.4. *The adjunction*
$$\mathcal{S}et_\Delta^{\mathrm{sc}} \underset{U}{\overset{\iota}{\rightleftarrows}} \mathrm{St}_2 \quad (\iota \dashv U)$$
is a Quillen adjunction between the model structure for ∞-bicategories and that of 2-trivial saturated complicial sets.

The proof of Proposition 7.4 will require the following two lemmas.

Lemma 7.5. *For $n \geq 3$ the map*
$$i_n\colon \big(\Lambda^n_0, \{\Delta^{\{0,1\}}, \Delta^{\{0,1,n\}}\}\big) \to \big(\Delta^n, \{\Delta^{\{0,1\}}, \Delta^{\{0,1,n\}}\}\big)$$
is a trivial cofibration in St_2.

Proof. It suffices to show that i_n has the left lifting property with respect to all fibrations between fibrant objects $p\colon (E, M_E) \to (B, M_B)$. We therefore consider a lifting problem of the form

$$\begin{array}{ccc} (\Lambda^n_0, \{\Delta^{\{0,1\}}, \Delta^{\{0,1,n\}}\}) & \xrightarrow{f} & (E, M_E) \\ \downarrow & \nearrow & \downarrow p \\ (\Delta^n, \{\Delta^{\{0,1\}}, \Delta^{\{0,1,n\}}\}) & \xrightarrow{g} & (B, M_B) \end{array} \tag{7.1}$$

Since E is fibrant its underlying scaled simplicial set $(E, M_E \cap E_2)$ admits filler for scaled inner horns (as those are also anodyne maps up to marking in dimension ≥ 3 in the context of stratified sets, see Definition 1.32). By Lemma 4.8 we may thus find a $(\sigma \dashv \rho)$-transformation $h\colon \Delta^1 \times \Lambda^n_0 \to E$ with respect to the scaling $M_E \cap E_2$, where $\rho\colon \Delta^n \to \Delta^2$ is the surjective map with $\rho^{-1}(0) = \{0\}, \rho^{-1}(1) = \{1, ..., n-1\}$ and $\rho^{-1}(2) = \{n\}$ and $\sigma\colon \Delta^2 \to \Delta^n$ is the associated minimal section. Let M be the set of simplices of $\Delta^1 \times \Lambda^n_0$ containing the degenerate simplices and the simplices of the following form:

- the simplices $\Delta^{\{1\}} \times \Delta^{\{0,1\}}$ and $\Delta^{\{1\}} \times \Delta^{\{0,1,n\}}$;
- the edges of the form $\Delta^1 \times \{i\}$ for $i = 0, 1, n$;
- the triangles of the form $\Delta^{\{(0,i),(1,i),(1,j)\}}$ for every $i, j \in [n]$;
- the triangles of the form $\Delta^{\{(0,i),(0,j),(1,j)\}}$ whenever $j = 0, 1, n$ or $j = i + 1$ and $i, j \in \{1, ..., n-1\}$;
- the triangles $\Delta^{\{0\}} \times \Delta^{\{0,1,i\}}$ for $i \in \{2, ..., n-1\}$;
- all the simplices of dimension ≥ 3;

By the definition of $(\sigma \dashv \rho)$-transformation and since (E, M_E) is 2-trivial we have that h extends to a map of stratified sets $(\Delta^1 \times \Lambda^n_0, M) \to (E, M_E)$. In particular, the second to last subset of M from the previous list is sent to thin triangles since, by point (2) in Definition 4.2 and by downward induction on i, we have that the whole simplex $h(\Delta^{\{0\}} \times \Delta^{\{0,1,\dots n-1\}})$ degenerates to $h(\Delta^{\{0\}} \times \Delta^{\{0,1\}})$. Let M' be the union of M with the simplices $\Delta^{\{0\}} \times \Delta^{\{0,1\}}$ and $\Delta^{\{0\}} \times \Delta^{\{0,1,n\}}$. By Remark 1.36 we see that h extends to a map of stratified sets $h \colon (\Delta^1 \times \Lambda^n_0, M') \to (E, M_E)$.

Now consider the composed map $ph \colon (\Delta^1 \times \Lambda^n_0, M') \to (B, M_B)$. Since the scaled simplicial set $(B, M_B \cap B_2)$ also admits fillers for scaled inner horns we may apply Lemma 4.8 again to extend the $(\sigma \dashv \rho)$-transformation ph to a $(\sigma \dashv \rho)$-transformation $H \colon (\Delta^1 \times \Delta^n, M') \to (B, M_B)$. restricting h and H to $\Delta^{\{0\}} \subseteq \Delta^1$ we now obtain a modified lifting problem of the form

$$\begin{array}{ccc} (\Lambda^n_0, M'_0|_{\Lambda^n_0}) & \xrightarrow{f'} & (E, M_E) \\ \downarrow & \overset{h'}{\nearrow} & \downarrow{\scriptstyle p} \\ (\Delta^n, M'_0) & \xrightarrow{g'} & (B, M_B) \end{array} \tag{7.2}$$

where $M'_0 := M'|_{\Delta^{\{0\}} \times \Delta^n}$ coincides with the set of all faces of Δ^n which contain the edge $\Delta^{\{0,1\}}$. Since the left vertical map in (7.2) is an anodyne extension and the right vertical map is a fibration the dotted lift h' exists. The maps h', h and H now determine a commutative diagram of the form

$$\begin{array}{ccc} (\Delta^{\{0\}} \times \Delta^n, M'_0) \coprod_{\Delta^{\{0\}} \times \Lambda^n_0} (\Delta^1 \times \Lambda^n_0, M') & \longrightarrow & (E, M_E) \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ (\Delta^1 \times \Delta^n, M') & \xrightarrow{H} & (B, M_B) \end{array} \tag{7.3}$$

