ON THE FIBRATION METHOD FOR ZERO-CYCLES AND RATIONAL POINTS

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ABSTRACT. Conjectures on the existence of zero-cycles on arbitrary smooth projective varieties over number fields were proposed by Colliot-Thélène, Sansuc, Kato and Saito in the 1980's. We prove that these conjectures are compatible with fibrations, for fibrations into rationally connected varieties over a curve. In particular, they hold for the total space of families of homogeneous spaces of linear groups with connected geometric stabilisers. We prove the analogous result for rational points, conditionally on a conjecture on locally split values of polynomials which a recent work of Matthiesen establishes in the case of linear polynomials over the rationals.

1. INTRODUCTION

Let X denote a smooth and proper algebraic variety over a number field k.

It is customary to embed the set $X(k)$ of rational points of X diagonally into the space of adelic points $X(\mathbf{A}_k) = \prod_{v \in \Omega} X(k_v)$. In his foundational paper [Man71], Manin combined local class field theory, global class field theory and Grothendieck's theory of Brauer groups of schemes to define a natural closed subset $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ of $X(\mathbf{A}_k)$ which contains the image of $X(k)$. The absence of any rational point on X is in many examples explained by the vacuity of $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$. The following conjecture, first formulated by Colliot-Thélène and Sansuc in 1979 in the case of surfaces, posits a partial converse.

Conjecture 1.1 ([CT03, p. 174])**.** *For any smooth, proper, geometrically irreducible, rationally connected variety X over a number field k, the set $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.*

By "rationally connected", we mean that for any algebraically closed field K containing k, two general K-points of X can be joined by a rational curve (*e.g.*, geometrically unirational varieties are rationally connected; see [Kol96, Chapter IV] for more on this notion).

In a parallel development, a conjecture concerning the image of the Chow group $\mathrm{CH}_0(X)$ of zero-cycles up to rational equivalence in the product of the groups $\mathrm{CH}_0(X \otimes_k k_v)$ over the places v of k was put forward by Colliot-Thélène, Sansuc, Kato and Saito (for rational surfaces in [CTS81, §4], for arbitrary varieties in [KS86, §7] and in [CT95, §1]). Let us recall its statement. Denote by Ω_f (resp. Ω_∞) the set of finite (resp. infinite) places of k and by $\mathrm{CH}_{0,\mathbf{A}}(X)$ the group

$$(1.1) \qquad \mathrm{CH}_{0,\mathbf{A}}(X) = \prod_{v \in \Omega_f} \mathrm{CH}_0(X \otimes_k k_v) \times \prod_{v \in \Omega_\infty} \frac{\mathrm{CH}_0(X \otimes_k k_v)}{N_{\overline{k}_v/k_v}(\mathrm{CH}_0(X \otimes_k \overline{k}_v))}.$$

By the reciprocity law of global class field theory, the pairings

$$\langle -, - \rangle_v : \mathrm{Br}(X \otimes_k k_v) \times \mathrm{CH}_0(X \otimes_k k_v) \to \mathrm{Br}(k_v) \hookrightarrow \mathbf{Q}/\mathbf{Z}$$

for $v \in \Omega$, characterised by the property that $\langle \alpha, P \rangle_v$ is the local invariant of $\alpha(P) \in \mathrm{Br}(k_v(P))$ whenever P is a closed point of $X \otimes_k k_v$, fit together in a complex

$$(1.2) \qquad \widehat{\mathrm{CH}_0(X)} \longrightarrow \widehat{\mathrm{CH}_{0,\mathbf{A}}(X)} \xrightarrow{\sum_{v \in \Omega} \langle -, - \rangle_v} \mathrm{Hom}(\mathrm{Br}(X), \mathbf{Q}/\mathbf{Z}),$$

where we denote $\widehat{M} = \varprojlim_{n \geq 1} M/nM$ for any abelian group M. In the formulation given by van Hamel [vH03, Theorem 0.2], the above-mentioned conjecture is then the following.

Conjecture 1.2 ([KS86, §7], [CT95, §1]; see also [Wit12, §1.1])**.** *For any smooth, proper, geometrically irreducible variety X over a number field k, the complex* (1.2) *is exact.*

Rationally connected varieties which satisfy Conjecture 1.1 over every finite extension of k are known to satisfy Conjecture 1.2 (cf. [Lia13a]).

Two general methods were elaborated to attack Conjecture 1.1 and Conjecture 1.2. The descent method (see [CTS87], [Sko99], [Sko01]; see [Sal03] for an adaptation to the context of zero-cycles) generalises the classical descent theory of elliptic curves to arbitrary varieties. The fibration method, with which we are concerned in the present paper, consists, under various sets of hypotheses, in exploiting the structure of a fibration $f : X \to Y$ to establish the conjectures for X when they are known for Y and for the fibers of f.

Our aim in this paper is twofold. First, in §§2–8, we prove the following general fibration theorem for the existence of zero-cycles, which is the main result of the paper.

Theorem 1.3. *Let X be a smooth, proper, geometrically irreducible variety over a number field k and let $f : X \to \mathbf{P}_k^1$ be a dominant morphism with rationally connected geometric generic fiber. If the smooth fibers of f satisfy Conjecture 1.2, then so does X.*

The precise results we prove on zero-cycles are stated in §8. They generalise Theorem 1.3 in the following respects:

(1) the base of the fibration is allowed to be any variety birationally equivalent to the product of a projective space with a curve C which satisfies Conjecture 1.2;

(2) only the smooth fibers of f above the closed points of a Hilbert subset (for instance, of a dense open subset) of the base are assumed to satisfy Conjecture 1.2;

(3) instead of requiring that the geometric generic fiber be rationally connected, it is enough to assume that it is connected, that its abelian étale fundamental group is trivial and that its Chow group of zero-cycles of degree 0 is trivial over any algebraically closed field extension;

(4) when the geometric generic fiber is rationally connected, the assumption that the smooth fibers (above the closed points of a Hilbert subset) satisfy Conjecture 1.2 may be replaced with the assumption that they satisfy Conjecture 1.1.

We note that a smooth, proper, geometrically irreducible curve C satisfies Conjecture 1.2 as soon as the divisible subgroup of the Tate–Shafarevich group of its Jacobian is trivial,

by a theorem of Saito [Sai89] (see [Wit12, Remarques 1.1 (iv)]). At least when $C(k) \neq \varnothing$, this condition is even equivalent to Conjecture 1.2 (see [Mil06, Ch. I, Theorem 6.26 (b)]).

To illustrate Theorem 1.3, let us consider the very specific example of the affine n-dimensional hypersurface V given by

$$(1.3) \qquad\qquad N_{K/k}(x_1\omega_1 + \cdots + x_n\omega_n) = P(t),$$

where $P(t) \in k[t]$ is a nonzero polynomial and where the left-hand side is the norm form associated to a finite extension K/k and to a basis $(\omega_1, \ldots, \omega_n)$ of the k-vector space K. The study of the arithmetic of a smooth and proper model X of V has a long history. Conjecture 1.2 was previously known for X only when K/k is cyclic (see [CTSSD98], which builds on [Sal88]), when K/k has prime degree or degree pq for distinct primes p, q (see [Wei14a, Theorem 4.3] and [Lia15, Example 3.1, Remark 3.7]), when K/k is quartic or is abelian with Galois group $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ under certain assumptions on $P(t)$ (see [Wei14a], [DSW14], [Lia14, Corollary 2.3] and [Lia15, Example 3.9]), and, finally, for arbitrary K/k, when $P(t) = ct^m(1-t)^n$ for some $c \in k^*$ and some integers m, n (see [SJ13], which builds on [HBS02] and [CTHS03] and applies descent theory and the Hardy–Littlewood circle method to establish Conjecture 1.1 for X over every finite extension of k, and see [Lia15, Example 3.10]). By contrast, it follows uniformly from Theorem 1.3 that X satisfies Conjecture 1.2 for any K/k and any $P(t)$. Indeed, on the one hand, the generic fiber of the projection map $V \to \mathbf{A}^1_k$ given by the t coordinate is a torsor under the norm torus associated to the finite extension K/k, and on the other hand, smooth compactifications of torsors under tori satisfy Conjecture 1.1 (see [Sko01, Theorem 6.3.1] and use (4) above; alternatively, these varieties satisfy Conjecture 1.2 by [Lia13a]). One can even go further and replace the "constant" field extension K/k with an arbitrary finite extension $K/k(t)$: Theorem 1.3 implies Conjecture 1.2 for smooth and proper models of the affine n-dimensional hypersurface defined by $N_{K/k(t)}(x_1\omega_1 + \cdots + x_n\omega_n) = P(t)$, if $(\omega_1, \ldots, \omega_n)$ now denotes a basis of the $k(t)$-vector space K.

More generally, we obtain the following result as a corollary of Theorem 1.3. This covers in particular all fibrations into toric varieties, into Châtelet surfaces or into Châtelet p-folds in the sense of [VAV12]. See §8.1 for a more detailed discussion.

Theorem 1.4. *Let X be a smooth, proper, irreducible variety over a number field k. Let Y be an irreducible variety over k, birationally equivalent to either $\mathbf{P}^n_k$, or C, or $\mathbf{P}^n_k \times C$, for some $n \geq 1$ and some smooth, proper, geometrically irreducible curve C over k which satisfies Conjecture 1.2. Let $f : X \to Y$ be a dominant morphism whose generic fiber is birationally equivalent to a homogeneous space of a connected linear algebraic group, with connected geometric stabilisers. Then X satisfies Conjecture 1.2.*

We then proceed to rational points in §9. We propose in §9.1 a conjecture on locally split values of polynomials in one variable ("locally split" means that the values are required to be products of primes which split in a given finite extension L/k) and prove, in §9.3, that this conjecture implies a general fibration theorem for the existence of rational points.

Theorem 1.5. *Let X be a smooth, proper, geometrically irreducible variety over a number field k and let $f : X \to \mathbf{P}^1_k$ be a dominant morphism with rationally connected geometric*

generic fiber. Suppose Conjecture 9.1 holds. If the smooth fibers of f above the rational points of $\mathbf{P}_k^1$ satisfy Conjecture 1.1, then so does X.

Theorem 1.5 admits various generalisations similar to those of Theorem 1.3 outlined in (1)–(4) above. See Corollary 9.23 and Corollary 9.25 for precise statements.

Various cases of Conjecture 9.1 can be established by algebro-geometric methods, by sieve methods or by additive combinatorics. We discuss these cases in §9.2 as well as the relation between Conjecture 9.1 and Schinzel's hypothesis (H). The most significant result towards Conjecture 9.1 is Matthiesen's recent work [Mat14], which renders Theorem 1.5 unconditional when $k = \mathbf{Q}$ and the singular fibers of f lie above rational points of $\mathbf{P}_\mathbf{Q}^1$. More precisely, we obtain the following theorem in §9.4.

Theorem 1.6. *Let X be a smooth, proper, geometrically irreducible variety over $\mathbf{Q}$. Let $f : X \to \mathbf{P}_\mathbf{Q}^1$ be a dominant morphism with rationally connected geometric generic fiber, whose non-split fibers all lie over rational points of $\mathbf{P}_\mathbf{Q}^1$. If $X_c(\mathbf{Q})$ is dense in $X_c(\mathbf{A}_\mathbf{Q})^{\mathrm{Br}(X_c)}$ for every rational point c of a Hilbert subset of $\mathbf{P}_\mathbf{Q}^1$, then $X(\mathbf{Q})$ is dense in $X(\mathbf{A}_\mathbf{Q})^{\mathrm{Br}(X)}$.*

This unconditional case of Theorem 1.5 was previously known only under the quite strong assumption that every singular fiber of f contains an irreducible component of multiplicity 1 split by an abelian extension of $\mathbf{Q}$ and that the smooth fibers of f above the rational points of $\mathbf{P}_\mathbf{Q}^1$ satisfy weak approximation (see [HSW14]), or otherwise under the assumption that f has a unique non-split fiber (see [Har97]). We note that the work [Mat14] relies on the contents of a recent paper of Browning and Matthiesen [BM13] which applied the descent method and additive combinatorics to prove Conjecture 1.1 for a smooth and proper model of the hypersurface (1.3) when $k = \mathbf{Q}$ and $P(t)$ is a product of linear polynomials, the number field K being arbitrary (a case now covered by Theorem 1.6).

The oldest instance of the use of the structure of a fibration to establish a particular case of Conjecture 1.1 or of Conjecture 1.2 can be traced back to Hasse's proof of the Hasse–Minkowski theorem for quadratic forms in four variables with rational coefficients starting from the case of quadratic forms in three variables with rational coefficients. It was based on Dirichlet's theorem on primes in arithmetic progressions and on the global reciprocity law. In 1979, Colliot-Thélène and Sansuc [CTS82] noticed that a variant of Hasse's proof yields Conjecture 1.1 for a large family of conic bundle surfaces over $\mathbf{P}_\mathbf{Q}^1$ if one assumes Schinzel's hypothesis (H), a conjectural generalisation of Dirichlet's theorem. Delicate arguments relying on the same two ingredients as Hasse's proof later allowed Salberger [Sal88] to settle Conjecture 1.2 unconditionally for those conic bundle surfaces over $\mathbf{P}_k^1$ whose Brauer group is reduced to constant classes. At the same time, building on the ideas of [CTSSD87], an abstract formalism for the fibration method was established in [Sko90b] in the case of fibrations, over $\mathbf{A}_k^n$, whose codimension 1 fibers are split and whose smooth fibers satisfy weak approximation. Further work of Serre [Ser92] and of Swinnerton-Dyer [SD94] on conic and two-dimensional quadric bundles over $\mathbf{P}_k^1$ led to the systematic study of fibrations, over the projective line and later over a curve of arbitrary genus or over a projective space, into varieties which satisfy weak approximation (see [CTSD94], [CTSSD98], [CT00], [Fro03], [vH03], [Wit12], [Lia13b], [HSW14]).

To deal with non-split fibers, all of these papers rely on the same reciprocity argument as Hasse's original proof, at the core of which lies the following well-known fact from class field theory: if L/k is a finite *abelian* extension of number fields, an element of k which is a local norm from L at all places of k except at most one must be a local norm at the remaining place too (see [AT09, Ch. VII, §3, Cor. 1, p. 51]). The failure of this statement for non-abelian extensions L/k turned out to be a critical hindrance to the development of the fibration method. On the one hand, it caused all of the papers mentioned in the previous paragraph to assume that every singular fiber of f over $\mathbf{A}_k^1$ contains an irreducible component of multiplicity 1 split by an abelian extension of the field over which the fiber is defined. (We note, however, that a trick allowing to deal with a few very specific non-abelian extensions whose Galois closure contains a nontrivial cyclic subextension, *e.g.*, cubic extensions, appeared in [Wei14a, Theorem 3.5] and [HSW14, Theorem 4.6].) On the other hand, it led to the restriction that the smooth fibers should satisfy weak approximation. Harari [Har94] [Har97] was able to relax this condition only in the following two situations: fibrations over $\mathbf{P}_k^1$ all of whose fibers over $\mathbf{A}_k^1$ are geometrically irreducible and fibrations, over more general bases, which admit a rational section. Further results in this direction were obtained in [Sme15] and in [Lia14] but were always limited by the assumption that some field extension built out of the splitting fields of the irreducible components of the fibers and of the residues, along these components, of a finite collection of classes of the Brauer group of the generic fiber, should be abelian.

The proofs of Theorem 1.3 and of Theorem 1.5 given below do not rely on Dirichlet's theorem on primes in arithmetic progressions or on any variant of it. As a consequence, they bypass the reciprocity argument alluded to in the previous paragraph, whose only purpose was to gain some control over the splitting behaviour of the unspecified prime output by Dirichlet's theorem. They bypass the ensuing abelianness assumptions as well. We replace Dirichlet's theorem by a simple consequence of strong approximation off a place for affine space minus a codimension 2 closed subset (see Lemma 5.2), or, in the context of rational points, by a certain conjectural variant of this lemma (see Conjecture 9.1, Proposition 9.9 and Corollary 9.10). Given a finite Galois extension L/k, these two statements directly give rise to elements of k which, outside a finite set of places of k, have positive valuation only at places splitting in L. To be applicable, however, they require that certain hypotheses be satisfied; the rest of the proofs of Theorem 1.3 and of Theorem 1.5 consists in relating these hypotheses to the Brauer–Manin condition, with the help of suitable versions of the Poitou–Tate duality theorem and of the so-called "formal lemma", a theorem originally due to Harari [Har94].

The paper is organised as follows. In §2, we give simple definitions for the groups $\mathrm{Pic}_+(C)$ and $\mathrm{Br}_+(C)$ first introduced in [Wit12, §5] and recall their properties and a consequence of the arithmetic duality theorem that they satisfy (Theorem 2.5). The group $\mathrm{Pic}_+(C)$ lies halfway between the usual Picard group $\mathrm{Pic}(C)$ and Rosenlicht's relative Picard group $\mathrm{Pic}(C, M)$ associated to a curve C endowed with a finite closed subset M. It plays a central role in the proof of Conjecture 1.2 for fibrations over a curve C of positive genus. When $C = \mathbf{P}_k^1$, one could equally well work with the algebraic tori which are implicit in the

definition of $\mathrm{Pic}_+(C)$ (see (2.1) below), but the formalism of $\mathrm{Pic}_+(C)$ and $\mathrm{Br}_+(C)$ turns out to be very convenient also in this case. In §3, we state and prove a version of the formal lemma for zero-cycles on a variety X fibered over a curve C of positive genus in which the zero-cycles under consideration are required to lie over a fixed linear equivalence class of divisors on C. In §4, we give a short proof of a theorem of Harari on the specialisation maps for the Brauer groups of the fibers of a morphism $f : X \to \mathbf{P}_k^1$ whose geometric generic fiber satisfies $H^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}) = 0$ and $H^2(X_{\bar{\eta}}, \mathscr{O}_{X_{\bar{\eta}}}) = 0$ and we adapt it to fibrations over a curve of positive genus. Building on the contents of §§2–3, we then prove in §5 the core existence theorem for effective zero-cycles on fibrations over curves from which Theorem 1.3 and all its refinements will be deduced (Theorem 5.1). In §6, we show that Theorem 5.1 remains valid if the arithmetic assumption on the smooth fibers is replaced by the same assumption over the closed points of a Hilbert subset of the base. To prove Conjecture 1.2, one needs to deal not only with effective cycles as in Theorem 5.1 but more generally with elements of completed Chow groups; the reduction steps required to bridge this gap are carried out in §7. Our main results on zero-cycles, including Theorem 1.3, are stated, established and compared with the literature in §8. Finally, we devote §9 to rational points. Conjecture 9.1 is stated in §9.1. Its relations with additive combinatorics, with Schinzel's hypothesis, with strong approximation properties and with sieve methods are all discussed in §9.2. Various results on rational points are deduced in §§9.3–9.4.

Notation and conventions. All cohomology groups appearing in this paper are étale (or Galois) cohomology groups. Following [Sko96], a variety X over a field k is said to be *split* if it contains a geometrically integral open subset. If X is a quasi-projective variety over k, we denote by $\mathrm{Sym}_{X/k}$ the disjoint union of the symmetric products $\mathrm{Sym}_{X/k}^d$ for $d \geq 1$. For any variety X over k, we denote by $\mathrm{Z}_0(X)$ the group of zero-cycles on X and by $\mathrm{CH}_0(X)$ its quotient by the subgroup of cycles rationally equivalent to 0. When X is proper, we let $\mathrm{A}_0(X) = \mathrm{Ker}\left(\deg : \mathrm{CH}_0(X) \to \mathbf{Z}\right)$. We write $\mathrm{Supp}(z)$ for the support of $z \in \mathrm{Z}_0(X)$ and denote by

(1.4) $\langle -, - \rangle : \mathrm{Br}(X) \times \mathrm{Z}_0(X) \to \mathrm{Br}(k)$

the pairing characterised by $\langle \beta, P \rangle = \mathrm{Cores}_{k(P)/k}(\beta(P))$ if P is a closed point of X and $\beta \in \mathrm{Br}(X)$ (see [Man71, Définition 7]). If X is proper, this pairing factors through rational equivalence and induces a pairing $\mathrm{Br}(X) \times \mathrm{CH}_0(X) \to \mathrm{Br}(k)$ which we still denote $\langle -, - \rangle$ (see *loc. cit.*, Proposition 8).

For any abelian group M, we set $\widehat{M} = \varprojlim_{n \geq 1} M/nM$.

When k is a number field, we let Ω denote the set of its places and Ω_f (resp. Ω_∞) the subset of finite (resp. infinite) places. For $v \in \Omega$, we denote by k_v the completion of k at v

and by $\mathcal{O}_v$ the ring of integers of k_v. For a finite subset $S \subset \Omega$, we denote by $\mathcal{O}_S$ the ring of S-integers of k (elements of k which are integral at the finite places not in S). If X is a variety over k, we denote by $Z_{0,\mathbf{A}}(X)$ the subset of $\prod_{v \in \Omega} Z_0(X \otimes_k k_v)$ consisting of those families $(z_v)_{v \in \Omega}$ such that if $\mathscr{X}$ is a model of X over $\mathcal{O}_S$ for some finite subset $S \subset \Omega$ (*i.e.*, a faithfully flat $\mathcal{O}_S$-scheme of finite type with generic fiber X), the Zariski closure of $\mathrm{Supp}(z_v)$ in $\mathscr{X} \otimes_{\mathcal{O}} \mathcal{O}_v$ is a finite $\mathcal{O}_v$-scheme for all but finitely many places v of k. (This property does not depend on the choice of $\mathscr{X}$ and S.) Note that $Z_{0,\mathbf{A}}(X) = \prod_{v \in \Omega} Z_0(X \otimes_k k_v)$ when X is proper. The sum of the local pairings $\langle -, - \rangle : \mathrm{Br}(X \otimes_k k_v) \times Z_0(X \otimes_k k_v) \to \mathrm{Br}(k_v)$ followed by the invariant map $\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbf{Q}/\mathbf{Z}$ of local class field theory gives rise to a well-defined pairing

(1.5) $$\mathrm{Br}(X) \times Z_{0,\mathbf{A}}(X) \to \mathbf{Q}/\mathbf{Z}.$$

If X is smooth and proper, we let $\mathrm{CH}_{0,\mathbf{A}}(X)$ denote the group defined in (1.1) and, for ease of notation, set

$$\mathrm{Pic}\widehat{}(X) = \widehat{\mathrm{Pic}(X)}, \quad \mathrm{CH}_0\widehat{}(X) = \widehat{\mathrm{CH}_0(X)}, \quad \mathrm{CH}_{0,\mathbf{A}}\widehat{}(X) = \widehat{\mathrm{CH}_{0,\mathbf{A}}(X)}.$$

The pairing (1.5) factors through rational equivalence and gives rise, as $\mathrm{Br}(X)$ is a torsion group for smooth X, to a pairing

(1.6) $$\mathrm{Br}(X) \times \mathrm{CH}_{0,\mathbf{A}}\widehat{}(X) \to \mathbf{Q}/\mathbf{Z}.$$

When X is a smooth curve, we write $\mathrm{Pic}_{\mathbf{A}}(X)$ and $\mathrm{Pic}_{\mathbf{A}}\widehat{}(X)$ for $\mathrm{CH}_{0,\mathbf{A}}(X)$ and $\mathrm{CH}_{0,\mathbf{A}}\widehat{}(X)$.

A *reduced* divisor on a smooth curve is a divisor all of whose coefficients belong to $\{0, 1\}$.

If $f : X \to Y$ is a morphism of schemes and Y is integral, we write $X_{\bar{\eta}} = X \times_Y \bar{\eta}$ for the geometric generic fiber of f, where $\bar{\eta}$ is a geometric point lying over the generic point of Y.

If X is a variety over a field k, endowed with a morphism $f : X \to C$ to a smooth proper curve over k, we denote by $Z_0^{\mathrm{eff},\mathrm{red}}(X)$ the set of effective zero-cycles z on X such that $f_* z$ is a reduced divisor on C. If k is a number field, we let $Z_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red}}(X) = Z_{0,\mathbf{A}}(X) \cap \prod_{v \in \Omega} Z_0^{\mathrm{eff},\mathrm{red}}(X \otimes_k k_v)$. If, in addition, a class $y \in \mathrm{Pic}(C)$ is fixed, we denote by $Z_0^{\mathrm{eff},\mathrm{red},y}(X \otimes_k k_v)$ the inverse image of $y \otimes_k k_v$ by the push-forward map $Z_0^{\mathrm{eff},\mathrm{red}}(X \otimes_k k_v) \to \mathrm{Pic}(C \otimes_k k_v)$ for $v \in \Omega$ and by $Z_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X)$ the inverse image of y by the push-forward map $Z_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red}}(X) \to \mathrm{Pic}_{\mathbf{A}}(C)$. In other words, the set $Z_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X)$ consists of those collections of local effective zero-cycles $(z_v)_{v \in \Omega}$ such that z_v is integral for all but finitely many $v \in \Omega$, such that $f_* z_v$ is reduced for all $v \in \Omega$, such that $f_* z_v$ is linearly equivalent to $y \otimes_k k_v$ for all $v \in \Omega_f$ and such that $f_* z_v$ is linearly equivalent to $y \otimes_k k_v$ up to a norm from $\bar{k}_v$ for all $v \in \Omega_\infty$.

For brevity, we adopt the following terminology.

Definition 1.7. A smooth and proper variety X over a number field k *satisfies* (E) if the complex (1.2) associated to X is exact; it *satisfies* (E$_1$) if the existence of a family $(z_v)_{v \in \Omega} \in \prod_{v \in \Omega} Z_0(X \otimes_k k_v)$ orthogonal to $\mathrm{Br}(X)$ with respect to (1.5) and such that $\deg(z_v) = 1$ for all $v \in \Omega$ implies the existence of a zero-cycle of degree 1 on X.

Any variety which satisfies (E) also satisfies (E$_1$) (see [Wit12, Remarques 1.1 (iii)]).

Following Lang [Lan83, Chapter 9, §5], we say that a subset H of an irreducible variety X is a *Hilbert subset* if there exist a dense open subset $X^0 \subseteq X$, an integer $n \geq 1$ and irreducible finite étale X^0-schemes $W_1, \ldots, W_n$ such that H is the set of points of X^0 above which the fiber of W_i is irreducible for all $i \in \{1, \ldots, n\}$. We stress that H is not a subset of $X(k)$; for example, the generic point of X belongs to every Hilbert subset. Hilbert subsets in this sense have also been referred to as *generalised Hilbert subsets* in the literature (see [Lia12a]).

Finally, for the sake of completeness, we include a proof of the following lemma, which was observed independently by Cao and Xu [CX14, Proposition 3.6] and by Wei [Wei14b, Lemma 1.1] and which will be used in §5.1 below. If v_0 is a place of a number field k, we say that a variety X over k satisfies *strong approximation off v_0* if either $X(k_{v_0}) = \varnothing$ or the following condition is satisfied: for any finite subset $S \subset \Omega$ containing $\Omega_\infty \cup \{v_0\}$ and any model $\mathscr{X}$ of X over $\mathcal{O}_S$, the diagonal map $\mathscr{X}(\mathcal{O}_S) \to \prod_{v \in S \setminus \{v_0\}} X(k_v) \times \prod_{v \notin S} \mathscr{X}(\mathcal{O}_v)$ has dense image.

Lemma 1.8. *Let $n \geq 1$ be an integer, let k be a number field, let v_0 be a place of k and let $F \subseteq \mathbf{A}_k^n$ be a closed subset of codimension ≥ 2. The variety $U = \mathbf{A}_k^n \setminus F$ satisfies strong approximation off v_0.*

Proof, after [Wei14b]. Let us fix a finite subset $S \subset \Omega$ containing $\Omega_\infty \cup \{v_0\}$, a model $\mathscr{U}$ of U over $\mathcal{O}_S$ and a family $(P_v)_{v \in \Omega} \in \prod_{v \in S} U(k_v) \times \prod_{v \notin S} \mathscr{U}(\mathcal{O}_v)$. Using weak approximation, we fix $Q \in U(k)$ arbitrarily close to P_v for $v \in S$. Let $S' \subset \Omega$ be a finite subset, containing S, such that $Q \in \mathscr{U}(\mathcal{O}_{S'})$. Let $Q' \in U(k)$ be arbitrarily close to P_v for $v \in S' \setminus S$ and general enough, in the Zariski topology, that the line $L \subset \mathbf{A}_k^n$ passing through Q and Q' is contained in U. As L satisfies strong approximation off v_0 and as the Zariski closure $\mathscr{L}$ of L in $\mathscr{U}$ possesses an $\mathcal{O}_{S'}$-point (namely Q) and an $\mathcal{O}_v$-point for each $v \in S' \setminus S$ (namely Q'), there exists $P \in \mathscr{L}(\mathcal{O}_S)$ arbitrarily close to Q, and hence to P_v, at the places of $S \setminus \{v_0\}$. $\quad\square$

2. REMINDERS ON THE GROUPS $\mathrm{Pic}_+(C)$ AND $\mathrm{Br}_+(C)$

The groups we denote $\mathrm{Pic}_+(C)$ and $\mathrm{Br}_+(C)$ were defined in [Wit12, §5] as étale hypercohomology groups of certain explicit complexes. We give down-to-earth definitions of these groups in §2.1 and recall in §2.2 a corollary of an arithmetic duality theorem established in *loc. cit.* which we shall use in §5 and in §9 and for the proof of which the hypercohomological point of view seems unavoidable when C has positive genus.

