

RATIONAL POINTS ON FIBRATIONS WITH FEW NON-SPLIT FIBRES

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ABSTRACT. We revisit the abstract framework underlying the fibration method for producing rational points on the total space of fibrations over the projective line. By fine-tuning its dependence on external arithmetic conjectures, we render the method unconditional when the degree of the non-split locus is ≤ 2 , as well as in various instances where it is 3. We are also able to obtain improved results in the regime that is conditionally accessible under Schinzel’s hypothesis, by incorporating into it, for the first time, a technique due to Harari for controlling the Brauer–Manin obstruction in families.

1. INTRODUCTION

In 1970, Manin [Man71] showed that an obstruction based on Brauer groups of schemes, now referred to as the Brauer–Manin obstruction, can often explain failures of the Hasse principle and weak approximation for the rational points of an algebraic variety X defined over a number field k . A conjecture of Colliot-Thélène predicts that the Brauer–Manin obstruction explains all such failures when X is smooth, proper and rationally connected—by which we mean that for any algebraically closed field extension K of k , two general K -points of X are joined by a rational curve defined over K (see [Kol96, Chapter IV]). In terms of the diagonal embedding of the set of rational points $X(k)$ in the space of adelic points $X(\mathbf{A}_k)$, this conjecture is stated as follows:

Conjecture 1.1 ([CT03]). *Let X be a smooth, proper and rationally connected variety over a number field k . The set $X(k)$ is dense in the Brauer–Manin set $X(\mathbf{A}_k)^{\text{Br}(X)}$.*

Though wide open, Conjecture 1.1 has been established for many special families of rationally connected varieties. The reader will find in [Wit18, §3] an almost up-to-date survey of known methods and results.

A common structure that can often be fruitfully exploited to study rational points on X is that of a fibration $f : X \rightarrow \mathbf{P}_k^1$. (We use the term “fibration” in a loose sense, to refer to a morphism whose generic fibre is geometrically irreducible.) By the Graber–Harris–Starr theorem [GHS03], if the generic fibre of f is rationally connected, then X is rationally connected as well. In the context of Conjecture 1.1, this naturally leads one to ask:

Question 1.2. Let X be a smooth, proper, irreducible variety over a number field k and $f : X \rightarrow \mathbf{P}_k^1$ be a dominant morphism whose geometric generic fibre is rationally connected.

Assume that $X_c(k)$ is dense in $X_c(\mathbf{A}_k)^{\mathrm{Br}(X_c)}$ for all but finitely many $c \in \mathbf{P}^1(k)$, where $X_c = f^{-1}(c)$. Does it follow that $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$?

Question 1.2 has been extensively studied. One approach consists in applying the theory of descent developed by Colliot-Thélène and Sansuc [CTS87a] to reduce the problem to certain torsors associated with the vertical Brauer group of X relative to \mathbf{P}_k^1 . This approach, systematically formalised by Skorobogatov [Sko96] and by Colliot-Thélène and Skorobogatov [CTS00], has been applied successfully in many special cases, including Châtelet surfaces [CTSSD87], other types of conic and quadric bundles [BMS14], and various normic (or more generally toric) bundles [HBS02, CTHS03, DSW15, Sko15, BM17]. See also [SD99, BHB12] for instances of implicit uses of this approach.

Another approach is the fibration method, the first instance of which was Hasse’s proof of the local-global principle for the isotropy of quadratic forms over number fields. Here, the argument is less concerned with the particular geometry of X , but on the other hand it is highly sensitive to the *rank* of f , defined as the degree of the finite locus in \mathbf{P}_k^1 consisting of the closed points m such that the fibre X_m is not split over $k(m)$, and to the possible finite extensions $L/k(m)$ that split X_m , i.e. that are such that the L -variety $X_m \otimes_{k(m)} L$ is split. (We recall that a variety is said to be *split* if it contains a geometrically integral open subset, see [Sko96, Definition 0.1].)

When the rank of f is ≤ 1 , a version of the fibration method yielding a positive answer to Question 1.2 was established by Skorobogatov [Sko90] under the additional assumption that the smooth fibres of f satisfy the Hasse principle and weak approximation. Under the same assumption on the smooth fibres, this result was extended by Colliot-Thélène and Skorobogatov [Sko96, CTS00] to the case where the rank of f is 2, and to the case where the rank is 3 and every non-split fibre X_m is split by a quadratic extension of $k(m)$. When the rank of f is ≤ 1 , the assumption that the fibres satisfy the Hasse principle and weak approximation was removed in Harari’s thesis [Har97], using a delicate argument to control the Brauer–Manin obstruction in the smooth fibres.

In situations more general than the above, one can still make the fibration method work conditionally if one assumes Schinzel’s hypothesis. This was first observed by Colliot-Thélène and Sansuc [CTS82] and later extended in [Ser92, SD94, CTSD94, CTSSD98], culminating in [CTSSD98, Theorem 1.1(e)] in a positive answer to Question 1.2 under the following assumptions:

- (1) Schinzel’s hypothesis holds.
- (2) Each non-split fibre X_m is split by an abelian extension of $k(m)$.
- (3) The smooth fibres of f satisfy the Hasse principle and weak approximation.

We recall that Schinzel’s hypothesis is a vast generalization of the twin prime conjecture, which says that a finite collection of irreducible polynomials in $\mathbf{Z}[t]$ infinitely often take prime values simultaneously, unless there is a local obstruction to this being so. Schinzel’s hypothesis is required in *loc. cit.*, when $k = \mathbf{Q}$, for the polynomials that encode the non-split fibres of f over the points of $\mathbf{A}_k^1 \subset \mathbf{P}_k^1$. (When $k \neq \mathbf{Q}$, see [CTSD94, Proposition 4.1].) The only known case of Schinzel’s hypothesis is the case of a single polynomial of degree 1 (i.e. Dirichlet’s theorem on primes in arithmetic progression), which makes the last theorem

unconditional when the locus of non-split fibres consists in either one or two rational points of \mathbf{P}_k^1 . However, as we described above, these small rank cases are also known under less restrictive conditions.

In a more recent development, the first- and third-named authors, in collaboration with Skorobogatov [HSW14], showed that the fibration method can also be set up using a variant of Schinzel’s hypothesis applied to homogeneous polynomials in two variables (still keeping conditions (2) and (3) above). This homogeneous variant, referred to as Schinzel’s hypothesis (HH_1), has the advantage of being known in a slightly wider set of circumstances. Most notably, by the seminal work of Green–Tao–Ziegler [GT10, GT12, GTZ12], it is known, over \mathbf{Q} , when all the homogeneous polynomials are linear (see [HSW14, Proposition 1.2]). Another known case, due to Heath-Brown and Moroz [HBM04], is that of a single homogeneous polynomial of degree 3, over \mathbf{Q} .

The use of the homogeneous variant of Schinzel’s hypothesis has led to more unconditional answers to Question 1.2 (see [HSW14, Theorem 3.3]); however, these were still subject to conditions (2) and especially (3) above. Later on, in [HW16, §9], the first- and third-named authors suggested a way of bypassing condition (2) by replacing Schinzel’s hypothesis (HH_1) with another hypothesis, namely [HW16, Conjecture 9.1], and showed that the latter implies a positive answer to Question 1.2 in complete generality, even in the absence of condition (3). This yields an unconditional positive answer to Question 1.2 when $k = \mathbf{Q}$ and all the non-split fibres of f lie above rational points of $\mathbf{P}_{\mathbf{Q}}^1$, as the corresponding special case of [HW16, Conjecture 9.1] was established by Matthiesen [Mat18], building on the work of Browning and Matthiesen [BM17].

Conjecture 9.1 of [HW16] mentioned above depends, among others, on a collection of closed points m_1, \dots, m_n of \mathbf{P}_k^1 and a collection of field extensions $L_1/k(m_1), \dots, L_n/k(m_n)$. Unfortunately, given a fibration $f : X \rightarrow \mathbf{P}_k^1$ with rationally connected geometric generic fibre, the precise choice of m_1, \dots, m_n and L_1, \dots, L_n needed to obtain a positive answer to Question 1.2 using the results of [HW16] is not completely straightforward: while one has to include in this list at least the closed points over which the fibre of f is non-split, additional points might be required in order to make it possible to describe the Brauer groups of the smooth fibres uniformly in terms of Brauer classes defined on the complement, in X , of the union of the fibres X_{m_1}, \dots, X_{m_n} . Similarly, each L_i needs to be not only sufficiently large as to split the fibre X_{m_i} , it must also split the “constant” part of the residues of these Brauer classes. All in all, this state of affairs is inconvenient since for a given fibration $f : X \rightarrow \mathbf{P}_k^1$, the precise choice of m_1, \dots, m_n and L_1, \dots, L_n to which Conjecture 9.1 needs to be applied is implicit and hard to determine in practice. A more serious consequence is that the applicability of [HW16, §9] finds itself hindered in the regime where unconditional cases of Question 1.2 are potentially within reach. For example:

- (i) Conjecture 9.1 of [HW16] is known in various instances where $r = \sum_{i=1}^n \deg(m_i)$ is small, e.g. when $r \leq 2$, or when $r \leq 3$ and each $L_i/k(m_i)$ is quadratic (see [HW16, Theorem 9.11] and Remark 6.5 below). However, due to the possible need for additional closed points, one cannot deduce from this an unconditional answer to Question 1.2 for fibrations of rank 2, or for fibrations of rank 3 whose non-split fibres

are split by quadratic extensions. One can still deduce a positive answer under the additional assumption that the smooth fibres of f satisfy the Hasse principle and weak approximation, thus (only) recovering [CTS00, Theorem A and Theorem B].

- (ii) When all the field extensions $L_i/k(m_i)$ are abelian (or, more generally, almost abelian in the sense of [HW16, Definition 9.4]), Conjecture 9.1 of [HW16] is implied by Schinzel’s hypothesis (HH₁) for the defining homogeneous polynomials of the points m_1, \dots, m_n . However, due to the possible need to increase the fields L_i , one cannot deduce that Schinzel’s hypothesis (HH₁) implies a positive answer to Question 1.2 when the fibres of f are split by, say, abelian extensions. One can still deduce a positive answer under the additional assumption that the smooth fibres of f satisfy the Hasse principle and weak approximation, a result already implicit in [HSW14]. By Heath-Brown and Moroz [HBM04], this yields an unconditional positive answer to Question 1.2, when $k = \mathbf{Q}$, in the case a single non-split fibre over a point of degree 3, but only when conditions (2) and (3) above both hold.

Our goal in this article is to address these issues by putting forth an improved form of [HW16, Conjecture 9.1]. To this end, we rely on a construction of auxiliary varieties that was introduced and exploited in [HW16] in connection with Conjecture 9.1 of *op. cit.* This construction, antecedents of which had previously come up in the context of descent theory (see [Sko96, §3.3], [CTS00, p. 391], [Sko01, §4.4]), involves associating with a choice of points m_1, \dots, m_n and of finite extensions $L_1/k(m_1), \dots, L_n/k(m_n)$ a certain family of non-proper varieties, typically denoted W , equipped with a morphism $W \rightarrow \mathbf{P}_k^1$ whose singular fibres lie over the closed points m_1, \dots, m_n and are respectively split by the field extensions $L_1/k(m_1), \dots, L_n/k(m_n)$. It is shown in [HW16, Proposition 9.9] that Conjecture 9.1 of *op. cit.* holds for the parameters m_1, \dots, m_n and $L_1/k(m_1), \dots, L_n/k(m_n)$ if and only if for every variety W in the family associated with these parameters, the subset $\bigcup_{c \in \mathbf{P}^1(k)} W_c(\mathbf{A}_k)$ of $W(\mathbf{A}_k)$ is a dense subset. In the present article, we study a strengthened version of this last property, which, on the one hand, becomes equivalent to the original version when stated for all possible choices of m_1, \dots, m_n and $L_1/k(m_1), \dots, L_n/k(m_n)$, and, on the other hand, is sufficiently strong to allow the incorporation of non-vertical Brauer classes into the fibration method. This leads to an equivalent, alternative approach to the framework of [HW16, §9] that has the practical advantage of unlocking many of the remaining unconditional cases which escaped *op. cit.* In particular, we obtain the following new cases as applications:

Theorem 1.3 (see Theorem 7.3 and Theorem 7.4). *Question 1.2 has a positive answer in each of the following cases:*

- (i) *the rank of f is at most 2;*
- (ii) *the rank of f is 3 and every fibre X_m is split by a quadratic extension of $k(m)$;*
- (iii) *the rank of f is 3, one fibre X_m lies above a rational point of \mathbf{P}_k^1 and every remaining fibre X_m is split by a quadratic extension of $k(m)$.*

In addition, using this improved framework, we are able to combine for the first time the arguments of Harari [Har97] for dealing with Brauer–Manin obstructions in the fibres with

the use of Schinzel’s hypothesis in the fibration method—a problem that had been open since the 1990’s. This leads to a version of the theorem of Colliot-Thélène, Skorobogatov and Swinnerton-Dyer [CTSSD98, Theorem 1.1(e)] in which the smooth fibres are not assumed any more to satisfy the Hasse principle or weak approximation (though at the expense of assuming that the splitting fields are cyclic, or, alternatively, almost abelian but not abelian):

Theorem 1.4 (see Theorem 7.2). *Question 1.2 has a positive answer when the following two conditions are both satisfied:*

- (1) *Schinzel’s hypothesis (HH₁) holds for the irreducible homogeneous two-variable polynomials vanishing on the closed points $m \in \mathbf{P}_k^1$ such that the fibre X_m is non-split;*
- (2) *every fibre X_m is split by an extension of $k(m)$ that is either cyclic or almost abelian but non-abelian (e.g. a cubic extension).*

When combined with the work of Heath-Brown and Moroz [HBM04], Theorem 1.4 yields an unconditional positive answer to Question 1.2, when $k = \mathbf{Q}$, in the case a single non-split fibre over a point of degree 3, subject to condition (2) of Theorem 1.4 but without assuming anything on the smooth fibres of f beyond their rational connectedness (see Theorem 7.4).

The article is organised as follows. We begin in §2 with a discussion of the unramified Brauer groups of norm 1 tori and their torsors, over arbitrary fields of characteristic 0. In §3, we discuss the variety W which plays a key rôle in this work. After verifying that the Brauer group of W is reduced to constant classes (Proposition 3.5), we discuss some canonical ramified Brauer classes on W (see §3.4) and use them to formulate a conjecture on the arithmetic of W , which we call Conjecture F_+ (see §3.6). It is a strengthened version of the property that the subset $\bigcup_{c \in \mathbf{P}^1(k)} W_c(\mathbf{A}_k)$ is dense in $W(\mathbf{A}_k)$, which we dub Conjecture F . In §4, we establish the main technical result of the article (Theorem 4.1), according to which Conjecture F_+ implies a positive answer to Question 1.2. The argument is similar to the proof of [HW16, Theorem 9.17], with the difference that the variety W for which one needs to assume Conjecture F_+ is under better control. In §5, we analyse the relationship between Conjectures F and F_+ in greater detail. The comparison goes through an auxiliary intermediate statement, which we call Conjecture F_{const} . The results of this section, and in particular those of §5.3, are the main input needed for Theorem 1.4. In §6, we collect all of the cases of Conjecture F_+ that we are able to prove, either by strong approximation arguments, from Schinzel’s hypothesis or from additive combinatorics. Finally, in §7, we deduce Theorem 1.3 and Theorem 1.4 and discuss some examples.

Acknowledgements.

The second-named author is supported by National Natural Science Foundation of China (Grant Nos. 11622111 and 11631009).

Notation and conventions. Let X be a variety, that is, a scheme of finite type, over a field k . We say that X is *split* if it contains a geometrically integral open subset. We say that X is *rationally connected* if for any algebraically closed field extension K of k , the variety $X \otimes_k K$ over K is rationally connected in the sense of Campana, Kollár,

Miyaoka and Mori. The Brauer group $\mathrm{Br}(X) = H_{\mathrm{ét}}^2(X, \mathbf{G}_m)$ contains the *algebraic Brauer group* $\mathrm{Br}_1(X) = \mathrm{Ker}(\mathrm{Br}(X) \rightarrow \mathrm{Br}(X \otimes_k K))$, where K denotes any algebraically closed field extension of k (the kernel does not depend on the choice of K), and the subgroup $\mathrm{Br}_0(X) = \mathrm{Im}(\mathrm{Br}(k) \rightarrow \mathrm{Br}(X))$ of constant classes. When X is smooth and k has characteristic 0, we denote by $\mathrm{Br}_{\mathrm{nr}}(X) \subseteq \mathrm{Br}(X)$ the *unramified Brauer group*, which coincides with the Brauer group of any smooth and proper variety birationally equivalent to X . If k is a number field and v is a place of k , we denote by k_v the completion of k at v and, if v is finite, by \mathbf{F}_v the residue field of v . Given a nonzero k -algebra L and a variety Y over L , we denote by $R_{L/k}Y$ the Weil restriction of Y from L down to k . If M is an abelian group and L is an étale k -algebra, say $L = \prod L_i$ where the L_i are fields, we write $H^n(L, M)$ for the group $\prod H^n(L_i, M)$, where $H^n(L_i, M)$ denotes the n -th Galois cohomology group of L_i with values in M . If G is a group and M is a G -module, we denote by $C^n(G, M)$ the group of n -cochains of G with values in M and by $Z^n(G, M)$ the subgroup of n -cocycles.

2. UNRAMIFIED BRAUER GROUPS OF TORSORS UNDER NORM TORI

We discuss three complementary aspects of unramified Brauer groups of norm tori and of their torsors over arbitrary fields of characteristic 0, in three logically independent sections §2.1, §2.2, §2.3. The first one provides a criterion for classes of the shape $\mathrm{Cores}_{L/k}(z, \chi)$ to be unramified (Proposition 2.1). This will play a key rôle in the proof of Theorem 1.4. The second one is devoted to norm tori T such that $\mathrm{Br}_{\mathrm{nr}}(T) = \mathrm{Br}_0(T)$, a condition which also turns up in the proof of Theorem 1.4 (see Corollary 5.8). The third one reinterprets the computation of the quotient group $\mathrm{Br}_{\mathrm{nr}}(T)/\mathrm{Br}_0(T)$ for norm tori T in terms of central extensions of a finite group G with \mathbf{Q}/\mathbf{Z} (Proposition 2.10). This last result is not used in the rest of the article but it may be of interest in a broader perspective. We illustrate it by analysing an example of Kunyavskii of a class of three-dimensional tori satisfying $\mathrm{Br}_{\mathrm{nr}}(T) \neq \mathrm{Br}_0(T)$ (Example 2.11) and by giving an independent proof of Bartels' theorem according to which $\mathrm{Br}_{\mathrm{nr}}(T) = \mathrm{Br}_0(T)$ if T is the norm torus associated with a degree n extension whose Galois closure has Galois group D_n (Example 2.12).

2.1. Unramifiedness of corestrictions. Let k be a field of characteristic 0 with algebraic closure \bar{k} . Let T be an algebraic torus over k . Let Z be a torsor, over k , under T .

The problem of calculating the unramified Brauer group of Z is classically addressed by noting, on the one hand, that the group $\mathrm{Br}_{\mathrm{nr}}(Z \otimes_k \bar{k})$ vanishes, so that $\mathrm{Br}_{\mathrm{nr}}(Z) \subseteq \mathrm{Br}_1(Z)$, and on the other hand, that the group $\mathrm{Pic}(Z \otimes_k \bar{k})$ vanishes, so that the Hochschild–Serre spectral sequence determines an isomorphism $\mathrm{Br}_1(Z) = H^2(k, \bar{k}[Z]^*)$ and hence an injection

$$(2.1) \quad \mathrm{Br}_1(Z)/\mathrm{Br}_0(Z) \hookrightarrow H^2(k, \bar{k}[Z]/\bar{k}) = H^2(k, \widehat{T}),$$

which is an isomorphism when either $Z(k) \neq \emptyset$ or $H^3(k, \bar{k}^*) = 0$, e.g. when k is a number field. Let us denote by $\mathrm{III}_{\omega}^2(k, \widehat{T})$ the subgroup of $H^2(k, \widehat{T})$ consisting of those classes which vanish in $H^2(k', \widehat{T})$ for any field extension k'/k such that the absolute Galois group of k' is procyclic. It is known that an element of $\mathrm{Br}_1(Z)$ belongs to $\mathrm{Br}_{\mathrm{nr}}(Z)$ if and only if its

image in $H^2(k, \widehat{T})$ belongs to $\text{III}_\omega^2(k, \widehat{T})$ (see [BDH13, Théorème 8.1]). This results in an injection

$$(2.2) \quad \text{Br}_{\text{nr}}(Z)/\text{Br}_0(Z) \hookrightarrow \text{III}_\omega^2(k, \widehat{T}),$$

which is an isomorphism when $Z(k) \neq \emptyset$ or $H^3(k, \bar{k}^*) = 0$, thus turning, in these cases, the determination of $\text{Br}_{\text{nr}}(Z)/\text{Br}_0(Z)$ into a finite and computable endeavour: indeed, the action of $\text{Gal}(\bar{k}/k)$ on \widehat{T} factors through a finite quotient, say G , and we can identify

$$(2.3) \quad \text{III}_\omega^2(k, \widehat{T}) \xleftarrow{\simeq} \text{III}_{\text{cyc}}^2(G, \widehat{T}),$$

where $\text{III}_{\text{cyc}}^2(G, \widehat{T})$ denotes the subgroup of $H^2(G, \widehat{T})$ consisting of those elements whose image in $H^2(H, \widehat{T})$ vanishes for every cyclic subgroup H of G .

Our goal in this section is to extract from the above an explicit description of certain classes in $\text{Br}_{\text{nr}}(Z)$ in the case where T is the norm 1 torus associated with a given nonzero étale algebra L over k and Z is the corresponding norm c torsor for some $c \in k^*$.