We now claim that the left vertical map in (7.3) is an anodyne extension, and hence the lifting problem has a solution. To see this, first we note that Δ^n contains two non-degenerate faces that are not in Λ^n_0, the simplex $\Delta^{\{0,\dots,n\}}$ and the simplex $\Delta^{\{1,\dots,n\}}$. Furthermore, by the choice of $\sigma \dashv \rho$ the restriction of $H \colon \Delta^1 \times \Delta^n \to B$ to $\Delta^1 \times \Delta^{\{1,\dots,n\}}$ is a $(\sigma' \dashv \rho')$-transformation, where $\rho' \colon \Delta^{\{1,\dots,n\}} \to \Delta^{\{1,2\}}$ and $\sigma' \colon \Delta^{\{1,2\}} \to \Delta^{\{1,\dots,n\}}$ are the restrictions of ρ and σ, respectively. We can then factor the left vertical map as a composition of two maps, each of which is a pushout of a map of the form

$$(\Delta^{\{0\}} \times \Delta^m, M) \coprod_{\Delta^{\{0\}} \times \partial \Delta^m} (\Delta^1 \times \partial \Delta^m, M) \to (\Delta^1 \times \Delta^m, M) \tag{7.4}$$

with $m = n-1, n$, where the marking M includes the edge $\Delta^1 \times \{0\}$, all the triangles of the form $\Delta^{\{(0,i),(1,i),(1,j)\}}$ and all the triangles of the form $\Delta^{\{(0,i),(0,i+1),(1,i+1)\}}$. We may then factor (7.4) into a sequence of anodyne extensions using the filtration of Remark 4.7. The dotted lift in (7.3) then yields a lift in (7.1), by restricting to $\{1\} \times \Delta^n$ as desired. □

Lemma 7.6. *Let $j\colon (\Delta^4, T) \to (\Delta^4, T')$ be the generating scaled anodyne appearing second in the list in Definition 1.16. Then j is sent by $\iota\colon \mathcal{S}et_\Delta^{\mathrm{sc}} \to \mathrm{St}_2$ to a trivial cofibration in St_2.*

Proof. It is enough to show that every fibration between fibrant objects satisfies the right lifting property against $\iota(j)$. In fact, it is enough to check the lifting property against fibrant objects, since the underlying simplicial map of j is an isomorphism. Let us hence suppose that X is a fibrant stratified set with underlying scaled simplicial set $X' := U(X)$ and let $\sigma\colon (\Delta^4, T) \to X'$ be a map. Set $x := \sigma(0)$ and $y := \sigma(4)$. Taking into account Lemma 7.5 above we see that X' satisfies the extension property with respect to maps of type (1) and (3) in Definition 1.16. It then follows from Corollary 2.18 that the map of scaled simplicial sets $p\colon \overline{X'}_{/y} \to X'$ is an outer cartesian fibration and every edge which is marked in $X'_{/y}$ is p-cartesian. Since σ sends the triangles $\Delta^{\{1,2,3\}}, \Delta^{\{0,1,3\}}$ and $\Delta^{\{0,1,2\}}$ to thin triangles and X admits fillers for thinness extensions it follows that σ must send $\Delta^{\{0,2,3\}}$ to a thin triangle as well. We may therefore consider the 4-simplex σ as encoding a 3-simplex $\tau\colon \Delta^3_\sharp \to \overline{X'}_{/y}$ such that $p\tau\colon \Delta^3_\sharp \to X'$ coincides with $\sigma|_{\Delta^{\{0,1,2,3\}}}$. Since σ sends the triangles $\Delta^{\{0,2,4\}}$ and $\Delta^{\{1,3,4\}}$ to thin triangles in X' it follows that τ sends the edges $\Delta^{\{0,2\}}$ and $\Delta^{\{1,3\}}$ to edges which are marked in $X'_{/y}$.

Let $H\colon \Delta^1_\flat \times \Delta^3_\sharp \to X'$ be the natural transformation from the constant map on x to $p\tau$ induced by the unique natural transformation $\Delta^1_\flat \times \Delta^3_\sharp \to \Delta^3_\sharp$ from the constant map on 0 to the identity. By Proposition 2.26 we may lift H to a pointwise p-cartesian natural transformation $\widetilde{H}\colon \Delta^1_\flat \times \Delta^3_\sharp \to \overline{X'}_{/y}$ from some τ' to τ. In fact, Corollary 2.18 also tells us that every edge in X' has a marked p-cartesian lift, and so by Proposition 2.26 we may assume that $\widetilde{H}$ is pointwise marked. By Remark 2.19 we have that the collection of marked edges in $\overline{X'}_{/y}$ satisfies a certain closure property: if $\sigma\colon \Delta^2 \to \overline{X'}_{/y}$ is a thin triangle such that $\sigma|_{\Delta^{\{1,2\}}}$ is marked in $X'_{/y}$ then $\sigma|_{\Delta^{\{0,1\}}}$ is marked if and only if $\sigma|_{\Delta^{\{0,2\}}}$ is marked. It then follows that $\tau'\colon \Delta^3_\sharp \to \overline{X'}_{/y}$ also sends the edges $\Delta^{\{0,2\}}$ and $\Delta^{\{1,3\}}$ to marked edges in $X'_{/y}$. In addition the image of τ' lies by construction in the fiber $(\overline{X}'_{/y})_x$ above x. This means that τ' corresponds to a 4-simplex $\sigma'\colon (\Delta^4, T) \to X'$ such that $\sigma'|_{\Delta^{\{0,1,2,3\}}}$ degenerates to the point. It then follows that σ' determines a map of *stratified sets* of the form $\sigma''\colon \Delta^3_{\mathrm{eq}} \ast \Delta^0 \to X$. Since X is fibrant σ'' extends to a map $\mathrm{th}(\Delta^3) \ast \Delta^0 \to X$, which implies in particular that σ' sends the triangles $\Delta^{\{0,1,4\}}$ and $\Delta^{\{0,3,4\}}$ to thin triangles in X', and so $\tau'\colon \Delta^3_\sharp \to \overline{X'}_{/y}$ sends the edges $\Delta^{\{0,1\}}$ and $\Delta^{\{0,3\}}$ to marked edges in $X'_{/y}$. By the closure property for marked edges invoked above we get that the same holds for τ. This, in turn, means that $\sigma\colon (\Delta^4, T) \to X'$ also sends $\Delta^{\{0,1,4\}}$ and $\Delta^{\{0,3,4\}}$ to thin triangles and hence extends to a map $(\Delta^4, T') \to X'$, yielding an extension $\iota(\Delta^3, T') \to X$, as desired. □