2.1. **Over an arbitrary field.** Let C be a smooth, proper, geometrically irreducible curve over a field k of characteristic 0. Let $C^0 \subseteq C$ be a dense open subset. Let $M = C \setminus C^0$. For each $m \in M$, let L_m be a nonzero finite étale algebra over the residue field $k(m)$ of m. (In all of the applications considered in this paper, the algebra L_m will be a finite field extension of $k(m)$ whenever k is a number field.)

Definition 2.1. To the data of C, of M and of the finite $k(m)$-algebras L_m, we associate two groups, denoted $\mathrm{Pic}_+(C)$ and $\mathrm{Br}_+(C)$:

- the group $\mathrm{Pic}_+(C)$ is the quotient of $\mathrm{Div}(C^0)$ by the subgroup of principal divisors $\mathrm{div}(f)$ such that for every $m \in M$, the rational function f is invertible at m and its value $f(m)$ belongs to the subgroup $N_{L_m/k(m)}(L_m^*) \subseteq k(m)^*$;
- the group $\mathrm{Br}_+(C)$ is the subgroup of $\mathrm{Br}(C^0)$ consisting of those classes whose residue at m belongs to the kernel of the restriction map

$$H^1(k(m), \mathbf{Q}/\mathbf{Z}) \xrightarrow{r_m} H^1(L_m, \mathbf{Q}/\mathbf{Z})$$

for every $m \in M$.

Thus $\mathrm{Pic}_+(C)$ and $\mathrm{Br}_+(C)$ depend on M and on the algebras L_m, though this dependence is not made explicit in the notation. The group $\mathrm{Pic}_+(C)$ fits into a natural exact sequence

$$(2.1) \qquad k^* \longrightarrow \bigoplus_{m \in M} k(m)^*/N_{L_m/k(m)}(L_m^*) \xrightarrow{\delta} \mathrm{Pic}_+(C) \longrightarrow \mathrm{Pic}(C) \longrightarrow 0,$$

where δ sends the class of $(a_m)_{m \in M} \in \bigoplus_{m \in M} k(m)^*$ to the class in $\mathrm{Pic}_+(C)$ of $\mathrm{div}(f)$ for any rational function f on C, invertible in a neighbourhood of M, such that $f(m) = a_m$ for every $m \in M$ (the existence of such f follows from the vanishing of $H^1(U, \mathscr{O}_U(-M))$ for an affine open neighbourhood U of M). Similarly, there is a natural exact sequence

$$(2.2) \qquad 0 \longrightarrow \mathrm{Br}(C) \longrightarrow \mathrm{Br}_+(C) \longrightarrow \bigoplus_{m \in M} \mathrm{Ker}(r_m) \longrightarrow H^3(C, \mathbf{G}_\mathrm{m})$$

(see [Wit12, p. 2148]).

Remarks 2.2. (i) The group $\mathrm{Pic}_+(C)$ is a quotient of the relative Picard group $\mathrm{Pic}(C, M)$, first considered by Rosenlicht. Recall that $\mathrm{Pic}(C, M)$ may be defined either as the quotient of $\mathrm{Div}(C^0)$ by the subgroup of principal divisors $\mathrm{div}(f)$ such that $f(m) = 1$ for all $m \in M$, or, in cohomological terms, as $\mathrm{Pic}(C, M) = H^1(C, \mathrm{Ker}(\mathbf{G}_\mathrm{m} \to i_* \mathbf{G}_\mathrm{m}))$, where $i : M \hookrightarrow C$ denotes the inclusion of M in C (see [SV96, §2]).

(ii) As $\mathrm{Ker}(r_m)$ is finite for each $m \in M$, the quotient $\mathrm{Br}_+(C)/\mathrm{Br}(C)$ is finite. The finiteness of this quotient will play a crucial role in §5 and §9, as the "formal lemma" (see §3) can only be applied to a finite subgroup of the Brauer group. This is the main reason why $\mathrm{Pic}_+(C)$ is needed in this paper instead of the more classical $\mathrm{Pic}(C, M)$. Using the latter would entail dealing with the usually infinite quotient $\mathrm{Br}(C^0)/\mathrm{Br}(C)$.

(iii) The groups $\mathrm{Pic}_+(C)$ and $\mathrm{Br}_+(C)$ were defined in a slightly more general context in [Wit12, §5], where the input was, for each $m \in M$, a finite collection $(K_{m,i})_{i \in I_m}$ of finite extensions of $k(m)$ together with a collection of integers $(e_{m,i})_{i \in I_m}$. The more restrictive situation considered here, in which I_m has cardinality 1 and $e_{m,i} = 1$ for all m, suffices for our purposes.

It is also possible to give cohomological definitions for the groups $\mathrm{Pic}_+(C)$ and $\mathrm{Br}_+(C)$. Namely, let $j : C^0 \hookrightarrow C$ and $\rho : C \to \mathrm{Spec}(k)$ denote the canonical morphisms, and for $m \in M$, let $i_m : \mathrm{Spec}(L_m) \to C$ denote the map induced by the $k(m)$-algebra L_m and by the inclusion of m in C. Endow the category of smooth k-schemes with the étale topology. Let $\mathscr{E}$ be the object of the bounded derived category of sheaves of abelian groups, on the

corresponding site, defined by $\mathscr{E} = \tau_{\leq 1} R\rho_* [j_* \mathbf{G}_{\mathrm{m}} \to \bigoplus_{m \in M} i_{m*} \mathbf{Z}]$, where the brackets denote a two-term complex placed in degrees 0 and 1.

Proposition 2.3. *There are canonical isomorphisms* $\mathrm{Pic}_+(C) = H^0(k, R\mathscr{H}om(\mathscr{E}, \mathbf{G}_{\mathrm{m}}))$ *and* $\mathrm{Br}_+(C) = H^2(k, \mathscr{E})$.

Proof. The second isomorphism is obtained in [Wit12, Remarques 5.2 (i)]. For the first isomorphism, we note that $\mathrm{Pic}(C, M) = H^0(k, R\mathscr{H}om(\tau_{\leq 1} R\rho_*(j_* \mathbf{G}_{\mathrm{m}}), \mathbf{G}_{\mathrm{m}}))$, as follows from Deligne's universal coefficient formula [Wit12, (5.4)], from the cohomological definition of $\mathrm{Pic}(C, M)$ and from the exact sequence $0 \to \mathbf{G}_{\mathrm{m}} \to j_* \mathbf{G}_{\mathrm{m}} \to i_* \mathbf{Z} \to 0$, where i denotes the inclusion of M in C. The natural distinguished triangle

$$(2.3) \qquad \mathscr{E} \longrightarrow \tau_{\leq 1} R\rho_*(j_* \mathbf{G}_{\mathrm{m}}) \longrightarrow \bigoplus_{m \in M} (\rho \circ i_m)_* \mathbf{Z} \xrightarrow{+1}$$

then presents $H^0(k, R\mathscr{H}om(\mathscr{E}, \mathbf{G}_{\mathrm{m}}))$ as the desired quotient of $\mathrm{Pic}(C, M)$. $\qquad \square$

The next proposition is an immediate consequence of Proposition 2.3 (see [KS06, Theorem 18.6.4 (vii)]). The interested reader may also deduce it directly from Definition 2.1 as an amusing exercise.

Proposition 2.4. *The natural pairing* $\langle -, - \rangle : \mathrm{Br}(C^0) \times \mathrm{Div}(C^0) \to \mathrm{Br}(k)$ *induces a pairing* $\mathrm{Br}_+(C) \times \mathrm{Pic}_+(C) \to \mathrm{Br}(k)$.

2.2. **Over a number field.** Let us now assume that k is a number field. We shall denote by $\mathrm{Pic}_+(C \otimes_k k_v)$, $\mathrm{Br}_+(C \otimes_k k_v)$ the groups associated to the curve, the open subset, and the finite algebras obtained by scalar extension from k to k_v. Let

$$\mathrm{Pic}_{+,\mathbf{A}}(C) = \prod_{v \in \Omega_f}{}' \mathrm{Pic}_+(C \otimes_k k_v) \times \prod_{v \in \Omega_\infty} \mathrm{Coker}\left(N_{\bar{k}_v/k_v} : \mathrm{Pic}_+(C \otimes_k \bar{k}_v) \to \mathrm{Pic}_+(C \otimes_k k_v) \right)$$

where the symbol $\prod'$ denotes the restricted product with respect to the subgroups $\mathrm{Im}(\mathrm{Pic}_+(\mathscr{C} \otimes \mathcal{O}_v) \to \mathrm{Pic}_+(C \otimes_k k_v))$, for a model $\mathscr{C}$ of C over a dense open subset of the spectrum of the ring of integers of k (see [Wit12, §5.3]). Composing the pairing $\mathrm{Pic}_+(C \otimes_k k_v) \times \mathrm{Br}_+(C \otimes_k k_v) \to \mathrm{Br}(k_v)$ given by Proposition 2.4 with the local invariant map $\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbf{Q}/\mathbf{Z}$ of local class field theory, and then summing over all $v \in \Omega$, yields a well-defined pairing

$$(2.4) \qquad\qquad \mathrm{Br}_+(C) \times \mathrm{Pic}_{+,\mathbf{A}}(C) \to \mathbf{Q}/\mathbf{Z},$$

since $\mathrm{Br}(\mathcal{O}_v) = 0$ for $v \in \Omega_f$ (see *loc. cit.*). By the global reciprocity law, the image of $\mathrm{Pic}_+(C)$ in $\mathrm{Pic}_{+,\mathbf{A}}(C)$ is contained in the right kernel of (2.4). Thus we obtain a complex of abelian groups

$$(2.5) \qquad\qquad \mathrm{Pic}_+(C) \longrightarrow \mathrm{Pic}_{+,\mathbf{A}}(C) \longrightarrow \mathrm{Hom}(\mathrm{Br}_+(C), \mathbf{Q}/\mathbf{Z}).$$

For topological reasons, this complex fails to be exact when C has positive genus. (Indeed, assuming, for simplicity, that $C(k) \neq \varnothing$ and that $L_m = k(m)$ for all $m \in M$, the group $\mathrm{Pic}_+(C) = \mathrm{Pic}(C)$ is finitely generated by the Mordell–Weil theorem, so that its degree 0

subgroup often fails to be closed in the group of adelic points of the Jacobian of C and therefore also in $\mathrm{Pic}_{+,\mathbf{A}}(C) = \mathrm{Pic}_{\mathbf{A}}(C)$.) The exactness of an analogous complex where the groups $\mathrm{Pic}_+(C)$ and $\mathrm{Pic}_{+,\mathbf{A}}(C)$ are replaced with suitable completions is investigated in [Wit12, Théorème 5.3]; we shall only need the following corollary of this result.

Theorem 2.5 ([Wit12, Corollaire 5.7])**.** *Any element of $\mathrm{Pic}_{+,\mathbf{A}}(C)$ which is orthogonal to $\mathrm{Br}_+(C)$ with respect to (2.4) and whose image in $\mathrm{Pic}_{\mathbf{A}}(C)$ comes from $\mathrm{Pic}(C)$ is itself the image of an element of $\mathrm{Pic}_+(C)$.*

When $C = \mathbf{P}^1_k$, Theorem 2.5 is a simple consequence of Poitou–Tate duality for the norm 1 tori associated to the étale algebras $L_m/k(m)$ (see [Mil06, Ch. I, Theorem 4.20] for the relevant statement).

3. A version of the formal lemma over a curve

The name "formal lemma" refers to a theorem of Harari according to which if X^0 is a dense open subset of a smooth variety X over a number field k and $B \subset \mathrm{Br}(X^0)$ is a *finite* subgroup, adelic points of X^0 orthogonal to B are dense in the set of adelic points of X orthogonal to $B \cap \mathrm{Br}(X)$ (see [Har94, Corollaire 2.6.1], [CTSD94, §3.3], [CTS00, Proposition 1.1], [CT03, Théorème 1.4]). This result plays a key role in all instances of the fibration method. In this section, we establish a variant of this theorem for effective zero-cycles on the total space of a fibration over a curve, in which the zero-cycles are constrained to lie over a fixed linear equivalence class of divisors on the curve.

Proposition 3.1. *Let C be a smooth, proper and geometrically irreducible curve of genus g over a number field k. Let $y \in \mathrm{Pic}(C)$ be such that $\deg(y) > 2g+1$. Let X be a smooth and irreducible variety over k. Let $f : X \to C$ be a morphism with geometrically irreducible generic fiber. Let $X^0 \subseteq X$ be a dense open subset and $B \subset \mathrm{Br}(X^0)$ be a finite subgroup. For any $(z_v)_{v \in \Omega} \in \mathrm{Z}^{\mathrm{eff,red},y}_{0,\mathbf{A}}(X^0)$ orthogonal to $B \cap \mathrm{Br}(X)$ with respect to (1.5) and any finite subset $S \subset \Omega$, there exists $(z'_v)_{v \in \Omega} \in \mathrm{Z}^{\mathrm{eff,red},y}_{0,\mathbf{A}}(X^0)$ orthogonal to B with respect to (1.5) such that $z'_v = z_v$ for $v \in S$.*

Proof. We start with two lemmas.

Lemma 3.2. *Let $\mathscr{X}^0$ be a model of X^0 over $\mathcal{O}_S$ for some finite subset $S \subset \Omega$. There exists a finite subset $S' \subset \Omega$ containing $S \cup \Omega_\infty$ such that for any $v \in \Omega \setminus S'$, any finite closed subset $M' \subset C \otimes_k k_v$ and any $y_0 \in \mathrm{Pic}(C \otimes_k k_v)$ with $\deg(y_0) > 2g$, there exists an effective zero-cycle z_0 on $X^0 \otimes_k k_v$ satisfying the following conditions:*

- *the divisor $f_* z_0$ is reduced and supported outside M';*
- *the class of $f_* z_0$ in $\mathrm{Pic}(C \otimes_k k_v)$ is equal to y_0;*
- *the Zariski closure of $\mathrm{Supp}(z_0)$ in $\mathscr{X}^0 \otimes_{\mathcal{O}_S} \mathcal{O}_v$ is finite over $\mathcal{O}_v$.*

Proof. This is a consequence of the moving lemma [Wit12, Lemme 4.2], of the Lang–Weil–Nisnevich estimates (cf. [LW54], [Nis54]) and of Hensel's lemma. See the proof of [Wit12, Lemme 4.3]. $\square$

Lemma 3.3. *Let $\beta \in \mathrm{Br}(X^0)$. If β does not belong to the subgroup $\mathrm{Br}(X) \subseteq \mathrm{Br}(X^0)$, the map $\mathrm{Z}_0^{\mathrm{eff},\mathrm{red},y}(X^0 \otimes_k k_v) \to \mathrm{Br}(k_v)$, $z \mapsto \langle \beta, z \rangle$ is nonzero for infinitely many $v \in \Omega$.*

Proof. Choose a model $\mathscr{X}^0$ of X^0 over $\mathcal{O}_S$ for some finite subset $S \subset \Omega$. After enlarging S, we may assume that β comes from $\mathrm{Br}(\mathscr{X}^0)$. Let S' be given by Lemma 3.2. According to [Har94, Théorème 2.1.1], there are infinitely many $v \in \Omega$ such that $\beta(a) \neq 0$ for some $a \in X^0(k_v)$. Pick $v \notin S'$ and $a \in X^0(k_v)$ such that $\beta(a) \neq 0$ and let $y_0 \in \mathrm{Pic}(C \otimes_k k_v)$ denote the class $y - f(a)$. By the definition of S', there exists an effective zero-cycle z_0 on $X^0 \otimes_k k_v$ satisfying the conditions of Lemma 3.2 with $M' = \{f(a)\}$. Letting $z = z_0 + a$, we have $z \in \mathrm{Z}_0^{\mathrm{eff},\mathrm{red},y}(X^0 \otimes_k k_v)$ and $\langle \beta, z \rangle = \langle \beta, z_0 \rangle + \beta(a)$. As β comes from $\mathrm{Br}(\mathscr{X}^0)$ and the closure of $\mathrm{Supp}(z_0)$ in $\mathscr{X}^0 \otimes_{\mathcal{O}_S} \mathcal{O}_v$ is finite over $\mathcal{O}_v$, we have $\langle \beta, z_0 \rangle = 0$, so $\langle \beta, z \rangle \neq 0$. $\square$

Proposition 3.1 follows from Lemma 3.3 by the formal argument used in the proof of [Har94, Corollaire 2.6.1]. For the convenience of the reader, we reproduce it briefly. Let $B^* = \mathrm{Hom}(B, \mathbf{Q}/\mathbf{Z})$. For $v \in \Omega$, let $\varphi_v : \mathrm{Z}_0^{\mathrm{eff},\mathrm{red},y}(X^0 \otimes_k k_v) \to B^*$ be defined by $\varphi_v(z)(\beta) = \mathrm{inv}_v\langle \beta, z \rangle$. Let $\Gamma \subseteq B^*$ be the subgroup generated by those elements which belong to the image of φ_v for infinitely many $v \in \Omega$. Recall that we are given a finite set S and a family $(z_v)_{v \in \Omega}$ in the statement of Proposition 3.1. After enlarging S, we may assume that φ_v takes values in Γ for any $v \notin S$ and that $\varphi_v(z_v) = 0$ for $v \notin S$. Consider the natural pairing $B \times B^* \to \mathbf{Q}/\mathbf{Z}$. Any element of B which is orthogonal to Γ belongs to $B \cap \mathrm{Br}(X)$ according to Lemma 3.3, hence is orthogonal to $w = \sum_{v \in S} \varphi_v(z_v)$. As Pontrjagin duality is a perfect duality among finite abelian groups, it follows that $w \in \Gamma$. Therefore there exist a finite set $T \subset \Omega$ disjoint from S and a family $(z'_v)_{v \in T} \in \prod_{v \in T} \mathrm{Z}_0^{\mathrm{eff},\mathrm{red},y}(X^0 \otimes_k k_v)$ such that $w = -\sum_{v \in T} \varphi_v(z'_v)$. Letting $z'_v = z_v$ for $v \in \Omega \setminus T$ gives the desired family $(z'_v)_{v \in \Omega}$. $\square$

4. Specialisation of the Brauer group

The following result is a theorem of Harari when $C = \mathbf{P}^1_k$ (see [Har97, Théorème 2.3.1], see also [Har94, §3]). The arguments of [Har97] still apply when C is an arbitrary curve, once one knows that $H^3(k(C), \mathbf{G}_\mathrm{m}) = 0$. For the sake of completeness, we nevertheless include a shorter, self-contained proof. In the next statement, we denote by η the generic point of C, we identify h and η with $\mathrm{Spec}(k(h))$ and $\mathrm{Spec}(k(C))$ and we denote by $f_h : X_h \to \mathrm{Spec}(k(h))$ the fiber of f above h.

Proposition 4.1. *Let C be a smooth irreducible curve over a number field k. Let X be an irreducible variety over k and $f : X \to C$ be a morphism whose geometric generic fiber $X_{\bar{\eta}}$ is smooth, proper and irreducible and satisfies $H^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}) = 0$ and $H^2(X_{\bar{\eta}}, \mathcal{O}_{X_{\bar{\eta}}}) = 0$. Let $C^0 \subseteq C$ be a dense open subset, let $X^0 = f^{-1}(C^0)$ and let $B \subseteq \mathrm{Br}(X^0)$ be a subgroup. If the natural map $B \to \mathrm{Br}(X_\eta)/f_\eta^*\mathrm{Br}(\eta)$ is surjective, there exists a Hilbert subset $H \subseteq C^0$ such that the natural map $B \to \mathrm{Br}(X_h)/f_h^*\mathrm{Br}(h)$ is surjective for all $h \in H$.*

Proof. By shrinking C, we may assume that $C^0 = C$ and that f is smooth and proper. Let $V \subseteq H^2(X, \mathbf{Q}/\mathbf{Z}(1))$ be the inverse image of B by the second map of the exact sequence

(4.1) $$ 0 \longrightarrow \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} \longrightarrow H^2(X, \mathbf{Q}/\mathbf{Z}(1)) \longrightarrow \mathrm{Br}(X) \longrightarrow 0 $$

(see [Gro68, II, Théorème 3.1]). Let $\mathrm{Gal}(\bar{\eta}/\eta)$ and $\mathrm{Gal}(\bar{h}/h)$ denote the absolute Galois groups of $k(C)$ and of $k(h)$, respectively, and consider the commutative diagram

(4.2)

$$
\begin{array}{ccc}
H^2(X_\eta, \mathbf{Q}/\mathbf{Z}(1))/f_\eta^* H^2(\eta, \mathbf{Q}/\mathbf{Z}(1)) & \xrightarrow{\ \alpha_\eta\ } & H^2(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}(1))^{\mathrm{Gal}(\bar{\eta}/\eta)} \\
{\scriptstyle\beta_\eta}\uparrow & & \uparrow{\scriptstyle\gamma_\eta} \\
V & \xrightarrow{\hspace{3cm}} & H^0(C, R^2 f_* \mathbf{Q}/\mathbf{Z}(1)) \\
{\scriptstyle\beta_h}\downarrow & & \downarrow{\scriptstyle\gamma_h} \\
H^2(X_h, \mathbf{Q}/\mathbf{Z}(1))/f_h^* H^2(h, \mathbf{Q}/\mathbf{Z}(1)) & \xrightarrow{\ \alpha_h\ } & H^2(X_{\bar{h}}, \mathbf{Q}/\mathbf{Z}(1))^{\mathrm{Gal}(\bar{h}/h)}
\end{array}
$$

for $h \in C$, where γ_η and γ_h are pull-back maps. By comparing (4.1) with the analogous exact sequence for X_η, we note that the surjectivity of the natural maps $B \to \mathrm{Br}(X_\eta)/f_\eta^* \mathrm{Br}(\eta)$ and $\mathrm{Pic}(X) \to \mathrm{Pic}(X_\eta)$ implies the surjectivity of β_η. We shall now deduce the existence of a Hilbert subset $H \subseteq C$ such that β_h is surjective for all $h \in H$.

Lemma 4.2. *The group* $\mathrm{Gal}(\bar{\eta}/\eta)$ *acts on* $H^2(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}(1))$ *through a finite quotient.*

Proof. By [Gro68, II, §3.5], the group $H^2(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}(1))$ is an extension of $\mathrm{Br}(X_{\bar{\eta}})$, which is finite as $H^2(X_{\bar{\eta}}, \mathscr{O}_{X_{\bar{\eta}}}) = 0$ (see, *e.g.*, [CTS13, Lemma 1.3]), by $\mathrm{NS}(X_{\bar{\eta}}) \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z}$, on which $\mathrm{Gal}(\bar{\eta}/\eta)$ acts through a finite quotient since $\mathrm{NS}(X_{\bar{\eta}})$ is finitely generated. $\square$

Lemma 4.3. *There exists a Hilbert subset* $H \subseteq C$ *such that* γ_h *is an isomorphism for every* $h \in H$. *In particular, the map* γ_η *is an isomorphism.*

Proof. As f is smooth and proper, the étale sheaf $R^2 f_* \mathbf{Q}/\mathbf{Z}(1)$ is a direct limit of locally constant sheaves with finite stalks (see [Mil80, Ch. VI, Corollary 4.2]). Therefore it gives rise, for any $h \in C$, to an action of the profinite group $\pi_1(C, \bar{h})$ on $H^2(X_{\bar{h}}, \mathbf{Q}/\mathbf{Z}(1))$. In terms of this action, the map γ_h can be naturally identified with the inclusion

$$H^2(X_{\bar{h}}, \mathbf{Q}/\mathbf{Z}(1))^{\pi_1(C, \bar{h})} \subseteq H^2(X_{\bar{h}}, \mathbf{Q}/\mathbf{Z}(1))^{\mathrm{Gal}(\bar{h}/h)}$$

(see [Mil80, p. 156]). By Lemma 4.2, the group $\pi_1(C, \bar{\eta})$ acts on $H^2(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}(1))$ through the automorphism group of some irreducible finite Galois cover $\pi : D \to C$. The map γ_h is then an isomorphism for any $h \in C$ such that $\mathrm{Gal}(\bar{h}/h)$ surjects onto the quotient of $\pi_1(C, \bar{h})$ corresponding to π, *i.e.*, for any $h \in C$ such that $\pi^{-1}(h)$ is irreducible. $\square$

As the groups $H^3(\eta, \mathbf{Q}/\mathbf{Z}(1))$, $H^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z})$ and therefore also $H^1(X_{\bar{h}}, \mathbf{Q}/\mathbf{Z})$ all vanish (by [CTP00, Lemma 2.6] for the first group and by the smooth base change theorem [Mil80, Ch. VI, Corollary 4.2] for the third group), we deduce from the Hochschild–Serre spectral sequence that α_η is surjective and α_h is injective (these two maps are in fact isomorphisms). If H is given by Lemma 4.3, we conclude that for all $h \in H$, the surjectivity of β_h, and hence of the natural map $B \to \mathrm{Br}(X_h)/f_h^* \mathrm{Br}(h)$, follows from that of β_η. $\square$

Remark 4.4. Proposition 4.1 and its proof go through for fibrations over an arbitrary base as long as the natural map $H^3(\eta, \mathbf{Q}/\mathbf{Z}(1)) \to H^3(X_\eta, \mathbf{Q}/\mathbf{Z}(1))$ is injective.

5. Existence of zero-cycles

The goal of §5 is to establish the following existence result for zero-cycles on the total space of a fibration over a curve of arbitrary genus.

Theorem 5.1. *Let C be a smooth, projective and geometrically irreducible curve over a number field k. Let X be a smooth, projective, irreducible variety over k and $f : X \to C$ be a morphism with geometrically irreducible generic fiber and no multiple fiber (in the sense that the gcd of the multiplicities of the irreducible components of each fiber is 1). Let $C^0 \subseteq C$ be a dense open subset over which f is smooth. Let $X^0 = f^{-1}(C^0)$. Let $B \subset \mathrm{Br}(X^0)$ be a finite subgroup. Let $y \in \mathrm{Pic}(C)$ and $(z_v)_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X^0)$. Let $S \subset \Omega_f$ be a finite subset. Let $\mathscr{U} \subseteq \prod_{v \in S} \mathrm{Sym}_{X^0/k}(k_v)$ be a neighbourhood of $(z_v)_{v \in S}$. Assume that*

(i) *the family $(z_v)_{v \in \Omega}$ is orthogonal to $(B + f_\eta^* \mathrm{Br}(\eta)) \cap \mathrm{Br}(X)$ with respect to (1.5);*

(ii) *$\deg(y) \geq \deg(M) + 2g + 2$, where $M = C \setminus C^0$ and g denotes the genus of C;*

(iii) *for each real place v of k, there exists a smooth, proper, geometrically irreducible curve $Z_v \subset X \otimes_k k_v$ which contains the support of z_v and dominates C, such that $\deg(y) \geq [k_v(Z_v) : k_v(C)] \deg(M) + 2g_v + 2$, where g_v denotes the genus of Z_v.*

Then there exist a family $(z_v')_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}(X^0)$ and an effective divisor c on C^0 whose class in $\mathrm{Pic}(C)$ is y, such that

(1) *$f_* z_v' = c$ in $\mathrm{Div}(C^0 \otimes_k k_v)$ for each $v \in \Omega$;*

(2) *the family $(z_v')_{v \in \Omega}$ is orthogonal to B with respect to (1.5);*

(3) *z_v' is effective for each $v \in S$ and $(z_v')_{v \in S}$ belongs to $\mathscr{U}$;*

(4) *for each real place v of k and each connected component $\mathscr{B}$ of $X^0(k_v)$, if we write $z_v - z_v' = \sum_{x \in X^0 \otimes_k k_v} n_x x$, then $\sum_{x \in \mathscr{B}} n_x$ is even.*

Furthermore, if every fiber of f possesses an irreducible component of multiplicity 1, one can ensure that z_v' is effective for all $v \in \Omega$.

Let us emphasise that as per the conventions spelled out in §1, the notation $\mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X^0)$ in the above statement refers to the morphism $X^0 \to C$ induced by f. Thus, for $v \in \Omega_f$, the linear equivalence class of $f_* z_v$ is assumed to coincide with y in $\mathrm{Pic}(C \otimes_k k_v)$, not solely in $\mathrm{Pic}(C^0 \otimes_k k_v)$ (and similarly, up to norms, for $v \in \Omega_\infty$). Moreover, let us stress that in (4), we identify $X^0(k_v)$ with a subset of $X^0 \otimes_k k_v$. Thus, for $x \in X^0 \otimes_k k_v$, one has $x \in \mathscr{B}$ if and only if $k_v(x) = k_v$ and the corresponding k_v-point of X^0 belongs to $\mathscr{B}$.

The proof of Theorem 5.1 occupies §§5.1–5.5. We shall deal with curves C of arbitrary genus directly, without reducing to the case of genus 0, using the Riemann–Roch theorem in a spirit closer to [CT00] than to [Wit12].

5.1. **A consequence of strong approximation.** Lemma 5.2 below, which we state with independent notation, will play a decisive role in the proof of Theorem 5.1. It should be compared with Dirichlet's theorem on primes in arithmetic progressions for general number fields. According to the latter, given a finite set S of finite places of k and elements $t_v \in k_v^*$ for $v \in S$, there exists a totally positive $t \in k^*$ arbitrarily close to t_v for $v \in S$ such that t is a unit outside S except at one place, at which it is a uniformiser (see [Neu86, Ch. V, Theorem 6.2]).

