

An integral model structure and truncation theory for coherent group actions

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Abstract

In this work we study the homotopy theory of coherent group actions from a global point of view, where we allow both the group and the space acted upon to vary. Using the model of Segal group actions and the model categorical Grothendieck construction we construct a model category encompassing all Segal group actions simultaneously. We then prove a global rectification result in this setting. We proceed to develop a general truncation theory for the model-categorical Grothendieck construction and apply it to the case of Segal group actions. We give a simple characterization of n -truncated Segal group actions and show that every Segal group action admits a convergent Postnikov tower built out of its n -truncations.

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1 Introduction

Let \mathcal{S} be the category of simplicial sets (to which we shall refer as **spaces**) and let G be a simplicial group. Homotopy theories of spaces equipped with an action of G are of fundamental interest in algebraic topology. The robust machinery of modern homotopy theory can be applied to this theory in various ways. For example, one can equip the category of (honest) G -spaces with a model structure whose weak equivalences are the G -equivariant maps which induce a weak equivalence on the underlying spaces. We may identify this category with the functor category $\mathcal{S}^{\mathbb{B}G}$, where $\mathbb{B}G$ is the simplicial groupoid with one object and automorphism space G . This can be considered as a strict model of the theory – each object in $\mathcal{S}^{\mathbb{B}G}$ corresponds to an actual space equipped with an honest G -action. Alternatively, if one wishes to work in a non-strict setting, one can replace G with its classifying space $\overline{W}(G)$ and use the model category $\mathcal{S}_{/\overline{W}(G)}$ of spaces over $\overline{W}(G)$ (see [DDK])

In algebraic topology one often wishes to study group actions for several different groups simultaneously. This setup should combine the homotopy theory of simplicial groups and the homotopy theory of the various equivariant spaces in a compatible way. In a previous paper [HP], the authors provide a general machinery to study this setup through a suitable model category. More specifically, given a family \mathcal{F} of model categories parametrized by a model category \mathcal{M} , the authors construct a model structure on the Grothendieck construction $\int_{\mathcal{M}} \mathcal{F} \rightarrow \mathcal{M}$, called the **integral model structure**. This model structure combines the model structures of \mathcal{M} and the various fibers $\mathcal{F}(A)$ in a compatible fashion and provides a model for the corresponding ∞ -categorical Grothendieck construction. Moreover, the integral model structure is shown to be invariant (up to Quillen equivalence) under replacing the pair $(\mathcal{M}, \mathcal{F})$ with a suitably Quillen equivalent one.

In [HP], the authors provide two examples of integral model structures in the setting of group actions. In the first model one integrates the functor which assigns to each simplicial group G the Borel model category $\mathcal{S}^{\mathbb{B}G}$. In the second model one replaces simplicial groups with the equivalent model of reduced simplicial sets, under which a simplicial group G is modelled by its classifying space $\overline{W}(G)$. One can then integrate the functor which assigns to each reduced simplicial set B the model category of spaces over B . The invariance of the integral model structure alluded to above is then used to show that the resulting model categories are Quillen equivalent. We refer to such integral model categories as **global homotopy theories** for group actions.

The two models described above both have advantages and disadvantages. In the strict case one has direct access to the simplicial group G and the G -space X . However, working only with strict models makes it difficult to form homotopical constructions which only preserve Cartesian products up to homotopy. On the other hand, the weak model of spaces over $\overline{W}(G)$ is flexible and amenable to homotopical constructions, but does not give a direct access to the actual group G or the underlying space on which it acts.

There exists a third model for groups and group actions which enjoys the advantages of both worlds. A famous result of Segal (see [Seg, Proposition 1.5]) essentially shows that the homotopy theory of simplicial groups is equivalent to the homotopy theory of **Segal groups** (called **special Δ -spaces** in [Seg]), a model in which the group structure is encoded in a homotopy coherent way. It is tempting to try and extend the correspondence between simplicial groups and Segal groups into group actions. This was indeed done in [Pra] where the author defines the notion of a **Segal group action** over a fixed Segal group A_\bullet and constructs a model category for such objects. This model category is then shown to be Quillen equivalent to the Borel model category $\mathcal{S}^{\mathbb{B}G}$, where G is a simplicial group model for A_\bullet .

In this paper, we will adapt the construction of [Pra] to the setting of the integral model structure. This will require setting up a good model category for Segal groups, and showing that the functor which associates to each Segal group its model category of Segal group actions can be integrated in the sense discussed above. The integral model structure of Segal group actions is, in our opinion, a very convenient framework to study coherent group actions. However, as with any point-set model, it is important to be able to pass from it to another model when needed. In the case at hand, passing from Segal group actions to ordinary, (simplicial) group actions can also be viewed as a rectification question. We will prove that the integral model structure of Segal group actions is equivalent to the two models constructed in [HP], namely those of ordinary group actions and of spaces over a given pointed and connected space.

Finally, we study truncation theory in integral model categories and characterize n -truncated objects and n -truncation maps in terms of their counterparts in the base and in the fibers. We apply these results to the case of Segal group actions where truncations are shown to take a particularly nice form. We will show that every Segal group action admits a convergent Postnikov tower. This gives a canonical filtration on any Segal group action, which can be used to obtain a corresponding filtration on strict group actions as well.

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2 Preliminaries

Working with categories, either in the classical and higher setting, often requires handling objects which are “large”, for example, categories whose collection of objects is too big to be considered a set. There are several standard ways to handle these issues. We refer the reader to [Lur1, §1.2.15] for a discussion of the various possibilities. As in *loc. cit.*, we will officially adopt the framework of **Grothendieck universes**, and assume, in particular, that for every cardinal κ_0 there exists a strongly inaccessible cardinal $\kappa > \kappa_0$. We then let $\mathcal{U}(\kappa)$ denote the collection of all sets having rank $< \kappa$, so that $\mathcal{U}(\kappa)$ is a Grothendieck universe, and in particular a model for set theory. We will refer to a mathematical object as **small** if it belongs to $\mathcal{U}(\kappa)$ (up to isomorphism). When encountering mathematical objects which are not necessarily small, one may choose another strongly inaccessible cardinal $\kappa' > \kappa$ and obtain a larger Grothendieck universe $\mathcal{U}(\kappa')$ (in which $\mathcal{U}(\kappa)$ becomes small), and this procedure can be iterated as much as needed.

Throughout, a **space** will always mean a simplicial set. We denote by \mathcal{S} the category of spaces.

2.1 Model categories

We shall use an adjusted version of Quillen’s original definition of a (closed) model category (see [Qui1]).

Definition 2.1.1. A **model category** is category \mathcal{M} with three distinguished classes of morphisms $\mathcal{W} = \mathcal{W}_{\mathcal{M}}$, $\mathcal{Fib} = \mathcal{Fib}_{\mathcal{M}}$ and $\mathcal{Cof} = \mathcal{Cof}_{\mathcal{M}}$ called **weak equivalences**, **fibrations** and **cofibrations** (respectively), satisfying the following axioms:

- MC1 The category \mathcal{M} is complete and cocomplete.
- MC2 Each of the classes \mathcal{W} , \mathcal{Fib} and \mathcal{Cof} contains all isomorphisms and is closed under composition and retracts.
- MC3 If f, g are composable maps such that two of f, g and gf are in \mathcal{W} then so is the third.
- MC4 Given the commutative solid diagram in \mathcal{M}

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 i \downarrow & \nearrow & \downarrow p \\
 B & \longrightarrow & Y
 \end{array}$$

in which $i \in \mathcal{Cof}$ and $p \in \mathcal{Fib}$, a dashed arrow exists if either i or p are in \mathcal{W} .

- MC5 Any map f in \mathcal{M} has two functorial factorizations:

- (i) $f = pi$ with $i \in \mathcal{Cof}$ and $p \in \mathcal{Fib} \cap \mathcal{W}$;
- (ii) $f = qj$ with $j \in \mathcal{Cof} \cap \mathcal{W}$ and $q \in \mathcal{Fib}$.

We shall refer to the maps in $\mathcal{Fib} \cap \mathcal{W}$ (resp. $\mathcal{Cof} \cap \mathcal{W}$) as **trivial fibrations** (resp. **trivial cofibrations**). For an object $X \in \mathcal{M}$ we denote by X^{fib} (resp. X^{cof}) the functorial fibrant (resp. cofibrant) replacement of X , obtained by factorizing the map to the terminal object $X \rightarrow *$ (resp. from the initial object $\emptyset \rightarrow X$) into a trivial cofibration followed by a fibration (resp. a cofibration followed by trivial fibration).

Definition 2.1.2. (cf. [HP, Definition 2.16]) We denote by ModCat the (suitably large) $(2, 1)$ -category whose objects are model categories and whose morphisms the Quillen pairs (composition is done in the direction of the left Quillen functor). The 2-morphisms are given by the pseudo-natural isomorphisms of (Quillen) adjunctions.

2.2 ∞ -Categories and ∞ -localizations

Let Set_Δ^+ denote the category of marked simplicial sets. The category Set_Δ^+ can be endowed with the coCartesian model structure (see [Lur1, Remark 3.1.3.9]) yielding a model for the theory of ∞ -categories. Note that a fibrant object in Set_Δ^+ encodes an ∞ -category which is **small** in the sense discussed in §2. In order to work with ∞ -categories which are not necessarily small, one may introduce the category $\widehat{\text{Set}}_\Delta^+$ of marked simplicial sets which are not necessarily small (i.e., their sets of n -simplices do not belong to the current universe, but to a given larger one). $\widehat{\text{Set}}_\Delta^+$ can then be considered as a model category in a sufficiently large Grothendieck universe. Fibrant objects in $\widehat{\text{Set}}_\Delta^+$ encode ∞ -categories which are not necessarily small.

Let $(\mathcal{C}, \mathcal{W})$ be a relative category. We may then consider $(\mathcal{N}(\mathcal{C}), \mathcal{N}(\mathcal{W}))$ as a marked simplicial set. We will denote by $\mathcal{L}(\mathcal{C}, \mathcal{W})$ the fibrant replacement of $(\mathcal{N}(\mathcal{C}), \mathcal{N}(\mathcal{W}))$ in either Set_Δ^+ or $\widehat{\text{Set}}_\Delta^+$, appropriately. The ∞ -category $\mathcal{L}(\mathcal{C}, \mathcal{W})$ satisfies the following universal property (see the last proposition of [Hin, §1.1.3]): if \mathcal{D} is any other ∞ -category, then the induced functor

$$\text{Fun}(\mathcal{L}(\mathcal{C}, \mathcal{W}), \mathcal{D}) \rightarrow \text{Fun}(\mathcal{N}(\mathcal{C}), \mathcal{D})$$

is fully-faithful, and its image coincides with the full subcategory of $\text{Fun}(\mathcal{N}(\mathcal{C}), \mathcal{D})$ spanned by those functors which send the arrows in \mathcal{W} to equivalences in \mathcal{D} . One may hence consider $\mathcal{L}(\mathcal{C}, \mathcal{W})$ as a model for the ∞ -**localization** of \mathcal{C} obtained by formally inverting the arrows of \mathcal{W} . We refer the reader to [Hin, §1] for further details. Now let \mathcal{M} be a model category. We will denote by

$$\mathcal{M}_\infty \stackrel{\text{def}}{=} \mathcal{L}(\mathcal{M}^{\text{cof}}, \mathcal{W} \cap \mathcal{M}^{\text{cof}}).$$

Here, $\mathcal{M}^{\text{cof}} \subseteq \mathcal{M}$ denotes the full subcategory of cofibrant objects. Following Lurie (see [Lur2, Definition 1.3.4.15]), we will refer to \mathcal{M}_∞ as the **underlying**

∞ -category of \mathcal{M} . In the case of $\mathcal{M} = \text{Set}_\Delta^+$ we will also denote by $\text{Cat}_\infty \stackrel{\text{def}}{=} (\text{Set}_\Delta^+)_\infty$, which by definition is the (large) ∞ -category of small ∞ -categories.

Remark 2.2.1. Since \mathcal{M} is typically not small, \mathcal{M}_∞ is generally not a small ∞ -category.

Given an ∞ -category \mathcal{C} and two objects $X, Y \in \mathcal{C}$ we will denote by $\text{Map}_{\mathcal{C}}(X, Y)$ the mapping space from X to Y (see [Lur1, p. 27–28]). If \mathcal{M} is a model category, or even just a relative category, then there is a canonical weak equivalence (see [Hin, Proposition 1.2.1])

$$\text{Map}_{\mathcal{M}}^{\text{h}}(X, Y) \simeq \text{Map}_{\mathcal{M}_\infty}(X, Y)$$

where the former is the **derived mapping space**, defined via Dwyer and Kan’s simplicial localization. A fundamental feature of model categories is that they support various concrete ways of computing the derived mapping space $\text{Map}_{\mathcal{M}}^{\text{h}}(X, Y)$. For example, if \mathcal{M} is a *simplicial* model category, then a model for this space is given by the simplicial mapping space $\text{Map}_{\mathcal{M}}(X^{\text{cof}}, Y^{\text{fib}})$.

2.3 The Grothendieck construction

Let AdjCat denote the (suitably large) $(2, 1)$ -category of categories and adjunctions. A morphism in AdjCat is an adjunction (having the direction of the left adjoint) and a 2-morphism is a pseudo-natural transformation of adjunctions which is an isomorphism in each component. Let \mathcal{C} be a category. We shall refer to pseudo-functors $\mathcal{F} : \mathcal{C} \rightarrow \text{AdjCat}$ simply as functors.

For a morphism $f : A \rightarrow B$ in \mathcal{C} , we denote the associated adjunction $\mathcal{F}(f)$ in AdjCat by

$$f_! : \mathcal{F}(A) \xrightleftharpoons[\perp]{} \mathcal{F}(B) : f^*.$$

The **Grothendieck construction** of \mathcal{F} is the category $\int_{\mathcal{C}} \mathcal{F}$ where an object is a pair (A, X) with $A \in \text{Obj } \mathcal{M}$ and $X \in \text{Obj } \mathcal{F}(A)$ and a morphism $(A, X) \rightarrow (B, Y)$ is a pair (f, φ) with $f : A \rightarrow B$ is a morphism in \mathcal{M} and $\varphi : f_! X \rightarrow Y$ is a morphism in $\mathcal{F}(B)$. In this case, we denote by $\varphi^{\text{ad}} : X \rightarrow f^* Y$ the adjoint map of φ . The category $\int_{\mathcal{C}} \mathcal{F}$ comes equipped with a natural projection $\pi : \int_{\mathcal{C}} \mathcal{F} \rightarrow \mathcal{C}$ given by $(A, X) \mapsto A$ which is both a Cartesian and a coCartesian fibration. A morphism (f, φ) as above is coCartesian if and only if φ is an isomorphism, and is Cartesian if and only if φ^{ad} is an isomorphism.

Limits in $\int_{\mathcal{C}} \mathcal{F}$ are computed by first pulling back to the limit of the associated diagram in \mathcal{C} and then computing the limit in the appropriate fiber. More precisely, suppose \mathcal{J} is a small category and $D : \mathcal{J} \rightarrow \int_{\mathcal{C}} \mathcal{F}$ is a diagram given by $D(i) = (A_i, X_i)$ for $i \in \mathcal{J}$ and $D(\theta) = (f_\theta, \varphi_\theta) : (A_i, X_i) \rightarrow (A_j, X_j)$ for $\theta : i \rightarrow j$ in \mathcal{J} . Then the limit of D over \mathcal{J} is given by

$$\lim_{\mathcal{J}} D = (\lim_{\mathcal{J}} A_i, \lim_{\mathcal{J}} (f_\theta)^*(X_j)).$$

Colimits in $\int_{\mathcal{C}} \mathcal{F}$ are computed in a dual way.

2.4 The integral model structure

In this subsection we will recall the construction of the integral model structure on the Grothendieck construction of a diagram of model categories indexed by a model category, as developed in [HP].

Definition 2.4.1. Let \mathcal{M} be a model category. We will say that a functor $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$ is **relative** if for every weak equivalence $f : A \rightarrow B$ in \mathcal{M} , the associated Quillen pair $f_! \dashv f^*$ is a Quillen equivalence.