Proof of Proposition 7.4. We have already observed that ι preserves cofibrations in Lemma 7.2, so we are left with proving it preserves trivial cofibrations. By

Proposition A.10 and Remark A.11 (or Remark A.9) it is enough to check this for maps which are pushout-products of maps in $\mathbf{S} \cup \{i_m\colon \{m\} \hookrightarrow J_\sharp\}_{m=0,1}$ and cofibrations. Better yet, by Proposition 7.2 we have that ι preserves pushout-products up to homotopy, and since St_2 is cartesian closed, it will just suffice to show that ι sends $\mathbf{S} \cup \{i_m\colon \{m\} \hookrightarrow J_\sharp\}_{m=0,1}$ to trivial cofibrations. We now verify this claim case by case:

- for the map $(\Lambda^n_i, \{\Delta^{\{i-1,i,i+1\}}\}|_{\Lambda^n_i}) \to (\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\})$ with $0 < i < n$, we note that it is sent by ι to an (inner) complicial horn inclusion (Definition 1.32) up to marking in dimension greater than 2, and is hence a trivial cofibration;
- for the map $j\colon (\Delta^4, T) \to (\Delta^4, T')$, this follows from Lemma 7.6;
- for the map $(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}|_{\Lambda^n_0}) \to (\Delta^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\})$ with $n \geq 1$, when $n = 1, 2$ this map becomes, after marking all simplices in dimension ≥ 3, a pushout of an outer complicial horn inclusion and is hence a trivial cofibration. When $n \geq 3$ it is instead a pushout of the map appearing in Lemma 7.5, and is hence again a trivial cofibration;
- for the maps $i_m\colon \{m\} \hookrightarrow J_\sharp$ with $m = 0, 1$, Lemma 7.3 shows that a fibrant replacement of $\iota J_\sharp$ is given by the fully marked walking isomorphism E, which is shown to be an interval object for the model structure on stratified sets in Observation 43 of [23]. □

We now are in a position to prove the following main result.

Theorem 7.7. *The adjunction*

$$\mathcal{S}et_\Delta^{\mathrm{sc}} \underset{U}{\overset{\iota}{\rightleftarrows}} \mathrm{St}_2 \quad (\iota \dashv U)$$

is a Quillen equivalence.

Proof. By Lemma 7.3 the derived unit of this adjunction can be represented by an isomorphism $(X, T_X) \to U\tilde{X} = (X, T_X)$ whenever (X, T_X) is fibrant, and so it will suffice to show that the derived counit is a weak equivalence. Given a fibrant stratified set (Y, M), we can factor its counit map as the composite

$$\iota(Y, M \cap Y_2) \xrightarrow{f} \tilde{Y} \xrightarrow{g} (Y, M)$$

where $\tilde{Y}$ is the fibrant replacement of $\iota(Y, M \cap Y_2)$ constructed in Lemma 7.3. Since f is an anodyne extension by Lemma 7.3 it will suffice to show that g is a weak equivalence. We now observe that since the counit and f are both isomorphisms at the level of the underlying scaled simplicial set, the same holds for g. In addition, the stratified sets $\tilde{Y}$ and (Y, M) have the same marked simplices in dimension ≥ 2 by construction (and since (Y, M) is assumed 2-trivial). It will hence suffice to show that $\tilde{Y}$ and (Y, M) have the same marked 1-simplices. In other words, we need to show that any marked edge in (Y, M) is an equivalence. Indeed, this follows from the fact that the map of stratified sets $\mathrm{th}(\Delta^1) \to \mathrm{th}(J)$ is an anodyne map, see [23, Observation 43]. □

The following corollary gives a pointwise characterization of natural equivalences, and we frame here for future use:

Corollary 7.8. *Let X, Y be scaled simplicial sets and let $h\colon \Delta^1_\flat \times X \to Y$ be a map such that $h|_{\Delta^1 \times \{x\}}$ is an equivalence in Y for every 0-simplex $x \in X_0$. Then $h_0\colon X \to Y$ is a bicategorical equivalence if and only if $h_1\colon X \to Y$ is. Moreover, if X and Y are ∞-bicategories, then h extends to a map $h\colon J_\sharp \times X \to Y$.*

Proof. Let $j\colon X \to \tilde{X}$ and $\phi\colon Y \to \tilde{Y}$ be fibrant replacement maps consisting of anodyne morphisms. Say that a natural transformation of scaled simplicial sets $h\colon \Delta^1_\flat \times W \to Z$ satisfies Property (P) if $h|_{\Delta^1 \times \{w\}}$ is an equivalence for every vertex $w \in W$. By solving the lifting problem depicted below, we get a map $\tilde{h}\colon \Delta^1_\flat \times \tilde{X} \to \tilde{Y}$ which again satisfies P, since j and ϕ do not alter the set of 0-simplices.

$$\begin{array}{ccccc} \Delta^1_\flat \times X & \xrightarrow{h} & Y & \xrightarrow{\phi} & \tilde{Y} \\ {\scriptstyle \Delta^1 \times j}\downarrow & & & \overset{\tilde{h}}{\nearrow} & \\ \Delta^1_\flat \times \tilde{X} & & & & \end{array}$$

Suppose the statement holds true for every pair of ∞-bicategories X, Y, then thanks to the following commutative square (for $i = 0, 1$):

$$\begin{array}{ccc} X & \xrightarrow{h_i} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \phi} \\ \tilde{X} & \xrightarrow{\tilde{h}_i} & \tilde{Y} \end{array}$$

we obtain that it also holds for arbitrary scaled simplicial sets X, Y. Therefore, we can assume without loss of generality that X and Y are ∞-bicategories.