Lemma 5.2. *Let L/k be a finite extension of number fields. Let S be a finite set of places of k. For $v \in S$, let $t_v \in k_v^*$. Suppose t_v is a norm from $(L \otimes_k k_v)^*$ for every $v \in S$. Then there exists $t \in k^*$, arbitrarily close to t_v for $v \in S$, such that for any finite place $v \notin S$, either t is a unit at v or L possesses a place of degree 1 over v. In addition, if v_0 is a place of k, not in S, over which L possesses a place of degree 1, one can ensure that t is integral outside $S \cup \{v_0\}$.*

Proof. The complement of the quasi-trivial torus $R_{L/k}(\mathbf{G}_{\mathrm{m},L})$ in the affine space $R_{L/k}(\mathbf{A}_L^1)$ is a divisor with normal crossings (geometrically isomorphic to the union of all coordinate hyperplanes). We denote it by D, we let F be its singular locus and set $W = R_{L/k}(\mathbf{A}_L^1) \setminus F$. Similarly, we let $\mathscr{T} = R_{\mathcal{O}_L/\mathcal{O}_k}(\mathbf{G}_{\mathrm{m},\mathcal{O}_L})$, $\mathscr{D} = R_{\mathcal{O}_L/\mathcal{O}_k}(\mathbf{A}_{\mathcal{O}_L}^1) \setminus \mathscr{T}$ and $\mathscr{W} = R_{\mathcal{O}_L/\mathcal{O}_k}(\mathbf{A}_{\mathcal{O}_L}^1) \setminus \mathscr{F}$, where $\mathscr{F}$ denotes the singular locus of $\mathscr{D}$.

Letting $t_v = 1$ for $v \notin S$, we may enlarge S and assume that L/k is unramified outside S. For each $v \in S$, fix $x_v \in (L \otimes_k k_v)^*$ such that $t_v = N_{L \otimes_k k_v/k_v}(x_v)$. For $v \notin S$, let $x_v = 1$. This defines an adelic point $(x_v)_{v \in \Omega} \in W(\mathbf{A}_k)$ which, with respect to $\mathscr{W}$, is integral outside S.

Let v_0 denote a place of k, not in S, over which L possesses a place of degree 1. If v_0 is not given, we fix it by Chebotarev's density theorem (see [Mum08, p. 250, Lemma 5] for an elementary proof of the existence of v_0 splitting completely in L).

Being the complement of a codimension 2 closed subset in an affine space, the variety W satisfies strong approximation off any given place (see Lemma 1.8). We can thus find a point $x \in W(k)$, arbitrarily close to x_v for $v \in S$, which is integral, with respect to $\mathscr{W}$, outside $S \cup \{v_0\}$. Let us consider x as an element of L^* and set $t = N_{L/k}(x)$. Let v be a finite place of k, not in S, such that t has nonzero valuation at v. We need to check that L possesses a place of degree 1 over v. If $v = v_0$, there is nothing to prove. Otherwise $x \in \mathscr{W}(\mathcal{O}_v)$. On the other hand, as $v(t) \neq 0$, we have $x \notin \mathscr{T}(\mathcal{O}_v)$. Thus the reduction of x mod v lies in $(\mathscr{D} \setminus \mathscr{F})(\mathbf{F}_v)$, which, by Hensel's lemma, implies that $(D \setminus F)(k_v) \neq \varnothing$. This, in turn, implies that L embeds k-linearly into k_v, as the structure morphism $D \setminus F \to \mathrm{Spec}(k)$ factors through $\mathrm{Spec}(L)$. $\square$

5.2. **Reduction to adelic zero-cycles orthogonal to B.** We resume the notation of Theorem 5.1 and keep it until the end of §5.

The fibers of f over C^0 are geometrically connected (see [Gro66, Proposition 15.5.9 (ii)]) and hence geometrically irreducible, as they are smooth.

For $m \in M$, let $(X_{m,i})_{i \in I_m}$ denote the family of irreducible components of X_m. Let $e_{m,i}$ denote the multiplicity of $X_{m,i}$ in X_m. For each $m \in M$ and $i \in I_m$, choose a finite extension $E_{m,i}/k(X_{m,i})$ such that the residue of any element of B at the generic point of $X_{m,i}$ belongs to the kernel of the restriction map $H^1(k(X_{m,i}), \mathbf{Q}/\mathbf{Z}) \to H^1(E_{m,i}, \mathbf{Q}/\mathbf{Z})$. Let $L_{m,i}$ denote the algebraic closure of $k(m)$ in $E_{m,i}$. Let $L_m/k(m)$ be a finite Galois extension in which $L_{m,i}/k(m)$ embeds for all $i \in I_m$.

Let $\mathrm{Pic}_+(C)$ and $\mathrm{Br}_+(C)$ denote the groups associated in Definition 2.1 to the curve C, to the finite set $M \subset C$ and to the finite extensions $L_m/k(m)$ for $m \in M$.

Lemma 5.3. *We have the inclusion*

$$(B + f_\eta^* \mathrm{Br}(\eta)) \cap \mathrm{Br}(X) \subseteq B + f^* \mathrm{Br}_+(C)$$

of subgroups of $\mathrm{Br}(X^0)$. *Moreover, the quotient* $((B + f_\eta^* \mathrm{Br}(\eta)) \cap \mathrm{Br}(X))/f^* \mathrm{Br}(C)$ *is finite.*

Proof. The second assertion, which will be used in §6 and in §9, follows from the first one and from Remark 2.2 (ii). To prove the first, it suffices to check that if $\beta \in B$ and $\gamma \in \mathrm{Br}(\eta)$ satisfy $\beta + f_\eta^* \gamma \in \mathrm{Br}(X)$, then $\gamma \in \mathrm{Br}_+(C)$. Fix $m \in C$. Let $\partial_\gamma \in H^1(k(m), \mathbf{Q}/\mathbf{Z})$ denote the residue of γ at m. As the fibers of f over C^0 are smooth and geometrically irreducible and as $f_\eta^* \gamma \in \mathrm{Br}(X^0)$, we have $\partial_\gamma = 0$ if $m \in C^0$. Suppose now $m \in M$. For $i \in I_m$, the residue of $\beta + f_\eta^* \gamma$ at the generic point of $X_{m,i}$ is trivial and its image in $H^1(E_{m,i}, \mathbf{Q}/\mathbf{Z})$ coincides with the image of $e_{m,i} \partial_\gamma$ (see [CTSD94, Proposition 1.1.1]). Therefore $e_{m,i} \partial_\gamma$ vanishes in $H^1(L_{m,i}, \mathbf{Q}/\mathbf{Z})$, hence in $H^1(L_m, \mathbf{Q}/\mathbf{Z})$. As the gcd of $(e_{m,i})_{i \in I_m}$ is equal to 1, it follows that ∂_γ vanishes in $H^1(L_m, \mathbf{Q}/\mathbf{Z})$. $\qquad\square$

Lemma 5.4. *There exists a family* $(z_v^1)_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X^0)$ *orthogonal to* $B + f^* \mathrm{Br}_+(C)$ *with respect to* (1.5) *such that* $z_v^1 = z_v$ *for* $v \in S \cup \Omega_\infty$.

Proof. As the group $\mathrm{Br}_+(C)$ is torsion and the quotient $\mathrm{Br}_+(C)/\mathrm{Br}(C)$ is finite (see Remark 2.2 (ii)), there exists a finite subgroup $\Lambda \subset \mathrm{Br}_+(C)$ such that $\mathrm{Br}_+(C) = \Lambda + \mathrm{Br}(C)$. The lemma then follows from Proposition 3.1 applied to the subgroup $B + f^* \Lambda \subset \mathrm{Br}(X^0)$, in view of the fact that any element of $\mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X^0)$ is orthogonal to $f^* \mathrm{Br}(C)$. $\qquad\square$

It follows from Lemmas 5.3 and 5.4 that in order to establish Theorem 5.1, we may assume that $(z_v)_{v \in \Omega}$ is orthogonal to $B + f^* \mathrm{Br}_+(C) \subseteq \mathrm{Br}(X^0)$ with respect to (1.5). From now on we make this additional assumption.

5.3. **Construction of c_1 and choice of v_0.** We are now in a position to pinpoint the class, in $\mathrm{Pic}_+(C)$, of the desired effective divisor c. The construction of c itself will ultimately rely on an application of strong approximation off a place, say v_0, on an affine space over k. The goal of §5.3 is to define this place v_0.

The condition that $(z_v)_{v \in \Omega}$ is orthogonal to $f^* \mathrm{Br}_+(C)$ with respect to (1.5) implies that the class of $(f_* z_v)_{v \in \Omega}$ in $\mathrm{Pic}_{+,\mathbf{A}}(C)$ is orthogonal to $\mathrm{Br}_+(C)$ with respect to the pairing (2.4). By Theorem 2.5, we deduce that there exists a divisor $c_1 \in \mathrm{Div}(C^0)$ whose class in $\mathrm{Pic}(C)$ is y and whose class in $\mathrm{Pic}_{+,\mathbf{A}}(C)$ coincides with that of $(f_* z_v)_{v \in \Omega}$.

After enlarging the finite subset $S \subset \Omega_f$, we may assume that $\langle \beta, z_v \rangle = 0$ for all $v \in \Omega_f \setminus S$ and all $\beta \in B$, that the order of the finite group B is invertible in $\mathcal{O}_S$, that $\mathrm{Pic}(\mathcal{O}_S) = 0$, that the finite extensions L_m/k for $m \in M$ are unramified outside S, that C and X extend to smooth and proper $\mathcal{O}_S$-schemes $\mathscr{C}$ and $\mathscr{X}$ and that $f : X \to C$ extends to a flat morphism, which we still denote $f : \mathscr{X} \to \mathscr{C}$.

For $m \in M$, we let $\tilde{m}$ denote the Zariski closure of m in $\mathscr{C}$. Let $\widetilde{M}$ denote the Zariski closure of M in $\mathscr{C}$. Let $\mathscr{C}^0 = \mathscr{C} \setminus \widetilde{M}$ and $\mathscr{X}^0 = \mathscr{X} \times_{\mathscr{C}} \mathscr{C}^0$. Whenever D is a divisor on C, we denote by $\tilde{D}$ the horizontal divisor on $\mathscr{C}$ which extends D, *i.e.*, the unique divisor on $\mathscr{C}$ whose support is flat over $\mathcal{O}_S$ and whose restriction to C is D. For $m \in M$ and $i \in I_m$, we

write $\mathscr{X}_{\widetilde{m},i}$ for the Zariski closure of $X_{m,i}$ in $\mathscr{X}$, endowed with the reduced scheme structure, and we set $\mathscr{X}_{\widetilde{m}} = \mathscr{X} \times_{\mathscr{C}} \widetilde{m}$. For each $m \in M$ and $i \in I_m$, let $\mathscr{Y}_{\widetilde{m},i} \subseteq \mathscr{X}_{\widetilde{m},i}$ be a dense open subset. Let $\mathscr{E}_{\widetilde{m},i}$ denote the normalisation of $\mathscr{Y}_{\widetilde{m},i}$ in the finite extension $E_{m,i}/k(X_{m,i})$. By shrinking $\mathscr{Y}_{\widetilde{m},i}$, we may assume that the finite morphism $\mathscr{E}_{\widetilde{m},i} \to \mathscr{Y}_{\widetilde{m},i}$ is étale and that the scheme $(\mathscr{X}_{\widetilde{m}})_{\mathrm{red}}$ is smooth over $\widetilde{m}$ at the points of $\mathscr{Y}_{\widetilde{m},i}$.

By enlarging S further, we may assume that the equality of effective divisors on $\mathscr{X}$

$$\mathscr{X}_{\widetilde{m}} = \sum_{i \in I_m} e_{m,i} \mathscr{X}_{\widetilde{m},i}$$

holds for each $m \in M$, that $\mathscr{X}^0$ is smooth over $\mathscr{C}^0$, that $B \subset \mathrm{Br}(\mathscr{X}^0)$, that $\mathrm{Supp}(\widetilde{c}_1)$ is disjoint from $\widetilde{M}$ and that $\widetilde{M}$ is étale over $\mathcal{O}_S$. In particular $\widetilde{M}$ is a regular scheme, so that we may (and will freely) identify, for $m \in M$, the set of finite places of $k(m)$ which do not lie above S with the set of closed points of $\widetilde{m}$.

After a final enlargement of S, we may assume, thanks to the Lang–Weil–Nisnevich bounds [LW54] [Nis54], that the following statements hold:

- the fiber of f above any closed point of $\mathscr{C}^0$ contains a rational point;
- for any $m \in M$, any $i \in I_m$ and any closed point $w \in \widetilde{m}$ which, as a place of $k(m)$, splits completely in $L_{m,i}$, the fiber of $\mathscr{E}_{\widetilde{m},i} \to \widetilde{m}$ above w contains a rational point.

By shrinking the subset $\mathscr{U} \subseteq \prod_{v \in S} \mathrm{Sym}_{X^0/k}(k_v)$, we may assume that $\langle \beta, z'_v \rangle = \langle \beta, z_v \rangle$ for any $v \in S$, any $\beta \in B$ and any $(z'_v)_{v \in S} \in \mathscr{U}$.

We now fix a place $v_0 \in \Omega_f \setminus S$ which splits completely in L_m for all $m \in M$.

5.4. **Construction of c.** As the classes of c_1 and of $(f_* z_v)_{v \in \Omega}$ in $\mathrm{Pic}_{+,\mathbf{A}}(C)$ are equal and the divisors $f_* z_v$ for $v \in S$ are effective, we can find, for each $v \in S$, a rational function $h_v \in H^0(C \otimes_k k_v, \mathscr{O}_C(c_1))$ such that $\mathrm{div}(h_v) = f_* z_v - c_1$ and such that $h_v(m) \in N_{L_m \otimes_k k_v / k(m) \otimes_k k_v}((L_m \otimes_k k_v)^*)$ for all $m \in M$. At the infinite places v of k, we fix $h_v \in H^0(C \otimes_k k_v, \mathscr{O}_C(c_1))$ using the following lemma.

Lemma 5.5. *For each $v \in \Omega_\infty$, there exist a rational function $h_v \in H^0(C \otimes_k k_v, \mathscr{O}_C(c_1))$ and a zero-cycle $z_v^2 \in \mathrm{Z}_0^{\mathrm{eff},\mathrm{red},y}(X^0 \otimes_k k_v)$ satisfying the following conditions:*
- (a) $\mathrm{div}(h_v) = f_* z_v^2 - c_1$;
- (b) $h_v(m) \in N_{L_m \otimes_k k_v / k(m) \otimes_k k_v}((L_m \otimes_k k_v)^*)$ *for all $m \in M$;*
- (c) *for each connected component $\mathscr{B}$ of $X^0(k_v)$, if we write $z_v - z_v^2 = \sum_{x \in X^0 \otimes_k k_v} n_x x$, then $\sum_{x \in \mathscr{B}} n_x$ is even.*

Proof. The existence of h_v and z_v^2 for complex v is clear as c_1 is a very ample divisor on C (recall that $\deg(y) \geq 2g + 1$). Let us fix a real place v. By the definition of c_1, there exist $h_v^1 \in k_v(C)^*$ and $\xi_v \in \mathrm{Div}(C^0 \otimes_k \bar{k}_v)$ such that $\mathrm{div}(h_v^1) = f_* z_v - c_1 + N_{\bar{k}_v/k_v}(\xi_v)$ and such that $h_v^1(m) \in (k(m) \otimes_k k_v)^*$ is a norm from $(L_m \otimes_k k_v)^*$ for every $m \in M$. Let $Z_v \subset X \otimes_k k_v$ denote the curve given by assumption (iii) of Theorem 5.1. Let $Z_v^0 = Z_v \times_C C^0$ and $M_v = Z_v \times_C M$. By lifting ξ_v to $\mathrm{Div}(Z_v^0 \otimes_{k_v} \bar{k}_v)$, we can find $z_v^1 \in \mathrm{Div}(Z_v^0)$ such that $z_v - z_v^1$ is a norm from $\mathrm{Div}(Z_v^0 \otimes_{k_v} \bar{k}_v)$ and such that $\mathrm{div}(h_v^1) = f_* z_v^1 - c_1$. According to assumption (iii) and to Serre duality, we have $H^1(Z_v, \mathscr{O}_{Z_v}(z_v^1 - M_v)) = 0$, hence there exists

a rational function $\eta_v \in H^0(Z_v, \mathscr{O}_{Z_v}(z_v^1))$ whose restriction to M_v is equal to 1. We have $\mathrm{div}(\eta_v) = z_v^2 - z_v^1$ for some effective $z_v^2 \in \mathrm{Div}(Z_v^0)$. By choosing η_v general enough in the sense of the Zariski topology, we may assume that $f_* z_v^2$ is reduced (see the proof of [CT00, Lemme 3.1]) and hence that $z_v^2 \in Z_0^{\mathrm{eff,red},y}(X^0 \otimes_k k_v)$. Let $h_v = N_{k_v(Z_v)/k_v(C)}(\eta_v) h_v^1$. Conditions (a) and (b) are clearly satisfied (note that $h_v(m) = h_v^1(m)$ for $m \in M$). Condition (c) also holds because on the one hand $z_v - z_v^1$ is a norm from $\bar{k}_v$ and on the other hand $z_v^1 - z_v^2$ is the divisor, on Z_v, of a rational function which takes positive values at the k_v-points of M_v. (The sign of a rational function on a smooth real algebraic curve changes exactly at the points where the function has a pole or zero of odd order.) $\quad\square$

For each $m \in M$, applying Lemma 5.2 to the finite extension $L_m/k(m)$ and to the set of places of $k(m)$ which lie above S now produces an element $t_m \in k(m)^*$ arbitrarily close to $h_v(m) \in (k(m) \otimes_k k_v)^*$ for $v \in S \cup \Omega_\infty$, integral at the places of $k(m)$ which do not lie above $S \cup \{v_0\}$, such that t_m has nonzero valuation only at places of $k(m)$ which lie above S and at places of $k(m)$ which split completely in L_m (recall that $L_m/k(m)$ is Galois).

Thanks to assumption (ii) and to Serre duality, if $\rho : \mathscr{C} \to \mathrm{Spec}(\mathcal{O}_S)$ denotes the structure morphism, we have $R^1\rho_* \mathscr{O}_\mathscr{C}(\widetilde{c}_1 - \widetilde{M}) = 0$ (see [Mum08, Ch. II, §5, Corollary 2]). As $\mathrm{Spec}(\mathcal{O}_S)$ is affine, it follows that $H^1(\mathscr{C}, \mathscr{O}_\mathscr{C}(\widetilde{c}_1 - \widetilde{M})) = 0$ and hence that the natural sequence of $\mathcal{O}_S$-modules

$$(5.1) \qquad 0 \longrightarrow H^0(\mathscr{C}, \mathscr{O}_\mathscr{C}(\widetilde{c}_1 - \widetilde{M})) \longrightarrow H^0(\mathscr{C}, \mathscr{O}_\mathscr{C}(\widetilde{c}_1)) \xrightarrow{\ r\ } H^0(\widetilde{M}, \mathscr{O}_{\widetilde{M}}) \longrightarrow 0$$

is exact. These modules are free since they are finitely generated and torsion-free over the principal ideal domain $\mathcal{O}_S$ (recall $\mathrm{Pic}(\mathcal{O}_S) = 0$). Choosing an $\mathcal{O}_S$-linear section s of r and using strong approximation off v_0 in the free $\mathcal{O}_S$-module $H^0(\mathscr{C}, \mathscr{O}_\mathscr{C}(\widetilde{c}_1 - \widetilde{M}))$, we can find $h_0 \in H^0(\mathscr{C} \otimes_{\mathcal{O}_S} \mathcal{O}_{S \cup \{v_0\}}, \mathscr{O}_\mathscr{C}(\widetilde{c}_1 - \widetilde{M}))$ arbitrarily close to $h_v - s(r(h_v)) \in H^0(C \otimes_k k_v, \mathscr{O}_C(c_1 - M))$ for $v \in S \cup \Omega_\infty$. Note that $\bigoplus_{m \in M} t_m \in H^0(\widetilde{M}, \mathscr{O}_{\widetilde{M}}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S \cup \{v_0\}}$ lies, by construction, arbitrarily close to $r(h_v) \in H^0(M \otimes_k k_v, \mathscr{O}_M)$ for $v \in S \cup \Omega_\infty$. Letting $h = h_0 + s(\bigoplus_{m \in M} t_m)$, we therefore obtain $h \in H^0(C, \mathscr{O}_C(c_1))$ arbitrarily close to $h_v \in H^0(C \otimes_k k_v, \mathscr{O}_C(c_1))$ for $v \in S \cup \Omega_\infty$, such that $h(m) = t_m$ for $m \in M$ and such that h, as a rational function on $\mathscr{C}$, has no pole outside $\mathrm{Supp}(\widetilde{c}_1) \cup (\mathscr{C} \otimes_{\mathcal{O}_S} \mathbf{F}_{v_0})$.

Let $c \in \mathrm{Div}(C^0)$ be the effective divisor defined by $\mathrm{div}(h) = c - c_1$. It remains to check the existence of a family $(z_v')_{v \in \Omega} \in Z_{0,\mathbf{A}}(X^0)$ satisfying (1)–(4).

5.5. **Construction of** $(z_v')_{v \in \Omega}$**.** The next lemma takes care of (3) and (4).

Lemma 5.6. *There exists* $(z_v')_{v \in S \cup \Omega_\infty} \in \prod_{v \in S \cup \Omega_\infty} \mathrm{Sym}_{X^0/k}(k_v)$ *such that* $(z_v')_{v \in S} \in \mathscr{U}$, *such that* $f_* z_v' = c$ *in* $\mathrm{Div}(C^0 \otimes_k k_v)$ *for* $v \in S \cup \Omega_\infty$ *and such that (4) holds.*

Proof. Note that for each $v \in S$ (resp. for each real place v of k), the divisor c is arbitrarily close to $f_* z_v$ (resp. to $f_* z_v^2$) as a point of $\mathrm{Sym}_{C^0/k}(k_v)$. As $f_* z_v$ (resp. $f_* z_v^2$) is a reduced divisor for any $v \in S$ (resp. for any real v), the morphism $f_* : \mathrm{Sym}_{X^0/k} \to \mathrm{Sym}_{C^0/k}$ is smooth at z_v (resp. at z_v^2) for such v. Thus the existence of $(z_v')_{v \in S \cup \Omega_\infty}$ satisfying the required conditions follows from the inverse function theorem. $\quad\square$

Let us turn to (1) and (2). To construct z'_v for $v \in \Omega_f \setminus S$, we shall need to consider the reduction of $f^{-1}(\mathrm{Supp}(c)) \to \mathrm{Supp}(c)$ modulo v.

For $v \in \Omega_f \setminus S$, the divisor $c \otimes_k k_v$ on $C^0 \otimes_k k_v$ decomposes uniquely as

$$c \otimes_k k_v = c_{v,0} + \sum_{m \in M} \sum_{\substack{w \in \widetilde{m} \\ w | v}} c_{w,m}$$

where $c_{v,0}$ and the $c_{w,m}$ are effective divisors on $C^0 \otimes_k k_v$ satisfying the following properties: for any $m \in M$, any $w \in \widetilde{m} \otimes_{\mathcal{O}_S} \mathbf{F}_v$ and any $y_v \in \mathrm{Supp}(c \otimes_k k_v)$, the point y_v belongs to $\mathrm{Supp}(c_{w,m})$ if and only if its Zariski closure in $\mathscr{C} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ contains w; the closure of $\mathrm{Supp}(c_{v,0})$ in $\mathscr{C} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ is contained in $\mathscr{C}^0 \otimes_{\mathcal{O}_S} \mathcal{O}_v$. This is in fact a partition of the support of $c \otimes_k k_v$. The points of $\mathrm{Supp}(c_{v,0})$ are those above which the fiber of f has good reduction in $\mathscr{X}$, while for each m and each w, the points of $\mathrm{Supp}(c_{w,m})$ are those above which the fiber of f has the same reduction modulo v as X_m modulo w.

Lemma 5.7. *Let $m \in M$ and $w \in \widetilde{m}$ be a closed point. If $w \in \mathrm{Supp}(\widetilde{c})$ then w, as a place of $k(m)$, splits completely in L_m.*

Proof. If w divides v_0, then w splits completely in L_m since v_0 does. Assume now that w does not divide v_0. Then the rational function h on $\mathscr{C}$ is regular in a neighbourhood of w, since $\widetilde{m} \cap \mathrm{Supp}(\widetilde{c_1}) = \varnothing$ and h has no pole outside $\mathrm{Supp}(\widetilde{c_1}) \cup (\mathscr{C} \otimes_{\mathcal{O}_S} \mathbf{F}_{v_0})$. As $w \in \mathrm{Supp}(\widetilde{c})$, this regular function moreover vanishes at w. Thus, the function obtained by restricting h to $\widetilde{m}$ vanishes at w. In other words $h(m)$ has positive valuation at the place w. It follows that w splits completely in L_m, since $h(m) = t_m$. $\square$

According to Lemma 5.7 and to the assumptions made at the end of §5.3, we can choose, for each $m \in M$, each $i \in I_m$ and each $w \in \widetilde{m} \cap \mathrm{Supp}(\widetilde{c})$, a rational point of the fiber of $\mathscr{E}_{\widetilde{m},i} \to \widetilde{m}$ above w. Let $\xi_{w,m,i}$ denote its image in $\mathscr{Y}_{\widetilde{m},i}$.

As the fibers of $f : \mathscr{X} \to \mathscr{C}$ above the closed points of $\mathscr{C}^0$ are smooth and contain rational points, we can lift such rational points by Hensel's lemma and find, for each $v \in \Omega_f \setminus S$, an effective zero-cycle $z'_{v,0} \in \mathrm{Z}_0(X^0 \otimes_k k_v)$ such that $f_* z'_{v,0} = c_{v,0}$ in $\mathrm{Div}(C^0 \otimes_k k_v)$. By the next lemma, we can also find, for each $v \in \Omega_f \setminus S$, each $m \in M$, each $i \in I_m$ and each $w \in \widetilde{m} \cap \mathrm{Supp}(\widetilde{c})$ which divides v, an effective zero-cycle $z'_{w,m,i} \in \mathrm{Z}_0(X^0 \otimes_k k_v)$ such that $f_* z'_{w,m,i} = e_{m,i} c_{w,m}$ and such that the Zariski closure of $\mathrm{Supp}(z'_{w,m,i})$ in $\mathscr{X} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ meets $\mathscr{X} \otimes_{\mathcal{O}_S} \mathbf{F}_v$ only at the point $\xi_{w,m,i}$.

Lemma 5.8. *Let $v \in \Omega_f \setminus S$, let $m \in M$, let $i \in I_m$, let $w \in \widetilde{m}$ be a closed point which divides v and let ξ be a rational point of the fiber of $\mathscr{Y}_{\widetilde{m},i} \to \widetilde{m}$ above w. Let $y_v \in C^0 \otimes_k k_v$ be a closed point whose Zariski closure in $\mathscr{C} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ contains w. There exists an effective zero-cycle $x_v \in \mathrm{Z}_0(X^0 \otimes_k k_v)$ such that $f_* x_v = e_{m,i} y_v$ in $\mathrm{Div}(C^0 \otimes_k k_v)$ and such that the Zariski closure of $\mathrm{Supp}(x_v)$ in $\mathscr{X} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ meets $\mathscr{X} \otimes_{\mathcal{O}_S} \mathbf{F}_v$ only at the point ξ.*

Proof. Let R denote the completion of $\mathscr{O}_{\mathscr{C},w}$. Let $p \in \mathcal{O}_v$ be a uniformiser and $s \in R$ be a generator of the ideal of $\widetilde{m} \times_{\mathscr{C}} \mathrm{Spec}(R) \subset \mathrm{Spec}(R)$. Let $U \subset \mathscr{X} \times_{\mathscr{C}} \mathrm{Spec}(R)$ be an affine open neighbourhood of ξ. Letting $U = \mathrm{Spec}(A)$, we choose U small enough that the ideal of A defined by the (reduced) closed subscheme $\mathscr{X}_{\widetilde{m},i} \times_{\mathscr{C}} \mathrm{Spec}(R) \subset U$ is principal. Let $t \in A$

be a generator of this ideal. As $(\mathscr{X}_{\widetilde{m}})_{\mathrm{red}}$ is smooth over $\widetilde{m}$ at ξ and $\widetilde{m}$ is étale over $\mathcal{O}_S$, the ring $\mathcal{O}_{U,\xi}/(p,t)$ is regular, hence there exist $n \geq 0$ and $f_1, \ldots, f_n \in \mathfrak{m}_{U,\xi}$ such that $(p, t, f_1, \ldots, f_n)$ is a regular system of parameters for $\mathcal{O}_{U,\xi}$ (see [Gro64, Proposition 17.1.7]). The image of s in $\mathcal{O}_{U,\xi}$ can be written as $ut^{e_{m,i}}$ for some $u \in \mathcal{O}_{U,\xi}^*$. By shrinking U, we may assume that $f_1, \ldots, f_n \in A$ and $u \in A^*$. On the other hand, as $\widetilde{m}$ is étale over $\mathcal{O}_S$, the sequence (p, s) is a regular system of parameters for R; hence the fiber above the closed point of $\mathrm{Spec}(R)$ of the finite type R-scheme $T = \mathrm{Spec}(A/(f_1, \ldots, f_n))$ contains ξ as an isolated point, with multiplicity $e_{m,i}$. Thus we may assume, after further shrinking U, that T is a closed subscheme of $\mathscr{X} \times_{\mathscr{C}} \mathrm{Spec}(R)$, finite and flat over R of degree $e_{m,i}$ (see [Gro61, Proposition 6.2.5], [Gro64, Proposition 15.1.21]). Pulling it back by the natural map $\widetilde{y_v} \to \mathrm{Spec}(R)$, where $\widetilde{y_v}$ denotes the Zariski closure of y_v in $\mathscr{C} \otimes_{\mathcal{O}_S} \mathcal{O}_v$, yields a closed subscheme of $\mathscr{X} \times_{\mathscr{C}} \widetilde{y_v}$, finite and flat of degree $e_{m,i}$ over $\widetilde{y_v}$, whose special fiber consists of the point ξ with multiplicity $e_{m,i}$. $\qquad\square$

For each $m \in M$, we choose integers $(f_{m,i})_{i \in I_m}$ such that $\sum_{i \in I_m} e_{m,i} f_{m,i} = 1$ and we set $z'_{w,m,i} = 0$ for any $i \in I_m$ and any closed point $w \in \widetilde{m}$ which does not belong to $\mathrm{Supp}(\widetilde{c})$. Finally, for each $v \in \Omega_f \setminus S$, we let

$$z'_v = z'_{v,0} + \sum_{m \in M} \sum_{i \in I_m} \sum_{\substack{w \in \widetilde{m} \\ w|v}} f_{m,i} z'_{w,m,i}.$$

We have now defined a family $(z'_v)_{v \in \Omega} \in \prod_{v \in \Omega} \mathrm{Z}_0(X^0 \otimes_k k_v)$. When X_m contains an irreducible component of multiplicity 1 for each $m \in M$, we can choose $f_{m,i} \in \{0, 1\}$ for all m and all i, so that z' is effective for all v. In any case, the family $(z'_v)_{v \in \Omega}$ satisfies (1); as a consequence, it belongs to $\mathrm{Z}_{0,\mathbf{A}}(X^0)$. Let us check that (2) is also satisfied.