We fix L and c until the end of §2. The torsor Z (resp. the torus T) can be explicitly described as the closed subvariety of $R_{L/k}(\mathbf{A}_L^1)$ defined by the equation $N_{L/k}(z) = c$ (resp. by the equation $N_{L/k}(z) = 1$), where z denotes a point of $R_{L/k}(\mathbf{A}_L^1)$. We shall again denote by z the invertible function on $Z \otimes_k L$ obtained by restricting the tautological regular function on $R_{L/k}(\mathbf{A}_L^1) \otimes_k L$. For $\chi \in H^1(L, \mathbf{Q}/\mathbf{Z})$, let us write $(z, \chi) \in \text{Br}(Z \otimes_k L)$ for the class obtained by identifying $H^1(L, \mathbf{Q}/\mathbf{Z})$ with $H^2(L, \mathbf{Z})$ and considering the image of the pair (z, χ) by the cup product map

$$H^0(Z \otimes_k L, \mathbf{G}_m) \times H^2(L, \mathbf{Z}) \rightarrow \text{Br}(Z \otimes_k L).$$

We note that $(z, \chi) \in \text{Ker}(\text{Br}(Z \otimes_k L) \rightarrow \text{Br}(Z \otimes_k L \otimes_k \bar{k}))$ since χ vanishes when pulled back to $L \otimes_k \bar{k}$. Hence its push-forward $\text{Cores}_{L/k}(z, \chi) \in \text{Br}(Z)$ along the projection $Z \otimes_k L \rightarrow Z$ belongs to the subgroup $\text{Br}_1(Z) \subseteq \text{Br}(Z)$.

The following proposition gives criteria for this class $\text{Cores}_{L/k}(z, \chi) \in \text{Br}_1(Z)$ to be constant or unramified. It builds on the ideas contained in [Wei14c, Theorem 5], in which L is assumed to be a Galois extension of k .

Proposition 2.1. *Let $\chi \in H^1(L, \mathbf{Q}/\mathbf{Z})$.*

- (1) $\text{Cores}_{L/k}(z, \chi) \in \text{Br}_0(Z)$ if and only if $\chi \in \text{Im}(H^1(k, \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(L, \mathbf{Q}/\mathbf{Z}))$.
- (2) $\text{Cores}_{L/k}(z, \chi) \in \text{Br}_{\text{nr}}(Z)$ if and only if for any field extension k'/k such that the absolute Galois group of k' is procyclic and for any two k -algebra homomorphisms $\iota_1, \iota_2 : L \rightarrow k'$, the two images of χ in $H^1(k', \mathbf{Q}/\mathbf{Z})$ via ι_1 and via ι_2 are equal.

Proof. Letting \bar{k} denote a fixed algebraic closure of k and Σ the set of k -algebra homomorphisms $L \rightarrow \bar{k}$, the character group \widehat{T} of the torus T sits in the commutative diagram of

Galois modules

$$(2.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}[\Sigma] & \longrightarrow & \widehat{T} \longrightarrow 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \wr \\ 0 & \longrightarrow & \bar{k}^* & \longrightarrow & \bar{k}[Z]^* & \longrightarrow & \bar{k}[Z]^*/\bar{k}^* \longrightarrow 0 \end{array}$$

whose rows are exact, whose leftmost vertical arrow sends 1 to c , and whose middle vertical arrow sends $\sigma \in \Sigma$ to the image of z by the morphism $L[Z]^* \rightarrow \bar{k}[Z]^*$ induced by σ . The rightmost vertical arrow is an isomorphism. Passing to cohomology and taking into account the canonical isomorphism $\mathrm{Br}_1(Z) = H^2(k, \bar{k}[Z]^*)$ given by the Hochschild–Serre spectral sequence, we obtain the commutative diagram with exact rows

$$(2.5) \quad \begin{array}{ccccc} H^1(k, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^1(L, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\beta} & H^2(k, \widehat{T}) \\ \downarrow & & \downarrow \alpha' & & \downarrow \wr \\ \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}_1(Z) & \longrightarrow & H^2(k, \bar{k}[Z]^*/\bar{k}^*). \end{array}$$

Lemma 2.2. *The map α' sends $\chi \in H^1(L, \mathbf{Q}/\mathbf{Z})$ to $\mathrm{Cores}_{L/k}(z, \chi) \in \mathrm{Br}_1(Z)$.*

Proof. We denote by $f : Z \rightarrow \mathrm{Spec}(k)$ and $f' : Z \otimes_k L \rightarrow \mathrm{Spec}(L)$ the structure morphisms and by $\nu : Z \otimes_k L \rightarrow Z$ the natural projection. Let $\gamma : \mathbf{Z} \rightarrow \mathbf{G}_m$ be the morphism of étale sheaves on $Z \otimes_k L$ defined by $\gamma(1) = z \in H^0(Z \otimes_k L, \mathbf{G}_m)$. Applying the functors $H^2(Z, -)$ and $H^2(k, f_* -)$, related by a natural transformation $H^2(k, f_* -) \rightarrow H^2(Z, -)$, to the morphisms of étale sheaves on Z

$$(2.6) \quad \nu_* \mathbf{Z} \xrightarrow{\nu_* \gamma} \nu_* \mathbf{G}_m \xrightarrow{N_{L/k}} \mathbf{G}_m,$$

we obtain a commutative diagram

$$(2.7) \quad \begin{array}{ccccc} H^2(k, f_* \nu_* \mathbf{Z}) & \longrightarrow & H^2(k, f_* \nu_* \mathbf{G}_m) & \longrightarrow & H^2(k, f_* \mathbf{G}_m) \\ \parallel & & & & \parallel \\ H^1(L, \mathbf{Q}/\mathbf{Z}) = H^2(L, \mathbf{Z}) & \longrightarrow & & \longrightarrow & H^2(k, \bar{k}[Z]^*) \\ f'^* \downarrow & & \xi \mapsto z \sim \xi & & \downarrow \\ H^2(Z \otimes_k L, \mathbf{Z}) & \longrightarrow & \mathrm{Br}(Z \otimes_k L) & \xrightarrow{\mathrm{Cores}_{L/k}} & \mathrm{Br}(Z), \end{array}$$

The map $(f_* \nu_* \mathbf{Z})(\bar{k}) \rightarrow (f_* \mathbf{G}_m)(\bar{k})$ obtained from the composition of the morphisms in (2.6) coincides, via the canonical isomorphism $(f_* \nu_* \mathbf{Z})(\bar{k}) = \mathbf{Z}[\Sigma]$, with the map α appearing in (2.4). Thus, the composition of the middle horizontal maps of (2.7) with the lower right-hand side vertical map of (2.7) coincides with α' . The lemma then follows from the commutativity of this diagram. \square

Lemma 2.2 and a diagram chase in (2.5) together imply Proposition 2.1 (1).

Let us now prove Proposition 2.1 (2). As discussed above, an element of $\mathrm{Br}_1(Z)$ belongs to the subgroup $\mathrm{Br}_{\mathrm{nr}}(Z)$ if and only if its image in $H^2(k, \widehat{T})$ by the map $\mathrm{Br}_1(Z) \rightarrow H^2(k, \widehat{T})$ extracted from the diagram (2.5) belongs to $\mathrm{III}_\omega^2(k, \widehat{T})$, see [BDH13, Théorème 8.1]. In view

of (2.5) and of Lemma 2.2, the proof will therefore be complete once we check that the following two conditions are equivalent:

- (a) for any field extension k'/k such that the absolute Galois group of k' is procyclic, the images of χ in $H^1(k', \mathbf{Q}/\mathbf{Z})$ via any two k -algebra morphisms $L \rightarrow k'$ are equal;
- (b) for any field extension k'/k such that the absolute Galois group of k' is procyclic, the image of χ in $H^1(L \otimes_k k', \mathbf{Q}/\mathbf{Z})$ comes from $H^1(k', \mathbf{Q}/\mathbf{Z})$.

Obviously (b) \Rightarrow (a). For the converse, we need the following lemma.

Lemma 2.3. *Let $N \geq 1$. Let n_1, \dots, n_N be nonzero integers and let $(\lambda_i)_{1 \leq i \leq N} \in (\mathbf{Q}/\mathbf{Z})^N$ satisfy the equality*

$$(2.8) \quad \frac{n}{n_i} \lambda_i = \frac{n}{n_j} \lambda_j$$

for all $i, j \in \{1, \dots, N\}$ and for all integers n divisible by both n_i and n_j . Then there exists $\lambda \in \mathbf{Q}/\mathbf{Z}$ such that $\lambda_i = n_i \lambda$ for all $i \in \{1, \dots, N\}$.

Proof. By induction on N , we may assume that $N = 2$. Then, as \mathbf{Q}/\mathbf{Z} is divisible, we may assume that n_1 and n_2 are coprime. Write $1 = a_1 n_1 + a_2 n_2$ for $a_1, a_2 \in \mathbf{Z}$. Applying the hypothesis to $n = n_1 n_2$ shows that $\lambda = a_1 \lambda_1 + a_2 \lambda_2$ satisfies the desired property. \square

Going back to the implication (a) \Rightarrow (b), let us write $L \otimes_k k' = E_1 \times \dots \times E_N$ where E_i is a finite extension of k' of degree n_i and let us choose an algebraic closure \bar{k}' of k' , a topological generator $\tau \in \text{Gal}(\bar{k}'/k')$ and, for each $i \in \{1, \dots, N\}$, a k' -linear isomorphism between E_i and the subfield of \bar{k}' fixed by τ^{n_i} . Using these choices to identify $H^1(L \otimes_k k', \mathbf{Q}/\mathbf{Z})$ with $(\mathbf{Q}/\mathbf{Z})^N$, the implication (a) \Rightarrow (b) now results from Lemma 2.3. \square

When L is a field, the condition appearing in Proposition 2.1 (2) can be reformulated in Galois theoretical terms, viewing χ as a group homomorphism $\text{Gal}(\bar{k}/L) \rightarrow \mathbf{Q}/\mathbf{Z}$.

Corollary 2.4. *Suppose that L is a field. Fix an algebraic closure \bar{k} of k and a k -linear embedding $L \hookrightarrow \bar{k}$. Let $\chi \in H^1(L, \mathbf{Q}/\mathbf{Z})$. The following conditions are equivalent:*

- (1) $\text{Cores}_{L/k}(z, \chi) \in \text{Br}_{\text{nr}}(Z)$;
- (2) for any $\tau_1, \tau_2 \in \text{Gal}(\bar{k}/L)$ that are conjugate in $\text{Gal}(\bar{k}/k)$, the equality $\chi(\tau_1) = \chi(\tau_2)$ holds, where we regard χ as a group homomorphism $\text{Gal}(\bar{k}/L) \rightarrow \mathbf{Q}/\mathbf{Z}$.

Proof. If the condition of Proposition 2.1 (2) holds, then for any $\tau_1, \tau_2 \in \text{Gal}(\bar{k}/L)$ and $\sigma \in \text{Gal}(\bar{k}/k)$ such that $\sigma^{-1} \tau_1 \sigma = \tau_2$, one checks that the condition of Corollary 2.4 also holds by letting $\iota_1 : L \rightarrow k'$ be the inclusion of L in the subfield k' of \bar{k} fixed by τ_1 and $\iota_2 : L \rightarrow k'$ the map induced by σ .

To prove the converse implication, it suffices to note that if two embeddings ι_1, ι_2 of the extension L/k into an extension k'/k with procyclic absolute Galois group are given, then, after choosing an algebraic closure \bar{k}' of k' , a topological generator $\tau \in \text{Gal}(\bar{k}'/k')$ and, for each $i \in \{1, 2\}$, an embedding $\bar{\iota}_i : \bar{k} \rightarrow \bar{k}'$ that extends ι_i , the images τ_1 and τ_2 of τ by the restriction maps $\bar{\iota}_i^* : \text{Gal}(\bar{k}'/k') \rightarrow \text{Gal}(\bar{k}/L)$ are conjugate in $\text{Gal}(\bar{k}/k)$. \square

Remark 2.5. If k is a number field, then $H^2(k, \mathbf{Q}/\mathbf{Z}) = 0$ (see [Har20, Corollary 18.17]), so that the map β appearing in (2.5) is onto. As a consequence, all elements of $\mathrm{Br}_1(Z)$ can be written as $\delta + \mathrm{Cores}_{L/k}(z, \chi)$ for some $\chi \in H^1(L, \mathbf{Q}/\mathbf{Z})$ and $\delta \in \mathrm{Br}_0(Z)$ (see (2.5)). Thus, in this case, Proposition 2.1 fully describes the unramified Brauer group of Z .

2.2. Some norm tori with $\mathrm{Br}_{\mathrm{nr}}(T) = \mathrm{Br}_0(T)$. For later use in §6.2, the next proposition collects a few cases in which norm tori satisfy $\mathrm{Br}_{\mathrm{nr}}(T) = \mathrm{Br}_0(T)$.

Proposition 2.6. *Let k be a field of characteristic 0 and L be a finite extension of k . After choosing a bijection $\mathrm{Spec}(L \otimes_k \bar{k}) = \{1, \dots, n\}$, we view the Galois group G of a Galois closure of L/k as a transitive subgroup of the symmetric group S_n . The torus $T = R_{L/k}^1 \mathbf{G}_m$ over k satisfies $\mathrm{Br}_{\mathrm{nr}}(T) = \mathrm{Br}_0(T)$ under any of the following assumptions:*

- (i) for every prime divisor p of n , the p -Sylow subgroups of G are cyclic;
- (ii) G is cyclic;
- (iii) n is prime;
- (iv) L/k is Galois and $H^3(G, \mathbf{Z}) = 0$;
- (v) $\mathrm{Ker}(H^{\mathrm{ab}} \rightarrow G^{\mathrm{ab}})$ has order prime to n (for example $H^{\mathrm{ab}} = 0$) and n is squarefree;
- (vi) $G = S_n$ is the symmetric group;
- (vii) $G = D_n$ is the dihedral group of order $2n$;
- (viii) $G = A_n$ is the alternating group and $n \geq 5$.

Cases (vi), (vii), (viii) of Proposition 2.6 are due to Kunyavskii–Voskresenskii [VK84], Bartels [Bar81] and Macedo [Mac20], respectively. For the convenience of the reader, we reproduce below the main points of the arguments, and provide a simple alternative proof of Macedo’s theorem in the exceptional case $G = A_6$, which in *op. cit.* relied on the use of a computer. The proof we give here was inspired by an argument of Drakokhrust and Platonov [DP86].

Proof of Proposition 2.6. Let $H \subseteq G$ denote the subgroup whose fixed field is L , so that $\widehat{T} = \mathbf{Z}^{G/H}/\mathbf{Z}$ and $H^2(G, \widehat{T})$ fits into an exact sequence

$$(2.9) \quad H^2(G, \mathbf{Z}) \longrightarrow H^2(H, \mathbf{Z}) \longrightarrow H^2(G, \widehat{T}) \longrightarrow H^3(G, \mathbf{Z}) \longrightarrow H^3(H, \mathbf{Z}),$$

in view of Shapiro’s lemma. We need to prove that $\mathrm{III}_{\mathrm{cyc}}^2(G, \widehat{T}) = 0$ (see (2.2) and (2.3)).

Lemma 2.7. *Let p be a prime number. The p -torsion subgroup of $\mathrm{III}_{\mathrm{cyc}}^2(G, \widehat{T})$ vanishes under any of the following assumptions:*

- (a) p does not divide n ;
- (b) the p -Sylow subgroups of G are cyclic;
- (c) $\mathrm{Ker}(H^{\mathrm{ab}} \rightarrow G^{\mathrm{ab}})$ has order prime to p ; moreover, if H_p denotes a p -Sylow subgroup of H and G_p a p -Sylow subgroup of G containing H_p , then, letting L' and k' be their respective fixed fields, the torus $T' = R_{L'/k'}^1 \mathbf{G}_m$ over k' satisfies $\mathrm{Br}_{\mathrm{nr}}(T') = \mathrm{Br}_0(T')$.

Proof. The H -module \widehat{T} is isomorphic to $\mathbf{Z}^{(G/H) \setminus \{H\}}$, hence it is a permutation H -module. We deduce, by Shapiro’s lemma, that $\mathrm{III}_{\mathrm{cyc}}^2(H, \widehat{T}) = 0$. It follows that $(G : H) = n$ kills

$\text{III}_{\text{cyc}}^2(G, \widehat{T})$ (see [NSW08, Chapter I, Corollary 1.5.7]). This takes care of (a). By the same token, the index $(G : S)$ kills $\text{III}_{\text{cyc}}^2(G, \widehat{T})$ for any cyclic subgroup S of G , since $\text{III}_{\text{cyc}}^2(S, \widehat{T})$ obviously vanishes. This takes care of (b). Now assume that (c) holds. We let $M\{p\}$ denote the p -primary torsion subgroup of an abelian group M and consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} H^2(G, \mathbf{Z})\{p\} & \xrightarrow{\alpha} & H^2(H, \mathbf{Z})\{p\} & \longrightarrow & H^2(G, \widehat{T})\{p\} & \longrightarrow & H^3(G, \mathbf{Z})\{p\} \\ & & \downarrow & & \downarrow \beta & & \downarrow \gamma \\ H^2(G_p, \mathbf{Z})\{p\} & \longrightarrow & H^2(H_p, \mathbf{Z})\{p\} & \longrightarrow & H^2(G_p, \widehat{T}')\{p\} & \longrightarrow & H^3(G_p, \mathbf{Z})\{p\} \end{array}$$

induced, thanks to Shapiro's lemma, by the natural morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}^{G/H} & \longrightarrow & \widehat{T} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}^{G_p/H_p} & \longrightarrow & \widehat{T}' \longrightarrow 0. \end{array}$$

The first part of (c) implies that α is surjective, since $H^2(G, \mathbf{Z})\{p\} = \text{Hom}(G^{\text{ab}}, \mathbf{Q}_p/\mathbf{Z}_p)$ and $H^2(H, \mathbf{Z})\{p\} = \text{Hom}(H^{\text{ab}}, \mathbf{Q}_p/\mathbf{Z}_p)$. In addition, a corestriction argument as above shows that γ is injective. We conclude that β is injective. On the other hand, the image of $\text{III}_{\text{cyc}}^2(G, \widehat{T})\{p\}$ by β is contained in $\text{III}_{\text{cyc}}^2(G_p, \widehat{T}')\{p\}$, which vanishes by the second part of (c), in view of (2.2) and (2.3). Hence $\text{III}_{\text{cyc}}^2(G, \widehat{T})\{p\} = 0$. \square

Proposition 2.6 in case (i) results from Lemma 2.7 (a)(b). Cases (ii) and (iii) are subcases of (i), as the p -Sylow subgroups of S_p are cyclic. In case (iv), even $H^2(G, \widehat{T})$ vanishes. Proposition 2.6 in case (v) results from Lemma 2.7 (a)(c) and from case (iii) of Proposition 2.6, once one notes that if n is squarefree, then for any prime p dividing n , the extension L'/k' appearing in Lemma 2.7 (c) has prime degree (namely, degree p).

We now turn to cases (vi)–(viii). When $G = S_n$ and $n \geq 6$ (so that $H = S_{n-1}$) or $G = A_n$ and $n \geq 8$ (so that $H = A_{n-1}$), the restriction map $H^2(G, \mathbf{Z}) \rightarrow H^2(H, \mathbf{Z})$ is onto since $\text{Ker}(H^{\text{ab}} \rightarrow G^{\text{ab}}) = 0$; on the other hand, the restriction map $H^3(G, \mathbf{Z}) \rightarrow H^3(H, \mathbf{Z})$ is injective, by [Mac20, Lemma 2.2] and [NSW08, Chapter III, Proposition 3.1.2] when $G = A_n$ and $n \geq 8$, and by similar arguments starting from [Mac20, Proposition 2.4] when $G = S_n$ and $n \geq 6$. Thus $H^2(G, \widehat{T}) = 0$ in these cases. When $G = D_n$ (so that $H = \mathbf{Z}/2\mathbf{Z}$), Lemma 2.7 (a)(b)(c) reduces one to the case where n is a power of 2; in this case, a direct computation using the semidirect product structure of D_n shows that the restriction map $H^2(G, \widehat{T}) \rightarrow H^2(H, \widehat{T})$ is injective (see [Bar81, Lemma 3] and apply [NSW08, Chapter III, Proposition 3.1.2]), hence $\text{III}_{\text{cyc}}^2(G, \widehat{T}) = 0$. When $n = 4$ and $G = S_4$, one has $\text{III}_{\text{cyc}}^2(G, \widehat{T}) = 0$ by Lemma 2.7 (a)(c) and by Proposition 2.6 (vii) for $n = 4$. When $n = 6$ and $G = A_6$, case (v) of Proposition 2.6 is applicable. Finally, when $n \in \{2, 3, 5, 7\}$, case (iii) of Proposition 2.6 is applicable. \square

Remarks 2.8. (i) When the extension L/k is cyclic or has prime degree, the torus T is even a direct factor of a rational variety (when n is prime, see [CTS87b, Proposition 9.1,

Proposition 9.5]; when L/k is cyclic, the torus T is itself rational, see [Vos98, Chapter 2, §4.8] and the proof of [CTS87b, Proposition 9.1]). We note that the case where n is prime includes the case where L/k is an almost abelian extension in the sense of [HW16, Definition 9.4] but is not abelian.

(ii) The equality $\mathrm{Br}_{\mathrm{nr}}(T) = \mathrm{Br}_0(T)$ fails whenever the extension L/k is abelian but is not cyclic, since in this case $\mathrm{III}_{\mathrm{cyc}}^2(G, \widehat{T}) = H^3(G, \mathbf{Z}) \neq 0$. Another situation in which this equality fails can be found in Example 2.11 below.

2.3. Central extensions by \mathbf{Q}/\mathbf{Z} . Keeping the set-up of §2.1, we now explain how the elements of the group $\mathrm{III}_{\mathrm{cyc}}^2(G, \widehat{T})$ appearing in (2.3) can be interpreted in terms of central extensions of G by \mathbf{Q}/\mathbf{Z} . This turns the practical computation of $\mathrm{Br}_{\mathrm{nr}}(T)/\mathrm{Br}_0(T)$ into a very concrete question on extensions and their trivialisations. We illustrate this point of view in Example 2.11, where we provide a simple explanation for an example due to Kunyavskii of a class of norm tori such that $\mathrm{Br}_{\mathrm{nr}}(T) \neq \mathrm{Br}_0(T)$, and in Example 2.12, where we give an independent proof of Proposition 2.6 (vii) (Bartels' theorem).