Definition 2.4.2. Let \mathcal{M} be a model category and $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$ a functor. We shall say that \mathcal{F} is **proper** if whenever $f : A \rightarrow B$ is a trivial cofibration in \mathcal{M} the associated left Quillen functor preserves weak equivalences (i.e., $f_!(\mathcal{W}_{\mathcal{F}(A)}) \subseteq \mathcal{W}_{\mathcal{F}(B)}$) and whenever $f : A \rightarrow B$ is a trivial fibration in \mathcal{M} the associated right Quillen functor preserves weak equivalences (i.e., $f^*(\mathcal{W}_{\mathcal{F}(A)}) \subseteq \mathcal{W}_{\mathcal{F}(B)}$).

Let us now fix a model category \mathcal{M} be a model category and a functor $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$.

Definition 2.4.3. Call a morphism $(f, \varphi) : (A, X) \rightarrow (B, Y)$ in $\int_{\mathcal{M}} \mathcal{F}$

1. a **weak equivalence** if $f : A \rightarrow B$ is a weak equivalence in \mathcal{M} and the composite $f_!(X^{cof}) \rightarrow f_!X \rightarrow Y$ is a weak equivalence in $\mathcal{F}(B)$;
2. a **cofibration** if $f : A \rightarrow B$ is a cofibration in \mathcal{M} and $\varphi : f_!X \rightarrow Y$ is a cofibration in $\mathcal{F}(B)$;
3. a **fibration** if $f : A \rightarrow B$ is a fibration and $\varphi^{ad} : X \rightarrow f^*Y$ is a fibration in $\mathcal{F}(A)$.

We denote these classes by \mathcal{W} , \mathcal{Cof} and \mathcal{Fib} respectively.

The following is one of the main results of [HP].

Theorem 2.4.4. *Let \mathcal{M} be a model category and $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$ a proper relative functor. The classes of weak equivalences \mathcal{W} , fibrations \mathcal{Fib} and cofibrations \mathcal{Cof} of 2.4.3 endow $\int_{\mathcal{M}} \mathcal{F}$ with the structure of a model category, called the **integral model structure**.*

In this paper, we will focus our attention on two cases where Theorem 2.4.4 is applicable. The first case, whose details will be reviewed in § 2.6, is where \mathcal{M} is the category of simplicial groups and \mathcal{F} is the functor which associates to a simplicial group G the category $\mathbb{S}^{\mathbb{B}G}$ of G -spaces endowed with the projective model structure. The second case is a coherent variant of the first, where \mathcal{M} models the homotopy theory of Segal groups and \mathcal{F} is the functor which associates to each Segal group A_{\bullet} the model category of Segal group actions over A_{\bullet} .

2.5 Invariance of the integral model structure

In this subsection we briefly recall the invariance property of the integral model structure, as appears in [HP, §2.2]. We consider a diagram of categories

$$\begin{array}{ccc}
 \mathcal{J} & \xrightleftharpoons[\mathcal{R}]{\mathcal{L}} & \mathcal{J} \\
 & \searrow \mathcal{F} & \swarrow \mathcal{G} \\
 & \text{AdjCat} &
 \end{array}$$

such that the horizontal pair forms an adjunction.

Definition 2.5.1. A **left morphism** from \mathcal{F} to \mathcal{G} over $\mathcal{L} \dashv \mathcal{R}$ is a pseudo-natural transformation $\mathcal{F} \Rightarrow \mathcal{G} \circ \mathcal{L}$, i.e., a compatible family of adjunctions.

$$\Sigma_A^{\mathcal{L}} : \mathcal{F}(A) \xrightleftharpoons[\leftarrow]{\rightarrow} \mathcal{G}(\mathcal{L}(A)) : \Sigma_A^{\mathcal{R}}$$

indexed by $A \in \mathcal{J}$. Similarly, a **right morphism** from \mathcal{F} to \mathcal{G} is a pseudo-natural transformation $\mathcal{F} \circ \mathcal{R} \Rightarrow \mathcal{G}$, i.e., a compatible family of adjunctions

$$\Theta_B^{\mathcal{L}} : \mathcal{F}(\mathcal{R}(B)) \xrightleftharpoons[\leftarrow]{\rightarrow} \mathcal{G}(B) : \Theta_B^{\mathcal{R}}$$

indexed by $B \in \mathcal{J}$.

We can now recall the model categorical counterpart.

Definition 2.5.2. Let \mathcal{M}, \mathcal{N} be model categories and

$$\begin{array}{ccc}
 \mathcal{M} & \xrightleftharpoons[\mathcal{R}]{\mathcal{L}} & \mathcal{N} \\
 & \searrow \mathcal{F} & \swarrow \mathcal{G} \\
 & \text{ModCat} &
 \end{array}$$

a diagram such that the horizontal pair is a Quillen adjunction and \mathcal{F}, \mathcal{G} are proper relative functors. We will say that a left morphism $\mathcal{F} \Rightarrow \mathcal{G} \circ \mathcal{L}$ is a **left Quillen morphism** if the associated adjunctions

$$\Sigma_A^{\mathcal{L}} : \mathcal{F}(A) \xrightleftharpoons[\leftarrow]{\rightarrow} \mathcal{G}(\mathcal{L}(A)) : \Sigma_A^{\mathcal{R}}$$

are Quillen adjunction. Similarly we define **right Quillen morphisms**.

Definition 2.5.3. Let $\mathcal{M}, \mathcal{N}, \mathcal{F}, \mathcal{G}, \mathcal{L}, \mathcal{R}$ be as above. We will say that a left Quillen morphism

$$\Sigma_A^{\mathcal{L}} : \mathcal{F}(A) \xrightleftharpoons[\leftarrow]{\rightarrow} \mathcal{G}(\mathcal{L}(A)) : \Sigma_A^{\mathcal{R}}$$

indexed by $A \in \mathcal{M}$ is a **left Quillen equivalence** if $\mathcal{L} \dashv \mathcal{R}$ is a Quillen equivalence and $\Sigma_A^{\mathcal{L}} \dashv \Sigma_A^{\mathcal{R}}$ is a Quillen equivalence for every cofibrant $A \in \mathcal{M}$. Similarly, we will say that a right Quillen morphism

$$\Theta_B^{\mathcal{L}} : \mathcal{F}(\mathcal{R}(B)) \xrightleftharpoons[\leftarrow]{\rightarrow} \mathcal{G}(B) : \Theta_B^{\mathcal{R}}$$

indexed by $B \in \mathcal{N}$ is a **right Quillen equivalence** if $\mathcal{L} \dashv \mathcal{R}$ is a Quillen equivalence and $\Theta_B^L \dashv \Theta_B^R$ is a Quillen equivalence for every fibrant $B \in \mathcal{N}$

Having established the necessary terminology, we can now state the invariance property:

Theorem 2.5.4. [HP, 4.4] *Let \mathcal{M}, \mathcal{N} be model categories and*

$$\begin{array}{ccc}
 \mathcal{M} & \begin{array}{c} \xrightarrow{\mathcal{L}} \\ \xleftarrow{\mathcal{R}} \end{array} & \mathcal{N} \\
 & \begin{array}{c} \searrow \mathcal{F} \\ \swarrow \mathcal{G} \end{array} & \\
 & \text{ModCat} &
 \end{array}$$

a diagram in which the horizontal pair is a Quillen adjunction and \mathcal{F}, \mathcal{G} are proper relative functors. Let $\mathcal{F} \Rightarrow \mathcal{G} \circ \mathcal{L}$ be a left Quillen morphism given by a compatible family of adjunctions $(\Sigma_A^L, \Sigma_A^R)_{A \in \mathcal{M}}$. Then the induced adjunction

$$\Phi^L : \int_{\mathcal{M}} \mathcal{F} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\perp} \end{array} \int_{\mathcal{N}} \mathcal{G} : \Phi^R.$$

is a Quillen adjunction. Furthermore, if the left Quillen morphism is a left Quillen equivalence then (Φ^L, Φ^R) is a Quillen equivalence. The same result holds for the adjunction induced by a right Quillen morphism

2.6 Integral model categories of group actions

Here we will recall two models for the global homotopy theory of group actions on spaces which were constructed in [HP]. In the first model we associated to each simplicial group G the Borel model category $\mathcal{S}^{\mathbb{B}G}$ and then used Theorem 2.4.4 to integrate the corresponding functor. In the second model we replaced simplicial groups with the equivalent model of **reduced simplicial sets**, under which a group corresponds to its classifying space. One can then assign to each reduced simplicial set B the model category of spaces over B and again use theorem 2.4.4 to integrate the associated functor. The equivalence between G -spaces and spaces over the classifying space of G , combined with Theorem 2.5.4, yields an Quillen equivalence between these two integral model categories.

Remark 2.6.1. The terminology “global homotopy theory” already appears in the literature, most notably in [Sch], with a different meaning. We chose to use it here since it describes a key idea that underlines this work: the study of equivariant homotopy theory of spaces for all coherent groups simultaneously.

Let sGr be the category of simplicial groups. This category admits a model structure which is transferred from the Kan-Quillen model structure on spaces via the adjunction

$$\mathcal{S} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{\perp} \\ \xleftarrow{U} \end{array} \text{sGr} \tag{2.6.1}$$

where U is the forgetful functor and F is the free group functor (see [GJ, Example II.5.2(1)]). In particular, a map of simplicial groups $f : G \rightarrow H$ is a weak equivalence (resp. fibration) if and only if the map $U(f) : U(G) \rightarrow U(H)$ is a weak equivalence (resp. fibration).

For a simplicial group G one can consider the category of spaces endowed with an action of G . This category can be identified with the simplicial functor category $\mathcal{S}^{\mathbb{B}G}$ where $\mathbb{B}G$ is the simplicial groupoid with one object having G as its automorphism group. As such, one can consider $\mathcal{S}^{\mathbb{B}G}$ with the **projective model structure**, which we will call here the **Borel model structure**. In this model structure a map of G -spaces is a weak equivalence (resp. fibration) if and only if the underlying map of spaces is a weak equivalence (resp. fibration). In addition, a G -space X is cofibrant if and only if the action of G on X is **free** on each simplicial level (see [DDK] Proposition 2.2).

Let $f : G \rightarrow H$ be a map of simplicial groups. Then we have a Quillen adjunction

$$\mathcal{S}^{\mathbb{B}G} \begin{array}{c} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^*} \end{array} \mathcal{S}^{\mathbb{B}H}$$

where $f_!(X) = H \times_G X$ is the quotient of $H \times X$ by the action of G given by $g(h, x) = (hg^{-1}, gx)$ and $f^*(X) = \text{res}_G^H(X)$ is the restriction functor.

Recall the following proposition, which appears without proof in [HP]

Proposition 2.6.2. *If $f : G \rightarrow H$ is a cofibration of simplicial groups then $U(f) : U(G) \rightarrow U(H)$ is a cofibration of simplicial sets.*

For the purpose of completeness we shall give below a detailed proof of Proposition 2.6.2. We wish to thank the anonymous referee for suggesting the idea of the proof. Relying on Proposition 2.6.2 one may show that the integral model structure on $\int_{G \in \text{sGr}} \mathcal{S}^{\mathbb{B}G}$ exists.

Corollary 2.6.3. *(cf. [HP, Proposition 6.4]) The functor $\mathcal{U} : \text{sGr} \rightarrow \text{ModCat}$ given by $\mathcal{U}(G) = \mathcal{S}^{\mathbb{B}G}$ is proper and relative so that we obtain an integral model structure on $\int_{G \in \text{sGr}} \mathcal{S}^{\mathbb{B}G}$.*

We shall refer to the model structure of Corollary 2.6.3 as the **integral Borel model structure**.

Let us now recall a second model for the global homotopy theory of group actions constructed in [HP]. As established in [Kan], the homotopy theory of simplicial groups may be equivalently described as the homotopy theory of **reduced simplicial sets**, i.e., simplicial sets $B \in \mathcal{S}$ with $B_0 = \{*\}$. We will denote by \mathcal{S}_0 the category of reduced spaces and by $\iota : \mathcal{S}_0 \rightarrow \mathcal{S}$ the full inclusion of reduced spaces in spaces. Then there exists a model structure on \mathcal{S}_0 in which a map $f : X \rightarrow Y$ in \mathcal{S}_0 is a weak equivalence (resp. cofibration) if and only if $\iota(f) : \iota(X) \rightarrow \iota(Y)$ is a weak equivalence (resp. cofibration) in \mathcal{S} ([Kan], see

also [GJ, VI.6.2]). One then has a Quillen equivalence (see [GJ, V.6.3])

$$\mathcal{S}_0 \begin{array}{c} \xrightarrow{\mathfrak{G}} \\ \perp \\ \xleftarrow{\overline{W}} \end{array} \mathbf{sGr} \quad (2.6.2)$$

where \mathfrak{G} is the Kan loop group functor. Furthermore, for each simplicial group G there exists a natural Quillen equivalence between $\mathcal{S}^{\mathbb{B}G}$ and $\mathcal{S}_{/\iota(\overline{W}(G))}$ (see [DDK]), where the latter is endowed with the corresponding slice model structure. In other words, the homotopy theory of G -spaces can be equivalently described as the homotopy theory of spaces over the classifying space $\overline{W}(G)$.

Now consider the functor $\mathcal{V} : \mathcal{S}_0 \rightarrow \mathbf{ModCat}$ given by

$$\mathcal{V}(B) = \mathcal{S}_{/\iota(B)}, \quad (2.6.3)$$

where $\mathcal{S}_{/\iota(B)}$ is the model category of spaces over $\iota(B)$ (equipped with the slice model structure). We then have the following:

Proposition 2.6.4 ([HP], Proposition 6.6). *The functor \mathcal{V} above is proper and relative and gives rise to an integral model structure on $\int_{B \in \mathcal{S}_0} \mathcal{S}_{/\iota(B)}$.*

We now wish to compare the two models constructed above. In [HP, §6], the authors showed that the Quillen equivalence 2.6.2, together with the compatible family of Quillen equivalences $\mathcal{S}_{/\iota(\overline{W}(G))} \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\perp} \end{array} \mathcal{S}^{\mathbb{B}G}$ described in [DDK], assemble to form a right Quillen equivalence $\mathcal{V} \circ \overline{W} \Rightarrow \mathcal{U}$ in the sense of Definition 2.5.3. In light of Theorem 2.5.4 we then have

Corollary 2.6.5. *There exists a Quillen equivalence of integral model structures*

$$\Phi^L : \int_{B \in \mathcal{S}_0} \mathcal{S}_{/\iota(B)} \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\perp} \end{array} \int_{G \in \mathbf{sGr}} \mathcal{S}^{\mathbb{B}G} : \Phi^R.$$

Let us finish this subsection by giving a proof of Proposition 2.6.2 above. We wish to thank the anonymous referee for suggesting the idea for the proof. The proof will require the following notion:

Definition 2.6.6. Let \mathcal{C} be a category and $f : X \rightarrow Y$ a morphism in \mathcal{C} . We will say that f is **split** if there exists a map $g : Y \rightarrow X$ such that $g \circ f = \text{Id}_X$. Given a map $f : X_\bullet \rightarrow Y_\bullet$ of simplicial objects in \mathcal{C} , we will say that f is **levelwise split** if $f_n : X_n \rightarrow Y_n$ is split for every n . We note that this condition is weaker, in general then f being split in $\mathcal{C}^{\Delta^{\text{op}}}$.

The main step needed for the proof of Proposition 2.6.2 is given by the following lemma. We phrase it in a rather general manner for the purpose of future applications.

Lemma 2.6.7. *Let \mathcal{C} be a category which admits small colimits. Then the class of split maps is weakly saturated.*

Proof. We need to show that the class of split maps is closed under pushouts, retracts and transfinite composition. Let us begin with pushouts. Consider a pushout diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{f'} & P \end{array}$$

We need to show that if f is split then so is f' . Choose a map $g : Y \rightarrow X$ such that $g \circ f = \text{Id}$ and consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & X \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow{f'} & P & \xrightarrow{g'} & Q \end{array}$$

where Q is defined to be the pushout $Q = P \amalg_Y X$. By the pasting lemma for pushout squares the external rectangle is a pushout diagram, and hence the map $g' \circ f' : Z \rightarrow Q$ is an isomorphism. Composing g' with the inverse of this isomorphism we see that f' is split.