Lemma 5.9. *Let $v \in \Omega_f \setminus S$. For any $\beta \in B$, any $m \in M$, any $i \in I_m$ and any $w \in \widetilde{m}$ which divides v, we have $\mathrm{inv}_v\langle \beta, z'_{v,0}\rangle = 0$ and $\mathrm{inv}_v\langle \beta, z'_{w,m,i}\rangle = 0$.*

Proof. As $B \subset \mathrm{Br}(\mathscr{X}^0)$, as the Zariski closure of $\mathrm{Supp}(z'_{v,0})$ in $\mathscr{X} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ is contained in $\mathscr{X}^0 \otimes_{\mathcal{O}_S} \mathcal{O}_v$ and as the Brauer group of any finite $\mathcal{O}_v$-algebra vanishes (see [Gro68, III, Proposition 1.5, Théorème 11.7 (2)]), we have $\langle \beta, z'_{v,0}\rangle = 0$. Let us now fix m, i and w as in the statement. If $w \notin \mathrm{Supp}(\widetilde{c})$, then $z'_{w,m,i} = 0$. Otherwise, let R' denote the completion of $\mathcal{O}_{\mathscr{X},\xi_{w,m,i}}$ and set $V = \mathrm{Spec}(R')$ and $V^0 = \mathscr{X}^0 \times_{\mathscr{X}} V$. The natural morphism $\mathrm{Supp}(z'_{w,m,i}) \to \mathscr{X}^0$ factors through the projection $\sigma : V^0 \to \mathscr{X}^0$ since the Zariski closure of $\mathrm{Supp}(z'_{w,m,i})$ in $\mathscr{X} \otimes_{\mathcal{O}_S} \mathcal{O}_v$ meets $\mathscr{X} \otimes_{\mathcal{O}_S} \mathbf{F}_v$ only at $\xi_{w,m,i}$. To prove that $\langle \beta, z'_{w,m,i}\rangle = 0$ for $\beta \in B$, it therefore suffices to check that B is contained in the kernel of $\sigma^* : \mathrm{Br}(\mathscr{X}^0) \to \mathrm{Br}(V^0)$. As the order n of B is invertible in $\mathcal{O}_S$, the class $\beta \in \mathrm{Br}(\mathscr{X}^0)$ has a well-defined residue $\partial_\beta \in H^1(\mathscr{Y}_{\widetilde{m},i}, \mathbf{Z}/n\mathbf{Z})$, represented by a cyclic étale cover of $\mathscr{Y}_{\widetilde{m},i}$ through which $\mathscr{E}_{\widetilde{m},i} \to \mathscr{Y}_{\widetilde{m},i}$ factors. Recall that the fiber of $\mathscr{E}_{\widetilde{m},i} \to \mathscr{Y}_{\widetilde{m},i}$ above $\xi_{w,m,i}$ possesses a rational point, by the definition of $\xi_{w,m,i}$. Hence $\partial_\beta(\xi_{w,m,i}) = 0$ in $H^1(\xi_{w,m,i}, \mathbf{Z}/n\mathbf{Z})$. As $H^1(\mathscr{Y}_{\widetilde{m},i} \times_{\mathscr{X}} V, \mathbf{Z}/n\mathbf{Z}) = H^1(\xi_{w,m,i}, \mathbf{Z}/n\mathbf{Z})$, it follows that the residue of $\sigma^*\beta \in \mathrm{Br}(V^0)$ along $\mathscr{Y}_{\widetilde{m},i} \times_{\mathscr{X}} V$ vanishes as well, so that $\sigma^*\beta \in \mathrm{Br}(V) \subset \mathrm{Br}(V^0)$. Finally, we have $\mathrm{Br}(V) = 0$ (see *loc. cit.*), hence the lemma. $\qquad\square$

Let $\beta \in B$. We have $\langle \beta, z'_v \rangle = \langle \beta, z_v \rangle$ for all $v \in S \cup \Omega_\infty$ in view of (4) and of the fact that $(z'_v)_{v \in S} \in \mathscr{U}$ (see §5.3). On the other hand, we have $\langle \beta, z_v \rangle = 0$ and $\langle \beta, z'_v \rangle = 0$ for all $v \in \Omega_f \setminus S$, by the assumptions made in §5.3 and by Lemma 5.9 respectively. As the family $(z_v)_{v \in \Omega}$ is orthogonal to B with respect to (1.5) (see §5.2), we conclude that $\sum_{v \in \Omega} \mathrm{inv}_v \langle \beta, z'_v \rangle = 0$, as desired.

6. Hilbert subsets

The following lemma was noted by Swinnerton-Dyer in the case where h is a rational point (see [Sme15, Proposition 6.1]).

Lemma 6.1. *Let V be a normal quasi-projective variety over a number field k. Let $H \subseteq V$ be a Hilbert subset. Let $h \in H$ be a closed point. For any finite subset $S \subset \Omega$, there exist a finite subset $T \subset \Omega$ disjoint from S and a neighbourhood $\mathscr{V} \subset \prod_{v \in T} \mathrm{Sym}_{V/k}(k_v)$ of h such that any effective zero-cycle on V corresponding to a point of $\mathrm{Sym}_{V/k}(k) \cap \mathscr{V}$ is in fact a closed point of V and belongs to H.*

Proof. Let $V^0 \subseteq V$ and $W_i \to V^0$, for $i \in \{1, \ldots, n\}$, be the open subset and the irreducible étale covers defining the Hilbert subset H. Let $\pi : E \to U$ denote the universal family of reduced effective zero-cycles of degree $\deg(h)$ on V^0 (thus U is an open subscheme of $\mathrm{Sym}_{V^0/k}$ and E is a reduced closed subscheme of $U \times_k V^0$; the morphism π is finite étale of degree $\deg(h)$). Let $E_i = E \times_{V^0} W_i$ and let $\pi_i : E_i \to U$ denote the projection to E composed with π. By assumption, the fibers of $\pi, \pi_1, \ldots, \pi_n$ above the k-point of U corresponding to h are irreducible. As U is normal and irreducible and the morphisms π, $\pi_1, \ldots, \pi_n$ are finite and étale, it follows that $E, E_1, \ldots, E_n$ are themselves irreducible. Applying [Sme15, Proposition 6.1] (whose statement is identical to Lemma 6.1 with the additional assumption that h is a rational point) to the Hilbert subset of U defined by these $n + 1$ irreducible étale covers of U now concludes the proof. $\square$

Building on Lemma 6.1, we now strengthen the conclusion of Theorem 5.1.

Theorem 6.2. *Let us keep the notation and assumptions of Theorem 5.1. For any Hilbert subset $H \subseteq C$, the conclusion of Theorem 5.1 still holds if the effective divisor $c \in \mathrm{Div}(C^0)$ is required, in addition, to be a closed point of C belonging to H.*

Proof. As $\deg(y) \geq 2g$, the complete linear system $|y|$ has no base point. Composing the morphism to projective space defined by $|y|$ with projection from a general codimension 2 linear subspace yields a finite map $\pi : C \to \mathbf{P}_k^1$ the linear equivalence class of whose fibers is y. According to Hilbert's irreducibility theorem applied to the irreducible covers of $\mathbf{P}_k^1$ obtained by composing π with the covers of C which define H, we can find a rational point of $\mathbf{P}_k^1$ whose inverse image by π is a closed point h of C^0 belonging to H.

By Lemma 5.3, there exists a finite subgroup $B_0 \subset \mathrm{Br}(X)$ such that

$$(B + f_\eta^* \mathrm{Br}(\eta)) \cap \mathrm{Br}(X) = B_0 + f^* \mathrm{Br}(C).$$

Let $S' \subset \Omega$ be a finite subset containing $S \cup \Omega_\infty$, large enough that $X_h(k(h)_w) \neq \varnothing$ for any place w of $k(h)$ which does not divide a place of S' (see [LW54], [Nis54]) and large enough that $\langle \beta, u_v \rangle = 0$ for any $v \in \Omega \setminus S'$, any $\beta \in B_0$ and any $u_v \in \mathrm{Z}_0(X \otimes_k k_v)$.

Let $T \subset \Omega \setminus S'$ and $\mathscr{V} \subset \prod_{v \in T} \mathrm{Sym}_{C/k}(k_v)$ be the finite subset and the neighbourhood of h given by Lemma 6.1. For each $v \in T$, as $v \notin S'$, we can choose an effective zero-cycle $z_v^1 \in \mathrm{Z}_0(X^0 \otimes_k k_v)$ such that $f_* z_v^1 = h$ in $\mathrm{Div}(C^0 \otimes_k k_v)$. For $v \in \Omega \setminus T$, we let $z_v^1 = z_v$. As the class of h in $\mathrm{Pic}(C)$ is y, this defines a family $(z_v^1)_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff,red},y}(X^0)$. This family is orthogonal to B_0 with respect to the pairing (1.5) since $S' \cap T = \varnothing$ and $(z_v)_{v \in \Omega}$ is orthogonal to B_0. As any element of $\mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff,red},y}(X^0)$ is orthogonal to $f^* \mathrm{Br}(C)$, we deduce that $(z_v^1)_{v \in \Omega}$ is orthogonal to $(B + f_\eta^* \mathrm{Br}(\eta)) \cap \mathrm{Br}(X)$. We can therefore apply Theorem 5.1 to the family $(z_v^1)_{v \in \Omega}$. This yields $c \in \mathrm{Div}(C^0)$ and $(z_v')_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}(X^0)$ satisfying (1)–(4), with c effective, and z_v' effective and arbitrarily close to z_v^1 for $v \in T$. By choosing z_v' close enough to z_v^1 for $v \in T$, we can ensure that $c \in \mathscr{V}$, so that c must be a closed point belonging to H. $\qquad\square$

7. From completed Chow groups to effective cycles

Theorem 5.1 takes, as input, a collection of local effective zero-cycles. On the other hand, the statement of Conjecture 1.2 involves elements of $\widehat{\mathrm{CH}}_{0,\mathbf{A}}(X)$. Theorem 7.1 will allow us to bridge the gap between $\widehat{\mathrm{CH}}_{0,\mathbf{A}}(X)$ and $\mathrm{CH}_{0,\mathbf{A}}(X)$ and to reduce to effective cycles.

Theorem 7.1. *Let C be a smooth, projective and geometrically irreducible curve over a number field k. Let X be a smooth, projective, irreducible variety over k and $f : X \to C$ be a morphism whose geometric generic fiber $X_{\bar\eta}$ is irreducible and satisfies $H^1(X_{\bar\eta}, \mathbf{Q}/\mathbf{Z}) = 0$ and $\mathrm{A}_0(X_{\bar\eta} \otimes K) = 0$, where K denotes an algebraic closure of the function field of $X_{\bar\eta}$. Fix a dense open subset $X^0 \subseteq X$, a map $\Phi : \mathbf{N}^2 \to \mathbf{N}$ and an element $\widehat{z}_{\mathbf{A}} \in \widehat{\mathrm{CH}}_{0,\mathbf{A}}(X)$ such that $f_* \widehat{z}_{\mathbf{A}}$ belongs to the image of the natural map $\mathrm{Pic}\widehat{\;}(C) \to \mathrm{Pic}_{\widehat{\mathbf{A}}}(C)$. There exist classes $y \in \mathrm{Pic}(C)$, $\widehat{z} \in \widehat{\mathrm{CH}}_0(X)$, $z_{\mathbf{A}}^{\mathrm{eff}} = (z_v)_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff,red},y}(X^0)$, a finite subset $S \subset \Omega_f$ and a neighbourhood $\mathscr{U} \subseteq \prod_{v \in S} \mathrm{Sym}_{X^0/k}(k_v)$ of $(z_v)_{v \in S}$ such that:*

- *$\widehat{z}_{\mathbf{A}} = z_{\mathbf{A}}^{\mathrm{eff}} + \widehat{z}$ in $\widehat{\mathrm{CH}}_{0,\mathbf{A}}(X)$;*
- *for all $v \in \Omega_\infty$, the support of z_v is contained in a smooth, proper, geometrically irreducible curve $Z_v \subset X \otimes_k k_v$ which dominates C and satisfies $\deg(y) \geq \Phi(g_v, d_v)$, where g_v denotes the genus of Z_v and $d_v = [k_v(Z_v) : k_v(C)]$;*
- *for any $z_{\mathbf{A}}' = (z_v')_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}(X^0)$ such that the images of $f_* z_{\mathbf{A}}'$ and of y in $\mathrm{Pic}_{\mathbf{A}}(C)$ coincide and such that properties (3) and (4) of Theorem 5.1 are satisfied, the equality $z_{\mathbf{A}}^{\mathrm{eff}} = z_{\mathbf{A}}'$ holds in $\mathrm{CH}_{0,\mathbf{A}}(X)$.*

We shall prove Theorem 7.1 in §7.2, by combining preliminary reduction steps for general fibrations established in §7.1 with the results of [Wit12, §2] on the kernel and cokernel of the natural map $f_* : \mathrm{CH}_0(X \otimes_k k_v) \to \mathrm{Pic}(C \otimes_k k_v)$ for fibrations whose generic fiber is subject to the hypotheses of Theorem 7.1. For ease of reference, we summarise these results in the next theorem. They rely on a variant of the "decomposition of the diagonal" argument for families of varieties and on theorems of Kato, Saito and Sato about Chow groups of zero-cycles over finite and local fields (see *loc. cit.* for more details).

Theorem 7.2 ([Wit12, Théorème 2.1, Lemme 2.3, Lemme 2.4])**.** *Let $f : X \to Y$ be a proper dominant morphism between smooth, irreducible varieties over a field k.*

(i) *There exists an integer $n > 0$ such that for any field extension k'/k, the cokernel of $f_* : \mathrm{CH}_0(X \otimes_k k') \to \mathrm{CH}_0(Y \otimes_k k')$ is killed by n.*

(ii) *Assume that the geometric generic fiber $X_{\bar{\eta}}$ of f is smooth and irreducible and satisfies $\mathrm{A}_0(X_{\bar{\eta}} \otimes K) = 0$, where K denotes an algebraic closure of the function field of $X_{\bar{\eta}}$. Then there exists an integer $n > 0$ such that for any field extension k'/k, the kernel of $f_* : \mathrm{CH}_0(X \otimes_k k') \to \mathrm{CH}_0(Y \otimes_k k')$ is killed by n.*

(iii) *Assume, in addition, that $H^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}) = 0$, that Y is a proper curve and that k is a number field. Then the map $f_* : \mathrm{CH}_0(X \otimes_k k_v) \to \mathrm{CH}_0(Y \otimes_k k_v)$ is an isomorphism for all but finitely many places v of k.*

We shall also need a few results on abelian groups essentially contained in [Wit12].

Lemma 7.3. *Let $h : A \to B$ be a homomorphism of abelian groups whose cokernel has finite exponent. Letting $\widehat{h} : \widehat{A} \to \widehat{B}$ denote the map induced by h, we have:*

(i) *The natural map $\mathrm{Coker}(h) \to \mathrm{Coker}(\widehat{h})$ is injective.*

(ii) *If the N-torsion subgroup of B is finite for all $N > 0$, the natural map*

$$\widehat{\mathrm{Ker}(h)} \to \mathrm{Ker}\left(\widehat{h}\right)$$

is surjective and the abelian group $\mathrm{Coker}\left(\mathrm{Ker}(h) \to \mathrm{Ker}\left(\widehat{h}\right)\right)$ is divisible.

Proof. The first assertion is [Wit12, Lemma 1.11]. The proof of [Wit12, Lemme 1.12] is in fact a proof of the first half of (ii). The second half of (ii) follows from the first half and from [Wit12, Lemme 1.10]. $\square$

7.1. **General reductions.** For the whole of §7.1, we fix a smooth, projective, irreducible variety X over a number field k, a smooth, projective and geometrically irreducible curve C over k and a morphism $f : X \to C$ with geometrically irreducible generic fiber. We shall not make any other assumption on the generic fiber of f until §7.2. The goal of §7.1 is to perform two general reduction steps towards Theorem 7.1, stated as Proposition 7.4 (from completed Chow groups to Chow groups) and Proposition 7.5 (from cycles to effective cycles).

Proposition 7.4. *Let $\widehat{z}_{\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}^{\widehat{}}(X)$ be such that $f_* \widehat{z}_{\mathbf{A}} \in \mathrm{Pic}_{\mathbf{A}}^{\widehat{}}(C)$ belongs to the image of the natural map $\mathrm{Pic}^{\widehat{}}(C) \to \mathrm{Pic}_{\mathbf{A}}^{\widehat{}}(C)$. For any $n > 0$, there exist $z_{\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}(X)$, $\widehat{z} \in \mathrm{CH}_0^{\widehat{}}(X)$, $\widehat{\xi}_{\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}^{\widehat{}}(X)$ such that the equality*

(7.1) $$\widehat{z}_{\mathbf{A}} = z_{\mathbf{A}} + n(\widehat{z} + \widehat{\xi}_{\mathbf{A}})$$

holds in $\mathrm{CH}_{0,\mathbf{A}}^{\widehat{}}(X)$ and such that

(1) *$f_* \widehat{\xi}_{\mathbf{A}} = 0$ in $\mathrm{Pic}_{\mathbf{A}}^{\widehat{}}(C)$;*

(2) *$f_* z_{\mathbf{A}} \in \mathrm{Pic}_{\mathbf{A}}(C)$ belongs to the image of the natural map $\mathrm{Pic}(C) \to \mathrm{Pic}_{\mathbf{A}}(C)$.*

Proof. Let $Z \subset X$ be a proper irreducible curve dominating C and let $d = [k(Z) : k(C)]$. As a consequence of the projection formula $f_*(f^*y \cdot [Z]) = y \cdot f_*[Z] = dy$ (see [Ful98, Example 8.1.7]), the cokernels of the maps $\mathrm{CH}_0^{\widehat{}}(X) \to \mathrm{Pic}^{\widehat{}}(C)$ and $\mathrm{CH}_{0,\mathbf{A}}(X) \to \mathrm{Pic}_{\mathbf{A}}(C)$

induced by f are killed by d. On the other hand, the natural map $\mathrm{Pic}(C) \to \mathrm{Pic}^\widehat{}(C)$ has a divisible cokernel (apply Lemma 7.3 (ii) with $A = \mathrm{Pic}(C)$ and $B = 0$). Thus, the cokernel of the map

$$\mathrm{Pic}(C) \times \widehat{\mathrm{CH}_0}(X) \to \mathrm{Pic}^\widehat{}(C), \quad (y, \widehat{z}) \mapsto y + nf_*\widehat{z}$$

is at the same time divisible and killed by d, hence it vanishes. As $f_*\widehat{z}_{\mathbf{A}}$ belongs to the image of $\mathrm{Pic}^\widehat{}(C)$, we conclude that there exist $y_0 \in \mathrm{Pic}(C)$ and $\widehat{z} \in \widehat{\mathrm{CH}_0}(X)$ such that $f_*\widehat{z}_{\mathbf{A}} = y_0 + nf_*\widehat{z}$ in $\widehat{\mathrm{Pic}_{\mathbf{A}}}(C)$.

Let φ denote the map $\mathrm{CH}_{0,\mathbf{A}}(X) \to \mathrm{Pic}_{\mathbf{A}}(C)$ induced by f. As $\mathrm{Coker}(\varphi)$ is killed by d, the natural map $\mathrm{Coker}(\varphi) \to \mathrm{Coker}(\widehat{\varphi})$ is injective (see Lemma 7.3 (i)). As the image of y_0 in $\mathrm{Coker}(\widehat{\varphi})$ vanishes, we deduce the existence of $z_{0,\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}(X)$ such that $f_*z_{0,\mathbf{A}} = y_0$ in $\mathrm{Pic}_{\mathbf{A}}(C)$. We then have

(7.2)
$$\widehat{z}_{\mathbf{A}} = z_{0,\mathbf{A}} + n\widehat{z} + \widehat{\xi}_{0,\mathbf{A}}$$

for some $\widehat{\xi}_{0,\mathbf{A}} \in \mathrm{Ker}(\widehat{\varphi})$. As the groups $\mathrm{Coker}(N_{\overline{k}_v/k_v} : \mathrm{Pic}(C \otimes_k \overline{k}_v) \to \mathrm{Pic}(C \otimes_k k_v))$ for real places v of k and the N-torsion subgroup of $\mathrm{Pic}(C \otimes_k k_v)$ for $v \in \Omega_f$ and $N > 0$ are all finite and the group $\mathrm{Coker}(f_* : \mathrm{CH}_0(X \otimes_k k_v) \to \mathrm{Pic}(C \otimes_k k_v))$ is killed by d for all $v \in \Omega$, Lemma 7.3 (ii) applied to the v-adic component φ_v of φ for each v implies that $\mathrm{Coker}(\mathrm{Ker}(\varphi) \to \mathrm{Ker}(\widehat{\varphi}))$ is divisible, since $\widehat{\varphi} = \prod_{v \in \Omega} \widehat{\varphi_v}$. Thus there exist $\widehat{\xi}_{\mathbf{A}} \in \mathrm{Ker}(\widehat{\varphi})$ and $z_{1,\mathbf{A}} \in \mathrm{Ker}(\varphi)$ such that $\widehat{\xi}_{0,\mathbf{A}} = n\widehat{\xi}_{\mathbf{A}} + z_{1,\mathbf{A}}$. Letting $z_{\mathbf{A}} = z_{0,\mathbf{A}} + z_{1,\mathbf{A}}$, we now have $\widehat{z}_{\mathbf{A}} = z_{\mathbf{A}} + n(\widehat{z} + \widehat{\xi}_{\mathbf{A}})$ and $f_*z_{\mathbf{A}} = y_0$. $\qquad\square$

Proposition 7.5. *Let $z_{\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}(X)$ be such that $f_*z_{\mathbf{A}} \in \mathrm{Pic}_{\mathbf{A}}(C)$ belongs to the image of the natural map $\mathrm{Pic}(C) \to \mathrm{Pic}_{\mathbf{A}}(C)$. For any dense open subset $X^0 \subseteq X$, any finite subset $S \subset \Omega$, any map $\Phi : \mathbf{N}^2 \to \mathbf{N}$ and any $n > 0$, there exist $y \in \mathrm{Pic}(C)$, $z_{\mathbf{A}}^{\mathrm{eff}} = (z_v)_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X^0)$, $z \in \mathrm{CH}_0(X)$, $\xi_{\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}(X)$ such that the equality*

(7.3)
$$z_{\mathbf{A}} = z_{\mathbf{A}}^{\mathrm{eff}} + z + \xi_{\mathbf{A}}$$

holds in $\mathrm{CH}_{0,\mathbf{A}}(X)$ and such that

(1) $f_*\xi_{\mathbf{A}} = 0$ *in* $\mathrm{Pic}_{\mathbf{A}}(C)$;
(2) *the v-adic component of the family $\xi_{\mathbf{A}}$ is zero for all $v \in S$;*
(3) *for all $v \in S$, the support of z_v is contained in a smooth, proper, geometrically irreducible curve $Z_v \subset X \otimes_k k_v$ which dominates C and satisfies $\deg(y) \geq \Phi(g_v, d_v)$, where g_v denotes the genus of Z_v and $d_v = [k_v(Z_v) : k_v(C)]$.*

Proof. Let $\mathscr{X}^0$ be a model of X^0 over $\mathcal{O}_S$. Let $S' \subset \Omega$ be the finite subset containing $S \cup \Omega_\infty$ given by Lemma 3.2. Let $P \in X^0$ be a closed point. For $v \in S'$, let $z_v^0 \in \mathrm{Z}_0(X \otimes_k k_v)$ be a representative of the v-adic component of $z_{\mathbf{A}}$. According to [CT00, Lemmes 3.1 et 3.2] (see also [Wit12, Lemme 4.4]), there exist an integer n_0 and, for each $v \in S'$, a smooth, proper and geometrically irreducible curve $Z_v \subset X \otimes_k k_v$ which dominates C and contains both P and the support of z_v^0, such that for any $n \geq n_0$ and any $v \in S'$, there exists an effective divisor $z_v \in \mathrm{Div}(Z_v)$ satisfying the following condition:

($*$) z_v is an effective divisor linearly equivalent to $z_v^0 + nP$ on Z_v, its support is contained in $X^0 \otimes_k k_v$ and $f_* z_v$ is a reduced divisor on $C \otimes_k k_v$.

We fix $y_0 \in \mathrm{Pic}(C)$ such that $f_* z_{\mathbf{A}} = y_0$ and choose an integer $n \geq n_0$ such that

$$(7.4) \qquad \deg(y_0) + n \deg(P) > 2g, \quad \deg(y_0) + n \deg(P) \geq \Phi(g_v, d_v)$$

for all $v \in S'$, where $d_v = [k_v(Z_v) : k_v(C)]$ and where g and g_v denote the genera of C and of Z_v respectively. Let $y \in \mathrm{Pic}(C)$ be the class of $y_0 + n f_* P$. For each $v \in S'$, let us fix a divisor z_v on Z_v satisfying ($*$). For each $v \in \Omega \setminus S'$, we fix an effective $z_v \in \mathrm{Z}_0(X^0 \otimes_k k_v)$ such that $f_* z_v$ is reduced, such that the class of $f_* z_v$ in $\mathrm{Pic}(C \otimes_k k_v)$ is equal to the image of y and such that the Zariski closure of $\mathrm{Supp}(z_v)$ in $\mathscr{X}^0 \otimes_{\mathcal{O}_S} \mathcal{O}_v$ is finite over $\mathcal{O}_v$ (see the conclusion of Lemma 3.2). The family $z_{\mathbf{A}}^{\mathrm{eff}} = (z_v)_{v \in \Omega} \in \prod_{v \in \Omega} \mathrm{Z}_0(X^0 \otimes_k k_v)$ then belongs to $\mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X^0)$ and the desired equality (7.3) holds if we let $z = -nP$ and $\xi_{\mathbf{A}} = (z_v^0 - z_v + nP)_{v \in \Omega} \in \mathrm{CH}_{0,\mathbf{A}}(X)$. $\qquad \square$

7.2. **Proof of Theorem 7.1.** Theorem 7.1 will be proved by combining Proposition 7.4, Proposition 7.5 and Theorem 7.2. We henceforth assume, as in the statement of Theorem 7.1, that $H^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}) = 0$ and $\mathrm{A}_0(X_{\bar{\eta}} \otimes K) = 0$. According to Theorem 7.2, there exists a finite subset $S \subset \Omega_f$ such that the map

$$(7.5) \qquad f_* : \mathrm{CH}_0(X \otimes_k k_v) \to \mathrm{Pic}(C \otimes_k k_v)$$

is an isomorphism for all $v \in \Omega \setminus (S \cup \Omega_\infty)$.

Lemma 7.6. *There exists an integer $n > 0$ such that the natural map*

$$(7.6) \qquad \mathrm{Ker}\left(\mathrm{CH}_{0,\mathbf{A}}(X) \xrightarrow{f_*} \mathrm{Pic}_{\mathbf{A}}(C) \right) \to \mathrm{CH}_{0,\mathbf{A}}(X)/n\mathrm{CH}_{0,\mathbf{A}}(X)$$

is injective. Moreover, the kernel of $f_* : \widehat{\mathrm{CH}}_{0,\mathbf{A}}(X) \to \widehat{\mathrm{Pic}}_{\mathbf{A}}(C)$ has finite exponent.