From now on, we assume that L is a field and we fix an algebraic closure \bar{k} of k and a k -linear embedding $L \hookrightarrow \bar{k}$. Let \tilde{L} be the Galois closure of L/k in \bar{k} . Set $G = \mathrm{Gal}(\tilde{L}/k)$ and $H = \mathrm{Gal}(\tilde{L}/L)$. To interpret $H^2(G, \widehat{T})$, we consider central extensions

$$(2.10) \quad 1 \longrightarrow \mathbf{Q}/\mathbf{Z} \xrightarrow{\iota} \tilde{G} \xrightarrow{\rho} G \longrightarrow 1$$

of G by \mathbf{Q}/\mathbf{Z} equipped with a splitting of their pull-back

$$(2.11) \quad 1 \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow \rho^{-1}(H) \longrightarrow H \longrightarrow 1.$$

along the inclusion $H \hookrightarrow G$. Such data can be encoded by a triple (\tilde{G}, ρ, r) formed by a group \tilde{G} , a surjective homomorphism $\rho : \tilde{G} \rightarrow G$ and a homomorphism $r : \rho^{-1}(H) \rightarrow \mathbf{Q}/\mathbf{Z}$ whose restriction to the kernel of ρ is an isomorphism. We denote by $\mathrm{Ext}(G, H; \mathbf{Q}/\mathbf{Z})$ the set of isomorphism classes of such triples (\tilde{G}, ρ, r) and by $\mathrm{Ext}_{\mathrm{nr}}(G, H; \mathbf{Q}/\mathbf{Z})$ the subset consisting of the isomorphism classes of unramified triples, in the following sense.

Definition 2.9. A triple (\tilde{G}, ρ, r) is *unramified* if the equality $r(\tilde{h}_1) = r(\tilde{h}_2)$ holds for all $\tilde{h}_1, \tilde{h}_2 \in \rho^{-1}(H)$ that are conjugate in \tilde{G} .

The next proposition justifies this terminology, in view of (2.2) and (2.3).

Proposition 2.10. *There is a canonical bijection $H^2(G, \widehat{T}) = \mathrm{Ext}(G, H; \mathbf{Q}/\mathbf{Z})$. It induces, by restriction, a bijection $\mathrm{III}_{\mathrm{cyc}}^2(G, \widehat{T}) = \mathrm{Ext}_{\mathrm{nr}}(G, H; \mathbf{Q}/\mathbf{Z})$.*

Proof. Let $M = \widehat{T} \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z}$. The exact sequence of G -modules

$$(2.12) \quad 0 \longrightarrow \widehat{T} \longrightarrow \widehat{T} \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow M \longrightarrow 0$$

induces isomorphisms $H^1(G, M) = H^2(G, \widehat{T})$ and $\mathrm{III}_{\mathrm{cyc}}^1(G, M) = \mathrm{III}_{\mathrm{cyc}}^2(G, \widehat{T})$. In addition, the exact sequence of G -modules

$$(2.13) \quad 0 \longrightarrow \mathbf{Q}/\mathbf{Z} \xrightarrow{\Delta} (\mathbf{Q}/\mathbf{Z})^{G/H} \longrightarrow M \longrightarrow 0,$$

where Δ is the diagonal map, yields an identification

$$(2.14) \quad H^1(G, M) = \frac{\{(\alpha, \beta) \in Z^2(G, \mathbf{Q}/\mathbf{Z}) \times C^1(G, (\mathbf{Q}/\mathbf{Z})^{G/H}); d\beta = \Delta \circ \alpha\}}{\{(d\gamma, d\delta + \Delta \circ \gamma); (\gamma, \delta) \in C^1(G, \mathbf{Q}/\mathbf{Z}) \times (\mathbf{Q}/\mathbf{Z})^{G/H}\}}.$$

Given a central extension (2.10), let us consider its push-forward by Δ , i.e. the extension

$$(2.15) \quad 1 \longrightarrow (\mathbf{Q}/\mathbf{Z})^{G/H} \longrightarrow \frac{(\mathbf{Q}/\mathbf{Z})^{G/H} \rtimes \tilde{G}}{((-\Delta) \times \iota)(\mathbf{Q}/\mathbf{Z})} \longrightarrow G \longrightarrow 1,$$

where \tilde{G} acts on $(\mathbf{Q}/\mathbf{Z})^{G/H}$ through ρ . The extension (2.15) is also the induction from H to G of (2.11) in the sense of [Sti10, §2.4], since (2.11) is the extension obtained from (2.15) by applying the functor sh^2 of *loc. cit.* (see §2.3.4 and §2.4.2 of *op. cit.*). By Corollary 15 of *op. cit.*, splittings of (2.11) are therefore in one-to-one correspondence with splittings of (2.15) up to conjugation by $(\mathbf{Q}/\mathbf{Z})^{G/H}$.

Recall that $H^2(G, \mathbf{Q}/\mathbf{Z})$ is in canonical bijection with the set of isomorphism classes of central extensions of G by \mathbf{Q}/\mathbf{Z} ; if $s : G \rightarrow \tilde{G}$ is a set-theoretic section of ρ , the class of (2.10) is given by the cocycle $\alpha \in Z^2(G, \mathbf{Q}/\mathbf{Z})$ defined by $\iota(\alpha(\sigma, \tau)) = s(\sigma)s(\tau)s(\sigma\tau)^{-1}$ (see [Bro94, Chapter IV, Theorem 3.12]). Once s is fixed, the datum of a splitting of (2.15) is equivalent to that of a map $\beta : G \rightarrow (\mathbf{Q}/\mathbf{Z})^{G/H}$ such that $(-\beta) \times s : G \rightarrow (\mathbf{Q}/\mathbf{Z})^{G/H} \rtimes \tilde{G}$ is a homomorphism modulo $((-\Delta) \times \iota)(\mathbf{Q}/\mathbf{Z})$, or in other words, such that $d\beta = \Delta \circ \alpha$. For $(\gamma, \delta) \in C^1(G, \mathbf{Q}/\mathbf{Z}) \times (\mathbf{Q}/\mathbf{Z})^{G/H}$, conjugating $(-\beta) \times s$ by δ and replacing s with $(\iota \circ \gamma)s$ amounts to adding $(d\gamma, d\delta + \Delta \circ \gamma)$ to the pair (α, β) .

Altogether, we have now obtained a canonical map $\text{Ext}(G, H; \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(G, M)$. By unwinding its construction and applying [Bro94, Chapter IV, Theorem 3.12], one readily checks that it is bijective, thus proving the first part of Proposition 2.10.

To verify the second part, let us fix a central extension (2.10), a set-theoretic section s of ρ and a splitting of (2.15), corresponding on the one hand to a triple (\tilde{G}, ρ, r) and on the other hand to $\alpha \in Z^2(G, \mathbf{Q}/\mathbf{Z})$ and $\beta \in C^1(G, (\mathbf{Q}/\mathbf{Z})^{G/H})$ as above, with $d\beta = \Delta \circ \alpha$. Denoting by $\beta(g)(g')$, for $g, g' \in G$, the image of $g'H$ by $\beta(g) : G/H \rightarrow \mathbf{Q}/\mathbf{Z}$, we shall now conclude the proof by establishing the equivalence of the following conditions:

- (i) the triple (\tilde{G}, ρ, r) is unramified;
- (ii) $\beta(h)(g') = \beta(h)(1)$ for all $h \in H$ and $g' \in G$ such that $g'^{-1}hg' \in H$;
- (iii) $\beta(g)(g') = \beta(g)(g'')$ for all $g, g', g'' \in G$ such that $g'^{-1}gg' \in H$ and $g''^{-1}gg'' \in H$;
- (iv) the image of β in $H^1(G, M)$ belongs to the subgroup $\text{III}_{\text{cyc}}^1(G, M)$.

The splitting of (2.11) induced by the given splitting of (2.15) is $h \mapsto \iota(-\beta(h)(1))s(h)$ (see [Sti10, §2.4.2]); hence $r(s(h)) = \beta(h)(1)$ for every $h \in H$. Writing elements of $\rho^{-1}(H)$ as $\iota(u)s(h)$ with $u \in \mathbf{Q}/\mathbf{Z}$ and $h \in H$, we deduce that (i) holds if and only if

$$(2.16) \quad \iota(\beta(h_2)(1) - \beta(h_1)(1)) = s(g)s(h_2)s(g)^{-1}s(h_1)^{-1}$$

for all $h_1, h_2 \in H$ and $g \in G$ such that $h_1 = gh_2g^{-1}$. As the right-hand side of (2.16) is equal to $\iota(\alpha(g, h_2) - \alpha(h_1, g))$ and as $d\beta = \Delta \circ \alpha$, we can rewrite (2.16) as $\beta(h_2)(1) = \beta(h_2)(g^{-1})$. This proves the equivalence (i) \Leftrightarrow (ii). The implications (iv) \Rightarrow (iii) \Rightarrow (ii) are immediate. To

see that (ii) \Rightarrow (iii), we remark that as $d\beta = \Delta \circ \alpha$, the image $\bar{\beta} \in C^1(G, M)$ of β is a cocycle, hence $\bar{\beta}(g) = \bar{\beta}(g') + g'(\bar{\beta}(g'^{-1}gg')) - g(\bar{\beta}(g'))$ for all $g, g' \in G$. Evaluating at g' and g'' and noting that $\beta(g')(g') = \beta(g')(g'^{-1}g')$ and $\beta(g')(g'') = \beta(g')(g'^{-1}g'')$ if g' and g'' satisfy the hypotheses of (iii), one deduces that (ii) \Rightarrow (iii).

It remains to prove that (iii) \Rightarrow (iv). We assume that (iii) holds, fix a cyclic subgroup $C \subseteq G$ and verify that the image of $\bar{\beta}$ in $H^1(C, M)$ vanishes. To this end, we choose a generator g of C and representatives $g_1, \dots, g_N \in G$ of the orbits of C on G/H . Let n_1, \dots, n_N be the lengths of these orbits. Condition (iii) implies the following:

(iii') $\beta(g^n)(g_i) = \beta(g^n)(g_j)$ for all n, i, j such that n_i and n_j both divide n .

On the other hand, the exact sequence (2.13) induces an exact sequence

$$(2.17) \quad H^1(C, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^1(C, (\mathbf{Q}/\mathbf{Z})^{G/H}) \longrightarrow H^1(C, M) \longrightarrow H^2(C, \mathbf{Q}/\mathbf{Z}).$$

As $H^2(C, \mathbf{Q}/\mathbf{Z}) = 0$, there exists $\beta' \in Z^1(C, (\mathbf{Q}/\mathbf{Z})^{G/H})$ whose image in $H^1(C, M)$ equals the image of $\bar{\beta}$. Condition (iii') still holds with β replaced with β' . Set $\lambda_i = \beta'(g^{n_i})(g_i)$. For any integer n divisible by n_i , we have $\beta'(g^n) = (1 + g^{n_i} + g^{2n_i} + \dots + g^{n-n_i})(\beta'(g^{n_i}))$ since β' is a cocycle, and hence $\beta'(g^n)(g_i) = \frac{n}{n_i} \lambda_i$. Thus, thanks to (iii'), we deduce from Lemma 2.3 the existence of $\lambda \in \mathbf{Q}/\mathbf{Z}$ such that $\lambda_i = n_i \lambda$ for all i .

Let m be the order of g . Decomposing $(\mathbf{Q}/\mathbf{Z})^{G/H}$ as a direct sum indexed by the orbits of C on G/H and using Shapiro's lemma on each summand, we rewrite (2.17) as

$$\left(\frac{1}{m}\mathbf{Z}\right)/\mathbf{Z} \longrightarrow \bigoplus_{i=1}^N \left(\left(\frac{n_i}{m}\mathbf{Z}\right)/\mathbf{Z}\right) \longrightarrow H^1(C, M) \longrightarrow 0,$$

where the first map is $x \mapsto (n_i x)_{1 \leq i \leq N}$. As the class of β' in the second group is $(\lambda_i)_{1 \leq i \leq N}$, we see that β' comes from the left, hence $\bar{\beta}$ vanishes in $H^1(C, M)$ and (iv) is proved. \square

Example 2.11 (Kunyavskii [Kun82]). Let k be a field of characteristic 0 and L be a quartic extension of k . If a Galois closure of L/k has Galois group $G = A_4$, then the torus $T = R_{L/k}^1 \mathbf{G}_m$ over k satisfies $\text{Br}_{\text{nr}}(T)/\text{Br}_0(T) = \mathbf{Z}/2\mathbf{Z}$.

To verify this claim, we first note that $H^2(G, \hat{T}) = \mathbf{Z}/2\mathbf{Z}$, as one deduces from (2.9). Proposition 2.10 now reduces us to showing that $\text{Ext}_{\text{nr}}(A_4, A_3; \mathbf{Q}/\mathbf{Z}) = \text{Ext}(A_4, A_3; \mathbf{Q}/\mathbf{Z})$. To this end, we fix a central extension (2.10) and a retraction $r : \rho^{-1}(H) \rightarrow \mathbf{Q}/\mathbf{Z}$ of (2.11), with $G = A_4$ and $H = A_3$, and let $\tilde{h}_1, \tilde{h}_2 \in \rho^{-1}(H)$ and $\tilde{g} \in \tilde{G}$ satisfy $\tilde{h}_1 = \tilde{g}\tilde{h}_2\tilde{g}^{-1}$. Set $h_i = \rho(\tilde{h}_i)$ and $g = \rho(\tilde{g})$. Now $h_1, h_2 \in A_3$ and $g \in A_4$ are such that $h_1 = gh_2g^{-1}$, and this implies that $h_1 = h_2 = 1$ or $g \in A_3$. In both cases, we deduce that \tilde{h}_1 and \tilde{h}_2 are conjugate in $\rho^{-1}(H)$, and hence that $r(\tilde{h}_1) = r(\tilde{h}_2)$, as desired.

Example 2.12. Taking up the notation of Proposition 2.6, let us assume that $G = D_n$ and explain how the point of view of central extensions leads to a short proof of Bartels' theorem that $\text{Br}_{\text{nr}}(T) = \text{Br}_0(T)$. According to Proposition 2.10, we have to check that any unramified triple (\tilde{G}, ρ, r) is isomorphic to the trivial triple $(\mathbf{Q}/\mathbf{Z} \times G, \text{pr}_2, \text{pr}_1)$. Here $H = \mathbf{Z}/2\mathbf{Z}$ and the inclusion $H \hookrightarrow G$ admits a retraction, so that the restriction map

$H^1(G, \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(H, \mathbf{Q}/\mathbf{Z})$ is surjective, which implies the desired conclusion when the central extension (2.10) is trivial (i.e. when $\tilde{G} = \mathbf{Q}/\mathbf{Z} \times G$ and $\rho = \text{pr}_2$) since in this case the splittings of (2.11) are parametrised by $H^1(H, \mathbf{Q}/\mathbf{Z})$ and the automorphisms of (2.10) by $H^1(G, \mathbf{Q}/\mathbf{Z})$. Let us now fix an unramified triple (\tilde{G}, ρ, r) such that (2.10) is nontrivial. As $H^2(D_n, \mathbf{Q}/\mathbf{Z}) = 0$ when n is odd and $H^2(D_n, \mathbf{Q}/\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ when n is even, the integer n must be even and (2.10) is the unique nontrivial central extension of D_n by \mathbf{Q}/\mathbf{Z} up to isomorphism. Presenting D_m as $\langle \sigma, \tau; \sigma^2 = \tau^m = 1, \sigma\tau\sigma = \tau^{-1} \rangle$ for any m , setting

$$(2.18) \quad \tilde{D}_n = (\mathbf{Q}/\mathbf{Z} \times D_{2n}) / \langle (1/2, \tau^n) \rangle$$

and letting $\pi : \tilde{D}_n \rightarrow D_n$ be induced by the second projection (noting that $D_{2n}/\langle \tau^n \rangle = D_n$), one checks that π does not admit a section, as n is even; hence the central extension given by \tilde{D}_n and π must be isomorphic to (2.10) and we may assume that $\tilde{G} = \tilde{D}_n$ and $\rho = \pi$. Now the classes $\tilde{h}_1, \tilde{h}_2 \in \tilde{D}_n$ of $(0, \sigma), (0, \sigma\tau^n) \in \mathbf{Q}/\mathbf{Z} \times D_{2n}$ are conjugate in \tilde{D}_n (indeed $\tau^{-n/2}\sigma\tau^{n/2} = \sigma\tau^n$ in D_{2n}) and belong to $\pi^{-1}(H)$, but differ by the class of $(1/2, 1)$ in \tilde{D}_n , so that $r(\tilde{h}_1) \neq r(\tilde{h}_2)$, a contradiction.

3. THE VARIETY W AND CONJECTURE F_+

We recall in §§3.1–3.2 the definition and geometry of the quasi-affine variety W which was introduced in [HW16, §9.2.2] and partial compactifications of which had previously appeared in [Sko96, §3.3], [CTS00, p. 391], and in [Sko01, §4.4]. We discuss in §§3.3–3.5 some of its basic properties, showing, in particular, that its Brauer group consists only of constant classes. When the ground field is a number field, the adelic points of W are the subject of two key conjectures studied in this article: Conjecture F and Conjecture F_+ . We state them in §3.6. Conjecture F is another name for [HW16, Conjecture 9.1], while the formulation of Conjecture F_+ is new. As we shall see later in §5, these two conjectures are equivalent when their parameters are allowed to vary.

3.1. Parameters and definition. Let k be a field of characteristic 0. We denote by \mathcal{P} the collection of all triples $\pi = (M, (L_m)_{m \in M}, (b_m)_{m \in M})$ consisting of a finite closed subset $M \subset \mathbf{A}_k^1$ together with the data, for each $m \in M$, of a finite extension L_m of the residue field $k(m)$ of m and of an element $b_m \in k(m)^*$. Given $\pi \in \mathcal{P}$, we let $a_m \in k(m)$ denote the value at m of the regular function t on $\mathbf{A}_k^1 = \text{Spec}(k[t])$ and let F_m denote the singular locus of the variety $R_{L_m/k}(\mathbf{A}_{L_m}^1) \setminus R_{L_m/k}(\mathbf{G}_{m, L_m})$. Following [HW16, §9.2.2], for $\pi \in \mathcal{P}$, we consider the morphism

$$(3.1) \quad \left(\mathbf{A}_k^2 \setminus \{(0, 0)\} \right) \times \prod_{m \in M} \left(R_{L_m/k}(\mathbf{A}_{L_m}^1) \setminus F_m \right) \rightarrow \prod_{m \in M} R_{k(m)/k}(\mathbf{A}_{k(m)}^1)$$

defined by $(\lambda, \mu, (z_m)_{m \in M}) \mapsto (b_m(\lambda - a_m\mu) - N_{L_m/k(m)}(z_m))_{m \in M}$, where λ, μ are the coordinates of \mathbf{A}_k^2 and z_m stands for a point of $R_{L_m/k}(\mathbf{A}_{L_m}^1)$.

Definition 3.1. The variety W over k associated with $\pi \in \mathcal{P}$ is the fibre, above the rational point 0, of the morphism (3.1).

Remark 3.2. Let $\pi = (M, (L_m)_{m \in M}, (b_m)_{m \in M}) \in \mathcal{P}$. Let $M' = \{m \in M; L_m \neq k(m)\}$. Let $\pi' = (M', (L_m)_{m \in M'}, (b_m)_{m \in M'})$. The varieties W and W' associated with π and π' are naturally isomorphic.

3.2. Geometry. For ease of reference, we collect in the next proposition some elementary facts about the geometry of W that already appear in the proof of [HW16, Proposition 9.9]. We set $U = \mathbf{P}_k^1 \setminus M$ and let $p : W \rightarrow \mathbf{P}_k^1$ denote the map $(\lambda, \mu, (z_m)_{m \in M}) \mapsto [\lambda : \mu]$.

Proposition 3.3. *Let $\pi \in \mathcal{P}$. The morphism p is smooth (hence W is a smooth variety). Its fibres over U are geometrically integral. For each $m \in M$, the fibre $W_m = p^{-1}(m)$ is integral and the algebraic closure of $k(m)$ in its function field is L_m .*

Remark 3.4. The variety W' defined in the same way as W except that one does not remove F_m from $R_{L_m/k}(\mathbf{A}_{L_m}^1)$ in (3.1) was first considered in [Sko96, §3.3]. It is a smooth variety (*loc. cit.*, Lemma 3.3.1) and it contains W as the complement of a codimension 2 closed subset. More precisely, the variety W is the locus where the faithfully flat morphism $p' : W' \rightarrow \mathbf{P}_k^1$ that extends p is smooth.

3.3. Brauer group. The vertical Brauer group of p is reduced to constant classes, as was shown in [HW16, Proof of Proposition 9.9]. More generally, so is the algebraic Brauer group of W , according to [CT15, §1] (which formally deals with W' , in the notation of Remark 3.4, but applies to W too since the smooth varieties W' and W differ by a codimension 2 closed subset) or to [CWX18, Lemma 5.1]. The following proposition generalises this assertion to the full Brauer group.

Proposition 3.5. *Let $\pi \in \mathcal{P}$. The pull-back map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(W)$ is onto.*

Proof. Only until the end of this proof, let us modify the definition of \mathcal{P} by allowing L_m to be an arbitrary nonzero finite étale $k(m)$ -algebra, not necessarily a field. We shall prove that the proposition holds in this slightly more general situation, with the same definition for W , by induction on the dimension of W . If $\dim(W) < 3$, then $L_m = k(m)$ for all $m \in M$, hence $W = \mathbf{A}_k^2 \setminus \{(0, 0)\}$, so that $\mathrm{Br}(k) = \mathrm{Br}(W)$ (see [CTS20, Theorem 3.7.1, Theorem 6.1.1]). Let us now assume that $\dim(W) \geq 3$ and that the conclusion of the proposition holds for lower dimensions of W (letting both k and π vary).

In order to prove that the pull-back map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(W)$ is onto, we may and will assume that k is algebraically closed, since the algebraic Brauer group of W is reduced to constant classes by [CT15, Proposition 1.1, Proposition 1.2 (ii)–(iii)].