Consider the following diagram, where $\iota\colon \mathcal{S}\mathrm{et}^{\mathrm{sc}}_\Delta \to \mathrm{St}_2$ is the left Quillen equivalence of §7 and for a scaled simplicial set W the term $\iota(W)_f$ denotes the explicit fibrant replacement of $\iota(W)$ of Lemma 7.3:

$$\begin{array}{ccccc} \iota(\Delta^1_\flat \times X) & \xrightarrow{\iota(h)} & \iota(Y) & \xrightarrow{\simeq} & \iota(Y)_f \\ \downarrow{\scriptstyle \simeq} & & & & \\ \iota(\Delta^1_\flat) \times \iota(X) & & & & \\ \downarrow{\scriptstyle \simeq} & & {\scriptstyle H} & & \\ \iota(\Delta^1_\flat) \times \iota(X)_f & & {\scriptstyle H'} & & \\ \downarrow & & {\scriptstyle H''} & & \\ \mathrm{th}(\Delta^1) \times \iota(X)_f & & & & \\ \downarrow{\scriptstyle \simeq} & & & & \\ \mathrm{th}(J) \times \iota(X)_f & & & & \end{array}$$

A lift H exists since the left-hand side composite is an anodyne morphism (thanks to Lemma 7.3). A lift H'' exists since the inclusion $\mathrm{th}(\Delta^1_\flat) \to \mathrm{th}(J)$ is an anodyne map, as shown in [23, Observation 43]. The lift denoted by H' exists by our assumption of Property (P): this amount to showing that given a map $q\colon \iota(\Delta^1_\flat) \times \mathrm{th}(\Delta^1) \to W$ with W fibrant in St_2, if $q|_{\Delta^1 \times \{i\}}$ is marked in W for every $i = 0, 1$, then q extends to a map from $\mathrm{th}(\Delta^1) \times \mathrm{th}(\Delta^1)$. This is a straightforward verification, which thus guarantees the existence of the natural transformation H'', which again satisfies (P). Since

$U(\iota W)_f = W$ for every ∞-bicategory W, we get a map $U(H'')\colon J_\sharp \times X \to Y$ from h_0 to h_1, which proves the claim, since $J_\sharp$ is a contractible ∞-bicategory (see the proof of Proposition 6.2). □

8. The homotopy 2-category and the scaled 2-nerve

In this section we construct a Quillen adjunction:

$$\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \underset{\mathcal{N}_2}{\overset{\mathrm{ho}_2}{\rightleftarrows}} 2\text{-}\mathcal{C}at \qquad (\perp)$$

and prove the right adjoint $\mathcal{N}_2$ is homotopy fully faithful, *i.e.*, fully faithful in the ∞-categorical sense. We then prove that if $\mathcal{C}$ is an ∞-bicategory then the unit map $\mathcal{C} \to \mathrm{N}^{\mathrm{sc}}\mathrm{ho}_2(\mathcal{C})$ is 2-conservative in the sense that it detects thin triangles.

Recall that in [21], Street defines the free ω-category on the n-simplex, called the n-th *oriental* and denoted by $\mathcal{O}_n$, which extends to a cosimplicial ω-category $\mathcal{O}_\bullet\colon \Delta \to \omega\text{-}\mathcal{C}at$. We may then apply to it the "intelligent" truncation functor $\tau_{\leq 2}\colon \omega\text{-}\mathcal{C}at \to 2\text{-}\mathcal{C}at$ which sends an ω-category X to the 2-category having the same 0-cells and 1-cells, and whose 2-cells are equivalence classes $[x]$ of 2-cells in X, where $[x] = [y]$ if there is a zig-zag of 3-cells connecting x and y. The resulting 2-categories then admit an explicit description (see, e.g., [2, Corollary A.6]): the objects of $\mathcal{O}_n^{\leq 2} \coloneqq \tau_{\leq 2}\mathcal{O}_n$ are the elements of ordered set $[n]$, and given $i, j \in [n]$, the category $\mathrm{Hom}_{\tau_{\leq 2}\mathcal{O}_n}(i,j)$ is the partially ordered set of subsets $S \subseteq [n]$ such that $\min(S) = i$ and $\max(S) = j$.

Definition 8.1. We define the *scaled 2-nerve* $\mathcal{N}_2(\mathcal{D})$ of a 2-category $\mathcal{D}$ by the formula

$$\mathcal{N}_2(\mathcal{D}) \coloneqq (\mathrm{Hom}_{2\text{-}\mathcal{C}at}(\mathcal{O}_\bullet^{\leq 2}, \mathcal{D}), T_{\mathcal{D}}) \in \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}},$$

where $T_{\mathcal{D}}$ denotes the triangles corresponding to those maps $\mathcal{O}_2^{\leq 2} \to \mathcal{D}$ which send the unique non-identity morphism in $\mathrm{Hom}_{\mathcal{O}_2^{\leq 2}}(0,2)$ to an isomorphism.