Proof. For $v \in S$, the kernel of (7.5) and the torsion subgroup of $\mathrm{Pic}(C \otimes_k k_v)$ have finite exponent, by Theorem 7.2 (ii) and [Mat55] respectively. Let $N > 0$ be an even integer which kills these two groups for all $v \in S$. The map (7.6) is then injective for $n = N^2$. The second assertion follows from the definition of S and from Lemma 7.3 (ii) applied to the map (7.5) for $v \in S$. $\qquad \square$

Using Lemma 7.6, we choose an integer $n > 0$ such that (7.6) is injective and such that $\mathrm{Ker}(f_* : \widehat{\mathrm{CH}}_{0,\mathbf{A}}(X) \to \widehat{\mathrm{Pic}}_{\mathbf{A}}(C))$ is killed by n. Let us apply Proposition 7.4 to $\widehat{z}_{\mathbf{A}}$ and n and Proposition 7.5 to the resulting class $z_{\mathbf{A}}$ and to X^0, $S \cup \Omega_\infty$, Φ, n. This yields y, $z_{\mathbf{A}}^{\mathrm{eff}} = (z_v)_{v \in \Omega}$, z, $\widehat{z}$, $\xi_{\mathbf{A}}$ and $\widehat{\xi}_{\mathbf{A}}$ satisfying the conclusions of Propositions 7.4 and 7.5. By properties (1) and (2) of Proposition 7.5 and by the definition of S, we have $\xi_{\mathbf{A}} = 0$. By property (1) of Proposition 7.4 and by the definition of n, we also have $n \widehat{\xi}_{\mathbf{A}} = 0$. We thus obtain the decomposition $\widehat{z}_{\mathbf{A}} = z_{\mathbf{A}}^{\mathrm{eff}} + (z + n\widehat{z})$ and it only remains to check that the classes of $z_{\mathbf{A}}^{\mathrm{eff}}$ and $z_{\mathbf{A}}'$ in $\mathrm{CH}_{0,\mathbf{A}}(X)$ are equal for any $z_{\mathbf{A}}'$ as in the statement of Theorem 7.1, if the neighbourhood $\mathscr{U}$ is chosen small enough. At the places $v \notin S \cup \Omega_\infty$, this holds by the definition of S. At the places $v \in S$, this follows from [Wit12, Lemme 1.8], in view of the injectivity of (7.6). Finally, for $v \in \Omega_\infty$, condition (4) of Theorem 5.1 implies that the class

of $z_v - z'_v$ in $\mathrm{CH}_0(X \otimes_k k_v)$ is a norm from $\mathrm{CH}_0(X \otimes_k \bar{k}_v)$ (see [CTI81, Proposition 3.2 (ii)] or [Wit12, Lemme 1.8]).

8. Main theorems on zero-cycles

We now have at our disposal all of the required tools to prove that if X is a smooth, proper, irreducible variety over a number field k and if $f : X \to \mathbf{P}^1_k$ is a dominant morphism with rationally connected geometric generic fiber whose smooth fibers satisfy (E), then X satisfies (E) (Theorem 1.3). As discussed in §1, the results we obtain refine this statement in several ways: $\mathbf{P}^1_k$ is replaced with a curve of arbitrary genus satisfying (E), the hypothesis on the smooth fibers is restricted to the fibers over a Hilbert subset, etc. Our main theorem on zero-cycles is Theorem 8.3 below. We give its precise statement and compare its corollaries with the existing literature in §8.1 before proceeding to the proofs in §§8.2–8.4.

8.1. **Statements.** To state Theorem 8.3, we shall need to consider the following condition, which depends on the choice of a smooth and proper variety V over a number field k.

Condition 8.1. The image of $\widehat{\mathrm{CH}_0}(V)$ in $\widehat{\mathrm{CH}_{0,\mathbf{A}}}(V)$ contains the image of $V(\mathbf{A}_k)^{\mathrm{Br}(V)}$ in the same group.

Condition 8.1 clearly holds if V satisfies (E). The following lemma records another interesting condition which implies it.

Lemma 8.2. *Condition 8.1 holds if $V(k)$ is dense in $V(\mathbf{A}_k)^{\mathrm{Br}(V)}$ and V is rationally connected.*

Proof. For any $n \geq 1$ and any finite set S of places of k, the image of the natural map

$$\mathrm{CH}_0(V) \longrightarrow \prod_{v \in S} \mathrm{CH}_0(V \otimes_k k_v)/n\mathrm{CH}_0(V \otimes_k k_v)$$

contains the image of $V(\mathbf{A}_k)^{\mathrm{Br}(V)}$ as $V(k)$ is dense in $V(\mathbf{A}_k)^{\mathrm{Br}(V)}$ (see [Wit12, Lemme 1.8]). On the other hand, as V is rationally connected, the group $\mathrm{A}_0(V \otimes_k k_v)$ vanishes for all but finitely many places v and has finite exponent for every v (see [KS03] or Theorem 7.2). The lemma follows from these two facts (see [Wit12, Remarques 1.1 (ii)]). $\qquad\square$

We are now in a position to state the main result on zero-cycles obtained in this paper.

Theorem 8.3. *Let C be a smooth, proper and geometrically irreducible curve over a number field k. Let X be a smooth, proper and irreducible variety over k. Let $f : X \to C$ be a morphism whose geometric generic fiber $X_{\bar{\eta}}$ is irreducible and satisfies $H^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}) = 0$ and $\mathrm{A}_0(X_{\bar{\eta}} \otimes K) = 0$, where K denotes an algebraic closure of the function field of $X_{\bar{\eta}}$.*

  (1) *If C satisfies (E) and the smooth fibers of f above the closed points of a Hilbert subset of C satisfy (E), then X satisfies (E).*

  (2) *If the smooth fibers of f above the closed points of a Hilbert subset of C satisfy (E), the complex*

$$\mathrm{CH}_0(X/C) \longrightarrow \mathrm{CH}_{0,\mathbf{A}}(X/C) \longrightarrow \mathrm{Hom}(\mathrm{Br}(X)/f^*\mathrm{Br}(C), \mathbf{Q}/\mathbf{Z})$$

is an exact sequence, where $\mathrm{CH}_0(X/C)$ and $\mathrm{CH}_{0,\mathbf{A}}(X/C)$ denote the kernels of $f_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(C)$ and of $f_* : \mathrm{CH}_{0,\mathbf{A}}(X) \to \mathrm{CH}_{0,\mathbf{A}}(C)$ respectively.

(3) *If every fiber of f possesses an irreducible component of multiplicity 1, assertions (1) and (2) above remain valid when the smooth fibers of f above the closed points of a Hilbert subset of C are only assumed to satisfy Condition 8.1 instead of* (E).

(4) *If there exists a family $(z_v)_{v\in\Omega} \in \mathrm{Z}_{0,\mathbf{A}}(X)$ orthogonal to $\mathrm{Br}(X)$ with respect to* (1.5), *such that $\deg(z_v) = 1$ for all $v \in \Omega$ and such that the class of $(f_*z_v)_{v\in\Omega}$ in $\mathrm{Pic}\widehat{_\mathbf{A}}(C)$ belongs to the image of $\mathrm{Pic}\widehat{}(C)$, and if the smooth fibers of f above the closed points of a Hilbert subset of C satisfy* (E$_1$), *then X possesses a zero-cycle of degree 1.*

We recall that C satisfies (E) if $C = \mathbf{P}_k^1$ or more generally if the divisible subgroup of the Tate–Shafarevich group of the Jacobian of C is trivial, by a theorem of Saito [Sai89] (see [Wit12, Remarques 1.1 (iv)]). In this case, the condition, in (4), that the class of $(f_*z_v)_{v\in\Omega}$ in $\mathrm{Pic}\widehat{_\mathbf{A}}(C)$ should belong to the image of $\mathrm{Pic}\widehat{}(C)$ automatically holds.

An easy induction argument will allow us to deduce a version of Theorem 8.3 over some simple higher-dimensional bases.

Corollary 8.4. *Let X be a smooth, proper, irreducible variety over a number field k. Let Y be an irreducible variety over k, birationally equivalent to either $\mathbf{P}_k^n$, or C, or $\mathbf{P}_k^n \times C$, for some $n \geq 1$ and some smooth, proper, geometrically irreducible curve C over k which satisfies* (E). *Let $f : X \to Y$ be a morphism whose geometric generic fiber $X_{\bar\eta}$ is irreducible and satisfies $H^1(X_{\bar\eta}, \mathbf{Q}/\mathbf{Z}) = 0$ and $\mathrm{A}_0(X_{\bar\eta} \otimes K) = 0$, where K denotes an algebraic closure of the function field of $X_{\bar\eta}$.*

(1) *If the smooth fibers of f above the closed points of a Hilbert subset of Y satisfy* (E), *then X satisfies* (E).

(2) *If the smooth fibers of f above the closed points of a Hilbert subset of Y satisfy Condition 8.1 and the fibers of f above the codimension 1 points of Y possess an irreducible component of multiplicity 1, then X satisfies* (E).

By the generalised Bloch conjecture, the vanishing of $\mathrm{A}_0(X_{\bar\eta} \otimes K)$ should be equivalent to that of $H^i(X_{\bar\eta}, \mathscr{O}_{X_{\bar\eta}})$ for all $i > 0$. We refer the reader to [Voi11, §4] for a discussion of this condition and for some nontrivial examples of simply connected surfaces of general type which satisfy it (Barlow surfaces).

When the geometric generic fiber of f is rationally connected, both of the assumptions $H^1(X_{\bar\eta}, \mathbf{Q}/\mathbf{Z}) = 0$ and $\mathrm{A}_0(X_{\bar\eta} \otimes K) = 0$ hold (see [Deb01, Corollary 4.18 (b)] for the first). In particular, Theorem 1.3 follows from Theorem 8.3 (1). Moreover, in this case, every fiber of f above a codimension 1 point of Y contains an irreducible component of multiplicity 1 (see [GHS03]). Combining Corollary 8.4 with Lemma 8.2 and with Borovoi's theorem on the rational points of homogeneous spaces of linear groups [Bor96, Corollary 2.5], we thus obtain the following corollary.

Corollary 8.5. *Let X be a smooth, proper, irreducible variety over a number field k. Let Y be an irreducible variety over k, birationally equivalent to either $\mathbf{P}_k^n$, or C, or $\mathbf{P}_k^n \times C$, for some $n \geq 1$ and some smooth, proper, geometrically irreducible curve C over k which*

satisfies (E). *Let $f : X \to Y$ be a dominant morphism whose generic fiber is birationally equivalent to a homogeneous space of a connected linear algebraic group, with connected geometric stabilisers. Then X satisfies* (E).

Corollary 8.5 applies to arbitrary families of toric varieties. Even this particular case of the corollary suffices to cover all of the examples of varieties satisfying (E) or (E$_1$) dealt with in [Lia15, §3], [Wei14a], [CL13], [Lia14]. Corollary 8.5 also applies to fibrations into Châtelet surfaces over Y, or, more generally, into Châtelet p-folds in the sense of [VAV12], since the total space of such a fibration is birationally equivalent to the total space of a fibration into torsors under tori over $Y \times_k \mathbf{P}_k^1$. Such fibrations were considered in [Poo10]. This answers the question raised at the very end of [CT10, Remarque 3.3]. (The case of fibrations into Châtelet surfaces all of whose fibers are geometrically integral was previously treated in [Lia12a]; see also [Lia13b, Proposition 7.1].) In the situation of a fibration into Châtelet surfaces over a curve of positive genus, Theorem 8.3 (4) immediately implies the unconditional result [CT10, Théorème 3.1]. A similar application of Corollary 8.5 is the validity of (E) for the total space of any fibration into del Pezzo surfaces of degree 6 over Y (recall that degree 6 del Pezzo surfaces are toric [Man66]).

By applying Corollary 8.5 to the trivial fibration $V \times \mathbf{P}_k^1 \to \mathbf{P}_k^1$ (a trick first used in [Lia13a]), we recover Liang's theorem (proved in *op. cit.*) that varieties V birationally equivalent to a homogeneous space of a connected linear algebraic group, with connected geometric stabilisers, satisfy (E).

We note, finally, that Theorem 8.3 and Corollary 8.4 also subsume [CTSD94, §§5–6], [CTSSD98, §4], [CT00], [Fro03], [vH03], [Wit12], [Lia12a], [Lia12b], [Sme15, §5], [Lia13b]. In these papers, one can find versions of Corollary 8.4 in which every fiber of f is assumed to contain an irreducible component of multiplicity 1 split by an abelian extension of the field over which the fiber is defined, and in which the Brauer group of the generic fiber is assumed to be generated by classes which are unramified along the singular fibers of f. Both of these assumptions turn out to be quite restrictive for applications.

8.2. **Preliminaries.** Let us first recall some well-known implications between the geometric conditions appearing in the statements of Theorem 8.3, Theorem 6.2, Theorem 5.1 and of Proposition 4.1.

Lemma 8.6. *The properties*

    (a) $X_{\bar{\eta}}$ *is rationally connected;*
    (b) $\mathrm{A}_0(X_{\bar{\eta}} \otimes K) = 0$, *where K denotes an algebraic closure of the function field of $X_{\bar{\eta}}$;*
    (c) $H^i(X_\eta, \mathscr{O}_{X_\eta}) = 0$ *for all $i > 0$;*
    (d) $H^2(X_\eta, \mathscr{O}_{X_\eta}) = 0$;
    (e) *if $H^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}) = 0$, the quotient $\mathrm{Br}(X_\eta)/f_\eta^* \mathrm{Br}(\eta)$ is finite;*
    (f) *the gcd of the multiplicities of the irreducible components of each fiber of f is 1*

satisfy the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) *and* (c) $\Rightarrow$ (f).

Proof. The implication (b) $\Rightarrow$ (c) may be checked by decomposing the diagonal in Hodge theory, see [CL04, p. 141] (see also Lemma 8.11 below). See [CTS13, Lemma 1.3 (i)] for

the implication (d) $\Rightarrow$ (e) and [CTV12, Proposition 7.3 (iii)] or [ELW13, Proposition 2.4] for (c) $\Rightarrow$ (f). $\square$

The next proposition, which combines the contents of §§2–7, is the core result from which the four assertions of Theorem 8.3 will be deduced.

Proposition 8.7. *Let us keep the notation and assumptions of Theorem 8.3. Fix a Hilbert subset $H \subseteq C$ and an element $\widehat{z}_{\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}^{\widehat{}}(X)$ orthogonal to $\mathrm{Br}(X)$ with respect to (1.6), such that $f_* \widehat{z}_{\mathbf{A}}$ belongs to the image of the natural map $\mathrm{Pic}^{\widehat{}}(C) \to \mathrm{Pic}_{\mathbf{A}}^{\widehat{}}(C)$. There exist an element $\widehat{z} \in \mathrm{CH}_0^{\widehat{}}(X)$, a closed point c of H and a family $z'_{\mathbf{A}} \in \mathrm{Z}_{0,\mathbf{A}}(X_c)$ orthogonal to $\mathrm{Br}(X_c)$ with respect to the pairing (1.5) relative to X_c, such that $z'_{\mathbf{A}}$, when considered as an element of $\mathrm{Z}_{0,\mathbf{A}}(X)$, satisfies $f_* z'_{\mathbf{A}} = c$ in $\mathrm{Z}_{0,\mathbf{A}}(C)$ and $\widehat{z}_{\mathbf{A}} = z'_{\mathbf{A}} + \widehat{z}$ in $\mathrm{CH}_{0,\mathbf{A}}^{\widehat{}}(X)$. Furthermore, if every fiber of f possesses an irreducible component of multiplicity 1, one can ensure that $z'_{\mathbf{A}}$ is a family of effective zero-cycles.*

Proof. By Chow's lemma and Hironaka's theorem, there exist an irreducible, smooth projective variety X' and a birational morphism $\pi : X' \to X$. The induced morphism $\pi_c : X'_c \to X_c$ is a birational equivalence for all but finitely many $c \in C$. As the groups $H^1(V, \mathbf{Q}/\mathbf{Z})$, $\mathrm{Br}(V)$, $\mathrm{CH}_0(V)$ and $\mathrm{A}_0(V)$ are birational invariants of smooth and proper varieties V over a field of characteristic 0 (see [Gro71, Corollaire 3.4], [Gro68, III, §7], [Ful98, Example 16.1.11]) and as the existence of an irreducible component of multiplicity 1 in every fiber of f is equivalent to the existence of an irreducible component of multiplicity 1 in every fiber of $f \circ \pi$ (see [Sko96, Corollary 1.2] and [Nis55]), we may replace X with X' to prove Proposition 8.7. Thus, from now on, we assume that X is projective, which allows us to use Theorem 7.1.

Thanks to Lemma 8.6, we can choose a finite subgroup $B \subset \mathrm{Br}(X_\eta)$ which surjects onto the quotient $\mathrm{Br}(X_\eta)/f_\eta^* \mathrm{Br}(\eta)$. Let $C^0 \subseteq C$ be a dense open subset above which f is smooth, small enough that $B \subset \mathrm{Br}(X^0)$, where $X^0 = f^{-1}(C^0)$. By Proposition 4.1, there exists a Hilbert subset $H' \subseteq C^0$ such that the natural map

(8.1) $$B \to \mathrm{Br}(X_c)/f_c^* \mathrm{Br}(c)$$

is surjective for all $c \in H'$. We define a map $\Phi : \mathbf{N}^2 \to \mathbf{N}$ by $\Phi(g, d) = \deg(M)d + 2g + 2$, where $M = C \setminus C^0$, and apply Theorem 7.1. This yields classes $y \in \mathrm{Pic}(C)$, $\widehat{z} \in \mathrm{CH}_0^{\widehat{}}(X)$, $z_{\mathbf{A}}^{\mathrm{eff}} = (z_v)_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}^{\mathrm{eff},\mathrm{red},y}(X^0)$ and subsets $S \subset \Omega_f$ and $\mathscr{U} \subseteq \prod_{v \in S} \mathrm{Sym}_{X^0/k}(k_v)$ satisfying the conclusion of Theorem 7.1. Note that $z_{\mathbf{A}}^{\mathrm{eff}}$ is orthogonal to $\mathrm{Br}(X)$ with respect to (1.5) since $\widehat{z}_{\mathbf{A}} = z_{\mathbf{A}}^{\mathrm{eff}} + \widehat{z}$ in $\mathrm{CH}_{0,\mathbf{A}}^{\widehat{}}(X)$ and $\widehat{z}_{\mathbf{A}}$ and $\widehat{z}$ are both orthogonal to $\mathrm{Br}(X)$. We can therefore apply Theorem 6.2 and obtain a closed point c of $H \cap H'$ and a family $z'_{\mathbf{A}} = (z'_v)_{v \in \Omega} \in \mathrm{Z}_{0,\mathbf{A}}(X^0)$ satisfying conditions (1)–(4) of Theorem 5.1. By condition (1), we may regard $z'_{\mathbf{A}}$ as an element of $\mathrm{Z}_{0,\mathbf{A}}(X_c)$. By condition (2) and the surjectivity of (8.1), this element is orthogonal to $\mathrm{Br}(X_c)$ with respect to the pairing (1.5). By conditions (3) and (4) and the conclusion of Theorem 7.1, we have $\widehat{z}_{\mathbf{A}} = z'_{\mathbf{A}} + \widehat{z}$ in $\mathrm{CH}_{0,\mathbf{A}}^{\widehat{}}(X)$. $\square$

8.3. **Proof of Theorem 8.3.**

8.3.1. *Assertion (1).* Let $\widehat{z}_{\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}^{\,\widehat{}}(X)$ be orthogonal to $\mathrm{Br}(X)$ with respect to (1.6). By functoriality, its image $f_* \widehat{z}_{\mathbf{A}} \in \mathrm{Pic}_{\mathbf{A}}^{\,\widehat{}}(C)$ is orthogonal to $\mathrm{Br}(C)$. As C satisfies (E), it follows that $f_* \widehat{z}_{\mathbf{A}}$ belongs to the image of the natural map $\mathrm{Pic}^{\,\widehat{}}(C) \to \mathrm{Pic}_{\mathbf{A}}^{\,\widehat{}}(C)$. We may therefore apply Proposition 8.7 to $\widehat{z}_{\mathbf{A}}$ and to a Hilbert subset $H \subseteq C$ such that for every closed point c of H, the fiber $X_c = f^{-1}(c)$ is smooth and satisfies (E). The class in $\mathrm{CH}_{0,\mathbf{A}}^{\,\widehat{}}(X_c)$ of the resulting family $z_{\mathbf{A}}'$ is then the image of an element of $\mathrm{CH}_0^{\,\widehat{}}(X_c)$. As a consequence, its image in $\mathrm{CH}_{0,\mathbf{A}}^{\,\widehat{}}(X)$ is the image of an element of $\mathrm{CH}_0^{\,\widehat{}}(X)$.

8.3.2. *Assertion (2).* By the Mordell–Weil theorem, the group $\mathrm{Pic}(C)$ is finitey generated. Moreover, by Theorem 7.2, the kernel and the cokernel of the map $f_* : \mathrm{CH}_0(X) \to \mathrm{Pic}(C)$ have finite exponent. By Lemma 7.3 (ii), we deduce that the natural complex

(8.2) $$\mathrm{CH}_0(X/C) \longrightarrow \mathrm{CH}_0^{\,\widehat{}}(X) \longrightarrow \mathrm{Pic}^{\,\widehat{}}(C)$$

is exact. Let $z_{\mathbf{A}} \in \mathrm{CH}_{0,\mathbf{A}}(X/C)$ be orthogonal to $\mathrm{Br}(X)/f^*\mathrm{Br}(C)$ with respect to (1.5). The argument of §8.3.1 shows that the image of $z_{\mathbf{A}}$ in $\mathrm{CH}_{0,\mathbf{A}}^{\,\widehat{}}(X)$ comes from $\mathrm{CH}_0^{\,\widehat{}}(X)$. As the natural maps $\mathrm{Pic}^{\,\widehat{}}(C) \to \mathrm{Pic}_{\mathbf{A}}^{\,\widehat{}}(C)$ and $\mathrm{CH}_{0,\mathbf{A}}(X/C) \to \mathrm{CH}_{0,\mathbf{A}}^{\,\widehat{}}(X)$ are injective (see [Wit12, Remarques 1.1 (v)] and Lemma 7.6) and the complex (8.2) is exact, we conclude that $z_{\mathbf{A}}$ is the image of an element of $\mathrm{CH}_0(X/C)$.

8.3.3. *Assertion (3).* According to the last sentence of Proposition 8.7, one can ensure, in §8.3.1 and §8.3.2, that $z_{\mathbf{A}}'$ is a family of effective zero-cycles. As $f_* z_{\mathbf{A}}' = c$ in $\mathrm{Z}_{0,\mathbf{A}}(C)$, it follows that $z_{\mathbf{A}}' \in X_c(\mathbf{A}_{k(c)})$; hence Condition 8.1 can be applied.

8.3.4. *Assertion (4).* By assumption there exists a Hilbert subset $H \subseteq C$ such that for every closed point c of H, the fiber X_c is smooth and satisfies (E_1). Applying Proposition 8.7 to $z_{\mathbf{A}} = (z_v)_{v\in\Omega}$, we find a closed point c of H, a family $z_{\mathbf{A}}' \in \mathrm{Z}_{0,\mathbf{A}}(X_c)$ orthogonal to $\mathrm{Br}(X_c)$ and a class $\widehat{z} \in \mathrm{CH}_0^{\,\widehat{}}(X)$ such that $f_* z_{\mathbf{A}}' = c$ in $\mathrm{Z}_{0,\mathbf{A}}(C)$ and $z_{\mathbf{A}} = z_{\mathbf{A}}' + \widehat{z}$ in $\mathrm{CH}_{0,\mathbf{A}}^{\,\widehat{}}(X)$. Considering $z_{\mathbf{A}}'$ as a collection of local zero-cycles of degree 1 on the variety X_c over $k(c)$, we deduce from (E_1) that X_c possesses a zero-cycle z' of degree 1 over $k(c)$. Regarding z' as a zero-cycle on X, we have $\deg(z' + \widehat{z}) = 1$ in $\widehat{\mathbf{Z}}$, hence the degree map $\mathrm{CH}_0^{\,\widehat{}}(X) \to \widehat{\mathbf{Z}}$ is surjective. By Lemma 7.3 (i), the degree map $\mathrm{CH}_0(X) \to \mathbf{Z}$ is surjective as well.

8.4. **Proof of Corollary 8.4.** When Y is a curve, this is Theorem 8.3 (1) and (3). In general, we argue by induction on the dimension of Y. As the hypotheses and the conclusion of Corollary 8.4 are invariant under birational equivalence over k (see [Wit12, Remarques 1.1 (vi)]), we may assume, when $\dim(Y) > 1$, that $Y = \mathbf{P}_k^n \times C$ for some $n \geq 1$ (setting $C = \mathbf{P}_k^1$ in the first of the three cases considered in the statement). Let $g : X \to C$ denote the composition of f with the second projection. For $c \in C$, let $X_c = g^{-1}(c)$. For the sake of brevity, we introduce the following definition.

Definition 8.8. We shall say that a proper variety V over an algebraically closed field *satisfies* $(*)$ if it is integral, if $H^1(V, \mathbf{Q}/\mathbf{Z}) = 0$ and if $\mathrm{A}_0(V \otimes K) = 0$ for an algebraic closure K of the function field of V.

Lemma 8.9. *The geometric generic fiber of g satisfies* $(*)$.

Proof. Let $\bar{\xi}$ denote the geometric generic point of C and $f_{\bar{\xi}} : X_{\bar{\xi}} \to \mathbf{P}^n_{k(\bar{\xi})}$ the morphism induced by f. The geometric generic fiber $X_{\bar{\xi}}$ of g is smooth and irreducible. The geometric generic fiber V of $f_{\bar{\xi}}$ satisfies $(*)$ since it is isomorphic to the geometric generic fiber of f.

Letting K denote an algebraic closure of the function field of $X_{\bar{\xi}}$, we deduce from Theorem 7.2 (ii) applied to $f_{\bar{\xi}}$ that the abelian group $\mathrm{A}_0(X_{\bar{\xi}} \otimes K)$ has finite exponent. It is also divisible, as K is algebraically closed (see [Blo80, Lemma 1.3]); hence $\mathrm{A}_0(X_{\bar{\xi}} \otimes K) = 0$.

It remains to check that $H^1(X_{\bar{\xi}}, \Lambda) = 0$ for any finite abelian group Λ. The morphism $f_{\bar{\xi}}$ has no multiple fiber in codimension 1 since V satisfies $(*)$ (see Lemma 8.6, (b)$\Rightarrow$(f)). As a consequence, the fundamental group of V surjects onto the fundamental group of the geometric fiber of $f_{\bar{\xi}}$ above any codimension 1 point of $\mathbf{P}^n_{k(\bar{\xi})}$ (see [Ray70, Proposition 6.3.5]). The constructible sheaf $R^1(f_{\bar{\xi}})_* \Lambda$ must therefore be supported on a closed subset $F \subset \mathbf{P}^n_{k(\bar{\xi})}$ of codimension ≥ 2. Letting $U = \mathbf{P}^n_{k(\bar{\xi})} \setminus F$, we have $H^1(U, (f_{\bar{\xi}})_* \Lambda) = H^1(U, \Lambda) = 0$ since $(f_{\bar{\xi}})_* \Lambda = \Lambda$ (see [Gro66, Proposition 15.5.3]). The Leray spectral sequence for $f_{\bar{\xi}}$ now shows that $H^1(f_{\bar{\xi}}^{-1}(U), \Lambda) = 0$. As $H^1(X_{\bar{\xi}}, \Lambda) \subseteq H^1(f_{\bar{\xi}}^{-1}(U), \Lambda)$, the proof is complete. $\square$

Lemma 8.10. *There exists a dense open subset $C^0 \subseteq C$ such that for all $c \in C^0$, the geometric generic fiber of the morphism $f_c : X_c \to \mathbf{P}^n_{k(c)}$ induced by f satisfies $(*)$.*

Proof. It suffices to prove the existence of a dense open subset $Y^0 \subseteq Y$ such that for any geometric point y of Y^0, the variety $V = f^{-1}(y)$ satisfies $(*)$.

As f is proper, the restriction of the étale sheaf $R^1 f_* \mathbf{Q}/\mathbf{Z}$ to any open subset of Y over which f is smooth is a direct limit of locally constant sheaves (see [Mil80, Ch. VI, Corollary 4.2]). Its geometric generic stalk vanishes; thus $H^1(V, \mathbf{Q}/\mathbf{Z})$ vanishes for any geometric fiber V of f along which f is smooth.

To check that the condition $\mathrm{A}_0(V \otimes K) = 0$ for the generic fiber of f implies the same condition for the fibers of f over a dense open subset of Y, we use an argument known as decomposition of the diagonal (see [Blo80, Appendix to Lecture 1]), summarised in the next lemma.

Lemma 8.11. *Let V be a geometrically integral variety of dimension d over an arbitrary field. Let $v \in V$ be a closed point. Let K be an algebraic closure of the function field of V. Let $[\Delta] \in \mathrm{CH}_d(V \times V)$ denote the class of the diagonal. The group $\mathrm{A}_0(V \otimes K)$ vanishes if and only if for some $N \geq 1$, one has a decomposition*

$$(8.3) \qquad\qquad N[\Delta] = [\Gamma_1] + [\Gamma_2]$$

in $\mathrm{CH}_d(V \times V)$ for some cycle Γ_1 supported on $V \times \{v\}$ and some cycle Γ_2 supported on $D \times V$ for a closed subset $D \subset V$ of codimension ≥ 1.