Let us now show that the class of split maps is closed under retracts. Consider a retract diagram of maps

$$\begin{array}{ccccc} X' & \xrightarrow{i} & X & \xrightarrow{r} & X' \\ \downarrow f' & & \downarrow f & & \downarrow f' \\ Y' & \xrightarrow{j} & Y & \xrightarrow{s} & Y' \end{array}$$

We need to show that if f is split then f' is split. Let $g : Y \rightarrow X$ be such that $g \circ f = \text{Id}_X$. Define $g' : Y' \rightarrow X'$ to be the composite $Y' \xrightarrow{j} Y \xrightarrow{g} X \xrightarrow{r} X'$. We then have $g' \circ f' = r \circ g \circ f \circ i = \text{Id}_{X'}$ and hence f' is split.

Finally, let us show that split maps are closed under transfinite composition. Let α be an ordinal considered as a category and let $\{X^\beta\}_{\beta \in \alpha}$ be a functor from α to \mathcal{C} . For each $1 \leq \gamma \leq \alpha$ let us denote by $Y^\gamma = \text{colim}_{\beta < \gamma} X^\beta$ and by $\varphi_X^\gamma : X^0 \rightarrow X^\gamma$, $\varphi_Y^\gamma : X^0 \rightarrow Y^\gamma$ and $f^\gamma : Y^\gamma \rightarrow X^\gamma$ the natural maps. We need to show that if f^γ is split for every $1 \leq \gamma \leq \alpha$ then φ_Y^α is split.

For each $1 \leq \gamma \leq \alpha$ let us choose a map $g^\gamma : X^\gamma \rightarrow Y^\gamma$ such that $g^\gamma \circ f^\gamma = \text{Id}_{Y^\gamma}$. We now define, via a process of transfinite induction, compatible families of maps $\psi_X^\gamma : X^\gamma \rightarrow X^0$ and $\psi_Y^\gamma : Y^\gamma \rightarrow X^0$ indexed by $1 \leq \gamma \leq \alpha$. For $\gamma = 1$ we simply set $\psi_X^1 = g^1$ and ψ_Y^1 to be the identity $Y^1 = X^0 \rightarrow X^0$. Now let $\gamma > 1$ be such that ψ_X^β and ψ_Y^β have been defined compatibly for every $1 \leq \beta < \gamma$. We then define

$$\psi_Y^\gamma : Y^\gamma = \text{colim}_{\beta < \gamma} X^\beta \cong \text{colim}_{1 \leq \beta < \gamma} X^\beta \rightarrow X^0$$

to be the map induced by the ψ_X^β , and define $\psi_X^\gamma = \psi_Y^\gamma \circ g^\gamma$. Since the family $\{\psi_X^\beta\}_{1 \leq \beta < \gamma}$ is compatible it is clear that $\{\psi_Y^\beta\}_{1 \leq \beta \leq \gamma}$ is compatible. On the other hand, $\{\psi_X^\beta\}_{1 \leq \beta \leq \gamma}$ is compatible because for any $\beta < \gamma$ the composition $X^\beta \rightarrow X^\gamma \xrightarrow{g^\gamma} Y^\gamma$ coincides with the natural map $X^\beta \rightarrow Y^\gamma$. Finally, setting $\gamma = \alpha$ the compatibility of ψ_Y^α and ψ_Y^1 means that $\psi_Y^\alpha \circ \varphi_Y^\alpha = \text{Id}_{X^0}$ and hence φ_Y^α is split, as desired. \square

Corollary 2.6.8. *Let \mathcal{C} be a category and let $\mathcal{C}^{\Delta^{\text{op}}}$ denote the corresponding category of simplicial objects. Then the class of levelwise split maps is weakly saturated in $\mathcal{C}^{\Delta^{\text{op}}}$.*

Proof. This follows directly from the fact that the evaluation functor $X_\bullet \mapsto X_n$ preserves colimits (and, of course, retracts). \square

Proof of Proposition 2.6.2. We first note that if $f : G \rightarrow H$ is a levelwise split map of simplicial groups then $U(f)$ is a levelwise split map of simplicial set and hence a cofibration. In light of Corollary 2.6.8 it will suffice to prove that the generating cofibrations in sGr are levelwise split. These maps are the maps of the form $F(\partial\Delta^n) \rightarrow F(\Delta^n)$ for $n \geq 0$. It will hence suffice to show that if $f : A \rightarrow B$ is an injective map of sets then the corresponding map of free groups $\langle f \rangle : \langle A \rangle \rightarrow \langle B \rangle$ is split. But this is clear, since one can define $g : \langle B \rangle \rightarrow \langle A \rangle$ to be the identity on the generators that belong to A and trivial on the generators in $B \setminus A$. \square

3 An integral model structure for Segal group actions

We shall now adapt the theory of Segal groups and Segal group actions to the setup of the model categorical Grothendieck construction. We will begin in §3.1 by establishing a convenient model category for Segal groups. The construction we will use is based on a similar construction using topological spaces which appears in [Ber]. We will continue in §3.2 by showing that the functor which associates to each Segal group its corresponding model category of Segal group actions is proper and relative. For this purpose it will be convenient to identify the model structure constructed in [Pra] with a suitable slice model structure arising from work of Schwede, Shipley and Rezk. Finally, in §3.3 we will use the invariance theorem 2.5.4 to show that the resulting integral model structure is equivalent to the two other model structures constructed in [HP].

3.1 A model structure for Segal groups

In this subsection we will construct a simplicial combinatorial model category whose fibrant-cofibrant objects are precisely the Segal groups. A similar construction (in the setting of topological spaces) was considered in [Ber].

Let $s\mathcal{S} = \mathcal{S}^{\Delta^{\text{op}}}$ be the category of simplicial spaces. This category can be endowed with the **injective model structure** in which weak equivalences (resp. cofibrations) are the maps which are weak equivalences (resp. cofibrations) in each simplicial degree. This model structure coincides with the relevant **Reedy model structure** (see [GJ, IV.3, Theorem 3.8]).

Notation 3.1.1. Henceforth, we will write $s\mathcal{S}$ for the model category of simplicial spaces endowed with the injective/Reedy model structure. Other model structures on the category of simplicial spaces will be indicated by a subscript notation.

We shall denote by $|A_\bullet| = \text{diag}(A_\bullet) \in \mathcal{S}$ the realization of A_\bullet in \mathcal{S} . We will say that a simplicial space A_\bullet is **reduced** if $A_0 = *$ is the one-pointed space. Let $s\mathcal{S}_0$ be the category of reduced simplicial spaces and $\iota : s\mathcal{S}_0 \rightarrow s\mathcal{S}$ the natural full inclusion. The functor ι admits a left adjoint

$$(-)^{\text{red}} : s\mathcal{S} \rightarrow s\mathcal{S}_0$$

given by $A_n^{\text{red}} = A_n/s^*(A_0)$ where s is the unique map in Δ from $[n]$ to $[0]$.

Theorem 3.1.2 ([Ber]). *There exists a combinatorial model structure on $s\mathcal{S}_0$, which we shall call the **reduced Reedy model structure**, such that a map $f : A_\bullet \rightarrow B_\bullet$ is*

1. *A weak equivalence if and only if $\iota(f) : \iota(A_\bullet) \rightarrow \iota(B_\bullet)$ is a weak equivalence in each simplicial degree.*
2. *A fibration if and only if $\iota(f) : \iota(A_\bullet) \rightarrow \iota(B_\bullet)$ is a Reedy fibration.*
3. *A cofibration if and only if $\iota(f) : \iota(A_\bullet) \rightarrow \iota(B_\bullet)$ is a Reedy cofibration, i.e., if and only if it is a monomorphism.*

Proof. We first claim that the composed functor $\iota((-)^{\text{red}}) : s\mathcal{S} \rightarrow s\mathcal{S}$ preserves cofibrations and trivial cofibrations. Let $f : A_\bullet \rightarrow B_\bullet$ be a (trivial) cofibration. By standard properties of the Reedy model structure on simplicial spaces the induced map

$$\begin{array}{ccc} & B_0 & \\ & \swarrow & \searrow \\ A_n \amalg_{A_0} B_0 & \xrightarrow{\quad} & B_n \end{array}$$

is a (trivial) cofibration of spaces under B_0 . Since the pushforward functor

$$\mathcal{S}_{B_0/} \rightarrow \mathcal{S}_{*/}$$

given by $K \mapsto K/B_0$ is a left Quillen functor we get that the induced map

$$A_n/A_0 \cong \left(A_n \amalg_{A_0} B_0 \right) / B_0 \rightarrow B_n/B_0$$

is a (trivial) cofibration for every n .

Combining this result with the fact that the inclusion $\iota : s\mathcal{S}_0 \rightarrow s\mathcal{S}$ preserves pushouts and filtered colimits one can use the transfer lemma (see [Ber, Proposition 1.12.1]) to transfer the Reedy model structure from $s\mathcal{S}$ to $s\mathcal{S}_0$. The resulting model structure will satisfy (1) and (2) above by definition. Furthermore, we will get in addition that the functor ι preserves cofibrations. To prove that ι also reflects cofibrations it is enough to note that the unit map

$$(\iota(A_\bullet))^{\text{red}} \rightarrow A_\bullet$$

is an isomorphism for every $A_\bullet \in s\mathcal{S}_0$. \square

Notation 3.1.3. Henceforth, we will write $s\mathcal{S}_0$ for the model category of simplicial spaces endowed with the transferred Reedy model structure. Other model structures on the category of simplicial spaces will be indicated by a subscript notation.

Definition 3.1.4. Given a reduced simplicial space A_\bullet and a space $K \in \mathcal{S}$ we shall define

$$K \otimes A_\bullet = (K \times A_\bullet)^{\text{red}},$$

where $K \times A_\bullet \in s\mathcal{S}$ is the simplicial space defined by $(K \times A)_n = K \times A_n$.

Remark 3.1.5. The n 'th space of $(K \times A_\bullet)^{\text{red}}$ can be naturally identified with

$$(K \times A_n)/K \times \{*\} = K_+ \wedge A_n$$

where $K_+ = K \amalg *$ and A_n is considered as a pointed space with respect to the base point $s(A_0)$.

Lemma 3.1.6. *The association $(K, A_\bullet) \mapsto K \otimes A_\bullet$ extends to a simplicial structure on $s\mathcal{S}_0$ which is compatible with the reduced Reedy model structure.*

Proof. First note that the adjoint to $K \otimes A_\bullet$ is given by the levelwise co-tensor A_\bullet^K (which is automatically reduced). We need to check that the pushout product axiom is satisfied. Recall that the Reedy model structure on $s\mathcal{S}$ is simplicial (with respect to the level-wise product). The result now follows directly from the fact that the reducification functor $(-)^{\text{red}} : s\mathcal{S} \rightarrow s\mathcal{S}_0$ is simplicial by definition and preserves Reedy cofibrations, trivial Reedy cofibrations and pushouts. \square

For a linearly ordered set J , let Δ^J denote the nerve of J . If $J = \{m_1 < \dots < m_k\} \subseteq [n]$ we will identify Δ^J with a sub-simplicial set of Δ^n .

Definition 3.1.7. For each $n \geq 1$ and $0 \leq i \leq n$ we will denote by

$$f_{n,i} : \Delta^{\{0, \dots, n\}} \coprod_{\Delta^{\{i\}}} \Delta^{\{i, n+1\}} \hookrightarrow \Delta^{n+1}$$

the corresponding inclusion of (level-wise discrete) simplicial spaces. We will denote by $f_{n,i}^{\text{red}}$ the map of reduced simplicial spaces obtained by applying the reducification functor $(-)^{\text{red}} : s\mathcal{S} \rightarrow s\mathcal{S}_0$. We will denote by

$$\mathbb{S} = \{f_{n,i}^{\text{red}}\}$$

this set of maps in $s\mathcal{S}_0$.

Definition 3.1.8. We will say that a reduced simplicial space A_\bullet is a **Segal group** if it is a Segal space in the sense of [Rez] in which every morphism is invertible (see [Rez, §5.5]). Note that in this definition a Segal group is always fibrant in the reduced Reedy model structure.

Remark 3.1.9. If A_\bullet is a reduced Segal space then $\pi_0(A_1)$ inherits a natural structure of a monoid. The condition that every morphism in A_\bullet is invertible is equivalent to the condition that this monoid is a group.

Remark 3.1.10. Note that any map $f : T_\bullet \rightarrow S_\bullet$ in \mathbb{S} is a reducification of a cofibration in $s\mathbb{S}$. It follows that if $A_\bullet \in s\mathbb{S}$ is fibrant then the map

$$f^* : \text{Map}_{s\mathbb{S}_0}(T_\bullet, A_\bullet) \rightarrow \text{Map}_{s\mathbb{S}_0}(S_\bullet, A_\bullet)$$

is a Kan fibration.

Proposition 3.1.11. *Let A_\bullet be a reduced Reedy fibrant simplicial space. Then A_\bullet is a Segal group if and only if A_\bullet is **local** with respect to \mathbb{S} , i.e., if for every map $f : S_\bullet \rightarrow T_\bullet$ in \mathbb{S} the map*

$$\text{Map}_{s\mathbb{S}_0}(T_\bullet, A_\bullet) \rightarrow \text{Map}_{s\mathbb{S}_0}(S_\bullet, A_\bullet)$$

is a trivial Kan fibration.

Proof. Given $0 \leq i < j \leq n$ the (i, j) -**spine** $\text{Sp}_{i,j} \subseteq \Delta^n$ is the simplicial subset given by

$$\text{Sp}_{i,j} = \Delta^{\{i,i+1\}} \coprod_{\Delta^{\{i+1\}}} \Delta^{\{i+1,i+2\}} \coprod_{\Delta^{\{i+2\}}} \dots \coprod_{\Delta^{\{j-1\}}} \Delta^{\{j-1,j\}} \subseteq \Delta^n.$$

Since A_\bullet is Reedy fibrant we note that A_\bullet is a Segal space if and only if it is local with respect to the inclusion $\text{Sp}_{0,n} \subseteq \Delta^n$ for every $n \geq 2$.

Now assume that A_\bullet is local with respect to \mathbb{S} . Consider the sequence of inclusions

$$\begin{aligned} \text{Sp}_{0,n+1} &= \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \text{Sp}_{1,n+1} \subseteq \Delta^{\{0,1,2\}} \coprod_{\Delta^{\{2\}}} \text{Sp}_{2,n+1} \subseteq \Delta^{\{0,1,2,3\}} \coprod_{\Delta^{\{3\}}} \text{Sp}_{3,n+1} \subseteq \dots \\ &\dots \subseteq \Delta^{\{0,\dots,n\}} \coprod_{\Delta^{\{n\}}} \Delta^{\{n,n+1\}} \subseteq \Delta^{n+1}. \end{aligned}$$

Then each map in this sequence is a pushout along $f_{j,j} \in \mathbb{S}$ for some $1 \leq j \leq n$. Since A_\bullet is \mathbb{S} -local we deduce that A_\bullet is local with respect to $\text{Sp}_{0,n+1} \subseteq \Delta^{n+1}$ for every $n \geq 1$ and so A_\bullet is a Segal space. Let us now show that the monoid $\pi_0(A_1)$ is a group. Since A_\bullet is local with respect to $f_{1,0}, f_{1,1} \in \mathbb{S}$ we obtain a span of weak equivalences

$$\begin{array}{ccc} & A_2 & \\ f_{1,1}^* \swarrow & & \searrow f_{1,0}^* \\ A_1 \times A_1 & \simeq & A_1 \times A_1 \end{array}$$

This implies that the shearing map

$$\pi_0(A_1) \times \pi_0(A_1) \longrightarrow \pi_0(A_1) \times \pi_0(A_1)$$

given by $(a, b) \mapsto (a, ab)$ is a bijection, so that $\pi_0(A_1)$ is a group.