The cosimplicial object $\mathcal{O}_\bullet^{\leq 2}$ can be extended to the category Δ_{sc} (see Remark 1.14) by sending $[2]_t$ to the 2-category obtained from $\mathcal{O}_2^{\leq 2}$ by universally inverting its unique non-invertible 2-cell. This results in a 2-category with objects $0, 1, 2$ and with the same mapping categories as $\mathcal{O}_2^{\leq 2}$ except that $\mathrm{Hom}_{\mathcal{O}_2^{\leq 2}}(0,2)$ is the "walking isomorphism", that is the trivial groupoid on two objects. By general considerations the functor $\mathcal{N}_2$ then admits a left adjoint given by left Kan extending $\mathcal{O}_\bullet^{\leq 2}\colon \Delta_{\mathrm{sc}} \to 2\text{-}\mathcal{C}at$ along $\Delta_{\mathrm{sc}} \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$. We denote this left adjoint by

$$\mathrm{ho}_2\colon \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \to 2\text{-}\mathcal{C}at.$$

We then have the following:

Proposition 8.2. *The functor* ho_2 *is naturally isomorphic to the composite*

$$\mathcal{S}et_\Delta^{\mathrm{sc}} \xrightarrow{\mathfrak{C}^{\mathrm{sc}}} \mathcal{S}et_\Delta^{+}\text{-}\mathcal{C}at \xrightarrow{\mathrm{ho}_*} 2\text{-}\mathcal{C}at.$$

In particular, ho_2 *is a weak equivalences preserving left Quillen functor, being a composite of such (see Proposition 1.11). Consequently, by uniqueness of adjoints the scaled* 2*-nerve* $\mathcal{N}_2$ *identifies with the composition* $\mathrm{N}^{\mathrm{sc}} \circ \mathrm{N}_*$.

Proof. Both ho_2 and the composite $\mathrm{ho}_*\mathfrak{C}$ are left Kan extensions of their restriction to Δ_{sc}, and so it will suffice to construct a natural isomorphism on their restriction to Δ_{sc}. Let us first consider the further restriction to the subcategory $\Delta \subseteq \Delta_{\mathrm{sc}}$, so that we are dealing with the simplicial objects $\mathcal{O}^{\leq 2}_{\bullet}$ and $\mathrm{ho}_*\Delta^{\bullet}$. Both these simplicial objects admit natural extensions from Δ to the category of all *partially ordered sets*, and these extensions coincide since they admit exactly the same explicit formula (see [2, Corollary A.6] and [16, Remark 3.7.5]).

To extend this natural isomorphism to Δ_{sc} we observe that Δ_{sc} is obtained from Δ by freely adding the object $[2]_t$ and factorizing the all degeneracy maps from $[2]$ to $[0],[1]$ through $[2]_t$ in a compatible manner. The desired extension of the natural isomorphism now follows from the fact that in both cases the arrow $[2] \to [2]_t$ is sent to the universal inversion of the unique non-invertible 2-cell, while the 2-categories associated to $[0]$ and $[1]$ have all their 2-cells invertible. □

We now show the scaled 2-nerve is homotopy fully faithful.

Proposition 8.3. *The counit* $\epsilon_{\mathcal{C}} \colon \mathrm{ho}_2\mathcal{N}_2\mathcal{C} \to \mathcal{C}$ *is an equivalence of* 2*-categories. More precisely, it is bijective on objects and an equivalence on hom-categories.*

Proof. By Proposition 8.2 the adjunction $\mathrm{ho}_2 \dashv \mathcal{N}_2$ can be identified with the composition of the adjunctions $\mathfrak{C} \dashv \mathrm{N}^{\mathrm{sc}}$ and $\mathrm{ho}_* \dashv \mathrm{N}$. It follows that the counit map $\epsilon_{\mathcal{C}}$ factors as a composition

$$\mathrm{ho}_*\mathfrak{C}\mathrm{N}^{\mathrm{sc}}\mathrm{N}_*(\mathcal{C}) \to \mathrm{ho}_*\mathrm{N}_*(\mathcal{C}) \to \mathcal{C} \tag{8.1}$$

where the first map is induced by the counit of the adjunction $\mathfrak{C} \dashv \mathrm{N}^{\mathrm{sc}}$ and the second is the counit of $\mathrm{ho}_* \dashv \mathrm{N}_*$. Since $\mathfrak{C} \dashv \mathrm{N}^{\mathrm{sc}}$ is a Quillen equivalence and $\mathrm{N}_*(\mathcal{C})$ is fibrant (since all objects in 2-$\mathcal{C}at$ are fibrant) the first map in (8.1) is the image under ho_* of a weak equivalence, and is hence a 2-categorical equivalence by Proposition 1.11. To show that the second map is a 2-categorical equivalence we note that it is bijective on objects by construction, and it will hence suffice to check that it is fully-faithful. Indeed, the counit of $L\overline{\mathrm{ho}} \dashv \mathrm{N}^+\iota$ is an isomorpism since by $\iota \colon \mathcal{C}at \to \mathcal{R}\mathcal{C}at$ and $\mathrm{N}^+ \colon \mathcal{R}\mathcal{C}at \to \mathcal{S}\mathrm{et}^+_{\Delta}$ are fully-faithful right adjoints. □

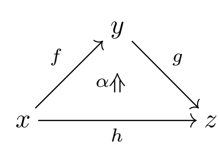
We now address the counit map of $\mathrm{ho}_2 \dashv \mathcal{N}_2$.