Proof. Letting $N[\Delta]$ act on $\mathrm{A}_0(V \otimes K)$ as a correspondence, we see that $\mathrm{A}_0(V \otimes K)$ is killed by N if the diagonal can be decomposed as above. On the other hand, the group $\mathrm{A}_0(V \otimes K)$ is divisible since K is algebraically closed (see [Blo80, Lemma 1.3]). This proves a half of the desired equivalence. The other half is established in [BS83, Proposition 1]. $\square$

Any rational equivalence which gives rise to the decomposition (8.3) on the generic fiber of f extends and specialises to the fibers of f over a dense open subset of Y. Thus, applying Lemma 8.11 twice concludes the proof of Lemma 8.10. $\square$

We can now complete the proof of Corollary 8.4. Let $C^0 \subseteq C$ be a dense open subset satisfying the conclusion of Lemma 8.10, small enough that X_c is smooth and irreducible for all $c \in C^0$ and that if the codimension 1 fibers of f contain an irreducible component of multiplicity 1, then so do the codimension 1 fibers of $f_c : X_c \to \mathbf{P}^n_{k(c)}$ for $c \in C^0$.

According to the next lemma, which we state with independent notation, there exists a Hilbert subset $H \subseteq C^0$ such that for every closed point c of H, the smooth fibers of f_c above the closed points of a Hilbert subset of $\mathbf{P}^n_{k(c)}$ satisfy (E) in case (1), satisfy Condition 8.1 in case (2).

Lemma 8.12. *Let V_1 and V_2 be geometrically irreducible varieties over a number field k. Let $H \subseteq V_1 \times_k V_2$ be a Hilbert subset. There exists a Hilbert subset $H_2 \subseteq V_2$ such that for every $h \in H_2$, the set $H \cap (V_1 \times_k \{h\})$ contains a Hilbert subset of $V_1 \times_k \{h\}$ (where $\{h\}$ denotes $\mathrm{Spec}(k(h))$).*

Proof. Let $W_1, \ldots, W_n$ denote irreducible étale covers of a dense open subset of $V_1 \times_k V_2$ which define H. By [Wit07, Lemme 3.12], there exists a Hilbert subset $H_2 \subseteq V_2$ such that for any $h \in H_2$, the scheme $W_i \times_{V_2} \{h\}$ is irreducible for all $i \in \{1, \ldots, n\}$. The projections $W_i \times_{V_2} \{h\} \to V_1 \times_k \{h\}$ then define a Hilbert subset of $V_1 \times_k \{h\}$ contained in H. $\square$

We deduce from the induction hypothesis applied to f_c that X_c satisfies (E) for every closed point c of H. In view of Lemma 8.9, and noting that the fibers of g contain an irreducible component of multiplicity 1 if the codimension 1 fibers of f do, we may now apply Theorem 8.3 to g and conclude that X satisfies (E).

9. RATIONAL POINTS IN FIBRATIONS

The goal of this last part of the paper is to explore the counterpart, in the context of rational points, of the ideas which enter into the proof of the main theorems of §8. The main unconditional result here is Theorem 9.28, stated as Theorem 1.6 in the introduction.

9.1. **A conjecture on locally split values of polynomials over number fields.** We propose the following conjectural variant of Lemma 5.2. Its statement parallels the consequence of Schinzel's hypothesis (H) first formulated by Serre in [Ser92] and referred to as Hypothesis (H_1) in [CTSD94]. Lemma 5.2 corresponds to the case $n = 1$, $\deg(P_1) = 1$, $b_1 = 1$ of Conjecture 9.1, the only difference between the two statements being that the element t output by Lemma 5.2 is required to be integral outside $S \cup \{v_0\}$, while no integrality condition on t_0 appears in Conjecture 9.1 (see also Remark 9.12). In fact, the absence of any integrality condition makes it more accurate to compare Conjecture 9.1 with the homogeneous variant (HH_1) of Schinzel's hypothesis, first considered by Swinnerton-Dyer [SD99, §2], than with (H_1) itself (see especially [HSW14, Proposition 2.1]). We refer the reader to §9.2.1 for more details on (HH_1).

Conjecture 9.1. *Let k be a number field. Let $n \geq 1$ be an integer and $P_1, \ldots, P_n \in k[t]$ denote pairwise distinct irreducible monic polynomials. Let $k_i = k[t]/(P_i(t))$ and let $a_i \in k_i$ denote the class of t. For each $i \in \{1, \ldots, n\}$, suppose given a finite extension L_i/k_i and an element $b_i \in k_i^*$. Let S be a finite set of places of k containing the real places of k and the finite places above which, for some i, either b_i is not a unit or L_i/k_i is ramified. Finally, for each $v \in S$, fix an element $t_v \in k_v$. Assume that for every $i \in \{1, \ldots, n\}$ and every $v \in S$, there exists $x_{i,v} \in (L_i \otimes_k k_v)^*$ such that*

$$b_i(t_v - a_i) = N_{L_i \otimes_k k_v / k_i \otimes_k k_v}(x_{i,v})$$

in $k_i \otimes_k k_v$. Then there exists $t_0 \in k$ satisfying the following conditions:

(1) *t_0 is arbitrarily close to t_v for $v \in S$;*
(2) *for every $i \in \{1, \ldots, n\}$ and every finite place w of k_i such that $w(t_0 - a_i) > 0$, either w lies above a place of S or the field L_i possesses a place of degree 1 over w.*

Conjecture 9.1 will be applied in §9.3 to the study of weak approximation on the total space of a fibration over $\mathbf{P}_k^1$. When one is only interested in the existence of a rational point or in weak approximation "up to connected components at the archimedean places" (as considered in [Sto07, §2, p. 351]), the following weaker variant of Conjecture 9.1 suffices.

Conjecture 9.2. *Same as Conjecture 9.1, except that* (1) *is weakened to:*

(1') *t_0 is arbitrarily close to t_v for finite $v \in S$; in addition, for every real place v of k, if i denotes an element of $\{1, \ldots, n\}$ and w denotes a real place of k_i dividing v, the sign of $(t_0 - a_i)(t_v - a_i)$ at w is independent of w and of i.*

Remarks 9.3. (i) Suppose all of the fields L_i are isomorphic, over k, to the same Galois extension L/k. This condition can be assumed to hold at no cost in all of the applications that we shall consider in this paper. Assertion (2) of Conjecture 9.1 may then be reformulated as follows: for every $i \in \{1, \ldots, n\}$, the value $P_i(t_0) \in k$ has nonpositive valuation at all places $v \notin S$, except perhaps at places v which split completely in L. Thus, Conjecture 9.1 may be regarded as a conjecture about locally split values of polynomials.

(ii) In order to prove Conjectures 9.1 and 9.2, one may assume, without loss of generality, that $t_v = 0$ for all $v \in S$ and $a_i \neq 0$ for all $i \in \{1, \ldots, n\}$. Indeed, one can choose $\tau \in k$ arbitrarily close to t_v for $v \in S$ and perform the change of variables $t \mapsto t + \tau$. Thanks to this remark, we see that Conjectures 9.1 and 9.2 for $k = \mathbf{Q}$ imply the same conjectures for arbitrary k (compare [CTSD94, Proposition 4.1]).

(iii) In order to prove Conjectures 9.1 and 9.2, one is free to adjoin any finite number of places to S. To see this, it is enough to check that for any finite $v \notin S$, there exists $t_v \in k_v$ such that $b_i(t_v - a_i)$ is a norm from $(L_i \otimes_k k_v)^*$ for every i and such that $w(t_v - a_i) \leq 0$ for every i and every place w of k_i dividing v. As b_i is a unit above v and L_i/k_i is unramified above v, the first condition is satisfied as soon as $w(t_v - a_i)$ is a multiple of $[L_i : k_i]$ for every place w of k_i dividing v. Thus, for finite $v \notin S$, any $t_v \in k_v$ with valuation $-N \prod_{i=1}^n [L_i : k_i]$ will satisfy the desired properties if N is a large enough positive integer.

(iv) The reader interested in a more geometric (albeit less elementary) formulation of Conjecture 9.1 will find one in Proposition 9.9 below.

9.2. **Known cases and relation with Schinzel's hypothesis.** In this paragraph, we collect some results which lend support to Conjectures 9.1 and 9.2 (see Theorem 9.6, Theorem 9.11, Theorem 9.14, Theorem 9.15).

9.2.1. *Relation with Schinzel's hypothesis.* We prove that when the extensions L_i/k_i are abelian, Conjecture 9.1 is a consequence of Schinzel's hypothesis (H). Extensive numerical evidence for (H) can be found in [BH62], [BH65]. Exploiting a trick first used by Colliot-Thélène in the context of cubic extensions (see [Wei14a, Theorem 3.5], [HSW14, Theorem 4.6]), we deal, slightly more generally, with extensions which are "almost abelian" in the following sense.

Definition 9.4. Let k be a number field, with algebraic closure $\overline{k}$. A finite extension L/k is *almost abelian* if it is abelian or if there exist a prime number p and a bijection $\mathrm{Spec}(L \otimes_k \overline{k}) \simeq \mathbf{F}_p$ such that the natural action of $\mathrm{Gal}(\overline{k}/k)$ on $\mathrm{Spec}(L \otimes_k \overline{k})$ gives rise, by transport of structure, to an action on $\mathbf{F}_p$ by affine transformations, *i.e.*, transformations of the form $x \mapsto ax + b$ for $a \in \mathbf{F}_p^*$, $b \in \mathbf{F}_p$.

Examples 9.5. Cubic extensions are almost abelian. For any $c \in k$ and any prime p, the extension $k(c^{1/p})/k$ is almost abelian.

Theorem 9.6. *Assume Schinzel's hypothesis* (H)*. If the extensions $L_1/k_1, \dots, L_n/k_n$ are almost abelian, the statement of Conjecture 9.1 holds true.*

In the proof of Theorem 9.6, we use (H) *via* the following homogeneous variant of Schinzel's hypothesis, which makes sense over an arbitrary number field.

Hypothesis (HH$_1$). *Let k be a number field, $n \geq 1$ be an integer and $P_1, \dots, P_n \in k[\lambda, \mu]$ be irreducible homogeneous polynomials. Let S be a finite set of places of k containing the archimedean places, large enough that for any $v \notin S$, there exists $(\lambda_v, \mu_v) \in \mathcal{O}_v \times \mathcal{O}_v$ such that $v(P_1(\lambda_v, \mu_v)) = \cdots = v(P_n(\lambda_v, \mu_v)) = 0$. Suppose given $(\lambda_v, \mu_v) \in k_v \times k_v$ for $v \in S$, with $(\lambda_v, \mu_v) \neq (0,0)$ when v is archimedean. There exists $(\lambda_0, \mu_0) \in k \times k$ such that*

   (1) *(λ_0, μ_0) is arbitrarily close to $(\lambda_v, \mu_v) \in k_v \times k_v$ for each finite place $v \in S$;*
   (2) *$[\lambda_0 : \mu_0]$ is arbitrarily close to $[\lambda_v : \mu_v] \in \mathbf{P}^1(k_v)$ for each archimedean place $v \in S$;*
   (3) *$\lambda_0 \lambda_v + \mu_0 \mu_v > 0$ for each real place $v \in S$;*
   (4) *λ_0 and μ_0 are integral outside S;*
   (5) *for each $i \in \{1, \dots, n\}$, the element $P_i(\lambda_0, \mu_0) \in k$ is a unit outside S except at one place, at which it is a uniformiser.*

We refer to [SD11, Lemma 7.1] for a proof that Schinzel's hypothesis (H) implies (HH$_1$) (noting that γ, in the proof of *loc. cit.*, can be chosen totally positive, so that (3) holds).

Remark 9.7. From the work of Heath-Brown and Moroz [HBM04] on primes represented by binary cubic forms, it is easy to deduce the validity of (HH$_1$) when $k = \mathbf{Q}$, $n = 1$ and $\deg(P_1) = 3$. Thus Conjecture 9.1 holds when $n = 1$, $k = \mathbf{Q}$, $[k_1 : \mathbf{Q}] = 3$ and L_1/k_1 is almost abelian.

Proof of Theorem 9.6. Let $P_i(\lambda, \mu) = \mu^{\deg(P_i)} P_i(\lambda/\mu)$. By Remark 9.3 (iii), we may assume that S is large enough for (HH$_1$) to be applicable and that a_i is integral outside S for all i.

Let $I = \{i \in \{1, \ldots, n\}; L_i/k_i \text{ is abelian}\}$ and $J = \{1, \ldots, n\} \setminus I$. Thus $J = \varnothing$ if the extensions $L_1/k_1, \ldots, L_n/k_n$ are all abelian; the reader interested in this case only may simply skip the next lemma and the paragraph which follows it.

Lemma 9.8. *For every $j \in J$, there exists a nontrivial cyclic subextension E_j/k_j of the Galois closure of L_j/k_j such that for any place w of k_j which is unramified in L_j, either w splits completely in E_j or L_j possesses a place of degree 1 over w.*

Proof. As L_j/k_j is almost abelian but is not abelian, its degree is a prime number p and the Galois group G_j of its Galois closure L'_j/k_j embeds into the group $G = \mathbf{F}_p \rtimes \mathbf{F}_p^*$ of affine transformations of $\mathbf{F}_p$. Let E_j be the subfield of L'_j fixed by $H_j = G_j \cap \mathbf{F}_p$. As G_j is not abelian, it is not contained in $\mathbf{F}_p$, hence its image in $\mathbf{F}_p^*$ is nontrivial. Thus G_j is an extension of a nontrivial subgroup of $\mathbf{F}_p^*$ by H_j. As a consequence, the extension E_j/k_j is cyclic and nontrivial. Let w be a place of k_j which is unramified in L_j and let $F \in G_j$ denote the Frobenius automorphism at w (well-defined up to conjugacy). If the image of F in $\mathbf{F}_p^*$ is trivial, then w splits completely in E_j. Otherwise, the action of F on $\mathrm{Spec}(L_j \otimes_{k_j} L'_j)$ admits a fixed point (any affine transformation of $\mathbf{F}_p$ which is not a translation admits a fixed point). Such a fixed point corresponds to a place of L_j of degree 1 over w. $\qquad\square$

By Chebotarev's density theorem, for every $j \in J$, there exist infinitely many places w of k_j of degree 1 over k which are inert in E_j (recall that the set of places of k_j of degree 1 over k has Dirichlet density 1; see, *e.g.*, [Lan94, Ch. VIII, §4]). If v denotes the trace on k of any such place w, there exists $t_v \in k_v$ satisfying the following properties:

(i) $t_v - a_j$ is a uniformiser at w and is a unit at all places of k_j above v different from w;

(ii) $t_v - a_i$ is a unit at all places of k_i above v, for all $i \in \{1, \ldots, n\} \setminus \{j\}$.

By adjoining to S finitely many places v as above and choosing t_v subject to (i) and (ii) at these places, we may thus assume that

$$(9.1) \qquad \sum_{v \in S} \sum_{w|v} \mathrm{inv}_w(E_j/k_j, t_v - a_j) \neq 0$$

for all $j \in J$, where the second sum ranges over the places w of k_j dividing v and the symbol $(E_j/k_j, t_v - a_j)$ denotes a cyclic algebra over $(k_j)_w$.

For $v \in S$, let $\lambda_v = t_v$ and $\mu_v = 1$. Let $(\lambda_0, \mu_0) \in k \times k$ be given by (HH_1). By choosing (λ_0, μ_0) close enough to (λ_v, μ_v) at the finite places $v \in S$ and $[\lambda_0 : \mu_0]$ close enough to $[\lambda_v : \mu_v] \in \mathbf{P}^1(k_v)$ at the real places $v \in S$, we may assume, in view of the fact that $\lambda_0 \lambda_v + \mu_0 \mu_v > 0$ for real v, that $b_i(\lambda_0 - a_i \mu_0) \in (k_i \otimes_k k_v)^*$ is a norm from $(L_i \otimes_k k_v)^*$ for all $i \in \{1, \ldots, n\}$ and all $v \in S$ and that

$$(9.2) \qquad \sum_{v \in S} \sum_{w|v} \mathrm{inv}_w(E_j/k_j, \lambda_0 - a_j \mu_0) \neq 0$$

for all $j \in J$. For $i \in \{1, \ldots, n\}$, let v_i denote the unique finite place of k outside S at which $P_i(\lambda_0, \mu_0)$ is not a unit. As λ_0, μ_0 and a_i are integral outside S, as $v_i(P_i(\lambda_0, \mu_0)) = 1$ and as $P_i(\lambda_0, \mu_0) = N_{k_i/k}(\lambda_0 - a_i \mu_0)$, there exists a unique place w_i of k_i above v_i such

that $w_i(\lambda_0 - a_i\mu_0) > 0$. Moreover, we have $w_i(\lambda_0 - a_i\mu_0) = 1$ and $w(\lambda_0 - a_i\mu_0) = 0$ for any place w of k_i which does not lie above $S \cup \{v_i\}$.

Let $t_0 = \lambda_0/\mu_0$. If w is a place of k_i which does not lie above S and if $w(t_0 - a_i) > 0$, then $w(\lambda_0 - a_i\mu_0) > 0$, since $w(\mu_0) \geq 0$; hence $w = w_i$. Therefore it only remains to be shown that L_i possesses a place of degree 1 over w_i for every $i \in \{1, \ldots, n\}$.

Suppose first $i \in I$. As b_i is a unit outside S and L_i/k_i is unramified outside S, the previous paragraphs imply that $b_i(\lambda_0 - a_i\mu_0) \in (k_i)_w^*$ is a norm from $(L_i \otimes_{k_i} (k_i)_w)^*$ for every place w of k_i different from w_i. As L_i/k_i is abelian, it follows, by the reciprocity law of global class field theory, that $b_i(\lambda_0 - a_i\mu_0)$ is a local norm from $(L_i \otimes_{k_i} (k_i)_{w_i})^*$ as well (see [AT09, Ch. VII, §3, Cor. 1, p. 51]). As $w_i(b_i(\lambda_0 - a_i\mu_0)) = 1$ and L_i/k_i is Galois, we conclude that w_i must split completely in L_i.

Suppose now $i \in J$. The reciprocity law of global class field theory, together with (9.2), implies that $\mathrm{inv}_{w_i}(E_i/k_i, \lambda_0 - a_i\mu_0) \neq 0$. It follows that w_i does not split completely in E_i, so that L_i must possess a place of degree 1 over w_i. $\qquad\square$

9.2.2. *A geometric criterion and cases of small degree.* Fix k, n, and k_i/k, L_i/k_i, a_i, b_i, for $i \in \{1, \ldots, n\}$, as in the statement of Conjecture 9.1. (We do not fix S or the t_v's.) To this data we associate an irreducible quasi-affine variety W over k endowed with a smooth morphism $p : W \to \mathbf{P}^1_k$ with geometrically irreducible generic fiber, as follows.

For $i \in \{1, \ldots, n\}$, let F_i denote the singular locus of $R_{L_i/k}(\mathbf{A}^1_{L_i}) \setminus R_{L_i/k}(\mathbf{G}_{\mathrm{m},L_i})$. This is a codimension 2 closed subset of the affine space $R_{L_i/k}(\mathbf{A}^1_{L_i})$. Let W denote the fiber, above 0, of the morphism

$$\left(\mathbf{A}^2_k \setminus \{(0,0)\}\right) \times \prod_{i=1}^n \left(R_{L_i/k}(\mathbf{A}^1_{L_i}) \setminus F_i\right) \to \prod_{i=1}^n R_{k_i/k}(\mathbf{A}^1_{k_i})$$

defined by $(\lambda, \mu, x_1, \ldots, x_n) \mapsto (b_i(\lambda - a_i\mu) - N_{L_i/k_i}(x_i))_{1 \leq i \leq n}$, where λ, μ are the coordinates of $\mathbf{A}^2_k$ and x_i denotes a point of $R_{L_i/k}(\mathbf{A}^1_{L_i})$. Finally, denote by $p : W \to \mathbf{P}^1_k$ the composition of projection to the first factor with the natural map $\mathbf{A}^2_k \setminus \{(0,0)\} \to \mathbf{P}^1_k$.

Varieties closely related to W (in fact, partial compactifications of W) first appeared in the context of descent in [Sko96, §3.3], see also [CTS00, p. 391], [Sko01, §4.4]. We now state an equivalent, more geometric form of Conjecture 9.1, in terms of W. We shall use it as a criterion to prove a few cases of Conjecture 9.1 in Theorem 9.11 below.

Proposition 9.9. *For $c \in \mathbf{P}^1(k)$, let $W_c = p^{-1}(c)$. The statement of Conjecture 9.1 holds true for all S and all $(t_v)_{v \in S}$ satisfying its hypotheses if and only if $\bigcup_{c \in \mathbf{P}^1(k)} W_c(\mathbf{A}_k)$ is dense in $W(\mathbf{A}_k)$.*

Proof. Let us first verify that the second condition is sufficient for Conjecture 9.1 to hold. Let S and $(t_v)_{v \in S}$ be as in the statement of Conjecture 9.1. We assume that $\Omega_\infty \subseteq S$. For all but finitely many finite places v of k, there exists $t_v \in k_v$ such that $w(t_v - a_i) = 0$ for every $i \in \{1, \ldots, n\}$ and every place w of k_i dividing v. By Remark 9.3 (iii), we may assume, after enlarging S, that such t_v's exist for $v \notin S$; let us fix them. For each $i \in \{1, \ldots, n\}$ and each $v \in \Omega$, let us then fix $x_{i,v} \in (L_i \otimes_k k_v)^*$ such that $b_i(t_v - a_i) = N_{L_i \otimes_k k_v/k_i \otimes_k k_v}(x_{i,v})$. When $v \notin S$, we may ensure that $x_{i,v}$ is a unit at every place of L_i dividing v.

Let $Q_v \in W(k_v)$ denote the point with coordinates $(t_v, 1, x_{1,v}, \ldots, x_{n,v})$. We have defined an adelic point $(Q_v)_{v \in \Omega} \in W(\mathbf{A}_k)$. By assumption, there exist $c \in \mathbf{A}^1(k)$ and an adelic point $(Q'_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)$ arbitrarily close to $(Q_v)_{v \in \Omega}$. Let $(\lambda'_v, \mu'_v, x'_{1,v}, \ldots, x'_{n,v})$ denote the coordinates of Q'_v. By choosing $(Q'_v)_{v \in \Omega}$ close enough to $(Q_v)_{v \in \Omega}$ for the adelic topology, we may assume that $v(\mu'_v) \geq 0$ for every $v \in \Omega \setminus S$ and that the projection of Q'_v to $R_{L_i/k}(\mathbf{A}^1_{L_i}) \setminus F_i$ extends to an $\mathcal{O}_v$-point of $R_{\mathcal{O}_{L_i}/\mathcal{O}_k}(\mathbf{A}^1_{\mathcal{O}_{L_i}}) \setminus \mathscr{F}_i$ for every $i \in \{1, \ldots, n\}$ and every $v \in \Omega \setminus S$, where $\mathscr{F}_i$ denotes the Zariski closure of F_i. As we have seen in the proof of Lemma 5.2, this implies that $x'_{i,v}$ is a unit at every place of L_i dividing v except at most at one place, which must then have degree 1 over v.

We now check that the coordinate $t_0 \in k$ of c satisfies the conclusion of Conjecture 9.1. We have $t_0 = \lambda'_v/\mu'_v$, hence t_0 is arbitrarily close to t_v for $v \in S$. Fix $i \in \{1, \ldots, n\}$ and a place w of k_i whose trace v on k does not belong to S. As $v(\mu'_v) \geq 0$ and $w(b_i) = 0$, the condition $w(t_0 - a_i) > 0$ implies $w(b_i(\lambda'_v - a_i \mu'_v)) > 0$ and hence $w(N_{L_i \otimes_k k_v / k_i \otimes_k k_v}(x'_{i,v})) > 0$, so that L_i possesses a place of degree 1 over w according to the previous paragraph.

Conversely, if one assumes Conjecture 9.1, the density of $\bigcup_{c \in \mathbf{P}^1(k)} W_c(\mathbf{A}_k)$ in $W(\mathbf{A}_k)$ is an immediate consequence of Theorem 9.17 below applied with $B = 0$ and $M'' = \varnothing$, in view of the remark that the inverse image map $\mathrm{Br}(k) \to \mathrm{Br}_{\mathrm{vert}}(W/\mathbf{P}^1_k)$ is onto. (Recall that the vertical Brauer group is defined as $\mathrm{Br}_{\mathrm{vert}}(W/\mathbf{P}^1_k) = \mathrm{Br}(W) \cap p^*_\eta \mathrm{Br}(\eta)$, if η denotes the generic point of $\mathbf{P}^1_k$.) To prove this remark, we first note that the morphism $p: W \to \mathbf{P}^1_k$ is smooth and that its fibers are irreducible. Let $m_i \in \mathbf{P}^1_k$ be the closed point defined by $P_i(t) = 0$. The fibers of p over $\mathbf{P}^1_k \setminus \{m_1, \ldots, m_n\}$ are geometrically irreducible, while for each $i \in \{1, \ldots, n\}$, the algebraic closure of $k_i = k(m_i)$ in the function field of $p^{-1}(m_i)$ is L_i. These observations on the fibers of p imply that any element of $\mathrm{Br}_{\mathrm{vert}}(W/\mathbf{P}^1_k)$ can be written as $p^*_\eta \gamma$, where $\gamma \in \mathrm{Br}(\eta)$ has the shape

$$\gamma = \sum_{i=1}^n \mathrm{Cores}_{k_i(t)/k(t)}(b_i(t - a_i), \chi_i) + \delta$$

for some $\delta \in \mathrm{Br}(k)$ and some $\chi_i \in \mathrm{Ker}\big(H^1(k_i, \mathbf{Q}/\mathbf{Z}) \to H^1(L_i, \mathbf{Q}/\mathbf{Z})\big)$, $i \in \{1, \ldots, n\}$ satisfying $\sum_{i=1}^n \mathrm{Cores}_{k_i/k}(\chi_i) = 0$ (see [CTSD94, Proposition 1.1.1, Proposition 1.2.1]). Now we have $b_i(t - a_i) = b_i(\lambda - a_i\mu)/\mu$ in $k_i(W)$, and $b_i(\lambda - a_i\mu)$ is a norm from $L_i(W)$ according to the defining equations of W. Thus $(b_i(t - a_i), \chi_i) = -(\mu, \chi_i)$ in $\mathrm{Br}(k_i(W))$ and hence $p^*_\eta \gamma = -(\mu, \sum_{i=1}^n \mathrm{Cores}_{k_i/k}(\chi_i)) + \delta = \delta$, as required. $\square$

Corollary 9.10. *Suppose W satisfies strong approximation off v_0 for any finite place v_0 of k. Then the statement of Conjecture 9.1 holds true for all S and all $(t_v)_{v \in S}$ satisfying its hypotheses.*

Proof. Fix an integral model $\mathscr{W}$ of W. A glance at the definition of W shows that the map $p: \mathscr{W}(\mathcal{O}_v) \to \mathbf{P}^1(\mathcal{O}_v)$ is onto for all but finitely many of the places v of k which split completely in L_i for all i. Let $(Q_v)_{v \in \Omega} \in W(\mathbf{A}_k)$. Let S be a finite set of places of k. We must prove that there exist $c \in \mathbf{P}^1(k)$ and $(Q'_v)_{v \in \Omega} \in W_c(\mathbf{A}_k)$ with Q'_v arbitrarily close to Q_v for $v \in S$ and Q'_v integral with respect to $\mathscr{W}$ for $v \notin S$. Fix a place $v_0 \notin S$ for which $p: \mathscr{W}(\mathcal{O}_{v_0}) \to \mathbf{P}^1(\mathcal{O}_{v_0})$ is onto. By assumption, we can find a point $Q \in W(k)$

arbitrarily close to Q_v for $v \in S$ and integral with respect to $\mathscr{W}$ for $v \notin S \cup \{v_0\}$. We then let $c = p(Q)$, choose an arbitrary $Q'_{v_0} \in \mathscr{W}(\mathcal{O}_{v_0})$ such that $p(Q'_{v_0}) = c$ and let $Q'_v = Q$ for $v \in \Omega \setminus \{v_0\}$. $\qquad\square$

Thanks to Corollary 9.10, we can prove Conjecture 9.1 for non-abelian extensions L_i/k_i when $\sum_{i=1}^{n}[k_i : k]$ is small. The underlying geometric arguments run parallel to those used in the proofs of [CTS00, Theorems A and B] (which themselves originate from [Sko90a]).

Theorem 9.11. *Conjecture 9.1 holds under each of the following sets of assumptions:*

(i) $\sum_{i=1}^{n}[k_i : k] \leq 2$;
(ii) $\sum_{i=1}^{n}[k_i : k] = 3$ *and* $[L_i : k_i] = 2$ *for every* i.