Now assume that A_\bullet is a Segal group. Then A_\bullet is a Segal space and is hence local with respect to $\mathrm{Sp}_{0,n+1} \subseteq \Delta^{n+1}$ for every $n \geq 1$. In particular, A_\bullet is local with respect to $f_{1,1} \in \mathbb{S}$. Now let $T_{n,i} \subseteq \Delta^{n+1}$ be the simplicial subset given by the edges $\Delta^{\{j,j+1\}}$ for $j = 0, \dots, i-1$ and the triangles $\Delta^{\{j,j+1,n+1\}}$ for $j = i, \dots, n-1$. Then $T_{n,i}$ contains $\mathrm{Sp}_{0,n+1}$ and can be obtained from $\mathrm{Sp}_{0,n+1}$ by a sequence of pushouts along $f_{1,1}$. This means that A_\bullet is local with respect to the inclusion $\mathrm{Sp}_{0,n+1} \subseteq T_{n,i}$ and so A_\bullet is local with respect to the inclusion $T_{n,i} \subseteq \Delta^{n+1}$ as well. Now consider the diagram

$$\begin{array}{ccc} \mathrm{Sp}_{0,n} \amalg_{\Delta^{\{i\}}} \Delta^{\{i,n+1\}} & \longrightarrow & T_{n,i} \\ \downarrow & & \downarrow \\ \Delta^{\{0,\dots,n\}} \amalg_{\Delta^{\{i\}}} \Delta^{\{i,n+1\}} & \xrightarrow{f_{n,i}} & \Delta^{n+1} \end{array}$$

From the above considerations we see that A_\bullet is local with respect to both vertical maps. Hence to show that A_\bullet is local with respect to $f_{n,i}$ is equivalent to showing that A_\bullet is local with respect to the upper horizontal map. Now $T_{n,i}$ can be obtained from $\mathrm{Sp}_{0,i} \amalg_{\Delta^{\{i\}}} \Delta^{\{i,n+1\}}$ by performing pushouts along $f_{1,0}$, and so it will suffice to show that A_\bullet is local with respect to $f_{1,0}$. Since $f_{1,0}$ can be identified with the inclusion $\Lambda_0^2 \hookrightarrow \Delta^2$, this follows directly from [Rez, Lemma 11.6]. □

Since the reduced Reedy model structure on sS_0 is left proper and combinatorial we can take its left Bousfield localization with respect to \mathbb{S} (see [Hir, §4]). We shall call the localized model structure the **Segal group model structure** and denote it by $(\mathrm{sS}_0)_{\mathrm{seg}}$. In light of Proposition 3.1.11, this (simplicial, combinatorial) model structure satisfies the following properties:

1. Weak equivalences in the Segal group model structure are the maps $f : X_\bullet \rightarrow Y_\bullet$ such that for every Segal group A_\bullet the induced map

$$\mathrm{Map}_{\mathrm{sS}_0}(X_\bullet, A_\bullet) \longrightarrow \mathrm{Map}_{\mathrm{sS}_0}(Y_\bullet, A_\bullet)$$

is a weak equivalence.

2. The cofibrations are the monomorphisms.
3. An object X_\bullet is fibrant if and only if it is a Segal group.

Remark 3.1.12. In light of Proposition 3.1.11, it follows from the general theory of left Bousfield localization that if $f : X_\bullet \rightarrow Y_\bullet$ is a weak equivalence in the Segal group model structure and both X_\bullet, Y_\bullet are Segal groups then f is a weak equivalence in \mathfrak{sS}_0 , i.e., f is a levelwise equivalence.

Note that if X_\bullet is a reduced simplicial space then $|X_\bullet|$ is a reduced simplicial set. We shall hence consider the realization functor also as a functor $|-| : \mathfrak{sS}_0 \rightarrow \mathfrak{S}_0$. As such, it admits a right adjoint

$$\Omega : \mathfrak{S}_0 \rightarrow \mathfrak{sS}_0 .$$

The functor $\Omega(-)$ can be written explicitly as the simplicial mapping space of pairs

$$\Omega(B)_n = \text{Map}((\Delta^n, (\Delta^n)_0), (B, *)) = \text{Map}_*(\Delta^n/(\Delta^n)_0, B)$$

for $B \in \mathfrak{S}_0$, where $(\Delta^n)_0$ is the set of vertices of Δ^n .

Remark 3.1.13. By the realization lemma (see, e.g., [Wal, Lemma 5.1]), a map $f : X_\bullet \rightarrow Y_\bullet$ of simplicial spaces which is a levelwise equivalence induces an equivalence

$$|X_\bullet| \xrightarrow{\simeq} |Y_\bullet| .$$

In other words, the functor $|-| : \mathfrak{sS}_0 \rightarrow \mathfrak{S}_0$ preserves weak equivalences.

In light of Remark 3.1.13 and since $|-|$ is a simplicial functor which preserves cofibrations we obtain a simplicial Quillen adjunction

$$\mathfrak{sS}_0 \begin{array}{c} \xrightarrow{|-|} \\ \perp \\ \xleftarrow{\Omega} \end{array} \mathfrak{S}_0 . \quad (3.1.1)$$

Now note that for any $f_{n,i}$ as in Definition 3.1.7 the induced map $|f_{n,i}|$ is a weak equivalence of (non-reduced) spaces, and induces an isomorphism on the sets of vertices. Since $|f_{n,i}|^{\text{red}} = |f_{n,i}^{\text{red}}|$ it follows that $|f_{n,i}^{\text{red}}|$ is a weak equivalence of reduced spaces. By adjunction we get that $\Omega(B)_\bullet$ is a Segal group for any fibrant reduced simplicial set B . Using adjunction again it follows that the realization functor sends every weak equivalence in $(\mathfrak{sS}_0)_{\text{seg}}$ to a weak equivalence in \mathfrak{S}_0 .

Corollary 3.1.14. *The adjunction 3.1.1 descends to a simplicial Quillen adjunction*

$$(\mathfrak{sS}_0)_{\text{seg}} \begin{array}{c} \xrightarrow{|-|} \\ \perp \\ \xleftarrow{\Omega} \end{array} \mathfrak{S}_0$$

Let us also recall the following:

Theorem 3.1.15 ([Seg]). *If A_\bullet is a Segal group then the natural map*

$$\varphi : A_\bullet \rightarrow \Omega(|A_\bullet|^{\text{fib}})_\bullet$$

is a level-wise equivalence.

Proof. Since both the domain and codomain of φ are Segal groups it is enough to show that

$$\varphi_1 : A_1 \longrightarrow \Omega(|A_\bullet|^{\text{fib}})_1 \simeq \Omega(|A_\bullet|^{\text{fib}})$$

is a weak equivalence. This in turn, is the simplicial analogue of Segal's well-known result [Seg, Proposition 1.5]. \square

Corollary 3.1.16. *A map $f : X_\bullet \longrightarrow Y_\bullet$ is a weak equivalence in the Segal group model structure if and only if it induces a weak equivalence*

$$|f| : |X_\bullet| \longrightarrow |Y_\bullet|$$

in \mathcal{S}_0 .

Proof. Since the Quillen adjunction in question is simplicial we see that $|f|$ is a weak equivalence in \mathcal{S}_0 if and only if f induces an equivalence

$$\text{Map}_{\text{s}\mathcal{S}_0}(Y_\bullet, \Omega(B)_\bullet) \longrightarrow \text{Map}_{\text{s}\mathcal{S}_0}(X_\bullet, \Omega(B)_\bullet)$$

for every fibrant reduced space $B \in \mathcal{S}_0$. The result now follows from Theorem 3.1.15. \square

Corollary 3.1.17. *The simplicial Quillen adjunction*

$$(\text{s}\mathcal{S}_0)_{\text{seg}} \begin{array}{c} \xrightarrow{|-|} \\ \perp \\ \xleftarrow{\Omega} \end{array} \mathcal{S}_0$$

is a Quillen equivalence.

Proof. From Corollary 3.1.16 we get that $| - |$ preserves and reflects weak equivalences. It is hence enough to check that for every fibrant object $A \in \mathcal{S}_0$ the co-unit map

$$\epsilon : |\Omega(A)_\bullet| \longrightarrow A$$

is a weak equivalence. Since $\Omega(A)_\bullet$ is a Segal group we get from Theorem 3.1.15 that the map ϵ induces an equivalence on loop spaces. Since A and $|\Omega(A)_\bullet|$ are reduced (and in particular connected) this implies that ϵ is a weak equivalence. \square

3.1.1 Relation with Bousfield's approach

We have seen that Segal groups constitute a model for pointed connected spaces and hence for simplicial groups. There is another approach, based on an unpublished manuscript of A. K. Bousfield, which we now describe.

Definition 3.1.18. Let $T_n \subseteq \Delta^n$ be the simplicial subset given by the edges $\Delta^{\{0,i\}}$ for $i = 1, \dots, n$. We shall refer to T_n as the n 'th **Bousfield spine**.

The terminology of the following definition is taken from [Bergn].

Definition 3.1.19 ([Bou]). A **reduced Bousfield-Segal space** is a Reedy fibrant reduced simplicial space A_\bullet which is local with respect to the inclusion $T_n \subseteq \Delta^n$ for every n . In particular, for every n the induced map

$$A_n = \text{Map}(\Delta^n, A) \longrightarrow \text{Map}(T_n, A) = A_1^n$$

is a weak equivalence.

In [Bou, Theorem 3.1] Bousfield proves that the adjunction

$$\text{sS}_0 \begin{array}{c} \xrightarrow{|-|} \\ \xleftarrow{\perp} \\ \xrightarrow{\Omega} \end{array} \mathcal{S}_0 \quad (3.1.2)$$

induces an equivalence between the homotopy category of Bousfield-Segal spaces and the homotopy category of Kan simplicial sets. In particular, he shows that a reduced simplicial space is a Bousfield-Segal space if and only if it is levelwise equivalent to a reduced simplicial space of the form $\Omega(B)_\bullet$ for some reduced Kan simplicial set B . It then follows from Corollary 3.1.17 and Theorem 3.1.15 that a reduced simplicial space is a Bousfield-Segal space if and only if it is a Segal group.

Since the reduced Reedy model structure on sS_0 is left proper and combinatorial one can take its left Bousfield localization with respect to the inclusions $T_n \hookrightarrow \Delta^n$. The resulting model category was considered in [Bergn]. However, in light of [BergnE], the argument appearing right after [Bergn, Proposition 6.1], comparing this model category to the model category of simplicial groups is only partial, as it refers back to the withdrawn [Bergn, §4]. On the other hand, it follows directly from the arguments above that this model category is in fact equal to the model category $(\text{sS}_0)_{\text{seg}}$, and hence equivalent to the model category of simplicial groups by Corollary 3.1.17.

3.2 Construction of the integral model structure

Our goal in this subsection is to show that the model categories of Segal group actions constructed in [Pra] can be assembled to form an integral model structure. Let us begin with the basic definitions.

Definition 3.2.1. Let $f : X_\bullet \longrightarrow Y_\bullet$ be a map of simplicial spaces and let $\rho : [m] \longrightarrow [n]$ be a map in Δ . We will say that ρ is **f -Cartesian** if the induced square

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \rho^* \downarrow & & \downarrow \rho^* \\ X_m & \xrightarrow{f_m} & Y_m \end{array}$$

is homotopy Cartesian.

Remark 3.2.2. Let $f : X_\bullet \rightarrow Y_\bullet$ be a map of simplicial spaces. The pasting lemma for pullback squares implies that if $\sigma, \tau \in \Delta$ are f -Cartesian then $\tau \circ \sigma$ is f -Cartesian, and that if σ and $\tau \circ \sigma$ are f -Cartesian then τ is f -Cartesian.

Definition 3.2.3. [Pra, Definition 3.1] (cf. [NSS, 3.1,3.4]) Let A_\bullet be a reduced simplicial space. A *Segal group action* over A_\bullet is a Reedy fibration of simplicial spaces

$$p : X_\bullet \rightarrow \iota(A_\bullet)$$

such that for every n , the maps

$$X_n \begin{array}{c} \xrightarrow{\sigma_{\{n\}}^* \times p_n} \\ \rightrightarrows \\ \xrightarrow{\sigma_{\{0\}}^* \times p_n} \end{array} X_0 \times A_n,$$

induced by the maps $\sigma_{\{0\}}, \sigma_{\{n\}} : [0] \rightarrow [n]$ (which send 0 to 0 and n respectively) are weak equivalences.

Remark 3.2.4. If A_\bullet is a Segal group then the 0'th space X_0 should be considered as an underlying space on which the loop space $A_1 \simeq \Omega|A_\bullet|$ coherently acts.

Remark 3.2.5. Let A_\bullet be a reduced simplicial space. Unwinding the definitions we see that a map $p : X_\bullet \rightarrow \iota(A_\bullet)$ is a Segal group action if and only if the maps $\sigma_{\{0\}}, \sigma_{\{n\}} \in \Delta$ are p -Cartesian.

Remark 3.2.6. We will be mostly interested in Segal group actions when the reduced simplicial space A_\bullet is a Segal group. As observed in [Pra], in this case both $\sigma_{\{0\}}, \sigma_{\{n\}} \in \Delta$ are p -Cartesian as soon as one of them is. However, it will be convenient to have the notion of a Segal group action defined for a general reduced simplicial space A_\bullet , in which case it is certainly possible for only one of $\sigma_{\{0\}}, \sigma_{\{n\}}$ to be p -Cartesian.

Let $\Delta_{\text{inj}} \subseteq \Delta$ denote the subcategory consisting of all objects and all injective maps.

Definition 3.2.7. A map of simplicial spaces $f : X_\bullet \rightarrow Y_\bullet$ is said to be **equifibred** if every map in Δ_{inj} is f -Cartesian.

Remark 3.2.8. Since any map in Δ can be written as a composition of an injective map followed by a surjective map, and since any surjective map in Δ has a right inverse which is injective, Remark 3.2.2 implies that if $f : A_\bullet \rightarrow B_\bullet$ is an equifibred map, then **every** map in Δ is f -Cartesian.

Proposition 3.2.9. *Let A_\bullet be a reduced simplicial space. A map $p : X_\bullet \rightarrow \iota(A_\bullet)$ of simplicial spaces is a Segal group action if and only if it is an equifibred Reedy fibration.*

Proof. By Remark 3.2.5 we see that every equifibred Reedy fibration is a Segal group action. Now assume that p is a Segal group action. We need to show that every map in Δ_{inj} is p -Cartesian. Since p is a Segal group action we have that $\sigma_{\{0\}} : [0] \rightarrow [n]$ and $\sigma_{\{n\}} : [0] \rightarrow [n]$ are p -Cartesian. Applying Remark 3.2.2 we can prove the desired result in three steps:

1. Since $\sigma_{\{0\}} : [0] \rightarrow [m]$ and $\sigma_{\{0\}} : [0] \rightarrow [n]$ are p -Cartesian we get that $\sigma_{\{0, \dots, m\}} : [m] \rightarrow [n]$ is p -Cartesian.
2. Since $\sigma_{\{m\}} : [0] \rightarrow [m]$ and $\sigma_{\{0, \dots, m\}} : [m] \rightarrow [n]$ are p -Cartesian we get that $\sigma_{\{m\}} : [0] \rightarrow [n]$ is p -Cartesian.
3. Since $\sigma_{\{0\}} : [0] \rightarrow [m]$ and $\sigma_{\{i\}} : [0] \rightarrow [n]$ are p -Cartesian it follows that every injective map $\sigma : [m] \rightarrow [n]$ such that $\sigma(0) = i$ is p -Cartesian.

□

We now wish to construct a model for coherent group actions using $(\mathbb{sS}_0)_{\text{seg}}$. For this, we shall consider the following model structure introduced by Rezk, Schwede and Shipley ([RSS]):

Theorem 3.2.10 ([RSS]).