Setting $d_m = \dim_k L_m$ and letting $\lambda, \mu, (z_{m,j})_{m \in M, 1 \leq j \leq d_m}$ denote the coordinates of $\mathbf{A}_k^2 \times \prod_{m \in M} \mathbf{A}_k^{d_m}$, the variety W is then isomorphic to the subvariety of this affine space defined by the following equations: $\prod_{j=1}^{d_m} z_{m,j} = b_m(\lambda - a_m \mu)$ for each $m \in M$; at most one of the coordinates $z_{m,1}, \dots, z_{m,d_m}$ vanishes for each $m \in M$; and $(\lambda, \mu) \neq (0, 0)$.

Let us fix $m \in M$ such that $d_m > 1$. The generic fibre V of the projection $W \rightarrow \mathbf{A}_k^1$ to the coordinate z_{m,d_m} is a variety of the form W associated with a parameter in \mathcal{P} over the function field of \mathbf{A}_k^1 . By the induction hypothesis, we deduce that $\mathrm{Br}(k(\mathbf{A}_k^1))$ surjects onto $\mathrm{Br}(V)$. Hence $\mathrm{Br}(V) = 0$, in view of Tsen's theorem. As $\mathrm{Br}(W)$ injects into $\mathrm{Br}(V)$, we conclude that $\mathrm{Br}(W) = 0$ and the proof is complete. \square

Remark 3.6. When k is a number field, it was proved in [HW16, Corollary 9.10] that if, for every $\pi \in \mathcal{P}$ and every finite place v_0 of k , the variety W satisfies strong approximation off v_0 , then Conjecture 9.1 of *op. cit.* is true. This strong approximation property holds in several cases (see [HW16, Theorem 9.11] and [BS19, Theorem 1.2]) and it is natural to hope that it may hold in general. Proposition 3.5 supports this hope, as it shows that the Brauer–Manin obstruction to strong approximation on W always vanishes.

3.4. Canonical ramified Brauer classes. Given $\pi \in \mathcal{P}$, the classes

$$(3.2) \quad \text{Cores}_{L_m/k}(z_m, \chi) \in \text{Br}(k(W))$$

for $m \in M$ and $\chi \in H^1(L_m, \mathbf{Q}/\mathbf{Z})$, where z_m denotes the regular function on $W \otimes_k L_m$ obtained by pulling back the tautological regular function on $(R_{L_m/k}(\mathbf{A}_{L_m}^1)) \otimes_k L_m$, will play a special rôle in the article.

Proposition 3.7. *Let $\pi \in \mathcal{P}$. Let $m \in M$ and $\chi \in H^1(L_m, \mathbf{Q}/\mathbf{Z})$.*

- (1) *The class $\text{Cores}_{L_m/k}(z_m, \chi)$ belongs to $\text{Br}(p^{-1}(\mathbf{P}_k^1 \setminus \{m\}))$.*
- (2) *Its residue at the generic point of $W_m = p^{-1}(m)$ is equal to the image of χ by the natural map $H^1(L_m, \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(k(W_m), \mathbf{Q}/\mathbf{Z})$ (see Proposition 3.3).*

Proof. Assertion (1) follows from the remark that the regular function z_m on $W \otimes_k L_m$ is invertible on $p^{-1}(\mathbf{P}_k^1 \setminus \{m\}) \otimes_k L_m$. Assertion (2) results from [CTSD94, Proposition 1.1.2 and Proposition 1.1.3]. \square

3.5. Local integral points. We now assume that k is a number field. Given a finite set S of places of k , we denote by \mathcal{O}_S the ring of S -integers of k and by $p : \mathcal{W} \rightarrow \mathbf{P}_{\mathcal{O}_S}^1$ the integral model of $p : W \rightarrow \mathbf{P}_k^1$ obtained by replacing, in the definition of W , all occurrences of the fields k , $k(m)$, L_m , respectively, by their rings of S -integers \mathcal{O}_S , $\mathcal{O}_{k(m),S}$, $\mathcal{O}_{L_m,S}$, and F_m by the singular locus \mathcal{F}_m of the scheme $R_{\mathcal{O}_{L_m,S}/\mathcal{O}_S}(\mathbf{A}_{\mathcal{O}_{L_m,S}}^1) \setminus R_{\mathcal{O}_{L_m,S}/\mathcal{O}_S}(\mathbf{G}_{m,\mathcal{O}_{L_m,S}})$. We will refer to \mathcal{W} as the *standard integral model* of W .

For $m \in \mathbf{P}_k^1$, we denote by \tilde{m} the Zariski closure of m in $\mathbf{P}_{\mathcal{O}_S}^1$, and we set $\tilde{M} = \bigcup_{m \in M} \tilde{m}$.

The next proposition records a useful interpretation for the integral local points of \mathcal{W} at the places of k outside of S , when S is large enough.

Proposition 3.8. *Assume that k is a number field. Let $S \subset \Omega$ be a finite subset containing the archimedean places, the finite places that ramify in K_m for some $m \in M$, the finite places above which one of the b_m for $m \in M$ fails to be a unit. Assume, finally, that S is large enough that $\tilde{M} \cup \infty$ is étale over \mathcal{O}_S . For any $v \in \Omega \setminus S$, the set $\mathcal{W}(\mathcal{O}_v)$ can be identified with the set of families $(\lambda_v, \mu_v, (z_{m,u})_{m \in M, u|v})$ consisting of two elements $\lambda_v, \mu_v \in \mathcal{O}_v$ at least one of which is a unit and, for each $m \in M$ and each place u of L_m dividing v , of an element $z_{m,u} \in \mathcal{O}_{(L_m)_u}$, such that for all $m \in M$, the following properties are satisfied:*

- *letting the product run over the places u of L_m dividing v and letting w denote the trace of u on $k(m)$, the equality*

$$\prod N_{(L_m)_u/k(m)_w}(z_{m,u}) = b_m(\lambda_v - a_m \mu_v)$$

holds in $k(m)_w$;

- there exists at most one place u of L_m dividing v such that $z_{m,u}$ is not a unit;
- if such a place u of L_m exists, then it has degree 1 over v .

Proof. We need only verify that for any \mathcal{O}_v -point $(z_{m,u})_{u|v}$ of $R_{\mathcal{O}_{L_m,S}/\mathcal{O}_S}(\mathbf{A}_{\mathcal{O}_{L_m,S}}^1)$, the reduction modulo v of $(z_{m,u})_{u|v}$ lies in $\mathcal{F}_m(\mathbf{F}_v)$ if and only if $z_{m,u}$ fails to be a unit at u either for at least two distinct u or for at least one u of degree > 1 over v . This local assertion can be checked after replacing k_v with any unramified extension of k_v , in particular it is enough to check it when v splits completely in L_m , in which case it is clear. \square

3.6. Statement of Conjecture F_+ . We take up the notation introduced in §§3.1–3.2 and assume, in addition, that k is a number field: with any $\pi \in \mathcal{P}$ are associated a variety W and a smooth morphism $p : W \rightarrow \mathbf{P}_k^1$, with split fibres above $U = \mathbf{P}_k^1 \setminus M$. For any $c \in \mathbf{P}_k^1$, we set $W_c = p^{-1}(c)$. The following conjecture on the arithmetic of W was put forward in [HW16, §9]. To be precise, it is equivalent to *op. cit.*, Conjecture 9.1, according to *op. cit.*, Proposition 9.9.

Conjecture F. *Let $\pi \in \mathcal{P}$. The subset $\bigcup_{c \in U(k)} W_c(\mathbf{A}_k)$ is dense in $W(\mathbf{A}_k)$.*

The goal of §3.6 is to propose a more general formulation, which we call Conjecture F_+ and view as an improved substitute for Conjecture F. Before stating it, we need to introduce some additional notation related to the parameters on which Conjecture F_+ depends.

We let \mathcal{P}_+ be the collection of all quadruples $\pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M})$ consisting of a triple $\pi = (M, (L_m)_{m \in M}, (b_m)_{m \in M})$ belonging to \mathcal{P} together with the data, for each $m \in M$, of a finite abelian extension K_m of L_m .

Given $\pi_+ \in \mathcal{P}_+$, we set

$$(3.3) \quad C_m = \text{Ker}(H^1(L_m, \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(K_m, \mathbf{Q}/\mathbf{Z}))$$

and denote by $B \subseteq \text{Br}(p^{-1}(U))$ the subgroup generated by the classes $\text{Cores}_{L_m/k}(z_m, \chi)$ for $m \in M$ and $\chi \in C_m$ (see Proposition 3.7 (1)). This is a finite subgroup. Finally, for $c \in U(k)$, we denote by $W_c(\mathbf{A}_k)^B$ the subset of $W_c(\mathbf{A}_k)$ consisting of those adelic points that are orthogonal to the image of B by the restriction map $\text{Br}(p^{-1}(U)) \rightarrow \text{Br}(W_c)$.

Conjecture F_+ . *Let $\pi_+ \in \mathcal{P}_+$. The subset $\bigcup_{c \in U(k)} W_c(\mathbf{A}_k)^B$ is dense in $W(\mathbf{A}_k)$.*

Conjecture F can be seen as the special case of Conjecture F_+ for those $\pi_+ \in \mathcal{P}_+$ that satisfy $K_m = L_m$ for all $m \in M$. In §§4–7, we shall see that Conjecture F_+ should be expected to be just as true as Conjecture F, and that the flexibility provided by allowing nontrivial abelian extensions K_m/L_m leads at the same time to a better theory and to more general results about rational points in fibrations into rationally connected varieties over the projective line.

4. FIBRATION THEOREM FOR RATIONAL POINTS

In this section, we show that the fibration method for proving the density of rational points in the Brauer–Manin set works for fibrations into proper smooth rationally connected varieties, if Conjecture F_+ holds for certain parameters π_+ associated with the fibration.

4.1. Main theorem. Theorem 4.1 should be compared with [HW16, Theorem 9.17], whose statement, based on Conjecture F , is very similar but contains an unpleasant technical assumption absent from Theorem 4.1 (namely, the surjectivity of [HW16, (9.9)]; this assumption is satisfied when k is totally imaginary or M contains a rational point, by [HW16, Remark 9.18 (ii)]). The proof of Theorem 4.1 refines ideas that were elaborated over a long series of works, among which [Har94, Har97, CTSD94, CTSSD98, HW16].

Theorem 4.1. *Let X be a smooth, irreducible variety over a number field k , endowed with a morphism $f : X \rightarrow \mathbf{P}_k^1$ with geometrically irreducible generic fibre. Let $U \subset \mathbf{P}_k^1$ be an open subset over which the fibres of f are split, with $\infty \in U$. Let $A \subset \text{Br}(f^{-1}(U))$ be a finite subgroup. Let $H \subset \mathbf{P}_k^1$ be a Hilbert subset. Let $M = \mathbf{P}_k^1 \setminus U$.*

For each $m \in M$, suppose given an irreducible component Y_m of multiplicity 1 of the fibre $X_m = f^{-1}(m)$ and a finite abelian extension $E_m/k(Y_m)$ such that the residue of any element of A at the generic point of Y_m belongs to the kernel of the restriction map $H^1(k(Y_m), \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(E_m, \mathbf{Q}/\mathbf{Z})$. Let L_m^0 (resp. K_m^0) denote the algebraic closure of $k(m)$ in $k(Y_m)$ (resp. in E_m). For each $m \in M$, suppose given a finite extension L_m of L_m^0 and a finite abelian extension K_m of L_m in which K_m^0 can be embedded L_m^0 -linearly.

Assume that for all choices of $(b_m)_{m \in M} \in \prod_{m \in M} k(m)^$, Conjecture F_+ holds for the parameter $\pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M})$. Then the subset*

$$(4.1) \quad \bigcup_{c \in U(k) \cap H} X_c(\mathbf{A}_k)^A$$

is dense in $X(\mathbf{A}_k)^{(A+f_\eta^ \text{Br}(\eta)) \cap \text{Br}(X)}$.*

Proof. The arguments used to deduce [HW16, Theorem 9.22] from [HW16, Theorem 9.17] also prove, in the exact same way, that Theorem 4.1 in the special case where $H = \mathbf{P}_k^1$ implies Theorem 4.1 in general. Hence we may, and will, assume that $H = \mathbf{P}_k^1$.

Let Ω denote the set of places of k . Let $(x_v)_{v \in \Omega} \in X(\mathbf{A}_k)^{(A+f_\eta^* \text{Br}(\eta)) \cap \text{Br}(X)}$ be the adelic point that we shall approximate. Our goal is to produce $c \in U(k)$ and $(x'_v)_{v \in \Omega} \in X_c(\mathbf{A}_k)^A$ arbitrarily close to $(x_v)_{v \in \Omega}$ in $X(\mathbf{A}_k)$. We break up the proof into five steps.

Step 1. *Choice of the parameter π_+ .*

In Step 3, we shall apply Conjecture F_+ to the parameter

$$(4.2) \quad \pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M})$$

for a certain choice of a family $(b_m)_{m \in M} \in \prod_{m \in M} k(m)^*$. Step 1 is devoted to specifying this choice, using the next lemma.

For its statement, we take up the notation of §3.1 and §3.6, where a variety W , a morphism $p : W \rightarrow \mathbf{P}_k^1$ and various groups $(C_m)_{m \in M}$ and $B \subseteq \text{Br}(p^{-1}(U))$ were associated

with the parameter π_+ defined by (4.2). In addition, we let $U^0 = U \setminus \{\infty\}$, $X^0 = f^{-1}(U^0)$ and $W^0 = p^{-1}(U^0)$. We stress that in the statement of Lemma 4.2, the variety W^0 , the morphism p and the group B depend on $(b_m)_{m \in M}$.

Lemma 4.2. *There exist an adelic point $(x'_v)_{v \in \Omega} \in X^0(\mathbf{A}_k)^A$ arbitrarily close to $(x_v)_{v \in \Omega}$ in $X(\mathbf{A}_k)$ and a family $(b_m)_{m \in M} \in \prod_{m \in M} k(m)^*$ such that if we let W , p and B be defined in terms of $(b_m)_{m \in M}$ as explained above, there exists an adelic point $(z'_v)_{v \in \Omega} \in W^0(\mathbf{A}_k)^B$ satisfying $p(z'_v) = f(x'_v)$ for all $v \in \Omega$. One can require, in addition, that the x'_v for $v \in \Omega$ all belong to smooth fibres of f .*

Proof. Let T be the torus over k defined by the exact sequence of tori

$$(4.3) \quad 1 \longrightarrow T \longrightarrow \mathbf{G}_m \times \prod_{m \in M} R_{K_m/k} \mathbf{G}_m \longrightarrow \prod_{m \in M} R_{k(m)/k} \mathbf{G}_m \longrightarrow 1,$$

where the second map is $(\nu, (z_m)_{m \in M}) \mapsto (N_{K_m/k(m)}(z_m)/\nu)_{m \in M}$. According to Shapiro's lemma and Hilbert's Theorem 90, this exact sequence induces a surjection

$$(4.4) \quad \prod_{m \in M} k(m)^* \twoheadrightarrow H^1(k, T).$$

Given $(b_m)_{m \in M} \in \prod_{m \in M} k(m)^*$, let $p_+ : W_+ \rightarrow \mathbf{P}_k^1$ denote the variety and the morphism associated in §3.1 with the triple $(M, (K_m)_{m \in M}, (b_m)_{m \in M})$ and let $W_+^0 = p_+^{-1}(U^0)$. The morphism $W_+^0 \rightarrow U^0$ induced by p_+ is a torsor under T . We denote by $\tau \in H_{\text{ét}}^1(U^0, T)$ its isomorphism class.

Starting with an arbitrary choice of $(b_m)_{m \in M}$ (for instance $b_m = 1$ for all m), let A^0 be the subgroup of $\text{Br}(X^0)$ generated by A and by the inverse images, by $f^* : \text{Br}(U^0) \rightarrow \text{Br}(X^0)$, of the cup products of τ with the elements of the finite group $H^1(k, \hat{T})$, where \hat{T} denotes the character group of T . As $(x_v)_{v \in \Omega} \in X(\mathbf{A}_k)^{A^0 \cap \text{Br}(X)}$, Harari's formal lemma [CT03, Théorème 1.4] provides $(x'_v)_{v \in \Omega} \in X^0(\mathbf{A}_k)^{A^0}$ arbitrarily close to $(x_v)_{v \in \Omega}$ in $X(\mathbf{A}_k)$. As A^0 is finite, we may assume, after modifying the x'_v using the implicit function theorem, that they all belong to smooth fibres of f .

Let us apply open descent theory to the projection $W_+ \times_{\mathbf{P}_k^1} X^0 \rightarrow X^0$, which is a torsor under T . According to [HS13, Theorem 8.4, Proposition 8.12], the adelic point $(x'_v)_{v \in \Omega}$ can be lifted to an adelic point of some twist of this torsor. Now, twisting the class τ by a class in $H^1(k, T)$ amounts, in view of the surjectivity of (4.4), to modifying the choice of $(b_m)_{m \in M}$. Thus, after modifying our choice of $(b_m)_{m \in M}$, we may assume that $(x'_v)_{v \in \Omega}$ comes from $(W_+ \times_{\mathbf{P}_k^1} X^0)(\mathbf{A}_k)$. In particular, there exists $(z'_{+,v})_{v \in \Omega} \in W_+^0(\mathbf{A}_k)$ such that $p_+(z'_{+,v}) = f(x'_v)$ for all $v \in \Omega$.

Let $q : W_+ \rightarrow W$ be defined by $q(\lambda, \mu, (z_m)_{m \in M}) = (\lambda, \mu, (N_{K_m/L_m}(z_m))_{m \in M})$ and set $z'_v = q(z'_{+,v})$, so that $p(z'_v) = f(x'_v)$ for all $v \in \Omega$. The projection formula implies that B is contained in the kernel of the pull-back map $q^* : \text{Br}(p^{-1}(U)) \rightarrow \text{Br}(p_+^{-1}(U))$ and hence that $\beta(z'_v) = 0$ for any $\beta \in B$ and any $v \in \Omega$. It follows that $(z'_v)_{v \in \Omega} \in W^0(\mathbf{A}_k)^B$. \square

We now apply Lemma 4.2 and leave the resulting $(x'_v)_{v \in \Omega}$, $(b_m)_{m \in M}$, π_+ , W , p , B and $(z'_v)_{v \in \Omega}$ fixed for the remainder of the proof of Theorem 4.1.

Step 2. *Choice of the finite set of bad places S .*

Let S be a finite set of places of k , containing the archimedean places and the places at which we want to approximate $(x_v)_{v \in \Omega}$, and large enough that $f : X \rightarrow \mathbf{P}_k^1$ extends to a flat morphism $f : \mathcal{X} \rightarrow \mathbf{P}_{\mathcal{O}_S}^1$, where \mathcal{O}_S denotes the ring of S -integers of k .

For $m \in \mathbf{P}_k^1$, we denote by \tilde{m} the Zariski closure of m in $\mathbf{P}_{\mathcal{O}_S}^1$ and by \mathcal{X}_m and \mathcal{Y}_m the Zariski closures of X_m and Y_m in \mathcal{X} , all endowed with the reduced scheme structures. When $m \in M$, we denote by $\tilde{m}' \rightarrow \tilde{m}$ the normalisation of \tilde{m} in the finite extension $L_m^0/k(m)$. We set $G_m = \text{Gal}(E_m/k(Y_m))$ and $H_m = \text{Gal}(E_m/k(Y_m)K_m^0) \subseteq G_m$ and fix, for each $m \in M$, a dense open subset $\mathcal{Y}_m^0 \subseteq \mathcal{Y}_m$, small enough that the normalisation $\mathcal{E}_m \rightarrow \mathcal{Y}_m^0$ of \mathcal{Y}_m^0 in the finite extension $E_m/k(Y_m)$ is a finite and étale morphism and that the restriction of f to \mathcal{Y}_m^0 factors through a smooth morphism $f_m : \mathcal{Y}_m^0 \rightarrow \tilde{m}'$.

Set $\tilde{M} = \bigcup_{m \in M} \tilde{m}$, $\mathcal{U} = \mathbf{P}_{\mathcal{O}_S}^1 \setminus \tilde{M}$ and $\mathcal{U}^0 = \mathbf{P}_{\mathcal{O}_S}^1 \setminus (\tilde{M} \cup \tilde{\infty})$. Let $p : \mathcal{W} \rightarrow \mathbf{P}_{\mathcal{O}_S}^1$ denote the standard integral model of $p : W \rightarrow \mathbf{P}_k^1$ (see §3.5). Let $\mathcal{W}^0 = p^{-1}(\mathcal{U}^0)$.

After enlarging S , we may assume that $\tilde{M} \cup \tilde{\infty}$ is étale over \mathcal{O}_S , that the morphisms $\tilde{m}' \rightarrow \tilde{m}$ for $m \in M$ are étale, that the b_m for $m \in M$ are units above the places of $\Omega \setminus S$, that the extensions K_m/k are unramified above the places of $\Omega \setminus S$, that $z'_v \in \mathcal{W}^0(\mathcal{O}_v)$ for all $v \in \Omega \setminus S$, that $A \subseteq \text{Br}(f^{-1}(\mathcal{U}))$, that $\sum_{v \in S} \text{inv}_v \alpha(x'_v) = 0$ for all $\alpha \in A$, that $\sum_{v \in S} \text{inv}_v \beta(z'_v) = 0$ for all $\beta \in B$, that S contains the finite places which divide the order of A , and, by the Lang–Weil–Nisnevich bounds [LW54] [Nis54] and by a geometric version of Chebotarev’s density theorem [Eke90, Lemma 1.2], that the following hold:

- every closed fibre of f above \mathcal{U} contains a smooth rational point;
- for every $m \in M$, every closed fibre of $f_m : \mathcal{Y}_m^0 \rightarrow \tilde{m}'$ contains a rational point;
- for every $m \in M$ and every place u of L_m^0 which splits in K_m^0 and does not lie above a place of S , any element of H_m can be realised as the Frobenius automorphism of the irreducible abelian étale cover $\mathcal{E}_m \rightarrow \mathcal{Y}_m^0$ at some rational point of the fibre of f_m above the closed point of \tilde{m}' corresponding to u .