Proof. When $\sum_{i=1}^{n}[k_i : k] \leq 2$, one checks that W is isomorphic to the complement of a codimension 2 closed subset in an affine space. (Specifically, when $\sum_{i=1}^{n}[k_i : k] = 2$, the projection $W \to \prod_{i=1}^{n} R_{L_i/k}(\mathbf{A}^1_{L_i})$ is an open immersion and the complement of its image has codimension 2.) As a consequence, the variety W satisfies strong approximation off any place of k (see Lemma 1.8), so that Corollary 9.10 applies. Suppose now $\sum_{i=1}^{n}[k_i : k] = 3$ and $[L_i : k_i] = 2$ for every i. In this case, just as in [CTS00, p. 392], the variety W is the punctured affine cone over the complement of a closed subset of codimension 2 in a smooth projective quadric of dimension 4. (The smooth projective quadric in question is the Zariski closure of the image of W in the projective space of lines of the 6-dimensional vector space $\prod_{i=1}^{n} R_{L_i/k}(\mathbf{A}^1_{L_i})$.) The same argument as in the proof of Lemma 1.8, with smooth projective conics replacing affine lines, shows that the complement of a codimension 2 closed subset in a smooth projective quadric satisfies strong approximation. We may therefore apply [CTX13, Proposition 3.1] and deduce that W satisfies strong approximation off any place of k. Applying Corollary 9.10 concludes the proof. $\qquad\square$

Remark 9.12. In hindsight, one may reinterpret the arguments of §5.4 as follows. Let $\mathbf{A}(V)$ denote the affine space whose underlying vector space is $V = H^0(C, \mathscr{O}_C(c_1))$ and let $\mathbf{P}(V)$ denote the projective space of lines in $\mathbf{A}(V)$. Let W be the fiber, above 0, of the morphism

$$(\mathbf{A}(V) \setminus \{0\}) \times \prod_{m \in M} \left(R_{L_m/k}(\mathbf{A}^1_{L_m}) \setminus F_m \right) \to \prod_{m \in M} R_{k(m)/k}(\mathbf{A}^1_{k(m)})$$

defined by $(h, (x_m)_{m \in M}) \mapsto (h(m) - N_{L_m/k(m)}(x_m))_{m \in M}$, where F_m stands for the singular locus of $R_{L_m/k}(\mathbf{A}^1_{L_m}) \setminus R_{L_m/k}(\mathbf{G}_{m,L_m})$. Let $p : W \to \mathbf{P}(V)$ denote the map induced by the first projection. Let $w_\mathbf{A} \in W(\mathbf{A}_k)$ be an adelic point whose image in $(\mathbf{A}(V) \setminus \{0\})(k_v)$, for each $v \in S$, is the function h_v of §5.4. The choice of a splitting of (5.1) determines an isomorphism between W and the complement of a codimension 2 closed subset in an affine space; therefore W satisfies strong approximation off any given place. In particular, by the same argument as in Corollary 9.10, one can approximate $w_\mathbf{A}$ by an adelic point of $p^{-1}(c)$ for some rational point c of $\mathbf{P}(V)$. This c, viewed as an effective divisor on C, is the divisor constructed in §5.4.

9.2.3. *Invariance under change of coordinates.* As a test for Conjectures 9.1 and 9.2, it is natural to ask whether they are compatible with linear changes of coordinates on $\mathbf{P}^1_k$. The following lemma expresses such an invariance property. It will be used in the proofs of Theorems 9.14 and 9.15 below. Its statement should be read as follows: given any $\alpha, \beta, \gamma, \delta$ which satisfy (i)–(iii), Conjectures 9.1 and 9.2 for k, n, k_i, L_i, a_i, b_i, S and $(t_v)_{v \in S}$ reduce to the same conjectures for k, n, k_i, L_i, a'_i, b'_i, S' and $(t'_v)_{v \in S'}$.

Lemma 9.13. *Let k, n, k_i, L_i, a_i, b_i, S and $(t_v)_{v \in S}$ satisfy the hypotheses of Conjecture 9.1. Let $\alpha, \beta, \gamma, \delta \in k$ be such that $\alpha\delta - \beta\gamma \neq 0$. We make the following assumptions:*

(i) $\gamma a_i + \delta \neq 0$ *for all $i \in \{1, \ldots, n\}$ and $\gamma t_v + \delta \neq 0$ for all $v \in S$;*

(ii) α *and $\alpha a_i + \beta$ for $i \in \{1, \ldots, n\}$ are units outside of S;*

(iii) *for every $i \in \{1, \ldots, n\}$ and every $v \in S$, there exists $y_{i,v} \in (L_i \otimes_k k_v)^*$ such that*
$$(\alpha\delta - \beta\gamma)/(\gamma t_v + \delta) = N_{L_i \otimes_k k_v / k_i \otimes_k k_v}(y_{i,v}).$$

Set $a'_i = (\alpha a_i + \beta)/(\gamma a_i + \delta)$ and $b'_i = b_i(\gamma a_i + \delta)$ for $i \in \{1, \ldots, n\}$. Let S'_0 be the union of S with the set of finite places of k above which one of α, γ, $a'_1, \ldots, a'_n$ is not integral or one of $b'_1, \ldots, b'_n$ is not a unit. Let S' be a finite set of places of k containing S'_0. Set $t'_v = 0$ for $v \in S' \setminus S$ and $t'_v = (\alpha t_v + \beta)/(\gamma t_v + \delta)$ for $v \in S$. Finally, when given $t'_0 \in k$ such that $\alpha \neq \gamma t'_0$, we denote by $t_0 \in k$ the unique solution to the equation $t'_0 = (\alpha t_0 + \beta)/(\gamma t_0 + \delta)$. We then have:

(1) *for every $i \in \{1, \ldots, n\}$ and every $v \in S'$, there exists $x'_{i,v} \in (L_i \otimes_k k_v)^*$ such that*
$$b'_i(t'_v - a'_i) = N_{L_i \otimes_k k_v / k_i \otimes_k k_v}(x'_{i,v})$$
in $k_i \otimes_k k_v$;

(2) *for $v \in S$, if t'_0 is arbitrarily close to t'_v, then t_0 is arbitrarily close to t_v;*

(3) *for $v \in S$, if i denotes an element of $\{1, \ldots, n\}$ and w denotes a real place of k_i dividing v, the sign of $(t'_0 - a'_i)(t'_v - a'_i)(t_0 - a_i)(t_v - a_i)$ at w is independent of w and of i;*

(4) *if t'_0 is arbitrarily close to t'_v for $v \in S' \setminus S$, then for every $i \in \{1, \ldots, n\}$ and every finite place w of k_i such that $w(t_0 - a_i) > 0$, either w lies above a place of S, or w does not lie above a place of S' and $w(t'_0 - a'_i) > 0$.*

Proof. It is straightforward to check (2) and (3), as well as (1) for $v \in S$ by setting $x'_{i,v} = y_{i,v}x_{i,v}$. To check (1) for $v \in S' \setminus S$, we note that above such v, the element $b'_i(t'_v - a'_i) = -a'_i b'_i = -(\alpha a_i + \beta)b_i$ is a unit, by (ii), and is therefore a norm from $L_i \otimes_k k_v$ since L_i/k_i is unramified above v. We now turn to (4) and fix i and a finite place w of k_i such that $w(t_0 - a_i) > 0$. It follows from (ii) that w cannot lie above a place of $S' \setminus S$, as $t_0 - a_i$ would then have to be a unit at w, being arbitrarily close to $-\beta/\alpha - a_i = -(\alpha a_i + \beta)/\alpha$. Thus we may assume that w lies above a place of S', so that $w(b_i) = w(b'_i) = 0$ and $w(\alpha) \geq 0$, $w(\gamma) \geq 0$, $w(a'_i) \geq 0$. A simple computation shows that
$$b'_i(t'_0 - a'_i) = b_i(t_0 - a_i)(\alpha - \gamma t'_0).$$

As $w(t_0 - a_i) > 0$, we deduce that if $w(t'_0 - a'_i) \leq 0$, then $w(\alpha - \gamma t'_0) < 0$, which implies that $w(t'_0 - a'_i) = w(t'_0) \leq w(\gamma t'_0) = w(\alpha - \gamma t'_0)$, a contradiction. $\square$

9.2.4. *Additive combinatorics.* The most significant result towards Conjecture 9.1 is the following theorem, whose proof relies on the methods initiated by Green and Tao.

Theorem 9.14 (Matthiesen [Mat14])**.** *Conjecture 9.1 holds when $k = k_1 = \cdots = k_n = \mathbf{Q}$.*

Proof. By Remarks 9.3 (ii) and (iii), we may assume that a_i is a unit outside S for all i. For $N \in \mathbf{N}$, let $(\alpha, \beta, \gamma, \delta) = (r^{2N}, 0, 0, r^N)$, where r denotes the product of the primes belonging to S. Letting N be a large enough multiple of $\prod_{i=1}^{n}[L_i : \mathbf{Q}]$, we may assume, by Lemma 9.13, that $a_i, b_i \in \mathbf{Z}$ for all $i \in \{1, \ldots, n\}$, that $v(t_v) \geq 0$ for every prime $v \in S$, that $b_i(t_v - a_i)$ is the norm of an integral element of $L_i \otimes_{\mathbf{Q}} \mathbf{Q}_v$ for every $i \in \{1, \ldots, n\}$ and every prime $v \in S$ and that S is large enough for the main theorem of [Mat14] to be applicable. Let $t_2 \in \mathbf{Q}$ be arbitrarily close to t_v for $v \in S$. We write $t_2 = \lambda_2/\mu_2$ with $\lambda_2, \mu_2 \in \mathbf{Z}$. Using strong approximation, we choose $\alpha \in \mathbf{Q}$ arbitrarily close to $1/\mu_2$ at the finite places of S, integral outside S and such that $\alpha\mu_2 > 0$.

The pair $(\lambda_1, \mu_1) = (\alpha\lambda_2, \alpha\mu_2)$ belongs to $\mathbf{Z}^2$. It lies arbitrarily close to $(t_v, 1)$ at the finite places of S. Moreover λ_1/μ_1 is arbitrarily close to t_v at the real place v of $\mathbf{Q}$, and $b_i(\lambda_1 - a_i\mu_1)$ is a local integral norm from L_i at the finite places of S and is a local norm from $L_i \otimes_{\mathbf{Q}} \mathbf{R}$, for all i. Matthiesen's theorem [Mat14] applied to the linear forms $f_i(\lambda, \mu) = b_i(\lambda - a_i\mu) \in \mathbf{Z}[\lambda, \mu]$ for $i \in \{1, \ldots, n\}$ and to the vector $(\lambda_1, \mu_1) \in \mathbf{Z}^2$ therefore produces a pair $(\lambda_0, \mu_0) \in \mathbf{Z}^2$ arbitrarily close to (λ_1, μ_1) at the finite places of S, such that λ_0/μ_0 is arbitrarily close to λ_1/μ_1 at the real place and such that $b_i(\lambda_0 - a_i\mu_0)$ is squarefree outside S and is the norm of an integral element of L_i, for all i. The rational number $t_0 = \lambda_0/\mu_0$ then satisfies all of the desired conclusions. Indeed, if v is a prime not in S such that $v(t_0 - a_i) > 0$ for some i, then, as $v(b_i) = 0$ and $v(\mu_0) \geq 0$, we must have $v(b_i(\lambda_0 - a_i\mu_0)) = 1$, which implies that L_i possesses a place of degree 1 over v since $b_i(\lambda_0 - a_i\mu_0)$ is a local integral norm from L_i at v and $L_i/\mathbf{Q}$ is unramified at v. $\qquad\square$

9.2.5. *Sieve methods.* Reformulating the work of Irving [Irv14], which rests on sieve methods, yields the first known case of Conjecture 9.2 for which $\sum_{i=1}^{n}[k_i : k] \geq 4$ and $k_i \neq k$ for some i (here the extensions L_i/k_i are once again almost abelian).

Theorem 9.15. *Let $K/\mathbf{Q}$ be a cubic extension. Let $q \geq 7$ be a prime number. If K has a unique real place and does not embed into the cyclotomic field $\mathbf{Q}(\zeta_q)$, then Conjecture 9.2 holds for $n = 2$, $k = \mathbf{Q}$, $k_1 = K$, $L_1 = K(2^{1/q})$, $k_2 = \mathbf{Q}$ and $L_2 = \mathbf{Q}(2^{1/q})$.*

Proof. Let a_1, a_2, b_1, b_2, S and $(t_v)_{v \in S}$ satisfy the hypotheses of Conjecture 9.2. When v is the real place of $\mathbf{Q}$, we also denote t_v by t_∞.

Lemma 9.16. *By a change of coordinates on $\mathbf{P}_k^1$, we may assume, in order to prove Theorem 9.15, that*

$$(9.3) \qquad\qquad b_2 t_\infty < b_2 a_1 < b_2 a_2$$

at any real place of K.

Proof. We choose $\beta \in \mathbf{Q}$ such that $b_2(a_2 + \beta) > 0$. After enlarging S, we may assume, by Remark 9.3 (iii), that $a_1 + \beta$ and $a_2 + \beta$ are units outside S. Then, using weak approximation, we let $\delta \in \mathbf{Q}$ be arbitrarily large at the finite places of S, arbitrarily

close to (but distinct from) a_2 at the real place of $\mathbf{Q}$, and such that $\delta - a_2$ has the same sign as $(t_\infty - a_1)(t_\infty - a_2)(a_2 - a_1)$ at the unique real place of K. By choosing δ large enough at the finite places of S, we may assume that the image of $(\delta + \beta)/(\delta - t_v)$ in $k_i \otimes_\mathbf{Q} \mathbf{Q}_v$ is a norm from $(L_i \otimes_\mathbf{Q} \mathbf{Q}_v)^*$ for all i and all $v \in S$ (at the real place, this condition is empty). Setting $\alpha = 1$ and $\gamma = -1$, all of the hypotheses of Lemma 9.13 are now satisfied. In the notation of that lemma, it is straightforward to check that $b_2'(t_\infty' - a_1') < 0 < b_2'(a_2' - a_1')$ at the real place of K. We may therefore replace a_i, b_i, S and $(t_v)_{v \in S}$ with a_i', b_i', S' and $(t_v')_{v \in S'}$ and assume that (9.3) holds. $\qquad\square$

Let c be a nonzero integer such that the binary cubic form

$$(9.4) \qquad f(x, y) = c^q N_{K/\mathbf{Q}}\left(b_1\left((a_2 - a_1)x + \frac{1}{b_2}y\right)\right)$$

has integral coefficients and the coefficient of x^3 is positive. By Remark 9.3 (iii), we may assume, after enlarging S, that a_1 is integral outside S, that the coefficients of x^3 and of y^3 and the discriminant of the polynomial $f(x, 1)$ are units outside S and that S contains all of the primes less than or equal to the constant denoted by P_1 in [Irv14] (see the end of §2).

Let $\tau \in \mathbf{Q}$ be arbitrarily close to t_v at the finite places of S. We choose integers λ, μ such that

$$(9.5) \qquad \frac{\mu}{\lambda} = b_2(\tau - a_2)$$

and such that λ is a q-th power. We then have

$$(9.6) \qquad \frac{f(\lambda, \mu)}{\lambda^3} = c^q N_{K/\mathbf{Q}}(b_1(\tau - a_1)).$$

The hypothesis of Conjecture 9.2 implies that $b_i(\tau - a_i)$ is a local norm in L_i/k_i at the places of S for $i \in \{1, 2\}$. As λ is a q-th power, it follows, in view of (9.5) and (9.6), that μ and $f(\lambda, \mu)$ are local norms in $\mathbf{Q}(2^{1/q})/\mathbf{Q}$ at the places of S. Note, furthermore, that the real roots of the polynomial $f(x, 1)$ are negative, by the assumption made in Lemma 9.16. We are therefore in the situation considered in [Irv14, §3–6] (the integers a_0, b_0 of *loc. cit.* being our λ, μ). Irving proves in [Irv14] the existence of positive integers x_0, y_0 with (x_0, y_0) arbitrarily close to (λ, μ) at the finite places of S, such that $y_0 f(x_0, y_0)$ has no prime factor $p \notin S$ with $p \equiv 1 \bmod q$. In particular 2 is a q-th power in $\mathbf{F}_p$ for any prime factor $p \notin S$ of $y_0 f(x_0, y_0)$; hence, for any such p and any place w of K dividing p, the number field L_1 (resp. L_2) possesses a place of degree 1 over w (resp. over p).

The unique solution $t_0 \in \mathbf{Q}$ to the equation

$$(9.7) \qquad \frac{y_0}{x_0} = b_2(t_0 - a_2)$$

is arbitrarily close to t_v for $v \in S \cap \Omega_f$ and satisfies

$$(9.8) \qquad \frac{f(x_0, y_0)}{x_0^3} = c^q N_{K/\mathbf{Q}}(b_1(t_0 - a_1)).$$

As x_0 and y_0 are positive, we deduce from (9.3) and (9.7) that $b_2 t_\infty < b_2 a_1 < b_2 a_2 < b_2 t_0$. Assertion (1') of Conjecture 9.2 follows. For any finite place $v \notin S$ such that $v(t_0 - a_2) > 0$, we have $v(y_0) > 0$, so that L_2 possesses a place of degree 1 over v. Let now w be a finite place of K above a place $v \notin S$ of $\mathbf{Q}$, such that $w(t_0 - a_1) > 0$. We remark that t_0 must be integral at p since a_1 is integral above p. As a consequence $t_0 - a_1$ is integral at every place of K dividing v. It follows that $v(N_{K/\mathbf{Q}}(b_1(t_0 - a_1))) > 0$ and hence $v(f(x_0, y_0)) > 0$, so that L_1 possesses a place of degree 1 over w according to the previous paragraph. $\square$

9.3. **Consequences for rational points in fibrations.** All of the results on rational points contained in this paper will be deduced from the following theorem. We adopt Poonen's notation $V(\mathbf{A}_k)_\bullet = \prod_{v \in \Omega_f} V(k_v) \times \prod_{v \in \Omega_\infty} \pi_0(V(k_v))$ for any proper variety V over a number field k (see [Sto07, §2]).

Theorem 9.17. *Let X be a smooth, irreducible variety over a number field k, endowed with a morphism $f : X \to \mathbf{P}^1_k$ with geometrically irreducible generic fiber. Assume that every fiber of f contains an irreducible component of multiplicity 1. Let $U \subseteq \mathbf{P}^1_k$ be a dense open subset over which the fibers of f are split, with $\infty \in U$. Let $B \subseteq \operatorname{Br}(f^{-1}(U))$ be a finite subgroup. Let $(x_v)_{v \in \Omega} \in X(\mathbf{A}_k)$ be orthogonal to $(B + f_\eta^* \operatorname{Br}(\eta)) \cap \operatorname{Br}(X)$ with respect to the Brauer–Manin pairing.*

Let M' be a nonempty subset of $\mathbf{P}^1_k \setminus U$ containing all of the points above which the fiber of f is not split, large enough that for each $m \in M'' = (\mathbf{P}^1_k \setminus U) \setminus M'$, the map

$$(9.9) \qquad \operatorname{Br}\left(\mathbf{P}^1_k \setminus (M' \cup \{m\})\right) \longrightarrow H^1(k(m), \mathbf{Q}/\mathbf{Z})$$

which sends a class to its residue at m is surjective. Let $P_1, \ldots, P_n \in k[t]$ denote the irreducible monic polynomials which vanish at the points of M'. Assume Conjecture 9.1 for $P_1, \ldots, P_n$ (resp. assume Conjecture 9.2 for $P_1, \ldots, P_n$ and assume that either $M'' = \varnothing$ and f is proper or k is totally imaginary).

Then there exist $c \in U(k)$ and $(x'_v)_{v \in \Omega} \in X_c(\mathbf{A}_k)$ such that X_c is smooth and $(x'_v)_{v \in \Omega}$ is orthogonal to B with respect to the Brauer–Manin pairing and is arbitrarily close to $(x_v)_{v \in \Omega}$ in $X(\mathbf{A}_k)$ (resp. in $X(\mathbf{A}_k)_\bullet$).

Remarks 9.18. (i) The reader who is willing to assume Conjectures 9.1 and 9.2 for the whole collection of polynomials vanishing on the points of $\mathbf{P}^1_k \setminus U$ may take $M' = \mathbf{P}^1_k \setminus U$. This makes the hypothesis on (9.9) trivially satisfied and also leads to simplifications in the proof of Theorem 9.17.

(ii) The map (9.9) is surjective for any m as soon as M' contains a rational point or k is totally imaginary. This follows from the Faddeev exact sequence (see [CTSD94, §1.2]), in view of the surjectivity of the corestriction map $H^1(k', \mathbf{Q}/\mathbf{Z}) \to H^1(k, \mathbf{Q}/\mathbf{Z})$ for any finite extension k'/k of totally imaginary number fields (see [Gra03, Note, p. 327] and note that $H^1(k, \mathbf{Q}/\mathbf{Z})$ is Pontrjagin dual to the group denoted C_k/D_k in *loc. cit.*).

(iii) Write $M' = \{m_1, \ldots, m_n\}$ and $k_i = k(m_i)$. When $B = 0$, the finite extension L_i/k_i to which Conjectures 9.1 and 9.2 are applied in the proof below can be taken to be the algebraic closure of k_i in the function field of an irreducible component of multiplicity 1 of $f^{-1}(m_i)$, for every $i \in \{1, \ldots, n\}$.

Proof of Theorem 9.17. Let us write $M' = \{m_1, \ldots, m_n\}$ and $M'' = \{m_{n+1}, \ldots, m_N\}$, with $N = n$ when $M'' = \varnothing$. For each $i \in \{1, \ldots, N\}$, let $k_i = k(m_i)$ and $X_i = f^{-1}(m_i)$, choose an irreducible component $Y_i \subseteq X_i$ of multiplicity 1, with the requirement that Y_i should be geometrically irreducible over k_i whenever $i > n$, and choose a finite abelian extension $E_i/k(Y_i)$ such that the residue of any element of B at the generic point of Y_i belongs to the kernel of the restriction map $H^1(k(Y_i), \mathbf{Q}/\mathbf{Z}) \to H^1(E_i, \mathbf{Q}/\mathbf{Z})$.

Thanks to our assumption on (9.9), we can choose, for each $i \in \{n+1, \ldots, N\}$, a finite subgroup $\Gamma_i \subset \mathrm{Br}\left(\mathbf{P}_k^1 \setminus (M' \cup \{m_i\})\right)$ which surjects, by the residue map at m_i, onto the kernel of the restriction map $H^1(k_i, \mathbf{Q}/\mathbf{Z}) \to H^1(E_i, \mathbf{Q}/\mathbf{Z})$. Put $\Gamma = \sum_{i=n+1}^N \Gamma_i \subset \mathrm{Br}(U)$.

For $i \in \{1, \ldots, N\}$, let K_i be the algebraic closure of k_i in E_i. For $i \in \{n+1, \ldots, N\}$, set $L_i = K_i' = K_i$ and note that this is an abelian extension of k_i as k_i is algebraically closed in $k(Y_i)$ for such values of i. For $i \in \{1, \ldots, n\}$, fix a finite abelian extension K_i' of k_i such that the residue at m_i of any element of Γ belongs to the kernel of the restriction map $H^1(k_i, \mathbf{Q}/\mathbf{Z}) \to H^1(K_i', \mathbf{Q}/\mathbf{Z})$ and let L_i/k_i be a compositum of the extensions K_i/k_i and K_i'/k_i. Finally, choose a finite extension L_∞/k such that the fields $L_1, \ldots, L_N$ embed k-linearly into L_∞.

Let $C^0 = U \setminus \{\infty\}$. Let $X^0 = f^{-1}(C^0)$. Let $\mathrm{Pic}_+(\mathbf{P}_k^1)$ and $\mathrm{Br}_+(\mathbf{P}_k^1)$ denote the groups associated in Definition 2.1 to the curve $\mathbf{P}_k^1$, to the finite set $M = \{m_1, \ldots, m_N, \infty\} \subset \mathbf{P}_k^1$ and to the finite extensions $L_1/k_1, \ldots, L_N/k_N, L_\infty/k$.

By Remark 2.2 (ii), the subgroup $B + f^*\mathrm{Br}_+(\mathbf{P}_k^1) \subseteq \mathrm{Br}(X^0)$ is finite modulo the inverse image of $\mathrm{Br}(k)$. Thus, by Harari's formal lemma, there exists $(x_v'')_{v \in \Omega} \in X^0(\mathbf{A}_k)$ orthogonal to $B + f^*\mathrm{Br}_+(\mathbf{P}_k^1)$ with respect to (1.5) and arbitrarily close to $(x_v)_{v \in \Omega}$ in $X(\mathbf{A}_k)$ (see [CT03, Théorème 1.4]). By the inverse function theorem, we may assume that x_v'' belongs to a smooth fiber of f for each v. As $(x_v'')_{v \in \Omega}$ is orthogonal to $f^*\mathrm{Br}_+(\mathbf{P}_k^1)$, the class of $(f(x_v''))_{v \in \Omega}$ in $\mathrm{Pic}_{+,\mathbf{A}}(\mathbf{P}_k^1)$ is orthogonal to $\mathrm{Br}_+(\mathbf{P}_k^1)$ with respect to the pairing (2.4). By Theorem 2.5, we deduce that there exists a divisor $c_1 \in \mathrm{Div}(C^0)$, necessarily of degree 1, whose class in $\mathrm{Pic}_{+,\mathbf{A}}(\mathbf{P}_k^1)$ coincides with that of $(f(x_v''))_{v \in \Omega}$. As $\mathrm{Pic}_+(\mathbf{P}_\mathbf{C}^1) = \mathrm{Pic}(\mathbf{P}_\mathbf{C}^1) = \mathbf{Z}$ and $\deg(c_1) = \deg(f(x_v''))$, the classes of c_1 and of $f(x_v'')$ in $\mathrm{Pic}_+(\mathbf{P}_{k_v}^1)$ must in fact be equal for all $v \in \Omega$, finite or infinite.

Let us write $\mathbf{A}_k^1 = \mathrm{Spec}(k[t])$. For $v \in \Omega$, let $t_v \in k_v$ denote the value of t at $f(x_v'') \in \mathbf{A}_{k_v}^1$. For $i \in \{1, \ldots, N\}$, let $a_i \in k_i$ denote the value of t at $m_i \in \mathbf{A}_k^1$. Evaluating the invertible function $t - a_i \in \mathbf{G}_\mathrm{m}(C^0 \otimes_k k_i)$ along the divisor $-c_1 \otimes_k k_i \in \mathrm{Div}(C^0 \otimes_k k_i)$ yields an element of k_i^* which we denote b_i (see [Ser75, Ch. III, §1.1]).

Lemma 9.19. *For every $i \in \{1, \ldots, N\}$ and every $v \in \Omega$, there exists $x_{i,v} \in (L_i \otimes_k k_v)^*$ such that the equality $b_i(t_v - a_i) = N_{L_i \otimes_k k_v / k_i \otimes_k k_v}(x_{i,v})$ holds in $k_i \otimes_k k_v$.*

Proof. For any i, any v and any place w of k_i dividing v, we must show that $b_i(t_v - a_i)$, as an element of $(k_i)_w$, is a norm from $L_i \otimes_{k_i} (k_i)_w$. Let us fix i, v and w. As the classes of c_1 and of $f(x_v'')$ in $\mathrm{Pic}_+(\mathbf{P}_{k_v}^1)$ are equal, there exists a rational function $h_v \in k_v(t)^*$ such that $\mathrm{div}(h_v) = f(x_v'') - c_1$ and such that the value of h_v at any $(k_i)_w$-point of $m_i \cup \{\infty\}$ is a norm from $L_i \otimes_{k_i} (k_i)_w$ (recall that L_i embeds into L_∞). Thus, evaluating h_v along the divisor of

the rational function $t - a_i$ on $\mathbf{P}^1_{(k_i)_w}$ yields an element of $N_{L_i \otimes_{k_i} (k_i)_w / (k_i)_w}((L_i \otimes_{k_i} (k_i)_w)^*)$. By Weil's reciprocity law (see [Ser75, Ch. III, §1.4, Proposition 7]), the lemma follows. □

Let S be a finite set of places of k containing the infinite places of k, the places at which we want to approximate $(x_v)_{v \in \Omega}$ and the finite places above which, for some i, either b_i is not a unit or L_i/k_i is ramified. We choose S large enough that the order of B is invertible in $\mathcal{O}_S$, that $\beta(x''_v) = 0$ for all $v \in \Omega \setminus S$ and all $\beta \in B + f^*\Gamma$, that X extends to a smooth scheme $\mathscr{X}$ over $\mathcal{O}_S$ and that $f : X \to \mathbf{P}^1_k$ extends to a flat morphism $f : \mathscr{X} \to \mathbf{P}^1_{\mathcal{O}_S}$. For $m \in \mathbf{P}^1_k$, let $\widetilde{m}$ denote the Zariski closure of m in $\mathbf{P}^1_{\mathcal{O}_S}$. For each i, let $\mathscr{X}_i$ and $\mathscr{Y}_i$ denote the Zariski closures of X_i and Y_i in $\mathscr{X}$, endowed with the reduced scheme structures, let $\mathscr{Y}_i^0 \subseteq \mathscr{Y}_i$ be a dense open subset and let $\mathscr{E}_i$ denote the normalisation of $\mathscr{Y}_i^0$ in the finite extension $E_i/k(Y_i)$. By shrinking $\mathscr{Y}_i^0$, we may assume that $\mathscr{E}_i$ is finite and étale over $\mathscr{Y}_i^0$ and that $\mathscr{Y}_i^0$ is smooth over $\mathcal{O}_S$. Let $\mathscr{U} = \mathbf{P}^1_{\mathcal{O}_S} \setminus (\bigcup_{i=1}^N \widetilde{m}_i)$. Finally, for $i \in \{n+1, \ldots, N\}$, let $G_i = \mathrm{Gal}(E_i/k(Y_i))$ and $H_i = \mathrm{Gal}(E_i/k(Y_i)K_i) \subseteq G_i$.