1. *There exists a model structure on \mathbb{sS} , denoted \mathbb{sS}_{equ} , such that:*
 - (a) *the weak equivalences are the maps $f : X_{\bullet} \rightarrow Y_{\bullet}$ such that the induced map $|f| : |X_{\bullet}| \rightarrow |Y_{\bullet}|$ is a weak equivalence in \mathcal{S} ;*
 - (b) *the cofibrations are the monomorphisms, and*
 - (c) *the fibrations are the equifibred Reedy fibrations.*
2. *The realization functor $|-| : \mathbb{sS} \rightarrow \mathcal{S}$ fits into a Quillen equivalence*

$$(\mathbb{sS})_{\text{equ}} \begin{array}{c} \xrightarrow{|-|} \\ \xleftarrow{C} \\ \xrightarrow{\perp} \end{array} \mathcal{S}$$

where $C(A)_n = \text{Map}(\Delta^n, A)$.

Proof. The first part is a particular case of Theorem 3.6 of [RSS]. As for part (2), it is clear that $|-|$ preserves cofibrations and trivial cofibrations. Furthermore, $|-|$ preserves and reflects weak equivalences. Since every object is cofibrant it will be enough to prove that for every fibrant $B \in \mathcal{S}$ the counit map

$$|C(B)_{\bullet}| \rightarrow B$$

is a weak equivalence. But this follows from the fact that if B is fibrant, $C(B)_{\bullet}$ is homotopy constant with value B . □

Now given an object $A_{\bullet} \in \mathbb{sS}_0$ we will consider the model category $\mathbb{sS}_{/\iota(A_{\bullet})}$ endowed with the slice model structure inherited from \mathbb{sS}_{equ} and denote it by $(\mathbb{sS}_{/\iota(A_{\bullet})})_{\text{act}}$. This notation is justified by the following Corollary of Proposition 3.2.9.

Corollary 3.2.11. *Let A_{\bullet} be a Segal group. Then the fibrant objects of $(\mathbb{sS}_{/\iota(A_{\bullet})})_{\text{act}}$ are precisely the Segal group actions, and all objects are cofibrant.*

Remark 3.2.12. By Remark 3.1.13 any levelwise equivalence in $(\mathfrak{sS})_{/\iota(A_\bullet)}$ is a weak equivalence in $(\mathfrak{sS}_{/\iota(A_\bullet)})_{\text{act}}$. The converse holds for maps between fibrant objects (i.e., maps between Segal group actions, see Corollary 3.2.11).

Proposition 3.2.13. *The functor $|-| : \mathfrak{S}_{\text{equ}} \rightarrow \mathfrak{sS}$ preserves fiber products and fibrations.*

Proof. It is well-known that the realization functor $|-| : \mathfrak{sS}_{\text{equ}} \rightarrow \mathfrak{S}$ coincides with the restriction along the diagonal $\Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}}$ (see, e.g., [Qui2, p. 86]), and hence preserves fiber products. Following [Jar], let us say that a map $X_\bullet \rightarrow Y_\bullet$ of simplicial spaces is a **diagonal fibration** if the induced map $|X_\bullet| \rightarrow |Y_\bullet|$ is a fibration. Our goal is hence to prove that every fibration in $\mathfrak{S}_{\text{equ}}$ is a diagonal fibration. According to [Jar, Theorem 2.14] it will suffice to check that every fibration in $\mathfrak{S}_{\text{equ}}$ has the right lifting property with respect to maps of the form (see [Jar, p.259]):

$$(\Lambda_k^r \widetilde{\times} \Delta^s) \cup (\Delta^r \widetilde{\times} \partial \Delta^s) \rightarrow \Delta^r \widetilde{\times} \Delta^s \quad (3.2.1)$$

and

$$(\partial \Delta^s \widetilde{\times} \Delta^r) \cup (\Delta^s \widetilde{\times} \Lambda_k^r) \rightarrow \Delta^s \widetilde{\times} \Delta^r \quad (3.2.2)$$

for $0 \leq k \leq r$ and $0 \leq s$, where $\widetilde{\times}$ denotes the external Cartesian product. Alternatively, we need to show that the maps 3.2.1 and 3.2.2 are trivial cofibrations in $\mathfrak{S}_{\text{equ}}$. Since they are clearly cofibrations, we just need to show that 3.2.1 and 3.2.2 become weak equivalences after realization. Observing that 3.2.1 and 3.2.2 are pushout products of a cofibration $\partial \Delta^s \hookrightarrow \Delta^s$ and a trivial cofibration $\Lambda_k^r \hookrightarrow \Delta^r$ with respect to the external Cartesian product the result now follows from the fact that $|-|$ sends external Cartesian product of simplicial sets to the corresponding Cartesian product, and that the Cartesian product on \mathfrak{S} satisfies the pushout-product axiom. \square

Corollary 3.2.14. *The model category $\mathfrak{sS}_{\text{equ}}$ is right proper.*

Proof. Since weak equivalences in $\mathfrak{sS}_{\text{equ}}$ are preserved and reflected by $|-|$, and since \mathfrak{S} is right proper, the result follows directly from Proposition 3.2.13. \square

Combining Corollary 3.2.14 with [HP, Corollary 6.2] we hence obtain the following

Corollary 3.2.15. *The slice functor $(\mathfrak{sS}_{\text{equ}})_{/(-)} : \mathfrak{sS}_{\text{equ}} \rightarrow \text{ModCat}$ is proper and relative.*

We warn the reader that the functor $\iota : \mathfrak{sS}_0 \rightarrow \mathfrak{sS}_{\text{equ}}$ is **not** a Quillen functor. However, it does preserve weak equivalences, trivial cofibrations and trivial fibrations (the latter consists of trivial Reedy fibrations in both cases). In light of this and the above remark, it follows that the association $A_\bullet \mapsto \mathfrak{sS}_{/\iota(A_\bullet)}$ does determine a proper relative functor

$$\mathcal{W} : \mathfrak{sS}_0 \rightarrow \text{ModCat}. \quad (3.2.3)$$

We hence obtain the following conclusion:

Corollary 3.2.16. *The integral model structure*

$$\int_{A_\bullet \in (\mathfrak{sS}_0)_{\text{seg}}} (\mathfrak{sS}_{/\iota(A_\bullet)})_{\text{act}}$$

exist and its fibrant objects are precisely the Segal group actions. Furthermore, every object in the integral model structure is cofibrant.

Remark 3.2.17. Note that a map of Segal group actions over different Segal groups is simply a commutative square of simplicial spaces

$$\begin{array}{ccc} X_\bullet & \longrightarrow & A_\bullet \\ \downarrow & & \downarrow \\ Y_\bullet & \longrightarrow & B_\bullet \end{array}$$

Under the Quillen equivalence of Corollary 3.3.4 such a map corresponds to a map of (ordinary) group actions $(G, X) \rightarrow (H, Y)$, namely a map of simplicial groups $G \rightarrow H$, and a map of spaces $X \rightarrow Y$ which is G -equivariant when Y is considered as a G -space via the map $G \rightarrow H$.

3.3 Rectification of Segal group actions

The purpose of this subsection is to prove that the integral model category constructed in §3.2 is equivalent to the two model structures recalled in §2.6. This can be viewed as a rectification theorem for Segal group actions in a global setting.

In light of Theorem 2.5.4, we need to provide a compatible family of Quillen adjunctions

$$\Xi_B^L : (\mathfrak{sS}_{/\iota(\Omega(B))})_{\text{act}} \xrightleftharpoons[\perp]{} \mathfrak{S}_{/\iota(B)} : \Xi_A^R$$

indexed by $B \in \mathfrak{S}_0$, which are Quillen equivalences whenever $B \in \mathfrak{S}_0$ is fibrant. This can be done as follows. Let $B \in \mathfrak{S}_0$ be a reduced space and X a space equipped with a map $X \rightarrow \iota(B)$. Define $\Xi_B^R(X)$ to be the pullback in the square

$$\begin{array}{ccc} \Xi_B^R(X) & \longrightarrow & C(X)_\bullet \\ \downarrow & & \downarrow \\ \iota(\Omega(B))_\bullet & \longrightarrow & C(\iota(B))_\bullet \end{array}$$

where $C(-)_\bullet = \text{Map}(\Delta^\bullet, -) : \mathfrak{S} \rightarrow \mathfrak{sS}$ is the functor discussed in Theorem 3.2.10 (2). Dually, if $f : X_\bullet \rightarrow \iota(\Omega(B))$ is an object of $\mathfrak{sS}_{/\iota(\Omega(B))}$ then $\Xi_B^L(X_\bullet)$ is given by

$$\Xi_B^L(X_\bullet) = |X_\bullet|$$

with the map $|X_\bullet| \rightarrow B$ given by the composition

$$|X_\bullet| \rightarrow |(X_\bullet^{\text{red}})| \rightarrow B$$

where the second map is the adjoint of the map $X_\bullet^{\text{red}} \rightarrow \Omega(B)$ which in turn is the adjoint of f . It is straightforward to verify that $\Xi_B^L \dashv \Xi_B^R$ forms an adjunction which depends functorially on $B \in \mathcal{S}_0$.

Proposition 3.3.1. *For every fibrant $B \in \mathcal{S}_0$, the adjunction*

$$\Xi_B^L : (\text{s}\mathcal{S}_{/\iota(\Omega(B))})_{\text{act}} \xrightleftharpoons[\perp]{} \mathcal{S}_{/\iota(B)} : \Xi_A^R$$

is a Quillen equivalence.

Proof. It is evident from the definitions that Ξ_B^L preserves cofibrations and trivial cofibrations. Furthermore, the functor Ξ_B^L preserves and reflects weak equivalences. Since every object in $\text{s}\mathcal{S}_{/\iota(\Omega(B))}$ is cofibrant it will now suffice to show that for every fibrant object $X \rightarrow \iota(B)$ in $\mathcal{S}_{/\iota(B)}$ the counit map

$$\Xi_B^L(\Xi_B^R(X)) \rightarrow X$$

is a weak equivalence. Consider the diagram

$$\begin{array}{ccccc} |\Xi_B^R(X)| & \longrightarrow & |C(X)_\bullet| & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ |\iota(\Omega(B))_\bullet| & \longrightarrow & |C(\iota(B))_\bullet| & \longrightarrow & \iota(B) \end{array}$$

To finish the proof we need to show that the composed map

$$\Xi_B^L(\Xi_B^R(X)) = |\Xi_B^R(X)| \rightarrow |C(X)_\bullet| \rightarrow X$$

is a weak equivalence. To this end we will prove that all horizontal maps in the diagram are weak equivalences. To begin, since X is fibrant in $\mathcal{S}_{/\iota(B)}$ we know that the map $X \rightarrow \iota(B)$ is a fibration in \mathcal{S} . Since B is fibrant in \mathcal{S}_0 it follows that $\iota(B)$ is fibrant in \mathcal{S} ([GJ, Lemma V.6.6]) and so X is fibrant in \mathcal{S} . From Theorem 3.2.10 (2) we may deduce that the horizontal maps on the right square are weak equivalences. By Corollary 3.1.17 the composition of the bottom horizontal maps is a weak equivalence and hence the bottom-left horizontal map is a weak equivalence by 2-out-of-3. By Proposition 3.2.13 the left square is a pullback square and the middle vertical map is a Kan fibration. Since \mathcal{S} is right proper we deduce that the top-left horizontal map is a weak equivalence, and hence all horizontal maps are weak equivalence. We may now conclude that the map

$$|\Xi_B^R(X)| \rightarrow X$$

is a weak equivalence, as desired. \square

In light of Corollary 3.1.17 we now obtain the following conclusion:

Corollary 3.3.2. *The compatible family of adjunctions*

$$\Xi_B^L : (\mathfrak{sS}_{/\iota(\Omega(B), \bullet)})_{\text{act}} \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\perp} \end{array} \mathfrak{S}_{/\iota(B)} : \Xi_B^R \quad (3.3.1)$$

forms a **right Quillen equivalence** from \mathcal{W} to \mathcal{V} (see Definition 2.5.3), where $\mathcal{V} : \mathfrak{S}_0 \rightarrow \text{ModCat}$ is the functor defined in 2.6.3 and $\mathcal{W} : \mathfrak{sS}_0 \rightarrow \text{ModCat}$ is the functor defined in 3.2.3.

Finally, combining Corollary 3.3.2 and Theorem 2.5.4 we obtain the following:

Corollary 3.3.3. *The right Quillen morphism 3.3.1 induces a Quillen equivalence*

$$\Psi^L : \int_{A_\bullet \in (\mathfrak{sS}_0)_{\text{seg}}} (\mathfrak{sS}_{/\iota(A_\bullet)})_{\text{act}} \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\perp} \end{array} \int_{B \in \mathfrak{S}_0} \mathfrak{S}_{/\iota(B)} : \Psi^R.$$

For future reference we record the following

Corollary 3.3.4. *The Quillen equivalences of Corollaries 2.6.5 and 3.3.3 compose and yield a Quillen equivalence*

$$\Lambda^L : \int_{A_\bullet \in (\mathfrak{sS}_0)_{\text{seg}}} (\mathfrak{sS}_{/\iota(A_\bullet)})_{\text{act}} \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\perp} \end{array} \int_{G \in \mathfrak{SGr}} \mathfrak{S}^{\mathbb{B}G} : \Lambda^R.$$

between the integral model structures for strict group actions and Segal group actions.

4 Truncation theory

4.1 Preliminaries

Recall that a space $X \in \mathfrak{S}$ is called **n -truncated** if it has no homotopy groups above dimension n . Following the approach of [Bie], which in turn is based on the work of Bousfield, one may study the homotopy theory of n -truncated spaces by putting a suitable model structure $\mathfrak{S}_{n\text{-tr}}$ on the category of simplicial sets, in which the fibrant objects are precisely the n -truncated Kan complexes. The model structure $\mathfrak{S}_{n\text{-tr}}$ can be obtained as a left Bousfield localization of the Kan-Quillen model structure, and inherits the simplicial structure of \mathfrak{S} (see [Bie, Theorem 2.3]). We may then identify the n -truncated spaces as those which are **local** with respect to a suitable class of maps. Since the model category $\mathfrak{S}_{n\text{-tr}}$ is simplicial it follows that if X is a Kan simplicial set which is fibrant in $\mathfrak{S}_{n\text{-tr}}$ (i.e., n -truncated), then X^K is fibrant in $\mathfrak{S}_{n\text{-tr}}$ (i.e., n -truncated) for every $K \in \mathfrak{S}$. This implies the following classical fact: a space X is n -truncated if and only if the derived mapping space $\text{Map}_{\mathfrak{S}}^{\text{h}}(K, X)$ is n -truncated for every space $K \in \mathfrak{S}$. The discussion above motivates the following definition (see also [Lur1, §5.5.6] for the corresponding generalization in the ∞ -categorical setting):

Definition 4.1.1. Let \mathcal{M} be a model category. We will say that an object $X \in \mathcal{M}$ is **n -truncated** if for every object $Y \in \mathcal{M}$ the derived mapping space $\mathrm{Map}_{\mathcal{M}}^h(Y, X)$ is an n -truncated space. We will say that a map $f : X \rightarrow X_n$ exhibits X_n as an **n -truncation** of X if X_n is n -truncated and for every n -truncated object Z the induced map

$$\mathrm{Map}_{\mathcal{M}}^h(X_n, Z) \rightarrow \mathrm{Map}_{\mathcal{M}}^h(X, Z)$$

is a weak equivalence.

Example 4.1.2. Let X_n be an n -truncated space. Then a map of spaces of the form $f : X \rightarrow X_n$ is an n -truncation if and only if it is an equivalence in the model category $\mathcal{S}_{n\text{-tr}}$ recalled above.

Remark 4.1.3. Since the collection of n -truncated objects can be described as the collection of local objects with respect to a suitable class of maps it follows that the full subcategory of \mathcal{S} spanned by n -truncated objects is closed under taking homotopy limits. It then follows immediately from the definition that in any model category \mathcal{M} , the full subcategory spanned by n -truncated objects is closed under homotopy limits.

4.2 Truncation in the slice category of spaces

Let us now examine truncation in the slice category of spaces.

Notation 4.2.1. Let $f : X \rightarrow A$ be a map in \mathcal{S} . We will denote by

$$X \xrightarrow{\simeq} \bar{X} \xrightarrow{f^{\mathrm{fib}}} A$$

the functorial factorization of f into a trivial cofibration followed by a fibration.

We now recall an explicit construction for the n -truncation in the slice model category of spaces (see [DK, 2.4]).