Step 3. *Application of Conjecture F_+ .*

Let us fix a collection $(v_m)_{m \in M}$ of places of k that are pairwise distinct and do not belong to S , such that v_m splits completely in K_m for all $m \in M$. For each $m \in M$, let us fix a place w_m of $k(m)$ dividing v_m , an element $t_{v_m} \in k_{v_m}$ such that $t_{v_m} - a_m$ is a uniformiser at w_m , and a point $z''_{v_m} \in \mathcal{W}(\mathcal{O}_{v_m})$ such that $p(z''_{v_m}) = t_{v_m}$, where we view t_{v_m} inside $k_{v_m} = \mathbf{A}^1(k_{v_m}) \subset \mathbf{P}^1(k_{v_m})$ (see Proposition 3.8 for the existence of z''_{v_m}). For every $v \in \Omega \setminus \{v_m; m \in M\}$, we set $z''_v = z'_v$. We have thus defined an adelic point $(z''_v)_{v \in \Omega} \in W(\mathbf{A}_k)$ such that $z''_v \in \mathcal{W}(\mathcal{O}_v)$ for all $v \in \Omega \setminus S$.

We now apply Conjecture F_+ to π_+ and $(z''_v)_{v \in \Omega}$ and deduce that there exist $c \in U(k)$ and $(z'''_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)^B$ arbitrarily close to $(z''_v)_{v \in \Omega}$ in $W(\mathbf{A}_k)$, in particular such that $z'''_v \in \mathcal{W}(\mathcal{O}_v)$ for all $v \in \Omega \setminus S$ and such that $\sum_{v \in S} \text{inv}_v \beta(z'''_v) = 0$ for all $\beta \in B$.

For each $v \in \Omega \setminus S$, let $(\lambda_v''', \mu_v''', (z_{m,u}''')_{m \in M, u|v})$ be the data corresponding to z_v''' in the notation of Proposition 3.8 and let w denote the closed point $\tilde{c} \cap \mathbf{P}_{\mathbf{F}_v}^1$ of $\mathbf{P}_{\mathbf{O}_S}^1$.

For $m \in M$, we set $\Omega_m = \{v \in \Omega \setminus S; w \in \tilde{m}\}$; equivalently, this is the set of places $v \in \Omega \setminus S$ such that $\lambda_v''' - a_m \mu_v'''$ has positive valuation at some place of $k(m)$ dividing v . The sets Ω_m for $m \in M$ are finite and pairwise disjoint. When $v \in \Omega_m$, we shall identify w with a place of $k(m)$ of degree 1 over v . For each $m \in M$, we may assume, by choosing z_{v_m}''' close enough to z_{v_m}'' , that $v_m \in \Omega_m$ and that $\lambda_{v_m}''' - a_m \mu_{v_m}'''$ is a uniformiser at w .

For later use, we note that according to Proposition 3.8, for each $m \in M$ and each $v \in \Omega_m$, there exists a unique place u of L_m dividing v such that $z_{m,u}'''$ is not a unit. Moreover, this place divides both w and v and it has degree 1 over them.

Step 4. *Construction of $(x_v''')_{v \in \Omega} \in X_c(\mathbf{A}_k)$ assuming given $(\sigma_m)_{m \in M} \in \prod_{m \in M} H_m$.*

For $v \in S$, we can ensure that c is arbitrarily close to $f(x'_v)$, by choosing z_v''' close enough to z_v'' . On the other hand, by the implicit function theorem, the map $X^0(k_v) \rightarrow \mathbf{P}^1(k_v)$ induced by f admits a local v -adic analytic section, around c , passing through x'_v . Thus, by choosing z_v''' close enough to z_v'' , we may assume that for every $v \in S$, there exists $x_v''' \in X_c(k_v)$ arbitrarily close to x'_v in $X(k_v)$. We fix x_v''' in this way for every $v \in S$. By ensuring that x_v''' is close enough to x'_v for $v \in S$, we may assume that

$$(4.5) \quad \sum_{v \in S} \text{inv}_v \alpha(x_v''') = 0$$

for all $\alpha \in A$.

For $v \in \Omega \setminus (S \cup \bigcup_{m \in M} \Omega_m)$, noting that $w \in \mathcal{U}$, we fix a smooth rational point of the fibre of $f : \mathcal{X} \rightarrow \mathbf{P}_{\mathbf{O}_S}^1$ above w and use Hensel's lemma to lift it to a k_v -point x_v''' of X_c .

For any $m \in M$ and any $v \in \Omega_m \setminus \{v_m\}$, let us consider the trace on L_m^0 of the unique place u of L_m dividing v such that $z_{m,u}'''$ is not a unit. It defines a closed point of \tilde{m}' . We fix a rational point ξ_v of the fibre of $f_m^0 : \mathcal{Y}_m^0 \rightarrow \tilde{m}'$ above this closed point and, viewing it as a smooth rational point of the fibre of $f : \mathcal{X} \rightarrow \mathbf{P}_{\mathbf{O}_S}^1$ above w , we lift it, using Hensel's lemma, to a k_v -point x_v''' of X_c .

Let us assume that we are given a family $(\sigma_m)_{m \in M} \in \prod_{m \in M} H_m$ (to be specified at the end of Step 5). Then, for $m \in M$, we construct x_v''' for $v = v_m$ in the exact same way as we constructed x_v''' for $v \in \Omega_m \setminus \{v_m\}$, except that we require, in addition, that the Frobenius automorphism of the irreducible abelian étale cover $\mathcal{E}_m \rightarrow \mathcal{Y}_m^0$ at ξ_v be equal to σ_m .

At this stage, we have now constructed an adelic point $(x_v''')_{v \in \Omega}$ of $X_c(\mathbf{A}_k)$, depending on the choice of $(\sigma_m)_{m \in M} \in \prod_{m \in M} H_m$. To conclude the proof of the theorem, it only remains to show that the family $(\sigma_m)_{m \in M}$ can be prescribed in such a way that $(x_v''')_{v \in \Omega}$ automatically belongs to $X_c(\mathbf{A}_k)^A$.

Step 5. *Evaluation of the Brauer–Manin obstruction.*

For $m \in M$ and $v \in \Omega_m$, let n_v denote the local intersection multiplicity of \tilde{c} and \tilde{m} at w inside $\mathbf{P}_{\mathbf{O}_S}^1$ (i.e. the length of the local ring of $\tilde{c} \cap \tilde{m}$ at w) and let $\text{Fr}_{\xi_v} \in G_m$ denote the Frobenius automorphism of the irreducible abelian étale cover $\mathcal{E}_m \rightarrow \mathcal{Y}_m^0$ at ξ_v .

Lemma 4.3. *For each $m \in M$, the element*

$$\sum_{v \in \Omega_m \setminus \{v_m\}} n_v \text{Fr}_{\xi_v}$$

of G_m belongs to the subgroup H_m .

Proof. Let us fix $m \in M$. We recall that $(z_v''')_{v \in \Omega} \in W_c(\mathbf{A}_k)^B$ and that $\sum_{v \in S} \text{inv}_v \beta(z_v''') = 0$ for all $\beta \in B$, so that $\sum_{v \in \Omega \setminus S} \text{inv}_v \beta(z_v''') = 0$ for all $\beta \in B$. Let us apply this equality to the class $\beta = \text{Cores}_{L_m/k}(z_m, \chi)$ for $\chi \in C_m$. For $v \in \Omega \setminus (S \cup \Omega_m)$, we have $\text{inv}_v \beta(z_v''') = 0$ since χ is unramified above v and $z_{m,u}'''$ is a unit at all places u of L_m above v (see Proposition 3.8). For $v \in \Omega_m$, we have $\text{inv}_v \beta(z_v''') = \text{inv}_u(z_{m,u}''', \chi)$ where u is the unique place of L_m dividing v such that $z_{m,u}'''$ fails to be a unit; moreover, the normalised valuation of $z_{m,u}'''$ is equal to n_v , so that $\text{inv}_u(z_{m,u}''', \chi) = n_v \chi(\text{Fr}_u)$, where we view χ as a homomorphism $\text{Gal}(K_m/L_m) \rightarrow \mathbf{Q}/\mathbf{Z}$ and $\text{Fr}_u \in \text{Gal}(K_m/L_m)$ denotes the Frobenius automorphism at u . Finally, we recall that $\text{Fr}_u = 0$ if $v = v_m$, since v_m splits completely in K_m . All in all, we conclude that

$$(4.6) \quad \sum_{v \in \Omega_m \setminus \{v_m\}} n_v \chi(\text{Fr}_u) = 0$$

for all $\chi \in C_m = \text{Hom}(\text{Gal}(K_m/L_m), \mathbf{Q}/\mathbf{Z})$, hence $\sum_{v \in \Omega_m \setminus \{v_m\}} n_v \text{Fr}_u = 0$ in $\text{Gal}(K_m/L_m)$ by Pontrjagin duality. Applying the natural map $\text{Gal}(K_m/L_m) \rightarrow \text{Gal}(K_m^0/L_m^0)$ to this equality now yields the statement of the lemma, as the image of Fr_u by this map coincides with the image of Fr_{ξ_v} by the quotient map $G_m \rightarrow G_m/H_m = \text{Gal}(K_m^0/L_m^0)$. \square

For $m \in M$ and $\alpha \in A$, let $\partial_{\alpha,m} \in H^1(G_m, \mathbf{Q}/\mathbf{Z}) \subset H^1(k(Y_m), \mathbf{Q}/\mathbf{Z})$ denote the residue of α at the generic point of Y_m . Viewing $\partial_{\alpha,m}$ as a homomorphism $G_m \rightarrow \mathbf{Q}/\mathbf{Z}$, we have

$$(4.7) \quad \text{inv}_v \alpha(x_v''') = n_v \partial_{\alpha,m}(\text{Fr}_{\xi_v})$$

for any $m \in M$ and any $v \in \Omega_m$, since $\alpha \in \text{Br}(f^{-1}(\mathcal{U}))$ (see [Har94, Corollaire 2.4.3]). Moreover, we have $\text{inv}_v \alpha(x_v''') = 0$ for all $v \in \Omega \setminus (S \cup \bigcup_{m \in M} \Omega_m)$, as $\alpha \in \text{Br}(f^{-1}(\mathcal{U}))$ and x_v''' is an \mathcal{O}_v -point of $f^{-1}(\mathcal{U})$ while $\text{Br}(\mathcal{O}_v) = 0$. In view of these remarks and of (4.5), we deduce that in order for $(x_v''')_{v \in \Omega}$ to be orthogonal to A with respect to the Brauer–Manin pairing, it suffices that the equality

$$(4.8) \quad \sum_{v \in \Omega_m} n_v \text{Fr}_{\xi_v} = 0$$

hold in G_m for all $m \in M$. Now, as $n_{v_m} = 1$ and as $\text{Fr}_{\xi_{v_m}} = \sigma_m$ for every $m \in M$, we can force (4.8) to hold by choosing $\sigma_m = -\sum_{v \in \Omega_m \setminus \{v_m\}} n_v \text{Fr}_{\xi_v}$ for all $m \in M$ in Step 4, thanks to Lemma 4.3. This concludes the proof of Theorem 4.1. \square

4.2. Stability of Conjecture F_+ . We first use Theorem 4.1 to show, in Corollary 4.4 below, that when the b_m are allowed to vary, Conjecture F_+ is stable under the operation of replacing the fields L_m and K_m by subfields.

Corollary 4.4. *Let k be a number field and $\pi_+^0 \in \mathcal{P}_+$ be a parameter for Conjecture F_+ . Write $\pi_+^0 = (M, (L_m^0)_{m \in M}, (b_m^0)_{m \in M}, (K_m^0)_{m \in M})$. For every $m \in M$, suppose given a finite extension L_m of L_m^0 and a finite abelian extension K_m of L_m in which K_m^0 can be embedded L_m^0 -linearly. Assume that for all choices of $(b_m)_{m \in M} \in \prod_{m \in M} k(m)^*$, Conjecture F_+ holds for $\pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M}) \in \mathcal{P}_+$. Then Conjecture F_+ holds for π_+^0 .*

Proof. Set $U = \mathbf{P}_k^1 \setminus M$. Let $p : W \rightarrow \mathbf{P}_k^1$ and $B \subset \text{Br}(p^{-1}(U))$ denote the morphism and the subgroup associated in §3.1 and §3.6 with the parameter π_+^0 . Taking Proposition 3.3 and Proposition 3.7 into account, we may apply Theorem 4.1 to $X = W$, $f = p$, $A = B$, $H = \mathbf{P}_k^1$, $Y_m = W_m$ and $E_m = k(W_m) \otimes_{L_m^0} K_m^0$. In view of Proposition 3.5, the desired conclusion follows. \square

4.3. Specialisation of the Brauer group. To obtain concrete corollaries for the fibration method, we shall apply Theorem 4.1 in §4.4 in conjunction with the following specialisation result for the Brauer group, which goes back to the work of Harari [Har94, Har97]. The version we state here simultaneously generalises [HW16, Proposition 4.1] (in which f was assumed to be proper) and [CTH16, Théorème 2.7] (in which the generic fibre of f was assumed to be a homogeneous space of a connected, semi-simple, simply connected linear algebraic group, with connected and reductive geometric stabilisers).

Proposition 4.5. *Let C be a smooth, irreducible curve over a number field k . Let X be a smooth, separated, irreducible variety over k , endowed with a morphism $f : X \rightarrow C$ whose geometric generic fibre $X_{\bar{\eta}}$ is irreducible. Assume that $H_{\text{ét}}^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}) = 0$ and that $\text{Br}(X_{\bar{\eta}})$ is finite. Let $C^0 \subseteq C$ be a dense open subset, let $X^0 = f^{-1}(C^0)$ and let $B \subseteq \text{Br}(X^0)$ be a subgroup. If the natural map $B \rightarrow \text{Br}(X_{\bar{\eta}})/f_{\bar{\eta}}^* \text{Br}(\bar{\eta})$ is surjective, there exists a Hilbert subset $H \subseteq C^0$ such that the natural map $B \rightarrow \text{Br}(X_h)/f_h^* \text{Br}(h)$ is surjective for all $h \in H$.*

Proof. The proof given in [HW16, Proposition 4.1] for the special case in which f is proper works verbatim under our hypotheses, once the first sentence of the proof is replaced with the following: by shrinking C , we may assume that $C^0 = C$ and, by the next lemma, that the étale sheaf $R^2 f_* \mathbf{Q}/\mathbf{Z}(1)$ is a direct limit of locally constant sheaves with finite stalks and that $H_{\text{ét}}^1(X_{\bar{h}}, \mathbf{Q}/\mathbf{Z}(1)) = 0$ for any geometric point \bar{h} of C . \square

Lemma 4.6. *For any $q \geq 0$, there exists a dense open subset $U \subseteq C$ satisfying the following properties: the restriction of the étale sheaf $R^q f_* \mathbf{Q}/\mathbf{Z}(1)$ to U is a direct limit of locally constant sheaves with finite stalks, and the stalk of $R^q f_* \mathbf{Q}/\mathbf{Z}(1)$ at any geometric point \bar{h} of U is naturally isomorphic to $H_{\text{ét}}^q(X_{\bar{h}}, \mathbf{Q}/\mathbf{Z}(1))$.*

Proof. After shrinking C , we may assume, thanks to Nagata [Nag62, Del10, Con07] and to Hironaka [Hir64, §3, Main theorem I, §5, Corollary 3], that there exist an open immersion $j : X \rightarrow X'$ and a smooth and proper morphism $g : X' \rightarrow C$ such that $f = g \circ j$ and that $X' \setminus X$ is a divisor with simple normal crossings on X' relatively to C (in the sense of [G⁺71, Exp. XIII, §2.1]). The morphism f is then cohomologically proper (with respect to μ_n for any $n > 0$, see [Del77, Appendice, §1.3.1, §1.3.3]), which already ensures the second part of the assertion. On the other hand, it is a locally acyclic morphism, since

it is smooth (see [Mil80, Theorem 4.15]). Hence the étale sheaf $R^q f_* \mathbf{Q}/\mathbf{Z}(1)$ is a direct limit of locally constant sheaves with finite stalks (see [Del77, Appendice, §2.4] and [Art73, Proposition 2.11]; or see the proof of [Mil80, Corollary 4.2]). \square

4.4. Main corollary. It is through Corollary 4.7 below, which depends on Proposition 4.5, that we shall apply Theorem 4.1 in §7.

Corollary 4.7. *Let X be a smooth, separated, irreducible variety over a number field k and $f : X \rightarrow \mathbf{P}_k^1$ be a morphism with irreducible geometric generic fibre $X_{\bar{\eta}}$. Assume that*

- (i) *the group $H_{\text{ét}}^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z})$ vanishes, the group $\text{Br}(X_{\bar{\eta}})$ is finite,*
- (ii) *every fibre of f contains an irreducible component of multiplicity 1,*
- (iii) *the geometric fibre $X_{\infty} = f^{-1}(\infty) \otimes_k \bar{k}$ is smooth and irreducible, and the group $H_{\text{ét}}^1(X_{\infty}, \mathbf{Q}/\mathbf{Z})$ vanishes.*

For each $m \in \mathbf{A}_k^1$, choose an irreducible component of multiplicity 1 of $f^{-1}(m)$ and let L_m denote the algebraic closure of $k(m)$ in its function field. Finally, assume that for all finite subsets $M \subset \mathbf{A}_k^1$ and for all choices of $(b_m)_{m \in M}$ and of $(K_m)_{m \in M}$, Conjecture F_+ holds for the parameter $\pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M})$. Then, for any Hilbert subset $H \subset \mathbf{P}_k^1$, the subset

$$(4.9) \quad \bigcup_{c \in U(k) \cap H} X_c(\mathbf{A}_k)^{\text{Br}(X_c)}$$

is dense in $X(\mathbf{A}_k)^{\text{Br}(X)}$. In particular, if $X_c(k)$ is dense in $X_c(\mathbf{A}_k)^{\text{Br}(X_c)}$ for all rational points c of a Hilbert subset of \mathbf{P}_k^1 , then $X(k)$ is dense in $X(\mathbf{A}_k)^{\text{Br}(X)}$.

Proof. By the Kummer exact sequence, we deduce from the vanishing of $H_{\text{ét}}^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z})$ that the group $\text{Pic}(X_{\bar{\eta}})$ is torsion-free and that every invertible function on $X_{\bar{\eta}}$ is constant. From these two facts and from the finiteness of $\text{Br}(X_{\bar{\eta}})$, it follows, by the Hochschild–Serre spectral sequence, that the quotient $\text{Br}(X_{\bar{\eta}})/f_{\bar{\eta}}^* \text{Br}(\eta)$ is finite, where $f_{\bar{\eta}} : X_{\bar{\eta}} \rightarrow \eta$ denotes the generic fibre of f . Thanks to (iii), we can therefore choose a finite subgroup $A \subset \text{Br}(X_{\bar{\eta}})$ which surjects onto $\text{Br}(X_{\bar{\eta}})/f_{\bar{\eta}}^* \text{Br}(\eta)$ and whose elements are unramified along $f^{-1}(\infty)$. Let $U \subset \mathbf{P}_k^1$ be a dense open subset containing ∞ such that $A \subset \text{Br}(f^{-1}(U))$. According to Proposition 4.5, there exists a Hilbert subset $H_0 \subseteq U$ such that the natural map $A \rightarrow \text{Br}(X_c)/f_c^* \text{Br}(k)$ is surjective for all $c \in H_0 \cap \mathbf{P}_k^1(k)$. To conclude the proof, we now apply Theorem 4.1 to the Hilbert subset $H \cap H_0$. \square

Remarks 4.8. (i) When assumption (i) of Corollary 4.7 is satisfied, assumption (iii) can always be made to hold by a change of coordinates on \mathbf{P}_k^1 .

(ii) If f is proper and $X_{\bar{\eta}}$ is rationally connected, assumptions (i) and (ii) of Corollary 4.7 are satisfied (see [Deb01, Corollary 4.18(b)], [CTS13, Lemma 1.3 (i)], [HW16, Lemma 8.6], [GHS03, Theorem 1.1]).

5. COMPARING CONJECTURES F AND F₊

This section is devoted to a detailed study of the relationship between Conjecture F and Conjecture F₊. In order to facilitate their comparison, we introduce an intermediate statement, Conjecture F_{const}. We prove that when the parameters are allowed to vary, the three conjectures are equivalent, and that in certain special circumstances, Conjecture F coincides with Conjecture F_{const}, while in certain other special circumstances, Conjecture F_{const} coincides with Conjecture F₊. One advantage of considering Conjecture F_{const} is that under some abelianness assumptions, it is implied by Schinzel's hypothesis (HH₁), as we will see in §6. This is what will eventually allow us to deduce Theorem 1.4.

5.1. Introduction. A number field k is fixed until the end of §5.3. Let us first formulate Conjecture F_{const}. Given $\pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M}) \in \mathcal{P}_+$, we define

$$C_{m, \text{const}} = C_m \cap \text{Im} \left(H^1(k(m), \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(L_m, \mathbf{Q}/\mathbf{Z}) \right)$$

for all $m \in M$ and let $B_{\text{const}} \subseteq B$ be the subgroup generated by the classes $\text{Cores}_{L_m/k}(z_m, \chi)$ for $m \in M$ and $\chi \in C_{m, \text{const}}$.

Conjecture F_{const}. Let $\pi_+ \in \mathcal{P}_+$. The subset $\bigcup_{c \in U(k)} W_c(\mathbf{A}_k)^{B_{\text{const}}}$ is dense in $W(\mathbf{A}_k)$.

We note right away, in the next proposition, that when the parameters are allowed to vary, the conjectures we have introduced are all equivalent. Thus, the existing evidence for Conjecture F (see [HW16, §9.2], [BS19]) lends support to Conjecture F₊ as well.

Proposition 5.1. *The following statements are equivalent:*

- (1) Conjecture F holds for all $\pi \in \mathcal{P}$;
- (2) Conjecture F_{const} holds for all $\pi_+ \in \mathcal{P}_+$;
- (3) Conjecture F₊ holds for all $\pi_+ \in \mathcal{P}_+$.