After enlarging S, we may assume that $B \subseteq \mathrm{Br}(f^{-1}(\mathscr{U}))$, that $\Gamma \subseteq \mathrm{Br}(\mathscr{U})$, that the Zariski closure $\widetilde{M}$ of M in $\mathbf{P}^1_{\mathcal{O}_S}$ is étale over $\mathcal{O}_S$ and, by the Lang–Weil–Nisnevich bounds [LW54] [Nis54] and by a geometric version of Chebotarev's density theorem [Eke90, Lemma 1.2], that the following statements hold:

- the fiber of f above any closed point of $\mathscr{U}$ contains a smooth rational point;
- for any $i \in \{1, \ldots, n\}$ and any place w of k_i which does not lie above a place of S, if L_i possesses a place of degree 1 over w then the fiber of $\mathscr{E}_i \to \widetilde{m}_i$ above the closed point corresponding to w contains a rational point;
- for any $i \in \{n+1, \ldots, N\}$, the closed fibers of $\mathscr{Y}_i^0 \to \widetilde{m}_i$ contain rational points;
- for any $i \in \{n+1, \ldots, N\}$ and any place w of k_i which splits completely in L_i and which does not lie above a place of S, any element of H_i can be realised as the Frobenius automorphism of the irreducible abelian étale cover $\mathscr{E}_i \to \mathscr{Y}_i^0$ at some rational point of the fiber of $\mathscr{Y}_i^0 \to \widetilde{m}_i$ above the closed point corresponding to w.

Using Chebotarev's density theorem, we fix pairwise distinct places $v_{n+1}, \ldots, v_N \in \Omega \setminus S$ such that v_i splits completely in L_i for each i. For each $i \in \{n+1, \ldots, N\}$, we also fix a place w_i of k_i lying over v_i and an element $t_{v_i} \in k_{v_i}$ such that $w_i(t_{v_i} - a_i) = 1$.

Thanks to Lemma 9.19, we may apply Conjecture 9.1 (resp. Conjecture 9.2) to the polynomials $P_1, \ldots, P_n$, to the finite extensions L_i/k_i for $i \in \{1, \ldots, n\}$ and to the set of places $S' = S \cup \{v_{n+1}, \ldots, v_N\}$: there exists $t_0 \in k$ satisfying conditions (1) and (2) (resp. (1') and (2)) of §9.1, arbitrarily close to t_{v_i} for $i \in \{n+1, \ldots, N\}$.

Let $c \in \mathbf{P}^1(k)$ denote the point with coordinate t_0. As at least one x''_v belongs to a smooth fiber of f above U, we may assume that $c \in U(k)$ and that X_c is smooth. By the inverse function theorem, for each $v \in S$ (resp. $v \in S \cap \Omega_f$), we can choose $x'_v \in X_c(k_v)$ arbitrarily close to x''_v. In the case in which only Conjecture 9.2 is assumed to hold for $P_1, \ldots, P_n$, let us define x'_v for $v \in \Omega_\infty$. If v is complex, we let x'_v be an arbitrary point of $X_c(k_v)$. If v is real, then k is not totally imaginary; recall that we have assumed, in this case, that $M'' = \varnothing$ and that f is proper. As $M'' = \varnothing$, condition (1') of Conjecture 9.2 implies that c and $f(x''_v)$ belong to the same connected component of $U(k_v)$. As f is flat and

proper, the map $f^{-1}(U)(k_v) \to U(k_v)$ induced by f is open and closed, so that it maps any connected component of $f^{-1}(U)(k_v)$ onto a connected component of $U(k_v)$. Thus, we can choose $x'_v \in X_c(k_v)$ such that x''_v and x'_v belong to the same connected component of $f^{-1}(U)(k_v)$.

Let us now construct x'_v for $v \in \Omega \setminus S$.

For $v \in \Omega \setminus S$, let $w \in \mathbf{P}^1_{\mathcal{O}_S}$ denote the closed point $w = \widetilde{c} \cap \mathbf{P}^1_{\mathbf{F}_v}$. For $i \in \{1, \ldots, N\}$, we define Ω_i to be the set of places $v \in \Omega \setminus S$ such that $w \in \widetilde{m}_i$. The sets Ω_i are finite and pairwise disjoint. When $v \in \Omega_i$, we may view w as a place of k_i dividing v; we then have $w(t_0 - a_i) > 0$. For $i > n$, we may assume that $w_i(t_0 - a_i) = 1$, and hence that $v_i \in \Omega_i$, by choosing t_0 close enough to t_{v_i}.

For each $v \in \Omega \setminus S$ which does not belong to any Ω_i, we use Hensel's lemma to lift an arbitrary smooth rational point of $f^{-1}(w)$ to a k_v-point x'_v of X_c. For each $i \in \{1, \ldots, n\}$ and each $v \in \Omega_i$, the field L_i possesses a place of degree 1 over w, by condition (2) of Conjecture 9.1. Therefore the fiber of $\mathscr{E}_i \to \widetilde{m}_i$ above w contains a rational point. We fix any such rational point, let $\xi_{w,i}$ denote its image in $\mathscr{Y}^0_i$ and use Hensel's lemma to lift $\xi_{w,i}$ to a k_v-point x'_v of X_c. For each $i \in \{n+1, \ldots, N\}$ and each $v \in \Omega_i \setminus \{v_i\}$, we fix an arbitrary rational point $\xi_{w,i}$ of the fiber of $\mathscr{Y}^0_i \to \widetilde{m}_i$ above w and again lift it to a k_v-point x'_v of X_c.

We have now defined x'_v for all $v \in \Omega \setminus \{v_{n+1}, \ldots, v_N\}$. Let us finally construct x'_v at the remaining places. For $i \in \{1, \ldots, N\}$ and $v \in \Omega_i$, let $n_{w,i} = w(t_0 - a_i)$. For $i \in \{n+1, \ldots, N\}$, let $\sigma_i = \sum_{v \in \Omega_i \setminus \{v_i\}} n_{w,i} \mathrm{Fr}_{\xi_{w,i}} \in G_i$, where $\mathrm{Fr}_{\xi_{w,i}}$ denotes the Frobenius automorphism of the irreducible abelian étale cover $\mathscr{E}_i \to \mathscr{Y}^0_i$ at $\xi_{w,i}$.

Lemma 9.20. *For each $i \in \{n+1, \ldots, N\}$, we have $\sigma_i \in H_i$.*

Proof. Let us fix $i \in \{n+1, \ldots, N\}$. Let γ denote an element of Γ_i. As c is arbitrarily close to $f(x''_v)$ for $v \in S \cap \Omega_f$, as c and $f(x''_v)$ belong to the same connected component of $U(k_v)$ for $v \in \Omega_\infty$, as $\sum_{v \in \Omega} \mathrm{inv}_v \gamma(f(x''_v)) = 0$ and as $\gamma(f(x''_v)) = 0$ for $v \in \Omega \setminus S$, we have $\sum_{v \in S} \mathrm{inv}_v \gamma(c) = 0$. As $\gamma \in \mathrm{Br}(\mathscr{U})$ and $\mathrm{Br}(\mathcal{O}_v) = 0$ for any finite place v, we have $\mathrm{inv}_v \gamma(c) = 0$ for $v \in \Omega \setminus (S \cup \Omega_1 \cup \cdots \cup \Omega_N)$. For $j \in \{1, \ldots, N\}$ and $v \in \Omega_j$, we have

$$(9.10) \qquad \mathrm{inv}_v \gamma(c) = n_{w,j} \partial_{\gamma, m_j}(\mathrm{Fr}_w),$$

where $\partial_{\gamma, m_j} \in \mathrm{Ker}(H^1(k_j, \mathbf{Q}/\mathbf{Z}) \to H^1(K'_j, \mathbf{Q}/\mathbf{Z})) = \mathrm{Hom}(\mathrm{Gal}(K'_j/k_j), \mathbf{Q}/\mathbf{Z})$ is the residue of γ at m_j and $\mathrm{Fr}_w \in \mathrm{Gal}(K'_j/k_j)$ denotes the Frobenius at w, since $\gamma \in \mathrm{Br}(\mathscr{U})$ (see [Har94, Corollaire 2.4.3]). By the global reciprocity law, we deduce from these remarks that

$$(9.11) \qquad \sum_{j=1}^{N} \sum_{v \in \Omega_j} n_{w,j} \partial_{\gamma, m_j}(\mathrm{Fr}_w) = 0.$$

We have $\mathrm{Fr}_w = 0$ when $j \leq n$ and when $v = v_j$ since L_j possesses a place of degree 1 over w in these two cases. Moreover, by the definition of Γ_i, we have $\partial_{\gamma, m_j} = 0$ if $j > n$ and $j \neq i$. All in all, we conclude that

$$(9.12) \qquad \sum_{v \in \Omega_i \setminus \{v_i\}} n_{w,i} \partial_{\gamma, m_i}(\mathrm{Fr}_w) = 0,$$

or in other words $\partial_{\gamma, m_i}(\bar{\sigma}_i) = 0$ if $\bar{\sigma}_i$ denotes the image of σ_i in $G_i/H_i = \mathrm{Gal}(K_i/k_i)$. As ∂_{γ, m_i} takes all possible values in $\mathrm{Hom}(\mathrm{Gal}(K_i/k_i), \mathbf{Q}/\mathbf{Z})$ when γ ranges over Γ_i, we conclude that $\bar{\sigma}_i = 0$. $\square$

By Lemma 9.20 and our hypotheses on S, we can choose, for each $i \in \{n+1, \ldots, N\}$, a rational point $\xi_{w_i, i}$ of the fiber of $\mathscr{Y}_i^0 \to \tilde{m}_i$ above w_i such that $\mathrm{Fr}_{\xi_{w_i, i}} = -\sigma_i$ and hence

$$(9.13) \qquad \qquad \sum_{v \in \Omega_i} n_{w,i} \mathrm{Fr}_{\xi_{w,i}} = 0$$

in G_i. We then lift $\xi_{w_i, i}$ to a k_{v_i}-point x'_{v_i} of X_c.

We have thus constructed an adelic point $(x'_v)_{v \in \Omega} \in X_c(\mathbf{A}_k)$ arbitrarily close to $(x_v)_{v \in \Omega}$ in $X(\mathbf{A}_k)$ (resp. in $X(\mathbf{A}_k)_\bullet$). It remains to check that $(x'_v)_{v \in \Omega}$ is orthogonal to B with respect to the Brauer–Manin pairing. By the orthogonality of $(x''_v)_{v \in \Omega}$ to B and by the definition of S, we have $\sum_{v \in S} \mathrm{inv}_v \beta(x''_v) = 0$ for all $\beta \in B$. On the other hand, as x'_v is arbitrarily close to x''_v for $v \in S \cap \Omega_f$ and as x'_v and x''_v belong to the same connected component of $f^{-1}(U)(k_v)$ for $v \in \Omega_\infty$, we have $\beta(x''_v) = \beta(x'_v)$ for all $v \in S$ and all $\beta \in B$, so that $\sum_{v \in S} \mathrm{inv}_v \beta(x'_v) = 0$ for all $\beta \in B$. For $v \in \Omega \setminus (S \cup \Omega_1 \cup \cdots \cup \Omega_N)$ we have $\beta(x'_v) = 0$ for all $\beta \in B$ since $B \subseteq \mathrm{Br}(f^{-1}(\mathscr{U}))$. For $v \in \Omega_1 \cup \cdots \cup \Omega_n$, the existence of a rational point in the fiber of $\mathscr{E}_i \to \mathscr{Y}_i^0$ above $\xi_{w,i}$ implies, by the same argument as in Lemma 5.9, that $\beta(x'_v) = 0$ for all $\beta \in B$. Thus

$$(9.14) \qquad \qquad \sum_{v \in \Omega} \mathrm{inv}_v \beta(x'_v) = \sum_{i=n+1}^{N} \sum_{v \in \Omega_i} \mathrm{inv}_v \beta(x'_v)$$

for all $\beta \in B$. Now, as $\beta \in \mathrm{Br}(f^{-1}(\mathscr{U}))$, we have $\mathrm{inv}_v \beta(x'_v) = n_{w,i} \partial_{\beta, Y_i}(\mathrm{Fr}_{\xi_{w,i}})$ for any $\beta \in B$, any $i \in \{n+1, \ldots, N\}$ and any $v \in \Omega_i$, if $\partial_{\beta, Y_i} \in \mathrm{Hom}(G_i, \mathbf{Q}/\mathbf{Z}) \subset H^1(k(Y_i), \mathbf{Q}/\mathbf{Z})$ denotes the residue of β at the generic point of Y_i (see [Har94, Corollaire 2.4.3]). In view of (9.13), we conclude that $\sum_{v \in \Omega} \mathrm{inv}_v \beta(x'_v) = 0$ for all $\beta \in B$. $\square$

Remark 9.21. The rather delicate arguments applied in the proof of Theorem 9.17 to deal with the split fibers $f^{-1}(m)$ for $m \in M''$ are those used by Harari in [Har94]. As explained above, these arguments can be avoided entirely by taking $M'' = \varnothing$, at the expense of losing some control over the polynomials to which Conjectures 9.1 and 9.2 are applied in the course of the proof.

Using arguments similar to those of §6, we now incorporate a Hilbert set into the conclusion of Theorem 9.17.

Theorem 9.22. *Let us keep the notation and assumptions of Theorem 9.17. For any Hilbert subset $H \subseteq \mathbf{P}_k^1$, the conclusion of Theorem 9.17 still holds if the rational point $c \in U(k)$ is required, in addition, to belong to H.*

Proof. By Lemma 5.3, there exists a finite subgroup $B_0 \subset \mathrm{Br}(X)$ such that

$$(B + f_\eta^* \mathrm{Br}(\eta)) \cap \mathrm{Br}(X) = B_0 + f^* \mathrm{Br}(k).$$

Let us fix a rational point $h \in H \cap U$. Such a point exists by Hilbert's irreducibility theorem. Let $(x_v)_{v \in \Omega} \in X(\mathbf{A}_k)$ be orthogonal to B_0. Let $S \subset \Omega$ be a finite subset. We must prove the existence of $c \in U(k)$ belonging to H and of $(x'_v)_{v \in \Omega} \in X_c(\mathbf{A}_k)$ orthogonal to B such that x'_v is arbitrarily close to x_v for $v \in S$ (resp. arbitrarily close to x_v for $v \in S \cap \Omega_f$ and in the same connected component as x_v for $v \in \Omega_\infty$). To this end, we may assume, after enlarging S, that $X_h(k_v) \neq \varnothing$ for any $v \in \Omega \setminus S$ (see [LW54], [Nis54]), that $\Omega_\infty \subseteq S$ and that any element of B_0 evaluates trivially on $X(k_v)$ for any $v \in \Omega \setminus S$. For $v \in \Omega \setminus S$, let x''_v be an arbitrary k_v-point of X_h. Let $x''_v = x_v$ for $v \in S$. By the definition of B_0, the adelic point $(x''_v)_{v \in \Omega}$ is orthogonal to $(B + f^*_\eta \mathrm{Br}(\eta)) \cap \mathrm{Br}(X)$ for the Brauer–Manin pairing. We may therefore apply Theorem 9.17 to it. The resulting point $c \in U(k)$ will then be arbitrarily close to h in $\prod_{v \in \Omega \setminus S} \mathbf{P}^1(k_v)$, which implies, by [Sme15, Proposition 6.1], that $c \in H$. □

In the remainder of §9.3, we spell out the most significant conditional corollaries of Theorem 9.22.

Corollary 9.23. *Let X be a smooth, proper, irreducible variety over a number field k, endowed with a morphism $f : X \to \mathbf{P}^1_k$ whose geometric generic fiber $X_{\bar{\eta}}$ is irreducible. Assume that*

(1) $H^1(X_{\bar{\eta}}, \mathbf{Q}/\mathbf{Z}) = 0$ *and* $H^2(X_{\bar{\eta}}, \mathscr{O}_{X_{\bar{\eta}}}) = 0$;
(2) *every fiber of f contains an irreducible component of multiplicity 1;*
(3) *Conjecture 9.1 (resp. Conjecture 9.2) holds;*
(4) *there exists a Hilbert subset $H \subseteq \mathbf{P}^1_k$ such that $X_c(k)$ is dense in $X_c(\mathbf{A}_k)^{\mathrm{Br}(X_c)}$ (resp. in $X_c(\mathbf{A}_k)^{\mathrm{Br}(X_c)}_\bullet$) for every rational point c of H.*

Then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ (resp. in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}_\bullet$).

Proof. After a change of coordinates on $\mathbf{P}^1_k$, we may assume that $f^{-1}(\infty)$ is split. Let U be a dense open subset of $\mathbf{P}^1_k$ over which the fibers of f are split, with $\infty \in U$. After shrinking U, we may assume that $U \neq \mathbf{P}^1_k$ and that there exists a finite subgroup $B \subset \mathrm{Br}(f^{-1}(U))$ such that $B + f^*_\eta \mathrm{Br}(\eta) = \mathrm{Br}(X_\eta)$. Indeed, it follows from (1) that $\mathrm{Br}(X_\eta)/f^*_\eta \mathrm{Br}(\eta)$ is finite (see Lemma 8.6). By Proposition 4.1, there exists a Hilbert subset $H' \subseteq U$ such that the natural map $B \to \mathrm{Br}(X_c)/f^*_c \mathrm{Br}(k)$ is surjective for all $c \in H'$. Applying Theorem 9.22 to the Hilbert subset $H \cap H'$, to $(x_v)_{v \in \Omega} \in X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ and to any subset $M' \subseteq \mathbf{P}^1_k \setminus U$ satisfying the hypothesis of Theorem 9.17 (for instance $M' = \mathbf{P}^1_k \setminus U$) produces a rational point c of $H \cap H'$ and an element of $X_c(\mathbf{A}_k)$ orthogonal to B, and therefore to $\mathrm{Br}(X_c)$, which is arbitrarily close to $(x_v)_{v \in \Omega}$ in $X(\mathbf{A}_k)$ (resp. in $X(\mathbf{A}_k)_\bullet$). In view of (4), we may approximate in the fiber X_c to conclude the proof. □

Corollary 9.24. *Let X be a smooth, proper, irreducible variety over a number field k, endowed with a morphism $f : X \to \mathbf{P}^1_k$ with geometrically irreducible generic fiber, such that the hypotheses (1), (2) and (4) of Corollary 9.23 are satisfied and $f^{-1}(\infty)$ is split. Let $M' \subset \mathbf{A}^1_k$ be a finite closed subset containing the points with non-split fiber. Assume M' contains a rational point or k is totally imaginary. If Conjecture 9.1 holds for the irreducible monic polynomials which vanish at the points of M', then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.*

Proof. Same proof as Corollary 9.23, in view of Remark 9.18 (ii).   □

Corollary 9.25. *Let $n \geq 1$. Let X be a smooth, proper, irreducible variety over a number field k and let $f : X \to \mathbf{P}_k^n$ be a dominant morphism with rationally connected geometric generic fiber. Assume Conjecture 9.1 (resp. Conjecture 9.2) holds. If there exists a Hilbert subset $H \subseteq \mathbf{P}_k^n$ such that $X_c(k)$ is dense in $X_c(\mathbf{A}_k)^{\mathrm{Br}(X_c)}$ (resp. in $X_c(\mathbf{A}_k)_\bullet^{\mathrm{Br}(X_c)}$) for every rational point c of H, then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ (resp. in $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}(X)}$).*

Proof. We argue by induction on n. If $n = 1$, Corollary 9.23 applies: its assumption (1) is satisfied by Lemma 8.6 and by [Deb01, Corollary 4.18(b)] while (2) follows from the Graber–Harris–Starr theorem [GHS03]. Let us now assume $n > 1$. As $\mathbf{P}_k^n$ is birationally equivalent to $\mathbf{P}_k^{n-1} \times \mathbf{P}_k^1$, we can find a smooth, proper, irreducible variety X' over k, a morphism $f' : X' \to \mathbf{P}_k^{n-1} \times \mathbf{P}_k^1$ and a Hilbert subset $H' \subseteq \mathbf{P}_k^{n-1} \times \mathbf{P}_k^1$ such that the generic fibers of f and of f' are isomorphic and such that $X'_c(k)$ is dense in $X'_c(\mathbf{A}_k)^{\mathrm{Br}(X'_c)}$ (resp. in $X'_c(\mathbf{A}_k)_\bullet^{\mathrm{Br}(X'_c)}$) for every rational point c of H'. Let $g : X' \to \mathbf{P}_k^1$ denote the composition of f' with the second projection. Let $H_1 \subseteq \mathbf{P}_k^1$ be a Hilbert subset satisfying the conclusion of Lemma 8.12 with respect to the Hilbert subset H'. After replacing H_1 with $H_1 \cap U$ for a small enough dense open subset U of $\mathbf{P}_k^1$, we may assume that for every rational point h of H_1, the variety $X'_h = g^{-1}(h)$ is smooth and irreducible and the morphism $f'_h : X'_h \to \mathbf{P}_k^{n-1}$ is dominant, with rationally connected geometric generic fiber (see [Kol96, Ch. IV, Theorem 3.5.3]). By the definition of H_1 and the hypothesis on H', the morphism $f'_h : X'_h \to \mathbf{P}_k^{n-1}$ then satisfies all of the assumptions of Corollary 9.25 if h is a rational point of H_1. Thus, by our induction hypothesis, the set $X'_h(k)$ is dense in $X'_h(\mathbf{A}_k)^{\mathrm{Br}(X'_h)}$ (resp. in $X'_h(\mathbf{A}_k)_\bullet^{\mathrm{Br}(X'_h)}$) for every rational point h of H_1. Now, according to [GHS03, Corollary 1.3], the geometric generic fiber of g is a rationally connected variety since it dominates $\mathbf{P}^{n-1}$ with rationally connected geometric generic fiber. We may therefore apply the case $n = 1$ of Corollary 9.25 to g and conclude that $X'(k)$ is dense in $X'(\mathbf{A}_k)^{\mathrm{Br}(X')}$ (resp. in $X'(\mathbf{A}_k)_\bullet^{\mathrm{Br}(X')}$). The desired result finally follows as X and X' are birationally equivalent and $\mathrm{Coker}(\mathrm{Br}(k) \to \mathrm{Br}(X))$ is finite (see [CTPS14, Proposition 6.1 (iii)]; for the finiteness assertion, see [CTS13, Lemma 1.3 (i)], [Deb01, Corollary 4.18 (b)], [GHS03, Corollary 1.3]).   □

Remark 9.26. In the situation of Corollary 9.23, let us assume that the generic fiber of f is birationally equivalent to a torsor under a torus T over the function field of $\mathbf{P}_k^1$. In this case, assumptions (1), (2) and (4) are satisfied (for (4), see [San81]). Thus $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ as soon as the varieties W associated to f in §9.2.2 satisfy strong approximation off any finite place, according to Corollary 9.10. This should be compared with [Sko15, Theorem 1.1], which asserts that when the torus T is defined over k, the same conclusion can be reached under the sole assumption that the varieties W satisfy weak approximation.

Combining Theorem 9.17 (with $M'' = \varnothing$ and $B = 0$), Theorem 9.6 and Remark 9.18 (iii) also yields the following conditional corollary, which extends the main results of [CTSSD98, §1] and recovers [Wei14a, Theorem 3.5].

Corollary 9.27. *Let X be a smooth, proper, irreducible variety over a number field k, endowed with a morphism $f : X \to \mathbf{P}^1_k$ with geometrically irreducible generic fiber. We make the following assumptions:*

- *Schinzel's hypothesis* (H) *holds;*
- *for every $m \in \mathbf{P}^1_k$, the fiber $f^{-1}(m)$ possesses an irreducible component of multiplicity 1 in the function field of which the algebraic closure of $k(m)$ is an almost abelian extension of $k(m)$ in the sense of Definition 9.4;*
- *the fibers of f above the rational points of a Hilbert subset of $\mathbf{P}^1_k$ satisfy weak approximation.*

Then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}(X/\mathbf{P}^1_k)}$, where $\mathrm{Br}_{\mathrm{vert}}(X/\mathbf{P}^1_k) = \mathrm{Br}(X) \cap f^*_\eta \mathrm{Br}(\eta)$.

9.4. **Some unconditional results.** Combining Corollary 9.24 with Matthiesen's Theorem 9.14 yields the following unconditional result.

Theorem 9.28. *Let X be a smooth, proper, irreducible variety over $\mathbf{Q}$ and $f : X \to \mathbf{P}^1_{\mathbf{Q}}$ be a dominant morphism, with rationally connected geometric generic fiber, whose non-split fibers all lie over rational points of $\mathbf{P}^1_{\mathbf{Q}}$. Assume that $X_c(\mathbf{Q})$ is dense in $X_c(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}(X_c)}$ for every rational point c of a Hilbert subset of $\mathbf{P}^1_{\mathbf{Q}}$. Then $X(\mathbf{Q})$ is dense in $X(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}(X)}$.*

Theorem 9.28 was previously known when f has a unique non-split fiber (see [Har97]) or when every singular fiber of f contains an irreducible component of multiplicity 1 split by an abelian extension of $\mathbf{Q}$ and the smooth fibers of f above the rational points of $\mathbf{P}^1_{\mathbf{Q}}$ satisfy weak approximation (see [HSW14]).

Thanks to Borovoi's theorem [Bor96, Corollary 2.5], Theorem 9.28 has the following corollary, whose statement should be compared with [CT98, Conjecture 1].

Corollary 9.29. *Let X be a smooth, proper, irreducible variety over $\mathbf{Q}$ endowed with a morphism $f : X \to \mathbf{P}^1_{\mathbf{Q}}$ whose generic fiber is birationally equivalent to a homogeneous space of a connected linear algebraic group, with connected geometric stabilisers. If the non-split fibers of f lie over rational points of $\mathbf{P}^1_{\mathbf{Q}}$, then $X(\mathbf{Q})$ is dense in $X(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}(X)}$.*

Remark 9.30. By the same induction argument as in the proof of Corollary 9.25, it follows from Theorem 9.28 that the statements of Theorem 9.28 and of Corollary 9.29 remain valid when $\mathbf{P}^1_{\mathbf{Q}}$ is replaced with $\mathbf{P}^n_{\mathbf{Q}}$ for some $n \geq 1$, if the non-split codimension 1 fibers of f all lie over generic points of *hyperplanes* of $\mathbf{P}^n_{\mathbf{Q}}$ defined over $\mathbf{Q}$.

Corollary 9.29 applies to the total space of an arbitrary pencil of toric varieties, as long as the non-split members of the pencil are defined over $\mathbf{Q}$. Thus, Corollary 9.29, together with Remark 9.30, subsumes all of the results of [BM13], [BMS14], [HSW14, §4], as well as [Sme15, §3.2] and [Sko15, Corollary 1.2, Corollary 1.3]. One should note, however, that Theorem 9.14 builds on the contents of [BM13], *via* [Mat14].

Other unconditional results can be obtained by combining Corollary 9.24 with Theorem 9.11. In the following statement, the *rank* of f is the sum of the degrees of the closed points of $\mathbf{P}^1_k$ above which the fiber of f is not split.

Theorem 9.31. *Let X be a smooth, proper, irreducible variety over a number field k and $f : X \to \mathbf{P}_k^1$ be a dominant morphism with rationally connected geometric generic fiber. Assume[1] that* $\operatorname{rank}(f) \leq 2$ *and that either k is totally imaginary or the non-split fibers of f lie over rational points of $\mathbf{P}_k^1$. If $X_c(k)$ is dense in $X_c(\mathbf{A}_k)^{\operatorname{Br}(X_c)}$ for every rational point c of a Hilbert subset of $\mathbf{P}_k^1$, then $X(k)$ is dense in $X(\mathbf{A}_k)^{\operatorname{Br}(X)}$.*

Theorem 9.31 is due to Harari [Har94] [Har97] when $\operatorname{rank}(f) \leq 1$. Fibrations over $\mathbf{P}_k^1$ with rank 2 had been dealt with (using the descent method) in [CTS00, Theorem A] under the assumption that the fibers above a Hilbert set of rational points satisfy weak approximation. Theorem 9.31 relaxes this assumption when k is totally imaginary. When k is not totally imaginary, one can recover Theorem A and Theorem B of *op. cit.* by combining Theorem 9.22 (with $M'' = \varnothing$) and Theorem 9.11. Thus, Theorem 9.31 answers the question raised at the end of [CT98, §2.2]. The results of [CTS00] were also extended in [HS03, Theorem 2.9] to cover the case of Theorem 9.31 in which the non-split fibers of f lie over rational points of $\mathbf{P}_k^1$ and are split by prime degree extensions of k.

Finally, we note that the statement obtained by combining Theorem 9.15 (a corollary of Irving's arguments [Irv14]) with Theorem 9.17 (with $M'' = \varnothing$) recovers the main result of [Irv14] on the existence of solutions to the equation (1.3) for cubic polynomials $P(t)$, and extends it to general fibrations with the same degeneration data, under the assumption that the cubic field K possesses a unique real place.

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[1]Added in proof: the hypothesis that either k is totally imaginary or the non-split fibers of f lie over rational points of $\mathbf{P}_k^1$ is now known to be superfluous. Details will be given elsewhere.

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