Definition 4.2.2. Let $f : X \rightarrow A$ be a map in \mathcal{S} . For $n \geq 1$ we will denote by $\mathrm{cosk}_n(f)$ the simplicial set whose k -simplices are the commutative squares of the form

$$\begin{array}{ccc} \mathrm{sk}_n(\Delta^k) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta^k & \longrightarrow & A. \end{array}$$

We have canonical maps

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{cosk}_n(f) \\ & \searrow f & \swarrow f_n \\ & & A \end{array}$$

We then define the **relative n 'th Postnikov section of X over A** to be

$$P_n(X/A) \stackrel{\mathrm{def}}{=} \mathrm{cosk}_{n+1}(f^{\mathrm{fib}}).$$

Remark 4.2.3. The functor $\text{cosk}_n : \mathcal{S}/A \rightarrow \mathcal{S}/A$ is right adjoint to the functor $\text{sk}_n : \mathcal{S}/A \rightarrow \mathcal{S}/A$ which sends an object $X \rightarrow A$ to the object $\text{sk}_n(X) \rightarrow A$.

Remark 4.2.4. If $A = *$ the above construction reproduces one of the standard models for the n 'th Postnikov section of X , namely, the coskeleton of its Kan replacement.

Remark 4.2.5. It is immediate from the definition that the functor $f \mapsto \text{cosk}_n(f)$ commutes with base change. In other words, if $g : Y \rightarrow A$ is any map and $h : X \times_A Y \rightarrow Y$ is the projection on the second coordinate, then the natural map $\text{cosk}_n(h) \xrightarrow{\cong} Y \times_A \text{cosk}_n(f)$ is an isomorphism. In particular, if a is any vertex of A then the natural map $\text{cosk}_n(f^{-1}(a)) \xrightarrow{\cong} \text{cosk}_n(f) \times_A \{a\}$ is an isomorphism.

Lemma 4.2.6. *If $f : X \rightarrow A$ is a fibration then $f_n : \text{cosk}_n(f) \rightarrow A$ is a fibration as well.*

Proof. We need to show that the dotted lift exist in any square of the form

$$\begin{array}{ccc} \Lambda_i^k & \longrightarrow & \text{cosk}_n(f) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^k & \longrightarrow & A \end{array}$$

By Remark 4.2.5 we may assume without loss of generality that $A = \Delta^k$ and that the lower vertical map is the identity, i.e., we need to show that for every fibration $f : X \rightarrow \Delta^k$, the dotted lift exist in any square of the form

$$\begin{array}{ccc} \Lambda_i^k & \longrightarrow & \text{cosk}_n(f) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^k & \xlongequal{\quad} & \Delta^k \end{array}$$

By adjunction (see Remark 4.2.3) it will suffice to prove that the dotted lift exist in any square of the form

$$\begin{array}{ccc} \text{sk}_n(\Lambda_i^k) & \xrightarrow{g} & X \\ h \downarrow & \nearrow \text{dotted} & \downarrow f \\ \text{sk}_n(\Delta^k) & \xrightarrow{u} & \Delta^k \end{array}$$

We now separate two possible cases. If $n < k - 1$ then h is an isomorphism and hence a lift exists. If $n \geq k - 1$ then $\text{sk}_n(\Lambda_i^k) = \Lambda_i^k$ and hence $u \circ h$ is a trivial cofibration. Since f is a fibration it follows that f admits a section $s : \Delta^k \rightarrow X$ such that $s \circ u \circ h = g$. We may then take the dotted lift to be $s \circ u$. \square

Corollary 4.2.7. *For any $X \rightarrow A$ in \mathcal{S}/A , the homotopy fiber of $P_n(X/A) \rightarrow A$ over any vertex $a \in A_0$ is naturally equivalent to $P_n(F)$, where F is the homotopy fiber of f over a .*

Proof. By Lemma 4.2.6 the map $P_n(X/A) \rightarrow A$ is a fibration, and hence its homotopy fiber coincides with its actual fiber. The result now follows from Remark 4.2.5. \square

Using the same argument as in [Bie] one can verify that the coaugmented functor $Q(-) = P_n(-/A)$ satisfies Bousfield's axioms **A.4**, **A.5** and **A.6** (see [Bie, Definition 2.2]) and hence one can left-Bousfield localize the slice model structure on $\mathcal{S}_{/A}$ so that the new weak equivalences are the maps $X \rightarrow Y$ over A such that $P_n(X/A) \rightarrow P_n(Y/A)$ is a weak equivalence in $\mathcal{S}_{/A}$. Let us denote the resulting model category by $(\mathcal{S}_{/A})_{n\text{-tr}}$. Note that for the terminal object $*_A \in \mathcal{S}_{/A}$, given by the identity map $A \rightarrow A$, the coaugmentation map $*_A \xrightarrow{\cong} P_n(*_A/A)$ is an isomorphism. It then follows from [Bie, Theorem 2.3] that a fibrant object $X \rightarrow A$ of $\mathcal{S}_{/A}$ (i.e., a fibration) is fibrant in $(\mathcal{S}_{/A})_{n\text{-tr}}$ if and only if the coaugmentation map $X \rightarrow P_n(X/A)$ is a weak equivalence in $\mathcal{S}_{/A}$.

Lemma 4.2.8. *Let $f : X \rightarrow A$ be a fibrant object of $\mathcal{S}_{/A}$ (i.e., a fibration). Then the following are equivalent:*

1. f is n -truncated in $\mathcal{S}_{/A}$.
2. The fibers of f over every $a \in A_0$ are n -truncated.
3. f is fibrant in $(\mathcal{S}_{/A})_{n\text{-tr}}$.

Proof. (1) \Rightarrow (2). If $f : X \rightarrow A$ is n -truncated in $\mathcal{S}_{/A}$ then $\text{Map}_{\mathcal{S}_{/A}}^h(a, f)$ is n -truncated for any vertex $a : \Delta^0 \rightarrow A$. Since f is a fibration we get that $\text{Map}_{\mathcal{S}_{/A}}^h(a, f) \simeq \text{Map}_A(*, X) \cong f^{-1}(a)$ and hence the fibers of f are n -truncated.

(2) \Rightarrow (3). If the fibers of f are n -truncated then by Corollary 4.2.7 the coaugmentation map $u : X \rightarrow P_n(X/A)$ induces a weak equivalence on homotopy fibers over any vertex of A . This implies that u is a weak equivalence (see Remark 4.2.9 below), and hence f is fibrant in $(\mathcal{S})_{n\text{-tr}}$.

(3) \Rightarrow (1). For this it will suffice to show that for every $f : X \rightarrow A$ in $\mathcal{S}_{/A}$ the object $P_n(X/A) \rightarrow A$ is n -truncated. To see this, observe that by Remark 4.2.5, for every $f : X \rightarrow A$ in $\mathcal{S}_{/A}$ the homotopy fibers of $P_n(X/A)$ are n -truncated. Combining Remark 4.2.5, Remark 4.1.3 with [Lur1, Corollary 3.3.3.4] we get that for every $g : Y \rightarrow A$ in $\mathcal{S}_{/A}$ the space

$$\text{Map}_{\mathcal{S}_{/A}}^h(g, \text{cosk}_n(f^{fib})) \simeq \text{Map}_A(Y, P_n(X/A)) \simeq \text{Map}_Y(Y, P_n(X \times_A Y/Y))$$

is n -truncated, and hence $P_n(X/A) = \text{cosk}_n(f^{fib})$ is n -truncated for every object $X \rightarrow A$ in $\mathcal{S}_{/A}$. \square

Remark 4.2.9. A map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \swarrow h \\ & & A \end{array}$$

in $\mathcal{S}_{/A}$ is a weak equivalence if and only if it induces a weak equivalence on homotopy fibers over every vertex $a \in A_0$. To see this, we may assume that A is a Kan complex (recall that \mathcal{S} is right proper) and that g, h are Kan fibrations, and apply the five lemma to the corresponding map of long exact sequences of homotopy groups, where special care is needed for the low degrees. Alternatively, this is a particular case of [Lur1, Remark 2.2.3.3], since categorical equivalences between Kan complexes are the same as weak equivalences (see [Lur1, Lemma 3.1.3.2]).

Corollary 4.2.10. *Let $f : X \rightarrow A$ be any object of $\mathcal{S}_{/A}$. Then f is n -truncated if and only if the homotopy fibers of f over any vertex of A are n -truncated.*

Corollary 4.2.11. *Let $Y \rightarrow A$ be an n -truncated object. Then a map $f : X \rightarrow Y$ over A is an n -truncation if and only if for every vertex $a \in A_0$ the induced map on homotopy fibers over a is an n -truncation map.*

Proof. It follows from Lemma 4.2.8 that a map in $\mathcal{S}_{/A}$ whose codomain is n -truncated is an n -truncation map if and only if it is a weak equivalence in $(\mathcal{S}_{/A})_{n\text{-tr}}$. In particular, for any $X \in \mathcal{S}_{/A}$ the natural map $X \rightarrow P_n(X/A)$ is an n -truncation in $\mathcal{S}_{/A}$. The desired result now follows from Corollary 4.2.7. \square

Corollary 4.2.12. *Let $f : X \rightarrow A$ be a map of spaces. Then the object $f_* : P_n(X) \rightarrow P_n(A)$ is n -truncated as an object of $\mathcal{S}_{/P_n(A)}$ and the map $u : X \rightarrow P_n(X)$ is an n -truncation map in $\mathcal{S}_{P_n(A)}$.*

Proof. By the long exact sequence of homotopy groups we see that the homotopy fibers of $f_* : P_n(X) \rightarrow P_n(A)$ are n -truncated, and hence f is n -truncated in $\mathcal{S}_{/P_n(A)}$ by Corollary 4.2.10. To see that u is an n -truncation we need, by Corollary 4.2.11, to show that the induced map on homotopy fibers is an n -truncation for every vertex $a \in P_n(A_0)$. In other words, we need to show that the induced map on homotopy fibers induces an isomorphism on homotopy groups up to dimension n . This now follows from the five lemma using the fact that the homotopy groups of $P_n(X)$ and $P_n(A)$ vanish in dimension $\geq n+1$. \square

4.3 Truncation in the slice category of simplicial spaces

Let us now apply the above ideas to the slice model category of **simplicial spaces** with respect to the injective model structure. We first consider the following general lemma.

Lemma 4.3.1. *Let \mathcal{M}, \mathcal{N} be simplicial model categories and let*

$$\mathcal{M} \begin{array}{c} \xrightarrow{\mathcal{L}} \\ \xleftarrow{\mathcal{R}} \\ \xrightarrow{\mathcal{R}} \end{array} \mathcal{N}$$

be a simplicial Quillen adjunction such that the model structure on \mathcal{N} is transferred from that of \mathcal{M} . Assume in addition that every object in \mathcal{M} is cofibrant. Then an object $X \in \mathcal{N}$ is n -truncated if and only if $\mathcal{R}(X)$ is n -truncated in \mathcal{M} .

Proof. First it is clear that if X is n -truncated then $\mathcal{R}(X)$ is n -truncated. Let us now assume that $\mathcal{R}(X)$ is n -truncated. Since the model structure on \mathcal{N} is transferred from that of \mathcal{M} it follows in particular that \mathcal{R} preserves weak equivalences. We may hence assume without loss of generality that X is fibrant. Let $\mathcal{C}_X \subseteq \text{Cof}_{\mathcal{N}}$ be the class of cofibrations $f : Y \rightarrow Z$ in \mathcal{N} for which the fibers of $f^* : \text{Map}_{\mathcal{N}}(Z, X) \rightarrow \text{Map}_{\mathcal{N}}(Y, X)$ are all n -truncated (where $\text{Map}_{\mathcal{N}}(-, -)$ denotes the simplicial mapping space of \mathcal{N}). Remark 4.1.3 implies that \mathcal{C}_X is weakly saturated. Since $\mathcal{R}(X)$ is n -truncated it follows by adjunction that \mathcal{C}_X contains all cofibrations of the form $\mathcal{L}(f')$ for some cofibration $f' : X' \rightarrow Y'$ in \mathcal{M} . Since the model structure on \mathcal{N} is transferred from that of \mathcal{M} these cofibrations generate $\text{Cof}_{\mathcal{N}}$, and we may hence conclude that $\mathcal{C}_X = \text{Cof}_{\mathcal{N}}$. This means that if $Y \in \mathcal{N}$ is cofibrant then the unique map $\emptyset \rightarrow Y$ belongs to \mathcal{C}_X and hence the simplicial mapping space $\text{Map}_{\mathcal{N}}(Y, X)$ is n -truncated. It now follows that X is n -truncated, as desired. \square

Corollary 4.3.2. *An object $X_{\bullet} \rightarrow A_{\bullet}$ is n -truncated in $(\text{sS})_{/A}$ if and only if $X_k \rightarrow A_k$ is n -truncated in $\mathcal{S}_{/A_k}$ for every k .*

Proof. Let sS_{inj} and sS_{proj} denote the injective and projective model structures on sS , respectively. Then the identity adjunction $\text{sS}_{\text{proj}} \rightleftarrows \text{sS}_{\text{inj}}$ is a Quillen equivalence and hence the identity adjunction $(\text{sS}_{/A_{\bullet}})_{\text{proj}} \rightleftarrows (\text{sS}_{/A_{\bullet}})_{\text{inj}}$ is a Quillen equivalence as well. It follows that an object $X_{\bullet} \rightarrow A_{\bullet}$ is n -truncated in $\text{sS}_{/A_{\bullet}} = (\text{sS}_{/A_{\bullet}})_{\text{inj}}$ if and only if it is n -truncated in $(\text{sS}_{/A_{\bullet}})_{\text{proj}}$. Now consider the Quillen adjunction

$$\prod_{[k] \in \Delta} \mathcal{S} \begin{array}{c} \xrightarrow{\mathcal{L}} \\ \perp \\ \xleftarrow{\mathcal{R}} \end{array} \text{sS}_{\text{proj}}$$

where \mathcal{R} is the restriction functor and \mathcal{L} is given by left Kan extension. We then have an induced Quillen adjunction

$$\prod_{[k] \in \Delta} \mathcal{S}_{/A_k} \begin{array}{c} \xrightarrow{\mathcal{L}'} \\ \perp \\ \xleftarrow{\mathcal{R}'} \end{array} (\text{sS}_{/A_{\bullet}})_{\text{proj}}$$

between the corresponding slice model structures. By definition a map $f : X_{\bullet} \rightarrow Y_{\bullet}$ over A_{\bullet} is a weak equivalence (resp. fibration) in $(\text{sS}_{/A_{\bullet}})_{\text{proj}}$ if and only if $f_k : X_k \rightarrow Y_k$ is a weak equivalence (resp. fibration) in $\mathcal{S}_{/A_k}$ for every k . It follows that the model structure on $(\text{sS}_{/A_{\bullet}})_{\text{proj}}$ is transferred from that of $\prod_{[k] \in \Delta} \mathcal{S}_{/A_k}$, and hence by Lemma 4.3.1 an object in $X_{\bullet} \rightarrow A_{\bullet}$ in $(\text{sS}_{/A_{\bullet}})_{\text{proj}}$ is n -truncated if and only if $X_k \rightarrow A_k$ is n -truncated in $\mathcal{S}_{/A_k}$ for every k . \square

Given a map $X_{\bullet} \rightarrow A_{\bullet}$ of simplicial spaces, define

$$(\text{P}_n(X/A))_k = \text{P}_n(X_k/A_k).$$

As above one can verify that the coaugmented functor $Q(-)_{\bullet} = \text{P}_n(-/A)_{\bullet}$ satisfies Bousfield's axioms **A.4**, **A.5** and **A.6** (see [Bie, Definition 2.2]) and hence one can left-Bousfield localize the slice model structure on $\text{sS}_{/A_{\bullet}}$ so that

the new weak equivalences are the maps $f : X_\bullet \rightarrow Y_\bullet$ over A_\bullet such that $f_* : P_n(X/A)_\bullet \rightarrow P_n(Y/A)_\bullet$ is a weak equivalence in $sS_{/A_\bullet}$. Let us denote the resulting model category by $(sS_{/A_\bullet})_{n\text{-tr}}$. Note that for the terminal object $*_A \in sS_{/A_\bullet}$, given by the identity map $A_\bullet \rightarrow A_\bullet$, the coaugmentation map $*_A \xrightarrow{\cong} P_n(*_A/A)_\bullet$ is an isomorphism. It then follows from [Bie, Theorem 2.3] that a fibrant object $X_\bullet \rightarrow A_\bullet$ of $sS_{/A_\bullet}$ (i.e., an injective fibration) is fibrant in $(sS_{/A_\bullet})_{n\text{-tr}}$ if and only if the coaugmentation map $X_\bullet \rightarrow P_n(X/A)_\bullet$ is a weak equivalence in $sS_{/A_\bullet}$. Combining Lemma 4.2.8 and Corollary 4.3.2 we now obtain the following corollary:

Corollary 4.3.3. *The fibrant objects in $(sS_{/A_\bullet})_{n\text{-tr}}$ are exactly the fibrant objects in $sS_{/A_\bullet}$ which are also n -truncated. Furthermore, for every object $X_\bullet \rightarrow A_\bullet$ of $(sS_{/A_\bullet})_{n\text{-tr}}$ the object $P_n(X/A)_\bullet \rightarrow A_\bullet$ is n -truncated and the map $X_\bullet \rightarrow P_n(X/A)_\bullet$ is an n -truncation map in $sS_{/A_\bullet}$.*

4.4 Truncation of Segal groups

In this section we will identify the truncated objects and truncation maps in the model category $(sS_0)_{\text{seg}}$ of Segal groups. We begin with the following general lemma.