Proof. The implications (3) \Rightarrow (2) \Rightarrow (1) being obvious, we need only prove that (1) \Rightarrow (3). Let us fix $\pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M}) \in \mathcal{P}_+$. Among the conditions

- (i) Conjecture F for $(M, (K_m)_{m \in M}, (b'_m)_{m \in M})$ for all choices of $(b'_m)_{m \in M}$,
- (ii) Conjecture F₊ for $(M, (K_m)_{m \in M}, (b'_m)_{m \in M}, (K_m)_{m \in M})$ for all choices of $(b'_m)_{m \in M}$,
- (iii) Conjecture F₊ for π_+ ,

we have (i) \Leftrightarrow (ii) by definition, and (ii) \Rightarrow (iii) by Corollary 4.4; hence (1) \Rightarrow (3). \square

From here onwards, we shall systematically consider the three conjectures with fixed parameters. Whenever a quadruple $\pi_+ \in \mathcal{P}_+$ is given, it will be understood that the notation $M, (L_m)_{m \in M}, (b_m)_{m \in M}$ and $(K_m)_{m \in M}$ refers to its components, unless specified otherwise, and that π denotes the underlying triple $(M, (L_m)_{m \in M}, (b_m)_{m \in M}) \in \mathcal{P}$.

It is immediate that Conjecture F₊ for π_+ implies Conjecture F_{const} for π_+ and that the latter in turn implies Conjecture F for π . We shall now attempt to reverse these trivial implications.

5.2. From Conjecture F to Conjecture F_{const} .

Proposition 5.2. *Let $\pi_+ \in \mathcal{P}_+$. Assume that the two groups*

$$(5.1) \quad \bigoplus_{m \in M} \text{Ker} (H^1(k(m), \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(L_m, \mathbf{Q}/\mathbf{Z}))$$

and

$$(5.2) \quad \bigoplus_{m \in M} \text{Ker} (H^1(k(m), \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(K_m, \mathbf{Q}/\mathbf{Z}))$$

have the same image by the “sum of corestrictions” map

$$(5.3) \quad \bigoplus_{m \in M} H^1(k(m), \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(k, \mathbf{Q}/\mathbf{Z}).$$

Then Conjecture F for π implies Conjecture F_{const} for π_+ .

Proof. Let us check that the assumption of the proposition is in fact equivalent to the inclusion $B_{\text{const}} \subset p^*\text{Br}(U)$ of subgroups of $\text{Br}(p^{-1}(U))$. This will prove the proposition, as this inclusion implies that $W_c(\mathbf{A}_k) = W_c(\mathbf{A}_k)^{B_{\text{const}}}$ for all $c \in U(k)$.

Assume first that $B_{\text{const}} \subset p^*\text{Br}(U)$ and let $(\chi_m)_{m \in M}$ belong to (5.2). Let $\gamma \in \text{Br}(U)$ be such that $\sum_{m \in M} \text{Cores}_{L_m/k}(z_m, \chi_m) = p^*\gamma$. By the Faddeev exact sequence, the family $(\partial_m \gamma)_{m \in M}$ given by the residues of γ belongs to the kernel of (5.3) (see [CTSD94, §1.2]). On the other hand, by Proposition 3.3, Proposition 3.7 and [CTSD94, Proposition 1.1.1], computing residues at the generic point of W_m for $m \in M$ shows that $(\partial_m \gamma - \chi_m)_{m \in M}$ belongs to (5.1). We have thus proved that (5.1) and (5.2) have the same image by (5.3).

Conversely, assuming that this last condition holds, let us fix $\beta \in B_{\text{const}}$ and write it as $\beta = \sum_{m \in M} \text{Cores}_{L_m/k}(z_m, \chi_m)$ with $(\chi_m)_{m \in M}$ in (5.2). Our assumption provides $(\delta_m)_{m \in M}$ in the kernel of (5.3) such that $(\delta_m - \chi_m)_{m \in M}$ belongs to (5.1). By the Faddeev exact sequence, there exists $\gamma \in \text{Br}(U)$ such that $\partial_m \gamma = \delta_m$ for all $m \in M$. By Proposition 3.3 and Proposition 3.7, the class $\beta - p^*\gamma \in \text{Br}(p^{-1}(U))$ belongs to the subgroup $\text{Br}(W)$. By Proposition 3.5, we conclude that $\beta \in p^*\text{Br}(U)$. \square

Corollary 5.3. *Suppose given a finite closed subset $M \subset \mathbf{A}_k^1$, a point $m_0 \in M$ and, for each $m \in M \setminus \{m_0\}$, a finite extension $L_m/k(m)$ and a finite abelian extension K_m/L_m . Assume that $k(m_0) = k$ or that k is totally imaginary. Then there exists a finite abelian extension $L_0/k(m_0)$ with the following property: for any $(b_m)_{m \in M} \in \prod_{m \in M} k(m)^*$ and any finite extension L_{m_0}/L_0 , if we set $K_{m_0} = L_{m_0}$, Conjecture F for $\pi = (M, (L_m)_{m \in M}, (b_m)_{m \in M})$ implies Conjecture F_{const} for $\pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M})$.*

Proof. The corestriction map $H^1(k(m_0), \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(k, \mathbf{Q}/\mathbf{Z})$ is surjective since $k(m_0) = k$ or k is totally imaginary (see [Gra03, Note, p. 327], [HW16, Remark 9.18 (ii)]). Hence we can choose a finite subgroup $C \subset H^1(k(m_0), \mathbf{Q}/\mathbf{Z})$ whose image by this map coincides with the image, by the map (5.3), of the finite group

$$(5.4) \quad \bigoplus_{m \in M \setminus \{m_0\}} \text{Ker} (H^1(k(m), \mathbf{Q}/\mathbf{Z}) \rightarrow H^1(K_m, \mathbf{Q}/\mathbf{Z})).$$

Let $L_0/k(m_0)$ be a finite abelian extension such that the image of C in $H^1(L_0, \mathbf{Q}/\mathbf{Z})$ vanishes. The assumption of Proposition 5.2 is now satisfied for any L_{m_0} and K_{m_0} as in the statement of the corollary. \square

Remarks 5.4. (i) More generally, one can verify that in the situation of [HW16, Theorem 9.17], if we set $K_m = L_m$ for $m \in M'$ and $L_m = k(m)$ for $m \in M''$ and if we are given finite abelian extensions $K_m/k(m)$ for $m \in M''$, the assumption that the map (9.9) of *loc. cit.* is onto for all $m \in M''$ (an assumption that is satisfied when M' contains a rational point or k is totally imaginary, by [HW16, Remark 9.18 (ii)]) implies that the assumption of Proposition 5.2 holds, with $M = M' \cup M''$, as soon as the fields L_m for $m \in M'$ are large enough (in the sense that they contain certain subfields, as in Corollary 5.3).

(ii) Conjecture F_{const} for π_+ is the same as Conjecture F_+ for π_+ when for all $m \in M$, at least one of the extensions K_m/L_m and $L_m/k(m)$ is trivial.

(iii) In view of Remark 3.2 and of Remarks 5.4 (i)–(ii), the statement of [HW16, Theorem 9.17] can be recovered by combining Theorem 4.1 with Proposition 5.2.

5.3. From Conjecture F_{const} to Conjecture F_+ . The next theorem, which gives an equivalent formulation for Conjecture F_+ , will allow us to pass from Conjecture F_{const} to Conjecture F_+ in more general situations than that of Remark 5.4 (ii).

Let $\pi_+ \in \mathcal{P}_+$. For $m \in M$, we denote by $C_{m,\text{nr}} \subseteq C_m$ the subgroup consisting of those $\chi \in C_m$ such that the class

$$(5.5) \quad \text{Cores}_{L_m/k(m)}(z, \chi) \in \text{Br}(R_{L_m/k(m)}^1 \mathbf{G}_m)$$

belongs to the subgroup $\text{Br}_{\text{nr}}(R_{L_m/k(m)}^1 \mathbf{G}_m)$, where z stands for the tautological invertible function on $R_{L_m/k(m)}^1 \mathbf{G}_m \otimes_{k(m)} L_m$. We denote by $B_{\text{nr}} \subseteq B$ the subgroup generated by the classes $\text{Cores}_{L_m/k}(z_m, \chi) \in \text{Br}(p^{-1}(U))$ for $m \in M$ and $\chi \in C_{m,\text{nr}}$.

Theorem 5.5. *Let $\pi_+ \in \mathcal{P}_+$. The following conditions are equivalent:*

- (1) *the subset $\bigcup_{c \in U(k)} W_c(\mathbf{A}_k)^{B_{\text{nr}}}$ is dense in $W(\mathbf{A}_k)$;*
- (2) *Conjecture F_+ holds for π_+ .*

Proof. The implication (2) \Rightarrow (1) being trivial, we assume (1) and prove (2). To this end, we fix an adelic point $(z_v)_{v \in \Omega} \in W(\mathbf{A}_k)$ and a finite subset $S \subset \Omega$ large enough that it satisfies the assumptions of Proposition 3.8. We take up the notation of §3.5 for the standard integral model $p: \mathcal{W} \rightarrow \mathbf{P}_{\mathcal{O}_S}^1$ of $p: W \rightarrow \mathbf{P}_k^1$, for the Zariski closure \tilde{m} of $m \in \mathbf{P}_k^1$ in $\mathbf{P}_{\mathcal{O}_S}^1$, and for $\tilde{M} = \bigcup_{m \in M} \tilde{m}$. We choose S large enough that $z_v \in \mathcal{W}(\mathcal{O}_v)$ for all $v \in \Omega \setminus S$. Our goal is then to produce $c \in U(k)$ and $(z'_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)^B$ with z'_v arbitrarily close to z_v for $v \in S$ and $z''_v \in \mathcal{W}(\mathcal{O}_v)$ for $v \in \Omega \setminus S$.

For each $m \in M$, fix a Galois closure J_m/k of K_m/k . We denote by Θ the set of triples (m, σ, τ) , where $m \in M$ and $(\sigma, \tau) \in \text{Gal}(J_m/L_m) \times \text{Gal}(J_m/k(m))$ are such that $\tau\sigma\tau^{-1} \in \text{Gal}(J_m/L_m)$. For $m \in M$, we denote by $\Theta_m \subseteq \Theta$ the subset of triples whose first component is m and by $\varphi_m: \Theta_m \rightarrow \text{Gal}(K_m/L_m)$ the map that sends (m, σ, τ) to the restriction, to K_m , of the automorphism $\sigma\tau\sigma^{-1}\tau^{-1}$ of J_m .

For $\theta = (m, \sigma, \tau) \in \Theta$, Chebotarev's density theorem ensures the existence of infinitely many places of J_m that do not lie over a place of S and whose Frobenius automorphism in $\text{Gal}(J_m/k)$ is equal to σ . For each θ , we choose such a place r_θ of J_m and let v_θ denote its trace on k . We choose the r_θ for $\theta \in \Theta$ in such a way that the v_θ are pairwise distinct, and let $T \subset \Omega \setminus S$ denote the subset consisting of the places v_θ for $\theta \in \Theta$.

Given $\theta = (m, \sigma, \tau) \in \Theta$, we denote by w_θ the trace of r_θ on $k(m)$ and by $u_{\theta,1}$ and $u_{\theta,2}$ the traces of r_θ and of $\tau(r_\theta)$ on L_m , respectively. We note that $u_{\theta,1}$ and $u_{\theta,2}$ both divide w_θ and that the Frobenius automorphisms of r_θ and of $\tau(r_\theta)$ in $\text{Gal}(J_m/k)$ are equal to σ and to $\tau\sigma\tau^{-1}$, respectively. In particular, letting $\text{Fr}_{u_{\theta,i}} \in \text{Gal}(K_m/L_m)$ denote the Frobenius automorphism of the abelian extension K_m/L_m at $u_{\theta,i}$, we have

$$(5.6) \quad \text{Fr}_{u_{\theta,1}} - \text{Fr}_{u_{\theta,2}} = \varphi_m(\theta)$$

in the abelian group $\text{Gal}(K_m/L_m)$.

Lemma 5.6. *Let $\theta = (m, \sigma, \tau) \in \Theta$. For ease of notation, set $v = v_\theta$ and $w = w_\theta$.*

- (1) *There exists $t_v \in k_v$ such that $t_v - a_m \in k(m)_w$ is a uniformiser.*
- (2) *For any $t_v \in k_v$ such that $t_v - a_m \in k(m)_w$ is a uniformiser and any $i \in \{1, 2\}$, there exists $z_{v,i} \in \mathscr{W}(\mathcal{O}_v)$ such that $p(z_{v,i}) = t_v$ (viewing t_v in $k_v = \mathbf{A}^1(k_v) \subset \mathbf{P}^1(k_v)$) and such that for all $m' \in M$ and all $\chi \in C_{m'}$, the following equality holds:*

$$(5.7) \quad \text{inv}_v(\text{Cores}_{L_{m'}/k}(z_{m'}, \chi))(z_{v,i}) = \begin{cases} \chi(\text{Fr}_{u_{\theta,i}}) & \text{if } m' = m, \\ 0 & \text{if } m' \neq m. \end{cases}$$

In the right-hand side of (5.7), we view χ as a homomorphism $\text{Gal}(K_m/L_m) \rightarrow \mathbf{Q}/\mathbf{Z}$.

Proof. As the Frobenius automorphisms of r_θ and of $\tau(r_\theta)$ in $\text{Gal}(J_m/k)$ lie in $\text{Gal}(J_m/L_m)$, the three places $u_{\theta,1}$, $u_{\theta,2}$ and w have degree 1 over v . Assertion (1) follows.

Let now t_v be as in (2), in particular $t_v \in \mathcal{O}_v$. As $\widetilde{M} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ is regular, we may interpret its closed points as pairs (m', w') where $m' \in M$ and w' is a place of $k(m')$ dividing v . For any $(m', w') \in \widetilde{M} \otimes_{\mathcal{O}_S} \mathcal{O}_v$, the element $t_v - a_{m'}$ has positive w' -adic valuation if and only if the Zariski closure of $t_v \in \mathbf{P}^1(k_v)$ in $\mathbf{P}^1_{\mathcal{O}_v}$ meets $\widetilde{M} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ at (m', w') . As $\widetilde{M} \cup \infty$ is étale over \mathcal{O}_S and as $t_v - a_m$ is a uniformiser at w , it follows that $t_v - a_{m'}$ is a unit at w' for all $(m', w') \in \widetilde{M} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ distinct from (m, w) .

Fix $i \in \{1, 2\}$ and set $\lambda_v = t_v$ and $\mu_v = 1$. By the conclusions of the last two paragraphs, there exists $(z_{m',u})_{m' \in M, u|v} \in \prod_{m' \in M, u|v} \mathcal{O}_{(L_{m'})_u}$, where u runs over the places of $L_{m'}$ dividing v , that satisfies the conditions of Proposition 3.8 as well as the following property: z_{m,u_i} is a uniformiser while $z_{m',u}$ is a unit for every $m' \in M$ and every place u of $L_{m'}$ dividing v such that $(m', u) \neq (m, u_i)$. By Proposition 3.8, the family $(\lambda_v, \mu_v, (z_{m',u})_{m' \in M, u|v})$ then gives rise to a point $z_{v,i} \in \mathscr{W}(\mathcal{O}_v)$ such that $p(z_{v,i}) = t_v$.

For any $m' \in M$ and $\chi \in C_{m'}$, the left-hand side of (5.7) is equal to $\sum_{u|v} \text{inv}_u(z_{m',u}, \chi)$, where the sum runs over the places u of $L_{m'}$ dividing v (see [Har20, Theorem 8.9]). As χ is unramified above v and as $z_{m',u}$ is a unit if $m' \neq m$ or if $u \neq u_i$, and is a uniformiser otherwise, the validity of (5.7) follows (see [Har20, Corollary 9.6]). \square

For $v \in \Omega \setminus T$, we set $z'_v = z_v$. For each $v \in T$, let us apply the two parts of Lemma 5.6, with $i = 1$, and denote by $z'_v \in \mathscr{W}(\mathcal{O}_v)$ the resulting local integral point.

Applying our assumption (1) to the adelic point $(z'_v)_{v \in \Omega} \in W(\mathbf{A}_k)$ thus constructed yields $c \in U(k)$ and $(z''_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)^{B_{\text{nr}}}$, with $(z''_v)_{v \in \Omega}$ arbitrarily close to $(z'_v)_{v \in \Omega}$. We may, in particular, assume that $c \neq \infty$, that z''_v is arbitrarily close to z_v for $v \in S$, that $z''_v \in \mathscr{W}(\mathcal{O}_v)$ for all $v \in \Omega \setminus S$ and that $\beta(z''_v) = \beta(z'_v)$ for all $v \in T$ and all $\beta \in B$ (so that $\text{inv}_v \beta(z''_v)$ is described by (5.7)).

Lemma 5.7. *There exists a subset $\Theta^\sharp \subseteq \Theta$ such that*

$$(5.8) \quad \sum_{v \in \Omega} \text{inv}_v (\text{Cores}_{L_m/k}(z_m, \chi))(z''_v) = \sum_{\theta \in \Theta^\sharp_m} \chi(\varphi_m(\theta))$$

for all $m \in M$ and all $\chi \in C_m$, where $\Theta^\sharp_m = \Theta^\sharp \cap \Theta_m$.

Proof. For $m \in M$, let $D_m \subseteq \text{Gal}(K_m/L_m)$ denote the subgroup generated by the image of φ_m . Under the perfect duality of finite abelian groups

$$(5.9) \quad C_m \times \text{Gal}(K_m/L_m) \rightarrow \mathbf{Q}/\mathbf{Z},$$

the subgroups $C_{m,\text{nr}}$ and D_m are exact orthogonal complements, according to Corollary 2.4. On the other hand, the homomorphism $C_m \rightarrow \mathbf{Q}/\mathbf{Z}$ that sends $\chi \in C_m$ to the left-hand side of (5.8) vanishes on $C_{m,\text{nr}}$, since $(z''_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)^{B_{\text{nr}}}$. It follows that there exists a family of integers $(n_\theta)_{\theta \in \Theta}$ such that for all $m \in M$ and all $\chi \in C_m$, the left-hand side of (5.8) is equal to $\sum_{\theta \in \Theta_m} n_\theta \chi(\varphi_m(\theta))$. Now one observes that for any $m \in M$ and any integer n , the image of φ_m is stable under multiplication by n in the abelian group $\text{Gal}(K_m/L_m)$. Hence we can choose the n_θ in $\{0, 1\}$, and the lemma is proved. \square

Let $T^\sharp \subseteq T$ denote the subset consisting of the places v_θ for $\theta \in \Theta^\sharp$. For each $v \in T^\sharp$, apply Lemma 5.6 (2) with $i = 2$ and with $t_v = c \in \mathbf{A}^1(k)$, and let $z'''_v \in \mathscr{W}(\mathcal{O}_v) \cap W_c(k_v)$ denote the resulting local integral point. Set $z'''_v = z''_v$ for all $v \in \Omega \setminus T^\sharp$. We have thus constructed an adelic point $(z'''_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)$ with $z'''_v \in \mathscr{W}(\mathcal{O}_v)$ for $v \in \Omega \setminus S$ and with z'''_v arbitrarily close to z_v for $v \in S$.

We finally check that $(z'''_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)^B$. It is enough to see that for any $m' \in M$ and any $\chi \in C_{m'}$, if we set $\beta = \text{Cores}_{L_{m'}/k}(z_{m'}, \chi)$, then $\sum_{v \in \Omega} \text{inv}_v \beta(z'''_v)$ vanishes. Now, as $z'''_v = z''_v$ for all $v \in \Omega \setminus T^\sharp$, this sum can be rewritten as

$$(5.10) \quad \sum_{v \in \Omega} \text{inv}_v \beta(z''_v) + \sum_{\theta \in \Theta^\sharp} \left(\text{inv}_{v_\theta} \beta(z'''_{v_\theta}) - \text{inv}_{v_\theta} \beta(z''_{v_\theta}) \right).$$

The first term of (5.10) is equal to $\sum_{\theta \in \Theta^\sharp_m} \chi(\varphi_{m'}(\theta))$, according to (5.8). On the other hand, the definition of z'_v and of z'''_v shows that for any $m \in M$ and any $\theta \in \Theta^\sharp_m$, the term $\text{inv}_{v_\theta} \beta(z'''_{v_\theta}) - \text{inv}_{v_\theta} \beta(z''_{v_\theta})$ vanishes if $m \neq m'$ and is equal to $\chi(\text{Fr}_{u_{\theta,2}}) - \chi(\text{Fr}_{u_{\theta,1}})$ otherwise (see (5.7)). In the latter case we have $\chi(\text{Fr}_{u_{\theta,2}}) - \chi(\text{Fr}_{u_{\theta,1}}) = -\chi(\varphi_{m'}(\theta))$ (see (5.6)). Hence (5.10) indeed vanishes. \square

Theorem 5.5 is especially useful when $B_{\text{nr}} = B_{\text{const}}$. The following corollary records situations in which this equality holds for purely algebraic reasons.

Corollary 5.8. *Let $\pi_+ \in \mathcal{P}_+$. Assume that for each $m \in M$, at least one of the following conditions is satisfied:*

- (1) *the torus $T = R_{L_m/k(m)}^1 \mathbf{G}_m$ over $k(m)$ satisfies $\text{Br}_{\text{nr}}(T) = \text{Br}_0(T)$;*
- (2) *the extension K_m/L_m is trivial.*

Then Conjecture F_{const} for π_+ implies Conjecture F_+ for π_+ .

Proof. Under either assumption, one has $C_{m,\text{nr}} = C_{m,\text{const}}$, according to Proposition 2.1 (1) in case (1), and because $C_m = 0$ in case (2). Hence $B_{\text{nr}} = B_{\text{const}}$. \square

Various conditions which ensure that $\text{Br}_{\text{nr}}(T) = \text{Br}_0(T)$ are listed in Proposition 2.6.

6. KNOWN CASES OF CONJECTURE F_+

The concrete cases in which Conjecture F is currently known to hold are those listed in [HW16, §9.2] and in [BS19]. In many of these cases, the underlying arguments can be enhanced to prove Conjecture F_+ as well, as we verify in this section.

6.1. From strong approximation. The following proposition strengthens [HW16, Corollary 9.10], where the analogous conclusion was obtained for Conjecture F .