Theorem 8.4. *Let* $\mathcal{C}$ *be an* ∞*-bicategory. Then the counit map*

$$\mathcal{C} \to \mathcal{N}_2\mathrm{ho}_2(\mathcal{C})$$

is 2-conservative *in the sense that a triangle in* $\mathcal{C}$ *is thin if and only if its image in* $\mathcal{N}_2\mathrm{ho}_2(\mathcal{C})$ *is thin.*

Remark 8.5. Triangles in $\mathcal{N}_2\mathrm{ho}_2(\mathcal{C})$ correspond to 2-functors $\mathcal{O}^{\leq 2}_2 \to \mathrm{ho}_2(\mathcal{C})$, and hence to lax commutative triangles



in $\mathrm{ho}_2\mathcal{C}$. By definition, such a triangle is thin in $\mathcal{N}_2\mathrm{ho}_2(\mathcal{C})$ if and only if the 2-cell α is invertible. We then interpret Theorem 8.4 as saying that the homotopy 2-category *detects invertibility of* 2*-cells.*

Before we come to the proof of Theorem 8.6 let us recall its $(\infty,1)$-categorical analogue:

Proposition 8.6. *Let $\mathcal{C}^\natural = (\mathcal{C}, \mathrm{Eq}(\mathcal{C}))$ be a fibrant marked simplicial set, that is an ∞-category $\mathcal{C}$ marked by its equivalences. Then the unit map $\mathcal{C}^\natural \to \mathrm{N}^+\overline{\mathrm{ho}}(\mathcal{C}^\natural)$ detects marked edges.*

Proof. Since all marked edges are equivalences we have that $\mathrm{hIm}(\mathcal{C}^\natural) \subseteq \mathrm{ho}(\mathcal{C})$ consists of isomorphisms and hence $\overline{\mathrm{ho}}(\mathcal{C}^\natural) = L(\mathrm{ho}(\mathcal{C}), \mathrm{hIm}(\mathcal{C}^\natural)) \cong \mathrm{ho}(\mathcal{C})$. The desired claim is then equivalent to saying that an arrow in $\mathcal{C}$ is an equivalence if and only if its corresponding arrow in $\mathrm{ho}(\mathcal{C})$ is an isomorphism. Indeed, this is simply the definition of equivalences. □

Proof of Theorem 8.4. We first argue that in order to prove the claim for $\mathcal{C}$ we may replace $\mathcal{C}$ by any equivalent model. To see this, suppose that $f\colon \mathcal{C} \to \mathcal{D}$ is a bicategorical equivalence of ∞-bicategories and consider the commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\simeq} & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{N}_2\mathrm{ho}_2\mathcal{C} & \xrightarrow{\simeq} & \mathcal{N}_2\mathrm{ho}_2\mathcal{D} \end{array}$$

Since the left Quillen functor ho_2 preserves weak equivalences By Proposition 8.2 and the right Quillen functor $\widetilde{\mathcal{N}}_2$ preserves weak equivalences since every 2-category is fibrant we have that the bottom horizontal map is a weak equivalence. By Proposition 3.3 it now follows that both horizontal maps detect thin triangles. The desired claim for $\mathcal{D}$ hence implies the same for $\mathcal{C}$. We may consequently assume without loss of generality that $\mathcal{C}$ is of the form $\mathrm{N}^{\mathrm{sc}}(\mathcal{E})$ for some fibrant marked-simplicial category $\mathcal{E}$. Now in this case the counit map

$$\mathfrak{C}^{\mathrm{sc}}\mathrm{N}^{\mathrm{sc}}(\mathcal{E}) \to \mathcal{E}$$

is a weak equivalence of marked-simplicial categories. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{N}^{\mathrm{sc}}(\mathcal{E}) & & \\ \downarrow & \searrow^{=} & \\ \mathrm{N}^{\mathrm{sc}}\mathfrak{C}^{\mathrm{sc}}\mathrm{N}^{\mathrm{sc}}(\mathcal{E}) & \longrightarrow & \mathrm{N}^{\mathrm{sc}}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{N}^{\mathrm{sc}}\mathcal{N}_*\mathrm{ho}_*\mathfrak{C}^{\mathrm{sc}}\mathrm{N}^{\mathrm{sc}}(\mathcal{E}) & \longrightarrow & \mathrm{N}^{\mathrm{sc}}\mathrm{N}_*\mathrm{ho}_*(\mathcal{E}) \end{array}$$

in which the top left vertical map is induced by the unit of $\mathfrak{C} \dashv \mathrm{N}^{\mathrm{sc}}$, the two horizontal maps by the counit of $\mathfrak{C} \dashv \mathrm{N}^{\mathrm{sc}}$ and the two bottom vertical maps by the unit of $\mathrm{ho}_* \dashv \mathrm{N}_*$. The diagonal equality is then a consequence of the triangle identities, and the composed vertical map is the one we wish to show detects thin triangles (see Proposition 8.2). We may hence show instead that the right vertical map detects thin triangles. But this map is obtained by applying N^{sc} to the unit map $\mathcal{E} \to \mathrm{N}_*\mathrm{ho}_*\mathcal{E}$. By the definition of thin triangles in scaled nerves of marked it will now suffice to show that for every $x, y \in \mathcal{E}$ the map $\mathrm{Map}_{\mathcal{E}}(x,y) \to \mathrm{Map}_{\mathrm{N}_*\mathrm{ho}_*\mathcal{E}}(x,y) = \mathrm{N}^+\overline{\mathrm{ho}}\,\mathrm{Map}_{\mathcal{E}}(x,y)$ detects marked edges. Indeed, since $\mathcal{E}$ is fibrant each $\mathrm{Map}_{\mathcal{E}}(x,y)$ is fibrant as a marked simplicial set and so the desired result follows from Proposition 8.6. □

Appendix A. Recollections on Cisinski's and Olschok's theory

Let $\mathcal{K}$ be a locally presentable category. Recall that an arrow $f\colon X \to Y$ in $\mathcal{K}$ is called a *monomorphism* if $\mathrm{Hom}_{\mathcal{K}}(Z,X) \to \mathrm{Hom}_{\mathcal{K}}(Z,Y)$ is injective for every $Z \in \mathcal{K}$. We will fix the following standing assumption:

Assumption A.1. The class of monomorphisms in $\mathcal{K}$ is weakly saturated, and is generated as a weakly saturated class by a *set* $\mathfrak{M}$ of monomorphisms. In addition, the map $\varnothing \to X$ from the initial object to any other object is a monomorphism.