Lemma 4.4.1. *Let*

$$\mathcal{M} \begin{array}{c} \xrightarrow{\mathcal{L}} \\ \perp \\ \xleftarrow{\mathcal{R}} \end{array} \mathcal{N}$$

be a Quillen adjunction such that the derived counit map

$$\mathcal{L}((\mathcal{R}(X))^{\text{cof}}) \rightarrow X$$

is a weak equivalence for every fibrant $X \in \mathcal{N}$. Then

1. *An object $X \in \mathcal{N}$ is n -truncated if and only if $\mathcal{R}(X^{\text{fib}})$ is n -truncated in \mathcal{M} .*
2. *Let $f : X \rightarrow Y$ be a map in \mathcal{N} where Y is n -truncated. Assume that the induced map $\mathcal{R}(X^{\text{fib}}) \rightarrow \mathcal{R}(Y^{\text{fib}})$ is an n -truncation. Then f is an n -truncation.*

Proof. Let us begin with (1). First by adjunction it follows that the derived functor $\mathcal{R}((-)^{\text{fib}})$ sends n -truncated objects to n -truncated objects. On the other hand, under the assumptions of the lemma the derived functor $\mathcal{R}((-)^{\text{fib}})$ induces an equivalence on derived mapping spaces

$$\text{Map}_{\mathcal{N}}^h(Y, X) \xrightarrow{\cong} \text{Map}_{\mathcal{M}}^h(\mathcal{R}(Y^{\text{fib}}), \mathcal{R}(X^{\text{fib}}))$$

for every $Y, X \in \mathcal{N}$. It hence follows that if $\mathcal{R}(X^{\text{fib}})$ is n -truncated then X is n -truncated.

We now proceed to prove (2). Let $Z \in \mathcal{N}$ be an n -truncated object. By the above we get that $\mathcal{R}(Z^{\text{fib}})$ is n -truncated as well. Furthermore we know that

$$\text{Map}^h(Y, Z) \simeq \text{Map}^h(\mathcal{R}(Y^{\text{fib}}), \mathcal{R}(Z^{\text{fib}}))$$

and

$$\text{Map}^h(X, Z) \simeq \text{Map}^h(\mathcal{R}(X^{\text{fib}}), \mathcal{R}(Z^{\text{fib}}))$$

Since we assumed that the map $\mathcal{R}(X^{\text{fib}}) \rightarrow \mathcal{R}(Y^{\text{fib}})$ is an n -truncation it follows that $f : X \rightarrow Y$ is an n -truncation as well. \square

Corollary 4.4.2. *Let X_\bullet be a reduced simplicial space. Then*

1. X_\bullet is n -truncated in sS_0 if and only if $\iota(X_\bullet)$ is n -truncated in sS , i.e., if and only if X_k is an n -truncated space for every k .
2. The natural map $X_\bullet \rightarrow P_n(X_\bullet)$ is an n -truncation in sS_0 .

Proof. Apply Lemma 4.4.1 to the Quillen adjunction

$$\text{sS} \begin{array}{c} \xrightarrow{(-)^{\text{red}}} \\ \perp \\ \xleftarrow{\iota} \end{array} \text{sS}_0$$

and use Corollary 4.3.2 with $A = *$. \square

Corollary 4.4.3. *Let A_\bullet be a reduced simplicial space and let $A_\bullet \rightarrow A_\bullet^{\text{fib}}$ be a fibrant replacement with respect to the model structure of $(\text{sS}_0)_{\text{seg}}$. Then*

1. A_\bullet is n -truncated in $(\text{sS}_0)_{\text{seg}}$ if and only if A_\bullet^{fib} level-wise n -truncated.
2. The natural map $A_\bullet \rightarrow P_n((A_\bullet)^{\text{fib}})$ is an n -truncation in $(\text{sS}_0)_{\text{seg}}$.

Proof. Apply Lemma 4.4.1 to the Quillen adjunction

$$\text{sS}_0 \begin{array}{c} \xrightarrow{\text{Id}} \\ \perp \\ \xleftarrow{\text{Id}} \end{array} (\text{sS}_0)_{\text{seg}}$$

and use Corollary 4.4.2. \square

Remark 4.4.4. Since P_n preserves Cartesian products up to weak equivalence we see that for any Segal group A_\bullet the n -truncated object $P_n(A_\bullet)$ is a Segal group up to level-wise weak equivalence.

4.5 Truncation in integral model structures

In this subsection we will give a description of n -truncated objects and n -truncation maps in a general integral model structure $\int_{\mathcal{M}} \mathcal{F}$. Recall that derived mapping spaces in a model category depend only on the underlying ∞ -category (see §2.2). Since the underlying ∞ -category of the integral model structure coincides with the ∞ -categorical Grothendieck construction of the underlying ∞ -functor, it will be convenient to prove the following proposition using the machinery of ∞ -categories.

Let us fix a model category \mathcal{M} and a proper relative functor $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$. Given a map $f : A \rightarrow B$ in \mathcal{M} let us denote by

$$\text{Map}_{\mathcal{M}}^{\text{h}}(A, B)_f \subseteq \text{Map}_{\mathcal{M}}^{\text{h}}(A, B)$$

the connected component containing the morphism f . Given $X \in \mathcal{F}(A)$ and $Y \in \mathcal{F}(B)$ we will denote by

$$\text{Map}_{\int_{\mathcal{M}} \mathcal{F}}^{\text{h}}((X, A), (Y, B))_f \subseteq \text{Map}_{\int_{\mathcal{M}} \mathcal{F}}^{\text{h}}((X, A), (Y, B))$$

the union of connected components which are mapped to $\text{Map}_{\mathcal{M}}^{\text{h}}(A, B)_f$ by the functor $\pi : \int_{\mathcal{M}} \mathcal{F} \rightarrow \mathcal{M}$. The following proposition describes the behaviour of derived mapping spaces in the integral model structure with respect to the derived mapping spaces of the base and of the fibers. We found it convenient to prove this claim using ∞ -categories, since we could then rely on [Lur1, Proposition 2.4.4.3].

Proposition 4.5.1. *Let \mathcal{M} be a model category and $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$ a proper relative functor. Consider the integral model structure on $\int_{\mathcal{M}} \mathcal{F}$ (see Theorem 2.4.4). Let $(A, X), (B, Y) \in \int_{\mathcal{M}} \mathcal{F}$ and consider a map $f : A \rightarrow B$. If X is cofibrant in $\mathcal{F}(A)$ then the sequence*

$$\text{Map}_{\mathcal{F}(B)}^{\text{h}}(f!X, Y) \rightarrow \text{Map}_{\int_{\mathcal{M}} \mathcal{F}}^{\text{h}}((A, X), (B, Y))_f \rightarrow \text{Map}_{\mathcal{M}}^{\text{h}}(A, B)_f$$

is a homotopy fibration sequence (with respect to the base point $f \in \text{Map}_{\mathcal{M}}^{\text{h}}(A, B)_f$).

Proof. Let $\mathcal{F}_{\infty} : \mathcal{M}_{\infty} \rightarrow \text{Cat}_{\infty}$ be the functor induced by \mathcal{F} (see [HP, §3]) and let $\int_{\mathcal{M}_{\infty}} \mathcal{F}_{\infty} \rightarrow \mathcal{M}_{\infty}$ be the coCartesian fibration classifying \mathcal{F}_{∞} . The homotopy fiber sequence of ∞ -categories (with respect to the base point $B \in \mathcal{M}_{\infty}$)

$$\mathcal{F}_{\infty}(B) \rightarrow \int_{\mathcal{M}_{\infty}} \mathcal{F}_{\infty} \rightarrow \mathcal{M}_{\infty}$$

induces a homotopy fiber sequence of spaces (with respect to the base point $\text{Id}_B \in \text{Map}_{\mathcal{M}_{\infty}}(B, B)$)

$$\text{Map}_{\mathcal{F}_{\infty}(B)}(f!X, Y) \rightarrow \text{Map}_{\int_{\mathcal{M}_{\infty}} \mathcal{F}_{\infty}}((B, f!X), (B, Y)) \rightarrow \text{Map}_{\mathcal{M}_{\infty}}(B, B).$$

Applying the dual statement of [Lur1, Proposition 2.4.4.3] to the p -coCartesian morphism $(f, \text{Id}) : (A, X) \rightarrow (B, f_!X)$ we may deduce that the square

$$\begin{array}{ccc} \text{Map}_{f_{\mathcal{M}_\infty} \mathcal{F}_\infty}((B, f_!X), (B, Y)) & \longrightarrow & \text{Map}_{f_{\mathcal{M}_\infty} \mathcal{F}_\infty}((A, X), (B, Y)) \\ \downarrow p_* & & \downarrow \\ \text{Map}_{\mathcal{M}_\infty}(B, B) & \longrightarrow & \text{Map}_{\mathcal{M}_\infty}(A, B) \end{array} \quad (4.5.1)$$

is homotopy Cartesian. It follows that the sequence

$$\text{Map}_{\mathcal{F}_\infty(B)}(f_!X, Y) \rightarrow \text{Map}_{f_{\mathcal{M}_\infty} \mathcal{F}_\infty}((A, X), (B, Y)) \rightarrow \text{Map}_{\mathcal{M}_\infty}(A, B)$$

is a homotopy fiber sequence of spaces with respect to the base point $f \in \text{Map}_{\mathcal{M}_\infty}(A, B)$. According to [HP, Proposition 3.10], there exists an equivalence of ∞ -categories over \mathcal{M}_∞

$$\begin{array}{ccc} (\int_{\mathcal{M}} \mathcal{F})_\infty & \xrightarrow{\cong} & \int_{\mathcal{M}_\infty} \mathcal{F}_\infty \\ \pi \searrow & & \swarrow p \\ & \mathcal{M}_\infty & \end{array} \quad (4.5.2)$$

and so we conclude that the sequence

$$\text{Map}_{\mathcal{F}(B)}^h(f_!X, Y) \rightarrow \text{Map}_{f_{\mathcal{M}} \mathcal{F}}^h((A, X), (B, Y))_f \rightarrow \text{Map}_{\mathcal{M}}^h(A, B)_f,$$

is a homotopy fiber sequence, as desired. \square

We shall now describe the behaviour of n -truncation in the integral model structure.

Proposition 4.5.2. *Let \mathcal{M} be a model category and $\mathcal{F} : \mathcal{M} \rightarrow \text{ModCat}$ a proper relative functor. Consider the integral model structure on $\int_{\mathcal{M}} \mathcal{F}$. Then*

1. *An object $(A, X) \in \int_{\mathcal{M}} \mathcal{F}$ is n -truncated if and only if A is n -truncated in \mathcal{M} and X is n -truncated in $\mathcal{F}(A)$.*
2. *Let $(f, \varphi) : (A, X) \rightarrow (B, Y)$ be a map in $\int_{\mathcal{M}} \mathcal{F}$ such that (B, Y) is n -truncated and X, Y are cofibrant in $\mathcal{F}(A), \mathcal{F}(B)$ respectively. Assume that $f : A \rightarrow B$ is an n -truncation in \mathcal{M} and that $\varphi : f_!X \rightarrow Y$ is an n -truncation in $\mathcal{F}(B)$. Then (f, φ) is an n -truncation in $\int_{\mathcal{M}} \mathcal{F}$.*

Proof. To prove claim (1) assume that (A, X) is n -truncated. Since $\pi : \int_{\mathcal{M}} \mathcal{F} \rightarrow \mathcal{M}$ is in particular a right Quillen functor (see [HP, Corollary 5.8]), the image $A = \pi(A, X)$ is n -truncated in \mathcal{M} . To see that X is n -truncated in $\mathcal{F}(A)$ consider a cofibrant object $X' \in \mathcal{F}(A)$. According to Proposition 4.5.1 we have a homotopy fibration sequence

$$\text{Map}_{\mathcal{F}(A)}^h(X', X) \rightarrow \text{Map}_{f_{\mathcal{M}} \mathcal{F}}^h((A, X'), (A, X))_{\text{Id}_A} \rightarrow \text{Map}_{\mathcal{M}}^h(A, A)_{\text{Id}_A}$$

with respect to the base point $\text{Id} \in \text{Map}_{\mathcal{M}}^h(A, A)_{\text{Id}}$. Since (A, X) is n -truncated and A is n -truncated in \mathcal{M} we conclude that $\text{Map}_{\mathcal{F}(A)}^h(X', X)$ is an n -truncated space for every $X' \in \mathcal{F}(A)$. This means that X is n -truncated in $\mathcal{F}(A)$.

Now assume that A is n -truncated in \mathcal{M} and X is n -truncated in $\mathcal{F}(A)$. Let (A', X') be a cofibrant object in $\int_{\mathcal{M}} \mathcal{F}$. For each $f : A' \rightarrow A$, consider the homotopy fibration sequence

$$\text{Map}_{\mathcal{F}(A)}^h(f!X', X) \longrightarrow \text{Map}_{\int_{\mathcal{M}} \mathcal{F}}^h((A', X'), (A, X))_f \longrightarrow \text{Map}_{\mathcal{M}}^h(A', A)_f$$

given by Proposition 4.5.1. By our assumptions both $\text{Map}_{\mathcal{M}}^h(A', A)_f$ and $\text{Map}_{\mathcal{F}(A)}^h(f!X', X)$ are n -truncated and so $\text{Map}_{\int_{\mathcal{M}} \mathcal{F}}^h((A', X'), (A, X))_f$ is n -truncated as well. This shows that (A, X) is an n -truncated object of $\int_{\mathcal{M}} \mathcal{F}$. This proves claim (1).