Proposition 6.1. *Let $\pi_+ \in \mathcal{P}_+$. If the variety W satisfies strong approximation off v_0 for any finite place v_0 of k , then Conjecture F_+ holds for π_+ .*

Proof. Let $(z_v)_{v \in \Omega} \in W(\mathbf{A}_k)$. Let $S \subset \Omega$ be a finite subset of places. Let $p : \mathcal{W} \rightarrow \mathbf{P}_{\mathcal{O}_S}^1$ be the standard integral model of $p : W \rightarrow \mathbf{P}_k^1$ (see §3.5). We choose S large enough that it satisfies the assumptions of Proposition 3.8 and that $z_v \in \mathcal{W}(\mathcal{O}_v)$ for all $v \in \Omega \setminus S$. To prove the theorem, it suffices to produce $c \in U(k)$ and $(z'_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)^B$ such that z'_v lies arbitrarily close to z_v for $v \in S$ and that $z'_v \in \mathcal{W}(\mathcal{O}_v)$ for $v \in \Omega \setminus S$.

Fix a place $v_0 \in \Omega \setminus S$ that splits completely in K_m for all $m \in M$. By assumption, there exists $z \in \mathcal{W}(\mathcal{O}_{S \cup \{v_0\}})$ lying arbitrarily close to z_v for $v \in S$. Set $c = p(z)$. As v_0 splits completely in L_m for all $m \in M$, there exists $z'_{v_0} \in \mathcal{W}(\mathcal{O}_{v_0})$ such that $p(z'_{v_0}) = c$ (see Proposition 3.8). Set $z'_v = z$ for $v \in \Omega \setminus \{v_0\}$. For any $\beta \in B$, one has $\sum_{v \in \Omega} \text{inv}_v \beta(z) = 0$ (by the global reciprocity law) and $\text{inv}_{v_0} \beta(z'_{v_0}) = \text{inv}_{v_0} \beta(z) = 0$ (since v_0 splits completely in K_m for all $m \in M$), hence $\sum_{v \in \Omega} \text{inv}_v \beta(z'_v) = 0$, as desired. \square

In the next corollary, case (iv) is a delicate theorem of Browning and Schindler [BS19] resting on methods of analytic number theory.

Corollary 6.2. *Let $\pi_+ \in \mathcal{P}_+$. Let $M' = \{m \in M; L_m \neq k(m)\}$. Conjecture F_+ holds for π_+ under any of the following sets of assumptions:*

- (i) $\sum_{m \in M'} [k(m) : k] \leq 2$;
- (ii) $\sum_{m \in M'} [k(m) : k] = 3$ and $[L_m : k(m)] = 2$ for all $m \in M'$;
- (iii) $\sum_{m \in M'} [k(m) : k] = 3$ and there exists $m_0 \in M'$ such that $k(m_0) = k$ and such that $[L_m : k(m)] = 2$ for all $m \in M' \setminus \{m_0\}$;

(iv) $\sum_{m \in M'} [k(m) : k] = 3$, the set M' has cardinality 2, and $k = \mathbf{Q}$.

Proof. We claim that in all cases, the variety W satisfies strong approximation off v_0 for any finite place v_0 of k . To prove this, we may assume that $M = M'$, by Remark 3.2. In case (iv), the claim is then [BS19, Corollary 2.1]. In the other cases, one observes, as in the proof of [HW16, Theorem 9.11], that the variety W is isomorphic, in case (i), to the complement of a codimension 2 closed subset in an affine space, in case (ii), to the punctured affine cone over the complement of a codimension 2 closed subset in a smooth projective quadric of dimension 4, and in case (iii), to a variety as in the statement of Lemma 6.3 below. Such a variety satisfies strong approximation off v_0 for any finite place v_0 of k , in case (i) by [CX18, Proposition 3.6] or [Wei14b, Lemma 1.1] (see also [HW16, Lemma 1.8]), in cases (ii) and (iii) by Lemma 6.3 and Lemma 6.4. \square

Lemma 6.3. *Let k be a field of characteristic 0, with algebraic closure \bar{k} . Let L be a nonzero étale algebra over k , of dimension m . Let $q \in k[x_1, \dots, x_n]$ be a non-degenerate quadratic form. Denote by X the smooth closed subvariety of $R_{L/k}(\mathbf{A}_L^1) \times (\mathbf{A}_k^n \setminus \{(0, \dots, 0)\})$ defined by the equation*

$$(6.1) \quad N_{L/k}(z) = q(x_1, \dots, x_n),$$

where z stands for a point of $R_{L/k}(\mathbf{A}_L^1)$ and x_1, \dots, x_n are the coordinates of \mathbf{A}_k^n . Let $F \subset X$ be a closed subset of codimension ≥ 2 and set $U = X \setminus F$.

- (i) If $n \geq 1$, then U is geometrically integral.
- (ii) If $n \geq 2$, then $\mathrm{Br}(U_{\bar{k}}) = 0$ and $\bar{k}[U]^* = \bar{k}^*$.
- (iii) If $n \geq 3$, then $\mathrm{Pic}(U_{\bar{k}}) = 0$. If $n = 2$, then $\mathrm{Pic}(U_{\bar{k}}) \simeq \mathbf{Z}^{m-1}$ (ignoring Galois actions).
- (iv) If $n \geq 3$, the pull-back map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(U)$ is onto;
- (v) If $n = 2$ and both L and q are split, then $\mathrm{Gal}(\bar{k}/k)$ acts trivially on $\mathrm{Pic}(U_{\bar{k}})$ and the pull-back map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(U)$ is onto.

Assume that k is a number field and let v_0 be a place of k .

- (vi) If $n = 2$, the subset $U(k)$ is dense off v_0 in $U(\mathbf{A}_k)^{\mathrm{Br}(U)}$ (see [Wit18, Definition 2.9]).
- (vii) If $n \geq 3$, the variety U satisfies strong approximation off v_0 .

Proof. Let us first prove (ii)–(v). As the restriction maps $H^q(X_{\bar{k}}, \mathbf{G}_m) \rightarrow H^q(U_{\bar{k}}, \mathbf{G}_m)$ and $H^q(X, \mathbf{G}_m) \rightarrow H^q(U, \mathbf{G}_m)$ are isomorphisms for $q \leq 2$ (see [CTS20, Theorem 3.7.2 (i)]), we may assume that $U = X$. We now argue by induction on m , with $n \geq 2$ fixed but letting k, L, q vary, to prove (ii)–(v) under this assumption. If $m = 1$, then $X = \mathbf{A}_k^n \setminus \{(0, \dots, 0)\}$ and (ii)–(v) are true (see [CTS20, Theorem 3.7.1, Theorem 6.1.1]). Let us fix $m > 1$ and assume that (ii)–(v) hold for lower values of m . The Hochschild–Serre spectral sequence shows that (ii) and (iii) imply (iv); hence, in order to prove (ii)–(v), we may assume, after extending the scalars, that both q and L are split over k . The variety X is then isomorphic to the subvariety of $\mathbf{A}_k^m \times (\mathbf{A}_k^n \setminus \{(0, \dots, 0)\})$, with coordinates $z_1, \dots, z_m, x_1, \dots, x_n$, defined by $\prod_{i=1}^m z_i = q(x_1, \dots, x_n)$. Let $f : X \rightarrow \mathbf{A}_k^1$ denote the projection to the coordinate z_m . When $n \geq 3$, its fibres are geometrically integral; when $n = 2$, its fibres are geometrically

integral except $f^{-1}(0)$, which is the disjoint union of two geometrically integral subvarieties. Letting X_η be the generic fibre of f , it follows that

$$(6.2) \quad \text{Pic}(X_{\bar{k}}) = \begin{cases} \text{Pic}(X_\eta \otimes_k \bar{k}) & \text{if } n \geq 3, \\ \mathbf{Z} \oplus \text{Pic}(X_\eta \otimes_k \bar{k}) & \text{if } n = 2, \end{cases}$$

compatibly with the actions of $\text{Gal}(\bar{k}/k)$. Now as $m > 1$, the variety X_η is one of the form appearing in Lemma 6.3, associated with a split algebra of rank $m-1$ and a split quadratic form of rank n , over the function field K of \mathbf{A}_k^1 . Letting \bar{K} be an algebraic closure of K , the induction hypothesis therefore guarantees that $\bar{K}[X_\eta]^* = \bar{K}^*$ (hence $\bar{k}[X]^* = \bar{k}^*$), that $\text{Br}(X_\eta \otimes_K \bar{K}) = 0$, that $\text{Pic}(X_\eta \otimes_K \bar{K}) = 0$ if $n \geq 3$, and that $\text{Pic}(X_\eta \otimes_K \bar{K}) \simeq \mathbf{Z}^{m-2}$, with trivial action of $\text{Gal}(\bar{K}/K)$, if $n = 2$. Using the Hochschild–Serre spectral sequence, one deduces that $\text{Pic}(X_\eta \otimes_k \bar{k}) = 0$ if $n \geq 3$, and $\text{Pic}(X_\eta \otimes_k \bar{k}) \simeq \mathbf{Z}^{m-2}$, with trivial action of $\text{Gal}(\bar{k}/k)$, if $n = 2$. The parts of (ii)–(v) that concern $\text{Pic}(X_{\bar{k}})$ follow from this and from (6.2). The last part of (v) follows from the rest of (ii)–(v), by the Hochschild–Serre spectral sequence. Only the vanishing of $\text{Br}(X_{\bar{k}})$ remains to be checked. For this, we note that $\text{Br}(X_{\bar{k}}) \subseteq \text{Br}(X_\eta \otimes_k \bar{k})$ and that $\text{Br}(X_\eta \otimes_k \bar{k}) = 0$ by the Hochschild–Serre spectral sequence, as $\text{Br}(X_\eta \otimes_K \bar{K}) = 0$ and $H^1(K, \text{Pic}(X_\eta \otimes_K \bar{K})) = H^1(K, \mathbf{Z}^{m-2}) = 0$ (by the induction hypothesis), while $\text{Br}(K \otimes_k \bar{k}) = 0$ (by Tsen’s theorem).

Let us prove (i). As U is smooth and nonempty, if it were not geometrically integral, it would not be geometrically connected, which would contradict the last part of (ii) if $n \geq 2$. When $n = 1$, the variety $X_{\bar{k}}$ is isomorphic to the subvariety of $\mathbf{A}_{\bar{k}}^{m+1}$, with coordinates z_1, \dots, z_m, x , defined by $\prod_{i=1}^m z_i = x^2 \neq 0$, which is indeed connected.

To prove (vi), we note that if $n = 2$, then X is a toric variety. Indeed, after a change of variables, we may assume that q is diagonal, i.e. $q(x_1, x_2) = b(x_1^2 - cx_2^2)$ for some $b, c \in k^*$; the equations $N_{L/k}(z) = x_1^2 - cx_2^2 \neq 0$ then define a torus which acts on X with a dense open orbit. As the variety X is toric and satisfies $\bar{k}[X]^* = \bar{k}^*$, the main theorem of [Wei14b] ensures the validity of (vi).

We shall now prove that the assertion of (vi) in fact holds for all $n \geq 2$, by induction on n . In view of (iv), this will establish (vii).

Let $n \geq 3$ be such that the assertion of (vi) holds for smaller values of n . Choose a codimension 2 linear subspace $D \subset \mathbf{A}_k^n$ containing $(0, \dots, 0)$ such that $X \cap (R_{L/k}(\mathbf{A}_L^1) \times D)$ is smooth and that $F \cap (R_{L/k}(\mathbf{A}_L^1) \times D)$ has codimension ≥ 1 in F . Write Λ for the projective line parametrising hyperplanes in \mathbf{A}_k^n containing D . Let $g : X' \rightarrow X$ be the blow-up of X along $X \cap (R_{L/k}(\mathbf{A}_L^1) \times D)$ and $f : X' \rightarrow \Lambda$ the morphism obtained by composing g , the second projection $X \rightarrow \mathbf{A}_k^n$ and the rational map $\mathbf{A}_k^n \dashrightarrow \Lambda$ given by projection from D . The fibres of f are the varieties $X \cap (R_{L/k}(\mathbf{A}_L^1) \times H)$ where H ranges over the hyperplanes of \mathbf{A}_k^n containing D . As the restriction of q to any hyperplane of \mathbf{A}_k^n is a quadratic form of rank $\geq n-2$, and as $n \geq 3$, we deduce that the fibres of f are geometrically integral. Let $F' = g^{-1}(F)$ and $U' = X' \setminus F'$. As F' has codimension ≥ 2 in X' , the geometric generic fibre U'_η of $f|_{U'} : U' \rightarrow \Lambda$ is a variety of the form appearing in Lemma 6.3 (with n

replaced by $n-1$). In particular, it has no non-constant invertible function, by (ii), and the abelian group $\text{Pic}(U'_\eta)$ is torsion-free, by (iii), so that $H_{\text{ét}}^1(U'_\eta, \mathbf{Q}/\mathbf{Z}) = 0$; and (ii) ensures that $\text{Br}(U'_\eta) = 0$. We can therefore apply Corollary 4.7 to $f|_{U'}$ (recalling that $\Lambda \simeq \mathbf{P}_k^1$). The parameters π_+ which appear in the statement of Corollary 4.7 satisfy $L_m = k(m)$ for all $m \in M$, so that Conjecture F_+ holds for them, by Corollary 6.2 (i). We conclude that any point of $U'(\mathbf{A}_k)^{\text{Br}(U')}$ can be approximated arbitrarily well by a point of $U'_c(\mathbf{A}_k)^{\text{Br}(U'_c)}$ for a rational point c of an arbitrary dense open subset of Λ . By the induction hypothesis, this point of $U'_c(\mathbf{A}_k)^{\text{Br}(U'_c)}$ can in turn be approximated, for the adelic topology off v_0 , by a rational point of U'_c . This shows that $U'(k)$ is dense off v_0 in $U'(\mathbf{A}_k)^{\text{Br}(U')}$. On the other hand, as the map $U' \rightarrow U$ induced by g is a blow-up with smooth centre, it induces a surjection $U'(\mathbf{A}_k)^{\text{Br}(U')} \rightarrow U(\mathbf{A}_k)^{\text{Br}(U)}$. Hence $U(k)$ is dense off v_0 in $U(\mathbf{A}_k)^{\text{Br}(U)}$. \square

Lemma 6.4. *Let X be the punctured affine cone over an n -dimensional smooth projective quadric, over a number field k . Let v_0 be a place of k . Let $U = X \setminus F$, where $F \subset X$ is a closed subset of codimension ≥ 2 . If $n = 2$, the subset $U(k)$ is dense off v_0 in $U(\mathbf{A}_k)^{\text{Br}(U)}$. If $n \geq 3$, the variety U satisfies strong approximation off v_0 .*

Remark 6.5. In the case $n = 4$, Lemma 6.4 was asserted and used in the proof of [HW16, Theorem 9.11]. However, a gap in the justification given there (as well as a very simple fix for the proof of that theorem, by working around Lemma 6.4) was kindly pointed out to us by Yang Cao. Thus, Lemma 6.4 is new and requires a proof, even for $n = 4$. The proof we give also fills the aforementioned gap.

Proof of Lemma 6.4. Choose, for the underlying quadric, a diagonal equation of the form $\sum_{i=1}^{n+2} a_i x_i^2 = 0$ with $a_{n+2} = -1$. Set $q(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i^2$ and $L = k[t]/(t^2 - a_{n+1})$. The variety X^0 associated with L and q by Lemma 6.3 is an open subvariety of X whose complement has codimension ≥ 2 in X . Hence the conclusions of Lemma 6.3 for $U^0 = X^0 \cap U$ imply the same conclusions for U . \square

6.2. From Schinzel's hypothesis. The next theorem strengthens [HW16, Theorem 9.6], where the same conclusion was obtained for Conjecture F rather than Conjecture F_{const} . In its statement, almost abelian extensions are meant in the sense of [HW16, Definition 9.4]. Let us recall that abelian extensions as well as cubic extensions are almost abelian. We refer the reader to [HW16, §9.2.1] for the statement of Schinzel's hypothesis in its homogeneous form (HH₁), which follows from the more common Schinzel's hypothesis (H) and which takes, as input, a finite collection of homogeneous polynomials in $k[\lambda, \mu]$. Given a closed point $m \in \mathbf{A}_k^1$, we set $P_m(\lambda, \mu) = N_{k(m)/k}(\lambda - a_m \mu)$.

Theorem 6.6. *Let $\pi_+ \in \mathcal{P}_+$. Assume that for each $m \in M$, the extension $L_m/k(m)$ is almost abelian. If Schinzel's hypothesis (HH₁) holds for the set of homogeneous polynomials $(P_m(\lambda, \mu))_{m \in M'}$, where $M' = \{m \in M; L_m \neq k(m)\}$, then Conjecture F_{const} holds for π_+ .*

Proof. Let $U = \mathbf{P}_k^1 \setminus M$ and let $q : W \rightarrow \mathbf{A}_k^2 \setminus \{(0, 0)\}$ and $r : \mathbf{A}_k^2 \setminus \{(0, 0)\} \rightarrow \mathbf{P}_k^1$ denote the natural projections, so that $p = r \circ q$. Under our assumptions, Conjecture F holds for π , by Remark 3.2 and [HW16, Theorem 9.6, Proposition 9.9]. The proof given in *loc. cit.* is

in fact a proof of the more precise result that the subset of $W(\mathbf{A}_k)$ consisting of the adelic points that lie in a fibre of q over a rational point of $\mathbf{A}_k^2 \setminus \{(0, 0)\}$ is dense in $W(\mathbf{A}_k)$. On the other hand, for any $m \in M$ and any $\chi \in H^1(k(m), \mathbf{Q}/\mathbf{Z})$, we have

$$\text{Cores}_{L_m/k}(z_m, \chi) = \text{Cores}_{k(m)/k}(N_{L_m/k(m)}(z_m), \chi) = \text{Cores}_{k(m)/k}(b_m(\lambda - a_m\mu), \chi),$$

hence $B_{\text{const}} \subseteq q^*\text{Br}(r^{-1}(U))$ and any adelic point of $p^{-1}(U)$ that lies in a fibre of q over a rational point of $\mathbf{A}_k^2 \setminus \{(0, 0)\}$ is therefore automatically orthogonal to B_{const} with respect to the Brauer–Manin pairing. \square

Corollary 6.7. *Let $\pi_+ \in \mathcal{P}_+$. Assume that for each $m \in M$, at least one of the following conditions is satisfied:*

- (1) *the extension $L_m/k(m)$ is cyclic, or it is almost abelian but not abelian;*
- (2) *the extension $L_m/k(m)$ is abelian and the extension K_m/L_m is trivial.*

Let $M' = \{m \in M; L_m \neq k(m)\}$ and assume that Schinzel’s hypothesis (HH₁) holds for the set of homogeneous polynomials $(P_m(\lambda, \mu))_{m \in M'}$. Then Conjecture F_+ holds for π_+ .

Proof. By Theorem 6.6, Conjecture F_{const} holds for π_+ . By Corollary 5.8, Proposition 2.6 and Remark 2.8, Conjecture F_{const} for π_+ implies Conjecture F_+ for π_+ . \square

The work of Heath-Brown and Moroz [HBM04] on primes represented by binary cubic forms implies the validity of Schinzel’s hypothesis (HH₁) for a single polynomial of degree 3 with coefficients in \mathbf{Q} (see [HW16, Remark 9.7]). We thus obtain the following corollary (to be compared with Corollary 6.2 (iv)):

Corollary 6.8. *Let $\pi_+ \in \mathcal{P}_+$. Assume that $k = \mathbf{Q}$, that there is a unique $m \in M$ such that $L_m \neq k(m)$, that this m is such that $[k(m) : k] = 3$, and that for this m , the extension $L_m/k(m)$ is cyclic or is almost abelian but non-abelian. Then Conjecture F_+ holds for π_+ .*

6.3. From additive combinatorics. Under the assumption that $k = \mathbf{Q}$, Conjecture F is known to hold for triples $\pi \in \mathcal{P}$ such that M only consists of rational points. This was proved by Matthiesen [Mat18] (see [HW16, Theorem 9.14]), following her work with Browning [BM17] and using the methods of additive combinatorics developed by Green, Tao and Ziegler [GT08, GT10, GT12, GTZ12]. The next theorem strengthens this result, by replacing Conjecture F with Conjecture F_+ while allowing non-rational points in M with trivial extension $L_m/k(m)$ (and arbitrary finite abelian extensions K_m/L_m).

Theorem 6.9. *Let $\pi_+ \in \mathcal{P}_+$. Assume that $k = \mathbf{Q}$ and that for each $m \in M$, at least one of the extensions $L_m/k(m)$ and $k(m)/k$ is trivial. Then Conjecture F_+ holds for π_+ .*

Proof. Let $\pi_+^0 = (M, (L_m^0)_{m \in M}, (b_m^0)_{m \in M}, (K_m^0)_{m \in M}) \in \mathcal{P}_+$ satisfy the assumptions of the theorem. We shall prove Conjecture F_+ for π_+^0 .

Set $M' = \{m \in M; L_m^0 \neq k(m)\}$. If $M' = \emptyset$, then Conjecture F_+ holds for π_+^0 by Corollary 6.2 (i). Otherwise, we choose $m_0 \in M'$ and note that $k(m_0) = k$ by assumption. For $m \in M \setminus M'$, let us set $K_m = K_m^0$ and $L_m = L_m^0 = k(m)$. For $m \in M' \setminus \{m_0\}$, let us set $K_m = L_m = K_m^0$. Let L_0/k denote the field extension given by Corollary 5.3 applied to the

extensions $K_m/L_m/k(m)$ for $m \in M \setminus \{m_0\}$. Finally, let us choose a finite extension L_{m_0} of $K_{m_0}^0$ in which L_0 can be embedded k -linearly, and set $K_{m_0} = L_{m_0}$.