Remark A.2. Assumption A.1 holds for any presheaf category and more generally any topos. Furthermore, if Assumption A.1 holds for $\mathcal{K}$ and $\mathcal{A} \subseteq \mathcal{K}$ is a reflective subcategory such that the reflector $r\colon \mathcal{K} \to \mathcal{A}$ preserves monomorphisms then Assumption A.1 also holds for $\mathcal{A}$.

We call *trivial fibration* any arrow of $\mathcal{K}$ which has the right lifting property with respect to all monomorphisms of $\mathcal{K}$. A class of arrows $\mathcal{W}$ is a *localiser* if it satisfies the following conditions:

(1) the class $\mathcal{W}$ has the 2-of-3 property;
(2) the class $\mathcal{W}$ is closed under retracts;
(3) the class $\mathcal{W}$ contains all trivial fibrations;
(4) the class of monomorphisms which are also elements of $\mathcal{W}$ is closed under pushout and transfinite compositions.

In this appendix, we shall review some basic elements of the theory developed by Cisinski [7] and Olschok [18] which studies the conditions under which, given a small set S of monomorphisms of $\mathcal{K}$, there is a model category structure on $\mathcal{K}$ where the cofibrations are the monomorphisms and the class $\mathcal{W}(S)$ of weak equivalences is the smallest localiser containing the set S (notice that localisers are closed under intersection). This theory can be made more general than how it is presented here, but we shall limit ourselves to the simpler framework in which our interest and examples lie.

Definition A.3. Let $\mathcal{K}$ be a locally presentable category with terminal object e. A *cylinder* of $\mathcal{K}$ is a pair $\mathcal{I} = (I, \partial)$ where I is an object of $\mathcal{K}$ and $\partial\colon e \coprod e \to I$ is a monomorphism of $\mathcal{K}$; for $\varepsilon = 0,1$ we denote by $\partial^{\varepsilon}\colon e \to I$ the composition of ∂ with each of the two canonical maps $e \to e \coprod e$ (which are always monos by our assumption A.1). This pair is subject to the following axioms:

(1) the functor $I \times -\colon \mathcal{K} \to \mathcal{K}$ has a right adjoint $\underline{\mathcal{K}}(I,-)\colon \mathcal{K} \to \mathcal{K}$, called the *path functor*;
(2) for any monomorphism i of $\mathcal{K}$, the arrows $\partial^{\varepsilon} \square\, i$, $\varepsilon = 0,1$, and $\partial \,\square\, i$ are monomorphisms.

Remark A.4. Conditions (1) and (2) above hold for any monomorphism $\partial\colon e \coprod e \to I$ if, for example, $\mathcal{K}$ is cartesian closed and the collection of monomorphisms is closed under pushout-products.

Notation A.5. Let $\mathcal{K}$ be a locally presentable category, S a set of monomorphisms of $\mathcal{K}$ and $\mathcal{I} = (I, \partial)$ a cylinder. For any set T of arrows of $\mathcal{K}$ we denote by $\mathbf{\Lambda}(T)$ the set of arrows of the form $\partial \,\square\, f$, for f in T. We set

(0): $\Lambda^0_{\mathcal{J}}(S) = S$,
(i): $\Lambda^{i+1}_{\mathcal{J}}(S) = \mathbf{\Lambda}(\Lambda^i_{\mathcal{J}})$, $i > 0$,
(∞): $\Lambda^\infty_{\mathcal{J}}(S) = \bigcup_{i\geq 0} \Lambda^i_{\mathcal{J}}$

and finally

$$\Lambda_{\mathcal{J}}(S) = \Lambda^\infty_{\mathcal{J}}(S) \cup \{\partial^\varepsilon \boxtimes i : \varepsilon = 0,1 \text{ and } i \in \mathfrak{M}\}. \tag{A.1}$$

We shall say that $\Lambda_{\mathcal{J}}(S)$ is a set of *generating* $(\mathcal{J}, S)$*-anodyne maps*, or simply *generating anodyne maps*, and the smallest saturated class of $\mathcal{K}$ containing $\Lambda_{\mathcal{J}}(S)$ will be denoted by $\mathrm{An}_{\mathcal{J}}(S)$, or simply An, and called the class of $(\mathcal{J}, S)$*-anodyne maps*, or simply *anodyne maps*.

Theorem A.6 ([7], [18])**.** *Let $\mathcal{K}$ be a locally presentable category, $\mathcal{J} = (I, \partial)$ a cylinder and S a subset of monomorphisms of $\mathcal{K}$. Then there exists a model category structure on $\mathcal{K}$ having the monomorphisms as cofibrations and such that the fibrant objects are precisely the objects of $\mathcal{K}$ injective (*i.e.*, weakly right orthogonal) with respect to the set $\Lambda_{\mathcal{J}}(S)$. Moreover, the fibrations between fibrant objects are the arrows having the right lifting property with respect to the set $\Lambda_{\mathcal{J}}(S)$ and the class of weak equivalences $\mathcal{W}(S)$ is the smallest localiser of $\mathcal{K}$ containing S.*

This is proven as Theorem 1.3.22 of [7] when $\mathcal{K}$ is a presheaves category and in general it is a particular case of Theorem 3.16 of [18]; see also Theorem 2.5 of [10].