Let us now prove claim (2). Let (C, Z) be an n -truncated fibrant object in $\int_{\mathcal{M}} \mathcal{F}$. According to (1) we see that C is n -truncated in \mathcal{M} and Z is n -truncated in $\mathcal{F}(C)$. We then obtain a homotopy commutative diagram of the form

$$\begin{array}{ccc} \text{Map}_{\mathcal{F}(B)}^h(Y, g^*Z) & \xrightarrow{\cong} & \text{Map}_{\mathcal{F}(C)}^h(g!Y, Z) \\ \cong \downarrow (-) \circ \varphi & & \downarrow (-) \circ (g! \varphi) \\ \text{Map}_{\mathcal{F}(B)}^h(f!X, g^*Z) & \xrightarrow{\cong} & \text{Map}_{\mathcal{F}(C)}^h(g!f!X, Z) \end{array}$$

where the left vertical map is an equivalence because φ is an n -truncation in $\mathcal{F}(B)$, and the horizontal maps are equivalences because $g! \dashv g^*$ is a Quillen adjunction, X, Y are cofibrant in $\mathcal{F}(A), \mathcal{F}(B)$ respectively and Z is fibrant in $\mathcal{F}(C)$. It then follows that the right vertical map $(-) \circ (g! \varphi)$ is an equivalence as well. Applying Proposition 4.5.1 we now obtain a map of homotopy fiber sequences

$$\begin{array}{ccccc} \text{Map}_{\mathcal{F}(C)}^h(g!Y, Z) & \longrightarrow & \text{Map}_{\int_{\mathcal{M}} \mathcal{F}}^h((B, Y), (C, Z))_g & \longrightarrow & \text{Map}_{\mathcal{M}}^h(B, C)_g \\ \cong \downarrow (-) \circ (g! \varphi) & & \downarrow (-) \circ (f, \varphi) & & \cong \downarrow (-) \circ f \\ \text{Map}_{\mathcal{F}(C)}^h(g!f!X, Z) & \longrightarrow & \text{Map}_{\int_{\mathcal{M}} \mathcal{F}}^h((A, X), (C, Z))_{g \circ f} & \longrightarrow & \text{Map}_{\mathcal{M}}^h(A, C)_{g \circ f} \end{array}$$

where the left vertical map is an equivalence by the above, and the right vertical map is an equivalence because f is an n -truncation. By Remark 4.2.9 the middle vertical map is an equivalence as well. Since this is true for every $g : B \rightarrow C$, the desired result follows. \square

4.6 Truncation of Segal group actions

Let $\text{Act} \stackrel{\text{def}}{=} \int_{A_\bullet \in (\text{sS}_0)_{\text{seg}}} (\mathcal{S}/\iota(A_\bullet))_{\text{act}}$ be the integral model category of Segal group actions (more precisely, the Segal group actions correspond to the fibrant objects of Act). In this section we will describe the n -truncation functors in Act . We begin with the following lemma:

Lemma 4.6.1. *Let A_\bullet be an object in $(\mathbb{S}\mathcal{S}_0)_{\text{seg}}$ and let $f : X_\bullet \rightarrow \iota(A_\bullet)$ be an object in $(\mathbb{S}\mathcal{S}_{/\iota(A_\bullet)})_{\text{act}}$. Then*

1. *f is n -truncated in $(\mathbb{S}\mathcal{S}_{/\iota(A_\bullet)})_{\text{act}}$ if and only if its fibrant replacement f^{fib} in $(\mathbb{S}\mathcal{S}_{/\iota(A_\bullet)})_{\text{act}}$ is n -truncated when considered as an object in the slice Reedy model structure $\mathbb{S}\mathcal{S}_{/\iota(A_\bullet)}$.*
2. *The natural map $X_\bullet \rightarrow P_n(f^{fib})$ is an n -truncation in $(\mathbb{S}\mathcal{S}_{/\iota(A_\bullet)})_{\text{act}}$.*

Proof. Apply Lemma 4.4.1 to the Quillen adjunction

$$\mathbb{S}\mathcal{S}_{/\iota(A_\bullet)} \begin{array}{c} \xrightarrow{\text{Id}} \\ \xleftarrow{\perp} \\ \xrightarrow{\text{Id}} \end{array} (\mathbb{S}\mathcal{S}_{/\iota(A_\bullet)})_{\text{act}}$$

□

Theorem 4.6.2. *Let A_\bullet be a Segal group and $p : X_\bullet \rightarrow A_\bullet$ a Segal group action. Then the map in Act determined by the diagram*

$$\begin{array}{ccc} X_\bullet & \longrightarrow & P_n(X_\bullet) \\ p \downarrow & & \downarrow p_* \\ A_\bullet & \xrightarrow{\tau_n} & P_n(A_\bullet) \end{array}$$

is an n -truncation.

Proof. According to Corollary 4.4.3, the map $\tau_n : A_\bullet \rightarrow P_n(A_\bullet)$ is an n -truncation in $(\mathbb{S}\mathcal{S}_0)_{\text{seg}}$. The object $(\tau_n)_!(X_\bullet) \in (\mathcal{S}_{/\iota(A_\bullet)})_{\text{act}}$ is simply given by the composed map

$$\tau_n \circ p : X_\bullet \rightarrow P_n(A_\bullet).$$

Let

$$q : Y_\bullet \rightarrow P_n(A_\bullet)$$

be a fibrant replacement of $\tau_n \circ p$ in $(\mathcal{S}_{/\iota(P_n(A_\bullet))})_{\text{act}}$, so that we have a commutative diagram

$$\begin{array}{ccccc} X_\bullet & \xlongequal{\quad} & X_\bullet & \longrightarrow & Y_\bullet \\ p \downarrow & & \downarrow & & \downarrow q \\ A_\bullet & \xrightarrow{\tau_n} & P_n(A_\bullet) & \xlongequal{\quad} & P_n(A_\bullet) \end{array}$$

Combining Proposition 4.5.2, Lemma 4.6.1 and Corollary 4.3.3 we see that the square

$$\begin{array}{ccc} X_\bullet & \longrightarrow & P_n(Y_\bullet/P_n(A_\bullet)) \\ p \downarrow & & \downarrow \\ A_\bullet & \longrightarrow & P_n(A_\bullet) \end{array} \quad (4.6.1)$$

determines an n -truncation map in Act.

By Corollary 4.2.12 the natural map $P_n(Y_\bullet/P_n(A_\bullet)) \rightarrow P_n(Y_\bullet)$ is a weak equivalence over $P_n(A_\bullet)$. Hence in order to prove the theorem we need to show that the map

$$P_n(X_\bullet) \rightarrow P_n(Y_\bullet)$$

is an equivalence of simplicial spaces (see Remark 3.2.12). In other words, we need to show that the maps

$$X_k \rightarrow Y_k$$

induce an isomorphism on homotopy groups up to dimension n . Since both $X_\bullet \rightarrow A_\bullet$ and $Y_\bullet \rightarrow P_n(A_\bullet)$ are Segal group actions and since the map $A_\bullet \rightarrow P_n(A_\bullet)$ is an isomorphism on homotopy groups up to dimension n we see that it will be enough to show that the map

$$X_0 \rightarrow Y_0$$

induces an isomorphism on homotopy groups up to dimension n .

Since the map $X_\bullet \rightarrow Y_\bullet$ is a weak equivalence in $sS_{/\iota(P_n(A_\bullet))}$ we see that the induced map

$$|X_\bullet| \rightarrow |Y_\bullet|$$

is a weak equivalence over $|P_n(A_\bullet)|$, and hence the homotopy fiber Y'_0 of $|X_\bullet| \rightarrow |P_n(A_\bullet)|$ is naturally equivalent to Y_0 . It will hence suffice to show that the natural map

$$X_0 \rightarrow Y'_0$$

induces an isomorphism on homotopy groups up to dimension n .

Let $W_n(A_\bullet) \rightarrow A_\bullet$ be the homotopy fiber of the map $A_\bullet \rightarrow P_n(A_\bullet)$ in $(sS_0)_{\text{seg}}$ (this coincides with the level-wise homotopy fiber of the corresponding map of reduced simplicial spaces, since both A_\bullet and $P_n(A_\bullet)$ are Segal groups up to Reedy fibrancy). Since the realization functor $|-| : (sS_0)_{\text{seg}} \rightarrow S_0$ is a left Quillen equivalence we see that the sequence

$$|W_n(A_\bullet)| \rightarrow |A_\bullet| \rightarrow |P_n(A_\bullet)|$$

is again a homotopy fibration sequence. Now consider the diagram

$$\begin{array}{ccccc} X_0 & \longrightarrow & Y'_0 & \longrightarrow & |W_n(A_\bullet)| \\ \downarrow & & \downarrow & & \downarrow \\ |X_\bullet| & \xlongequal{\quad} & |X_\bullet| & \longrightarrow & |A_\bullet| \\ \downarrow & & \downarrow & & \downarrow \\ |A_\bullet| & \longrightarrow & |P_n(A_\bullet)| & \xlongequal{\quad} & |P_n(A_\bullet)| \end{array}$$

where the right top horizontal map is the induced map on homotopy fibers. According to Lemma 4.6.3 below, the sequence

$$X_0 \rightarrow Y'_0 \rightarrow |W_n(A_\bullet)|$$

is a homotopy fibration sequence. Since $W_n(A_\bullet)$ is levelwise n -connected (i.e., has no homotopy groups in dimension $\leq n$), and is a Segal group up to Reedy fibrancy, we conclude that $|W_n(A_\bullet)|$ is $(n+1)$ -connected (see [Seg, Proposition 1.5(a)]). The desired result now follows from the long exact sequence in homotopy groups. \square

Lemma 4.6.3. *Consider a commutative diagram of pointed spaces of the form*

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \longrightarrow & A \\ \downarrow f & & \downarrow & & \downarrow g \\ A & \xrightarrow{g} & P & \xlongequal{\quad} & P \end{array}$$

Then the sequence

$$C \longrightarrow D \longrightarrow E$$

obtained by passing to homotopy fibers of the vertical maps is itself a homotopy fibration sequence.

Proof. We may assume without loss of generality that the maps $f : X \rightarrow A$ and $g : A \rightarrow P$ are Kan fibrations (which means that $h = g \circ f$ is a Kan fibration as well), so that the homotopy fibers C, D coincide with the actual fibers $C = f^{-1}(*), D = h^{-1}$ and $E = g^{-1}(*)$. It is then clear that the natural map

$$D \longrightarrow E \times_A X$$

is an isomorphism. Furthermore, since the map $X \rightarrow A$ is a fibration we see that the fiber product on the right hand side coincides with the associated homotopy fiber product $E \times_A^h X$. It is then clear that the resulting sequence

$$C \simeq * \times_A^h X \longrightarrow E \times_A^h X \longrightarrow E$$

is a homotopy fibration sequence. \square

Remark 4.6.4. Since P_n preserves Cartesian products up to weak equivalence we see that for any Segal group action $f : X_\bullet \rightarrow A_\bullet$ the n -truncated object $P_n(X_\bullet) \rightarrow P_n(A_\bullet)$ is a Segal group action up to levelwise weak equivalence (see Remark 4.4.4).

4.7 Convergence of the Postnikov tower

Given a simplicial group G we will denote by $G \curvearrowright X$ a space X equipped with a strict action of G . Similarly, given a Segal group A_\bullet we will denote a Segal group action of the form $X_\bullet \rightarrow A_\bullet$ by $A_1 \overset{h}{\curvearrowright} X_0$. The latter notation comes from viewing a Segal group action as coherent action of the loop space $A_1 \simeq \Omega|A_\bullet|$ on the space X_0 . Recall the Quillen equivalence

$$\Lambda^L : \int_{A_\bullet \in (\mathbb{S}S_0)_{\text{seg}}} (\mathbb{S}S/\nu(A_\bullet))_{\text{act}} \xrightleftharpoons[\perp]{} \int_{G \in \text{esGr}} \mathbb{S}^{\mathbb{B}G} : \Lambda^R$$

established in Corollary 3.3.4. For a G -space X we will denote its corresponding Segal group action by

$$B(G, X)_\bullet \longrightarrow B(G)_\bullet \stackrel{\text{def}}{=} \mathbb{R}\Lambda^R(G, X).$$

Now suppose X is a G -space and consider its Segal group action

$$B(G, X)_\bullet \longrightarrow B(G)_\bullet.$$

Using Theorem 4.6.2 and Remark 4.6.4 we obtain a tower of truncated Segal group actions

(4.7.1)

$$\begin{array}{ccc}
 & & \vdots \\
 & & P_n G^{\text{h}} \rightsquigarrow P_n X \\
 & & \downarrow \\
 & & \vdots \\
 & & P_1 G^{\text{h}} \rightsquigarrow P_1 X \\
 & & \downarrow \\
 G^{\text{h}} \rightsquigarrow X & \nearrow & P_0 G^{\text{h}} \rightsquigarrow P_0 X \\
 & \searrow & \\
 & &
 \end{array}$$

in which the maps are maps between Segal group actions.

Let us first observe that the Quillen equivalence of Corollary 3.3.4 induces a Quillen equivalence

$$\left(\int_{A_\bullet \in \text{ss}\mathcal{S}_0} \text{ss}\mathcal{S}_{/l(A_\bullet)} \right)^{\mathbb{N}^{\text{op}}} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} \left(\int_{G \in \text{ss}\mathcal{G}\text{r}} \mathcal{S}^{\mathbb{B}G} \right)^{\mathbb{N}^{\text{op}}}$$

where \mathbb{N} is the poset of natural numbers and where the model structure we use on the diagram categories is the Reedy model structure. Applying the (derived) left Quillen functor above on the tower 4.7.1, will now yield a tower of (ordinary) group actions

(4.7.2)

$$\begin{array}{c}
\vdots \\
\tau_n G \curvearrowright \tau_n X \\
\downarrow \\
\vdots \\
\tau_1 G \curvearrowright \tau_1 X \\
\downarrow \\
\tau_0 G \curvearrowright \tau_0 X \\
\uparrow \\
G \curvearrowright X
\end{array}$$

where $\tau_n G \simeq P_n G$ and $\tau_n X \simeq P_n X$. This means that the coherent tower 4.7.1 and the strict tower 4.7.2 contain the same homotopy-theoretical information.

Recall that the tower 4.7.1 can be built for any Segal group action $X_\bullet \rightarrow A_\bullet$ without reference to a strict group action $G \curvearrowright X$. We now claim that

Theorem 4.7.1. *For any Segal group action $X_\bullet \rightarrow A_\bullet$, viewed as an object in*

$$\int_{A_\bullet \in (\mathbb{S}\mathcal{S}_0)_{\text{seg}}} (\mathbb{S}\mathcal{S}/\iota(A_\bullet))_{\text{act}},$$

the tower of 4.7.1 converges in that

$$(X_\bullet \rightarrow A_\bullet) \simeq \text{holim}_n (P_n X_\bullet \rightarrow P_n A_\bullet).$$

Proof. Since each $P_n(A_\bullet)$ is a Segal group up to a level weak equivalence (see Remark 4.4.4), and since weak equivalences in $(\mathbb{S}\mathcal{S}_0)_{\text{seg}}$ between such objects are level weak equivalences, we can compute the homotopy limit of $\{P_n(A_\bullet)\}$ separately in each simplicial degree, yielding

$$\text{holim}_n P_n A_\bullet \simeq A_\bullet$$

in $(\mathbb{S}\mathcal{S}_0)_{\text{seg}}$. Thus, in order to compute $\text{holim}_n (P_n X_\bullet \rightarrow P_n A_\bullet)$ in

$$\int_{A_\bullet \in (\mathbb{S}\mathcal{S}_0)_{\text{seg}}} (\mathbb{S}\mathcal{S}/\iota(A_\bullet))_{\text{act}}$$

we first pull back each

$$P_n X_\bullet \rightarrow P_n A_\bullet$$

to the fiber $\mathbb{S}\mathcal{S}/\iota(A_\bullet)$ over A_\bullet and compute the homotopy limit there (see §2.3).

For a fixed n , the homotopy pullback

$$\begin{array}{ccc} L_{\bullet}^{(n)} & \longrightarrow & P_n X_{\bullet} \\ \downarrow & & \downarrow \\ A_{\bullet} & \longrightarrow & P_n A_{\bullet} \end{array}$$

is taken separately in each simplicial degree and by the axioms of a Segal group action we get

$$L_k^{(n)} \simeq P_n X_0 \times A_k.$$

We then obtain natural maps

$$X_{\bullet} \longrightarrow L_{\bullet}^{(n)}$$

(over A_{\bullet}) inducing a map $X_{\bullet} \rightarrow \operatorname{holim}_n L_{\bullet}^{(n)}$ of Segal group actions over A_{\bullet} . This map is an equivalence in each simplicial degree since $\operatorname{holim} P_n X_0 \simeq X_0$ and hence an equivalence of Segal group actions over A_{\bullet} . The result now follows. \square

By the Quillen equivalence of Corollary 3.3.4, we get

Corollary 4.7.2. *For any simplicial group G and any G -space X , the tower 4.7.2 converges in that $\operatorname{holim}_n (G_n \curvearrowright X_n) \simeq G \curvearrowright X$.*

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