According to Corollary 4.4, we will be done if we prove Conjecture F_+ for the parameter $\pi_+ = (M, (L_m)_{m \in M}, (b_m)_{m \in M}, (K_m)_{m \in M})$ for any choice of $(b_m)_{m \in M} \in \prod_{m \in M} k(m)^*$. Let us fix $(b_m)_{m \in M}$. By Matthiesen's theorem [HW16, Theorem 9.14, Proposition 9.9] and Remark 3.2, Conjecture F holds for π_+ . By the definition of L_0 (see Corollary 5.3), it follows that Conjecture F_{const} holds for π_+ . By Remark 5.4 (ii), we conclude that Conjecture F_+ holds for π_+ , as desired. \square

7. APPLICATIONS

As was the case for Conjecture F in [HW16], our motivation for Conjecture F_+ ultimately lies in the following question, which slightly refines Question 1.2 by incorporating a Hilbert subset into its statement:

Question 7.1. Let X be a smooth, proper, irreducible variety over a number field k . Let $f : X \rightarrow \mathbf{P}_k^1$ be a dominant morphism whose geometric generic fibre is rationally connected. Assume that $X_c(k)$ is dense in $X_c(\mathbf{A}_k)^{\text{Br}(X_c)}$ for all rational points c of a Hilbert subset of \mathbf{P}_k^1 . Does it follow that $X(k)$ is dense in $X(\mathbf{A}_k)^{\text{Br}(X)}$?

Question 7.1 admits an affirmative answer if Conjecture F (or Conjecture F_+) holds true, by [HW16, Corollary 9.25] (or Corollary 4.7 and Remarks 4.8 (i)–(ii)), and unconditionally, in various special cases listed in [HW16, §9.4] and [BS19], the most notable one being when $k = \mathbf{Q}$ and the non-split fibres of f lie over rational points of \mathbf{P}_k^1 (using Matthiesen's theorem, see [HW16, Theorem 9.28]).

7.1. Statements. We now turn to the new affirmative answers to Question 7.1 that can be obtained by combining the main results of §§4–6. In the statements below, we fix X and f as in Question 7.1, and let $M \subset \mathbf{P}_k^1$ denote the locus of non-split fibres of f . For each $m \in M$, we choose an irreducible homogeneous polynomial $P_m(\lambda, \mu)$ that vanishes on m , where λ, μ denote homogeneous coordinates of \mathbf{P}_k^1 . Following a terminology introduced by Skorobogatov [Sko96], the *rank* of f , denoted $\text{rank}(f)$, is the degree of M over k (viewing M as a reduced closed subscheme of \mathbf{P}_k^1). Finally, for $m \in M$, we say that a finite extension L_m of $k(m)$ *splits the fibre* X_m if the variety $X_m \otimes_{k(m)} L_m$ over L_m is split.

Theorem 7.2. *Assume that the following two conditions are satisfied:*

- (1) *Schinzel's hypothesis (HH₁) holds for the homogeneous polynomials $(P_m(\lambda, \mu))_{m \in M}$.*
- (2) *For each $m \in M$, there exists an extension of $k(m)$ that splits the fibre X_m and that is either cyclic or almost abelian but non-abelian (e.g. a cubic extension).*

Then Question 7.1 admits an affirmative answer.

Proof. Combine Corollary 4.7, Remarks 4.8 (i) and (ii), the invariance of (HH₁) under changes of coordinates of \mathbf{P}_k^1 (if X_∞ is singular), and Corollary 6.7. \square

Theorem 7.2 recovers and generalises a theorem of Smeets [Sme15, Corollaire 1.5], who dealt with the special case where the generic fibre of f is a torsor under a torus defined

over k and quasi-split by a cyclic extension of k . Apart from this case, Theorem 7.2 was previously known only under the assumption that the smooth fibres of f satisfy weak approximation (in which case X_m can be allowed to be split by an abelian extension; see [HW16, Corollary 9.27], which expanded on [CTSSD98] and on [Wei14a, Theorem 4.6]).

Theorem 7.3. *Question 7.1 admits an affirmative answer if $\text{rank}(f) \leq 2$.*

Proof. Combine Corollary 4.7, Remarks 4.8 (i) and (ii), and Corollary 6.2 (i). \square

Theorem 7.3 was previously known only under the assumption that k is totally imaginary or that M consists of rational points of \mathbf{P}_k^1 (see [HW16, Theorem 9.31]).

Theorem 7.4. *Assume that $\text{rank}(f) = 3$ and that at least one of the following holds:*

- (i) $k = \mathbf{Q}$ and f has at least two non-split fibres;
- (ii) $k = \mathbf{Q}$, the morphism f has a unique non-split fibre, say over m , and X_m is split by an extension of $k(m)$ that is either cyclic or almost abelian but non-abelian;
- (iii) for every $m \in M$, the fibre X_m is split by a quadratic extension of $k(m)$;
- (iv) there exists $m_0 \in M$ such that $k(m_0) = k$ and such that for every $m \in M \setminus \{m_0\}$, the fibre X_m is split by a quadratic extension of $k(m)$.

Then Question 7.1 admits an affirmative answer.

Proof. Combine Corollary 4.7 and Remarks 4.8 with Corollary 6.2 (iv) (which builds on the work of Browning and Schindler) in case (i) if f has two non-split fibres, with Theorem 6.9 (which builds on the work of Matthiesen) in case (i) if f has three non-split fibres, with Corollary 6.8 (which builds on the work of Heath-Brown and Moroz) in case (ii), with Corollary 6.2 (ii) in case (iii) and with Corollary 6.2 (iii) in case (iv). \square

The above theorem collects everything that can be proved to this day about Question 7.1 when f has rank 3. Assertion (i) is [BS19, Theorem 1.1] and is only stated for the record. Assertions (ii), (iii) and (iv), on the other hand, are entirely new. The only previously known case of these three assertions was a theorem of Colliot-Thélène and Skorobogatov [CTS00, Theorem B], who had established Theorem 7.4 (iii) under the assumption that the smooth fibres of f satisfy weak approximation.

7.2. Examples. We now describe some concrete examples of varieties for which one can prove the density of rational points in the Brauer–Manin set by applying Theorem 7.4 to rank 3 fibrations meeting the requirements (ii), (iii) or (iv) of its statement.

7.2.1. An example for Theorem 7.4 (ii). Consider a number field k and a nonzero étale algebra $L = \prod_i L_i$ over k , where the L_i are number fields. The arithmetic of smooth and proper models X of the affine closed subvariety of $R_{L/k}(\mathbf{A}_L^1) \times \mathbf{A}_k^1$ defined by the equation

$$N_{L/k}(z) = p(t),$$

where z and t are coordinates in $R_{L/k}(\mathbf{A}_L^1)$ and in \mathbf{A}_k^1 respectively, and where $p \in k[t]$ is a polynomial in one variable, has been extensively studied in the literature (see e.g. [HBS02, CTHS03, VAV12, BHB12, SJ13, Wei14a, DSW15, BM17, Irv17]). One can always

choose X so that the projection $(z, t) \mapsto t$ extends to a morphism $f : X \rightarrow \mathbf{P}_k^1$ each of whose smooth fibres is a compactification of a torsor under the norm torus defined by $N_{L/k}(z) = 1$. Then the generic fibre of f is rationally connected and $X_c(k)$ is dense in $X_c(\mathbf{A}_k)^{\mathrm{Br}(X_c)}$ for all $c \in \mathbf{P}^1(k)$ such that X_c is smooth, by a theorem of Colliot-Thélène and Sansuc (see [Sko01, Theorem 6.3.1]). Applying Theorem 7.4 (ii) thus yields the following:

Corollary 7.5. *Let $b \in \mathbf{Q}^*$. Let $e \geq 1$ be an integer. Let $q \in \mathbf{Q}[t]$ be an irreducible cubic polynomial. Set $E = \mathbf{Q}[t]/(q(t))$. Let $L = \prod L_i$ be a nonzero étale \mathbf{Q} -algebra, where the L_i are number fields. Let X be a smooth and proper model over \mathbf{Q} of the closed subvariety of $R_{L/\mathbf{Q}}(\mathbf{A}_L^1) \times \mathbf{A}_{\mathbf{Q}}^1$ defined by the equation*

$$N_{L/\mathbf{Q}}(z) = bq(t)^e,$$

where z denotes a point of $R_{L/\mathbf{Q}}(\mathbf{A}_L^1)$ and t is the coordinate of $\mathbf{A}_{\mathbf{Q}}^1$. Suppose that the following conditions hold:

- (1) *The gcd of the degrees $[L_i : \mathbf{Q}]$ divides $3e$.*
- (2) *Writing the étale E -algebra $L \otimes_{\mathbf{Q}} E$ as a product of fields, at least one of the factors is an extension of E that is either cyclic or almost abelian but non-abelian.*

Then the subset $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.

The rôle of condition (1) in the above corollary is to ensure that when X is chosen in such a way that the projection $(z, t) \mapsto t$ extends to a morphism $f : X \rightarrow \mathbf{P}_{\mathbf{Q}}^1$ (which we can assume, as the conclusion of the corollary is a birational invariant [Wit18, Remark 2.4 (iv)]), the fibre $f^{-1}(\infty)$ is split. The following homogeneous variant of this example permits one to dispense with this condition:

Corollary 7.6. *Let $b \in \mathbf{Q}^*$. Let $e \geq 1$ be an integer. Let $q \in \mathbf{Q}[\lambda, \mu]$ be an irreducible homogeneous cubic polynomial. Let $L = \prod L_i$ be a nonzero étale \mathbf{Q} -algebra, where the L_i are number fields. Let Y be a smooth and proper model over \mathbf{Q} of the closed subvariety of $R_{L/\mathbf{Q}}(\mathbf{A}_L^1) \times (\mathbf{A}_{\mathbf{Q}}^2 \setminus \{(0, 0)\})$ defined by the equation*

$$N_{L/\mathbf{Q}}(z) = bq(\lambda, \mu)^e,$$

where z denotes a point of $R_{L/\mathbf{Q}}(\mathbf{A}_L^1)$ and λ, μ are the coordinates of $\mathbf{A}_{\mathbf{Q}}^2 \setminus \{(0, 0)\}$. Setting $E = \mathbf{Q}[t]/(q(t, 1))$ and writing the étale E -algebra $L \otimes_{\mathbf{Q}} E$ as a product of fields, suppose that at least one of the factors is an extension of E that is either cyclic or almost abelian but non-abelian. Then the subset $Y(k)$ is dense in $Y(\mathbf{A}_k)^{\mathrm{Br}(X)}$.

Remark 7.7. When the gcd of the degrees $[L_i : \mathbf{Q}]$ divides $3e$, the variety Y considered in Corollary 7.6 is birationally equivalent to $X \times \mathbf{G}_m$, where X is the variety associated in Corollary 7.5 with the polynomial $q(t, 1)$. As a result, Corollary 7.5 is in fact equivalent to a special case of Corollary 7.6. We have nevertheless opted for stating Corollary 7.5 separately in view of the considerable attention that the variety X has received in the literature (see the references at the beginning of §7.2.1).

7.2.2. *An example for Theorem 7.4 (iv).* We keep the set-up of §7.2.1 and assume that L is a quartic extension of k and that $p \in k[t]$ is an irreducible quadratic polynomial having its roots in L . In this case, using techniques from analytic number theory, Browning and Heath-Brown [BHB12, Theorem 1] established the Hasse principle and weak approximation for X when $k = \mathbf{Q}$. Derenthal, Smeets and the second-named author then provided a second proof based on the descent method, which led, in [DSW15, Theorem 1], to the validity of the same result over an arbitrary number field k . These authors also verified the equality $\mathrm{Br}(X) = \mathrm{Br}_0(X)$ in the case under consideration (see [DSW15, Theorem 4]). The theorem of Browning and Heath-Brown was subsequently understood to fit into the framework of the fibration method: indeed it can be seen as an application of Browning and Schindler's Theorem 7.4 (i), in view of the equality $\mathrm{Br}(X) = \mathrm{Br}_0(X)$. Until the present work, however, its generalisation to arbitrary number fields had remained outside of the scope of the fibration method. We remedy this gap with Theorem 7.4 (iv). Applied to $f : X \rightarrow \mathbf{P}_k^1$ (with $m_0 = \infty$), the latter immediately yields the density of $X(k)$ in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$, thus recovering [DSW15, Theorem 1] since $\mathrm{Br}(X) = \mathrm{Br}_0(X)$.

7.2.3. *An example for Theorem 7.4 (iii).* Let us start with a field k of characteristic 0 and a reduced closed subscheme $M \subset \mathbf{P}_k^1$ of degree 3 over k . Let $U = \mathbf{P}_k^1 \setminus M$. Let C be a smooth projective curve over k (which we do not assume to be connected or geometrically connected) and $\pi : C \rightarrow \mathbf{P}_k^1$ be a finite morphism satisfying the following condition:

(*) the morphism π is étale over U and for every $m \in M$, the gcd of the ramification indices of π at the points of $\pi^{-1}(m)$ is equal to 2.

For $b \in k^*$, we consider the closed subvariety X^0 of $R_{C/\mathbf{P}_k^1}(\mathbf{A}_k^1 \times C)$ defined by

$$N_{C/\mathbf{P}_k^1}(z) = b$$

and let $f^0 : X^0 \rightarrow \mathbf{P}_k^1$ denote the projection.

Corollary 7.8. *For any M, C, π, b as above, and any smooth and proper model X of X^0 , if k is a number field, the subset $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.*

Proof. The variety X^0 is smooth (see Lemma 7.9 below). As the conclusion of the corollary is a birational invariant, we may therefore assume that X contains X^0 as a dense open subset and that f^0 extends to a morphism $f : X \rightarrow \mathbf{P}_k^1$. The corollary then results from Theorem 7.4 (iii) in view of the following description of the fibres of f^0 . \square

Lemma 7.9. *The morphism f^0 is smooth. Its fibres over U are geometrically integral. For $m \in M$, the fibre $(f^0)^{-1}(m)$ is split by the extension $k(m)(\sqrt{b})/k(m)$.*

Proof. As the morphism f^0 is obtained by base change from the norm map

$$(7.1) \quad N_{C/\mathbf{P}_k^1} : R_{C/\mathbf{P}_k^1}(\mathbf{G}_m \times C) \rightarrow \mathbf{G}_m \times \mathbf{P}_k^1$$

it suffices to prove that the latter is smooth, and to describe its fibres.

The fibre of (7.1) above an arbitrary point (b, m) of $\mathbf{G}_m \times \mathbf{P}_k^1$ is the closed subvariety of $R_{\pi^{-1}(m)/m}(\mathbf{A}_k^1 \times \pi^{-1}(m))$ defined by the equation $N_{\pi^{-1}(m)/m}(z) = b$. Let us choose

an isomorphism $\pi^{-1}(m) = \coprod_{i=1}^s \text{Spec}(k_i[v]/(v^{e_i}))$, where k_1, \dots, k_s are finite extensions of $k(m)$ and e_1, \dots, e_s are the ramification indices. There results an isomorphism

$$(7.2) \quad R_{\pi^{-1}(m)/m}(\mathbf{A}_k^1 \times \pi^{-1}(m)) = \prod_{i=1}^s \prod_{j=0}^{e_i-1} R_{k_i/k(m)} \mathbf{A}_{k_i}^1.$$

Letting $z_{i,j}$ stand for a point of the corresponding factor $R_{k_i/k(m)} \mathbf{A}_{k_i}^1$ in the right-hand side of (7.2), the equation $N_{\pi^{-1}(m)/m}(z) = b$ is rewritten, through this isomorphism, as

$$(7.3) \quad \prod_{i=1}^s N_{k_i/k(m)}(z_{i,0})^{e_i} = b.$$

All in all, the fibre of (7.1) above (b, m) is isomorphic to $Z \times \mathbf{A}_{k(m)}^{\deg(\pi) - \sum_{i=1}^s [k_i:k(m)]}$, where Z denotes the closed subvariety of $\prod_{i=1}^s R_{k_i/k(m)} \mathbf{G}_m$ defined by (7.3).

Let e denote the gcd of the e_i . It is easy to see that Z is a torsor under a group of multiplicative type over $k(m)$ which is an extension of μ_e by a torus, and that the torsor under μ_e induced by Z is the closed subvariety of $\mathbf{G}_{m,k(m)}$ defined by $z^e = b$. Hence Z is geometrically integral if $e = 1$, and in any case it is split by $k(m)(b^{1/e})/k(m)$.

Thanks to (*), we have now proved the second and third assertions of the lemma. Our description of the fibres of (7.1) also shows that they are smooth and all have the same dimension (namely $\deg(\pi) - 1$). As in addition (7.1) is a finite type morphism between regular schemes (indeed, between smooth \mathbf{P}_k^1 -schemes, see [BLR90, 7.6/5]), it follows that it is smooth (flatness being ensured by [Gro65, Proposition 6.1.5]). \square

It remains to give examples of covers π satisfying (*).

Example 7.10. Condition (*) holds if $\pi : C \rightarrow \mathbf{P}_k^1$ is a connected Galois cover with branch locus equal to M and with Galois group $(\mathbf{Z}/2\mathbf{Z})^2$. Such covers exist if M consists of three rational points (e.g. take $k(C) = k(t)(\sqrt{t(t+1)}, \sqrt{t(t-1)})$ if $M = \{-1, 0, 1\}$).

Example 7.11. More generally, consider the algebraic group S over k defined as the kernel of the norm map $N_{M/k} : R_{M/k}(\mathbf{Z}/2\mathbf{Z}) \rightarrow \mathbf{Z}/2\mathbf{Z}$. This is a twisted form of $(\mathbf{Z}/2\mathbf{Z})^2$. Let t denote the parameter of \mathbf{A}_k^1 and, assuming for simplicity that $M \subset \mathbf{A}_k^1$, let $a \in H^0(M, \mathcal{O}_M)$ denote the restriction of t to M . As M has degree 3 over k , the exact sequence

$$(7.4) \quad 1 \longrightarrow S \longrightarrow R_{M/k}(\mathbf{Z}/2\mathbf{Z}) \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0$$

induces an isomorphism $H_{\text{ét}}^1(U, S) \xrightarrow{\sim} \text{Ker}(N_{M/k} : H_{\text{ét}}^1(U \times_k M, \mathbf{Z}/2\mathbf{Z}) \rightarrow H_{\text{ét}}^1(U, \mathbf{Z}/2\mathbf{Z}))$; in particular, any invertible function on $U \times_k M$ whose norm down to U is a square defines a class in $H_{\text{ét}}^1(U, S)$. Applying this to the function $N_{M/k}(t-a)/(t-a)$, we obtain the isomorphism class of a torsor $C_U \rightarrow U$ under S . Let $\pi : C \rightarrow \mathbf{P}_k^1$ be the cover obtained by compactifying this torsor. We claim that (*) holds and that C is geometrically connected over k . To check this, we may freely extend the scalars and thus assume that k is algebraically closed and that $M = \{-1, 0, 1\}$, in which case we can identify S with $(\mathbf{Z}/2\mathbf{Z})^2$ and π with the cover considered in Example 7.10.

Remarks 7.12. (i) In the situation of Example 7.11, the Galois group of a Galois closure of the quartic extension $k(C)/k(t)$, when viewed as a subgroup of S_4 , is $(\mathbf{Z}/2\mathbf{Z})^2$ if M consists of three rational points, or D_4 if M consists of a rational point and a quadratic point, or A_4 if M consists of a cubic point with cyclic residue field, or else S_4 .

To see this, let k' be a minimal Galois extension of k that splits M completely and a_1, a_2, a_3 be the values of t at the k' -points of M . Let $p_i = \prod_{j \neq i} (t - a_j) \in k'[t]$ for every i . As the set $\{p_1, p_2, p_3\}$ is stable under $\text{Gal}(k'/k)$ and as $k'(C) = k'(t)(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$, the extension $k'(C)/k(t)$ is Galois. Viewing $\text{Gal}(k'/k)$ both as the quotient $\text{Gal}(k'(t)/k(t))$ of $G = \text{Gal}(k'(C)/k(t))$ and as its subgroup $\text{Gal}(k'(C)/k(C))$, and noting that $k'(C)/k'(t)$ is biquadratic, we find that $G \simeq (\mathbf{Z}/2\mathbf{Z})^2 \rtimes \text{Gal}(k'/k)$, from which the claim follows easily.

(ii) In the situation of Lemma 7.9, even though f and X are not explicit, it is possible to get a hold on the splitting behaviour of the fibres of f (rather than f^0) by considering the points of the generic fibre of f^0 with values in complete discretely valued fields with pseudo-algebraically closed residue field (see [CT11, Proposition 3.8]). In this way, one can check that for every $m \in M$, the fibre $f^{-1}(m)$ is split if and only if b becomes a square in the field k' of Remark 7.12 (while it follows from the proof of Lemma 7.9 that $(f^0)^{-1}(m)$ is split if and only if b becomes a square in $k(m)$, a slightly stronger condition in general). Thus, in Examples 7.10 and 7.11, all of the fibres of f over M are truly non-split if b does not become a square in k' .

7.2.4. Further comments. In all of the examples given in §7.2, the smooth fibres of the fibrations we construct are compactifications of torsors under algebraic tori. Specifically, in Corollary 7.5, the algebraic torus in question is $R_{L/\mathbf{Q}}^1 \mathbf{G}_m$; in Corollary 7.6, it is the subtorus of $\mathbf{G}_m \times R_{L/\mathbf{Q}} \mathbf{G}_m$ defined by the equation $y^{3e} N_{L/k}(z) = 1$; and in Corollary 7.8, the fibre over $c \in U$ is a compactification of a torsor under the norm torus $R_{\pi^{-1}(c)/c}^1 \mathbf{G}_m$.

Such torsors can have non-constant unramified Brauer classes and generally fail to satisfy the Hasse principle or weak approximation. Non-constant unramified Brauer classes in the fibres do exist, in the case of Corollary 7.5 (and therefore also of Corollary 7.6, see Remark 7.7), when E/\mathbf{Q} is a cyclic extension and L/\mathbf{Q} is a Galois extension with Galois group $G = \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ that contains E , since in this case the norm torus T associated with L/\mathbf{Q} satisfies $\text{III}_{\text{cyc}}^2(G, \hat{T}) \neq 0$ by Remark 2.8 (ii). (Recall that the injection (2.1) is an isomorphism since k is a number field.) In the case of Corollary 7.8, the same phenomenon occurs when M consists either of three rational points or of one cubic point with cyclic residue field, according to Remark 7.12 (i), Remark 2.8 (ii) and Example 2.11.

Because of the presence of non-constant unramified Brauer classes in the fibres, the various examples we have given are not covered by [CTS00, Theorem B].

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