Remark A.7. The weak equivalences of the model category structure described above are the morphisms $p: X \to Y$ of $\mathcal{K}$ such that for any fibrant object Z we have that the induced function $p^*: \mathcal{K}(Y, Z) \to \mathcal{K}(X, Z)$ becomes a bijection when mod out by the usual relation of $\mathcal{J}$-homotopy given by the interval $I \times -$.

Remark A.8. In the proof of the above theorem, a careful analysis of the small object argument is needed. In particular, one shows that the small object argument applied to $\Lambda_{\mathcal{J}}(S)$ gives a fibrant replacement functor $L: \mathcal{K} \to \mathcal{K}$ with the following properties: for any object X of $\mathcal{K}$ the morphism $X \to L(X)$ is a $(\mathcal{J}, S)$-anodyne map and a morphism $f: X \to Y$ of $\mathcal{K}$ is a weak equivalence if and only if the morphism $L(f): L(X) \to L(Y)$ is a $\mathcal{J}$-homotopy equivalence.

Remark A.9. If $\mathcal{K}$ is a category of presheaves, or more generally a reflective subcategory $\mathcal{A}$ of a category of presheaves $\mathcal{K}$ closed under coproducts and such that the reflector $r: \mathcal{K} \to \mathcal{A}$ preserves monomorphisms, it is possible to show that the functor L preserves monomorphisms, so that f is a trivial cofibration if and only if $L(f)$ is a $(\mathcal{J}, S)$-anodyne map (see [7, Proposition 1.2.35]).

Proposition A.10. *Let $\mathcal{K}$ be a locally presentable category, $\mathcal{J}$ a cylinder, S a subset of monomorphisms of $\mathcal{K}$ and suppose that the fibrant replacement $L: \mathcal{K} \to \mathcal{K}$ preserves monomorphisms. Consider a model category $\mathcal{M}$ and a functor $F: \mathcal{K} \to \mathcal{M}$ preserving small colimits and mapping monomorphisms of $\mathcal{K}$ to cofibrations of $\mathcal{M}$. Then F is a left Quillen functor if and only if it maps $(\mathcal{J}, S)$-anodyne maps to trivial cofibrations; the latter property is further equivalent to ask that F maps every generating $(\mathcal{J}, S)$-anodyne map to a trivial cofibration of $\mathcal{M}$*

Proof. This is Proposition 2.4.40 of [8]. □

Remark A.11. The above proposition holds true with weaker assumptions. In fact, it not necessary to assume that the fibrant replacement $L: \mathcal{K} \to \mathcal{K}$ preserves monomorphisms. Indeed, the functor F maps $(\mathcal{J}, S)$-anodyne maps to trivial cofibrations if and only if its right adjoint R maps fibrations between fibrant objects of $\mathcal{M}$ to fibrations of $\mathcal{K}$. By virtue of [13, Proposition E.2.14], this is equivalent to the fact that (F, G) is a Quillen pair.

References

1. Dimitri Ara, *Higher quasi-categories vs higher Rezk spaces*, J. K-Theory **14** (2014), no. 3, 701–749.
2. Dimitri Ara and Georges Maltsiniotis, *The homotopy type of the ∞-category associated to a simplicial complex*, arxiv:1503.02720, 2015, preprint.
3. Clark Barwick and Christopher Schommer-Pries, *On the unicity of the homotopy theory of higher categories*, arxiv:1112.0400, 2011, preprint.
4. Julia E. Bergner and Charles Rezk, *Comparison of models for (∞, n)-categories, I*, Geom. Topol. **17** (2013), no. 4, 2163–2202.
5. Julia E. Bergner and Charles Rezk, *Comparison of models for (∞, n)-categories, II*, arxiv:1406.4182, 2014.
6. John Michael Boardman and Rainer M Vogt, *Homotopy invariant algebraic structures on topological spaces*, vol. 347, Springer, 2006.
7. Denis-Charles Cisinski, *Les préfaisceaux comme modèles des types d'homotopie*, Astérisque, Soc. Math. Français, 2006.
8. ———, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019.
9. Andrea Gagna, Yonatan Harpaz, and Edoardo Lanari, *Fibrations of $(\infty, 2)$-categories pt. i*, In preparation.
10. Philippe Gaucher, *Left determined model categories*, New York J. Math. **21** (2015), 1093–1115.
11. Harry Gindi, *Rigidification and the coherent nerve for enriched quasicategories*, arxiv:1810.10075, 2018.
12. André Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222, Special volume celebrating the 70th birthday of Professor Max Kelly.
13. ———, *The theory of quasi-categories and its applications*, CRM Quaderns **45 (II)** (2008), 147–496.
14. André Joyal and Myles Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326.
15. Stephen Lack, *A Quillen model structure for bicategories*, K-Theory **33** (2004), no. 3, 185–197.
16. Jacob Lurie, *$(\infty, 2)$-categories and the Goodwillie Calculus I*, Preprint. Available at author's website.
17. ———, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.
18. Marc Olschok, *Left determined model structures for locally presentable categories*, Appl. Categ. Structures **19** (2011), no. 6, 901–938.
19. Victoriya Ozornova and Martina Rovelli, *Model structures for (∞, n)-categories on (pre)stratified simplicial sets and prestratified simplicial spaces*, arxiv:1809.10621, 2018, preprint.
20. Emily Riehl, *Complicial sets, an overture*, 2016 MATRIX annals, MATRIX Book Ser., vol. 1, Springer, Cham, 2018, pp. 49–76. MR 3792516
21. Ross Street, *The algebra of oriented simplexes*, Journal of Pure and Applied Algebra **49** (1987), no. 3, 283–335.
22. Dominic Verity, *Complicial sets characterising the simplicial nerves of strict ω-categories*, Mem. Amer. Math. Soc. **193** (2008), no. 905, xvi+184.
23. ———, *Weak complicial sets. I. Basic homotopy theory*, Adv. Math. **219** (2008), no. 4, 1081–1149.

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