The Einstein Equation

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1 Einstein's Equation in Vacuum

Consider a particle with unit mass falling from a static position $\overline{x}^0 = (x_1^0, x_2^0, x_3^0)$ in a force field $\overline{F}(\overline{x})$. We shall denote its position at times t with $\overline{x}(\overline{x}^0, t)$. We will assume that all other masses are far away (but they determine the force field $\overline{F}$) and that the effect of our mass it self on F is negligible.

Now consider the $(t, \overline{x})$ space as a semi-Riemannian manifold with a metric of the form

$$g = -g_{0,0}dt^2 + \sum_{i=1}^{3} g_{i,i}dx_i^2$$

with $g_{0,0}(t, x), g_{i,i}(t, \overline{x}) > 0$ for every $t, \overline{x}$. We wonder if one can choose the functions $g_{0,0}$ and $g_{i,i}$ such that the paths $\tau \mapsto (\tau, \overline{x}(\tau))$ in $\mathbb{R}^4$ will be geodesics. Thinking about it again we realize that since the original newtonian theory might have been inaccurate we should replace this question with whether we can choose $g_{0,0}$ and $g_{i,i}$ such that the paths $(t, \overline{x}(t))$ are **approximately** geodesics.

Since this is a very vague question let us add some assumptions: suppose that $g_{0,0}$ vary very slowly with $\overline{x}, t$, so we can neglect their derivatives. Second we're going to make the speed of light scale assumption - the newtonian mechanics are only valid when the speed $\frac{\partial \overline{x}}{\partial t}$ is small compared to the speed of light. Since we normalized our scale and set $c = 1$ we get that $|v| = \left|\frac{\partial \overline{x}}{\partial t}\right| << 1$.

Now return to the family of paths $\gamma_{\overline{x}^0}(\tau) = (\tau, \overline{x}(\tau, \overline{x}^0))$. Then the norm (measured using the metric g) of $\gamma'_{\overline{x}^0} = \frac{d\gamma_{\overline{x}^0}}{d\tau}$ is

$$g(\gamma'_{\overline{x}^0}, \gamma'_{\overline{x}^0}) = g((1, \overline{x}'), (1, \overline{x}')) = g_{0,0} + \sum_i g_{i,i}v_i \simeq -g_{0,0}$$

which by our assumptions is approximately constant. Hence this path has a chance of being approximately a geodesic (with non-unit speed parametrization) under our assumptions. Now if $\gamma_{\overline{x}^0}$ is approximately a family of geodesics then for each $i = 1, 2, 3$ the field

$$J^i(\tau) = \frac{\partial \gamma_{\overline{x}^0}}{\partial x_i^0}(\tau) = \left(0, \frac{\partial x_1}{\partial x_i^0}, \frac{\partial x_2}{\partial x_i^0}, \frac{\partial x_3}{\partial x_i^0}\right)$$

along γ is approximately a **Jacobi field** and hence should approximately satisfy the Jacobi equation

$$\frac{\nabla^2 J^i}{\nabla \tau^2} = -R(J^i(\tau), \gamma'_{\overline{x}^0})\gamma'_{\overline{x}^0}$$

where ∇ is the Levi-Civita connection associated with g. From our assumptions on the staticness of the metric coefficient $g_{0,0}$ and because $\gamma'_{\overline{x}^0} \simeq \frac{\partial}{\partial t}$ it is not hard to show that $\frac{\nabla^2 J^i}{\nabla \tau^2}$ is approximately $\frac{\partial^2 J^i}{\partial \tau^2}$ (just think what would happen if $g_{0,0}$ was actually a constant function). Hence we can do an approximate computation:

$$\frac{\nabla^2 J^i}{\nabla \tau^2} \simeq \frac{\partial^2}{\partial t^2}\frac{\partial \overline{x}}{\partial x_i^0} = \frac{\partial}{\partial x_i^0}\frac{\partial^2 \overline{x}}{\partial t^2} = \frac{\partial}{\partial x_i^0}\overline{F}(\overline{x}) = \sum_j \frac{\partial \overline{F}}{\partial x_j}\frac{\partial x_j}{\partial x_i^0}$$

and so we get that

$$\sum_j \frac{\partial \overline{F}}{\partial x_j}\frac{\partial x_j}{\partial x_i^0} \simeq -R\left(\frac{\partial \overline{x}}{\partial x_i^0}, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}$$

Writing in coordinates

$$R\left(e_i, e_j\right) e_k = \sum_l R^l_{k,i,j} e_l$$

where

$$e_0 = \frac{\partial}{\partial t}$$

and

$$e_i = \frac{\partial}{\partial x_i}$$

for $i = 1, 2, 3$, we get that

$$\sum_{j,l} \frac{\partial F_l}{\partial x_j}\frac{\partial x_j}{\partial x_i^0} e_l \simeq -R\left(\sum_j \frac{\partial x_j}{\partial x_i^0} e_j, e_0\right) e_0 = -\sum_{j,l}\frac{\partial x_j}{\partial x_i^0} R^l_{0,j,0} e_l$$

which means that

$$\frac{\partial F_l}{\partial x_j} \simeq -R^l_{0,j,0}$$

Now we assumed that the particle was moving in an area that was clear of other masses and that it's own mass is gravitationally negligible. Hence we expect that divergence of the force field to vanish, i.e.

$$0 = \sum_{l=1}^{3} \frac{\partial F_l}{\partial x_l} = -\sum_{l=1}^{3} R^l_{0,l,0}$$

Note that $R^l_{k,i,j}$ is anti-symmetric in i, j and so in particular $R^0_{0,0,0} = 0$. This means that we can also write

$$\sum_{l=0}^{3} R^l_{0,l,0} = 0$$

The tensor $\mathrm{Ric}_{i,j} = \sum_{l=0}^{3} R^l_{0,l,0}$ is called the **Ricci** tensor. It is defined in a coordinate free way as the tensor which associates to any two vectors $v, u \in T_pM$ the trace of the linear map $w \mapsto R_p(w, v)u$. We get that in an area with no mass we expect the semi-Riemannian metric to satisfy

$$\mathrm{Ric}_{0,0} = 0$$

Since the physical equations should be invariant to Lorentz transformations we see that such a constraint imply many others on the Ricci tensor (the physical universe doesn't isolate a well defined direction of time). In particular it will not surprise us that Einstien's equations in vaccum are actually

$$\mathrm{Ric}_{i,j} = 0$$

or simply

$$\mathrm{Ric} = 0$$

i.e. **the Ricci tensor in vacuum vanishes**.

2 The Stress-Energy Tensor

In the previous section we saw a heuristic argument for why in vacuum we should expect the Rici curvature of our metric to vanish. The argument was based on an equation of the $R_{0,0}$ component of the Rici tensor with the gradient of the gravitational field, which vanishes in vacuum. To be more specific , in newtonian gravitation theory the gradient of the gravitation force field is proportional to the **mass density**. Hence we should suspect that in general we expect the Rici tensor to equal some kind of tensor which would be a relativistic generalization of mass density.

The quantity of mass density is a ratio between mass and 3-dimensional volume. We know that mass is not a Lorentz scalar (i.e. it is not preserved by changing inertial frames). Instead it is part of the Lorentz energy-momentum 4-vector:

$$\overline{P} = (m, mv_1, mv_2, mv_3) = m_0\overline{u}$$

where m_0 is rest mass and $m = \gamma m_0$ is the observed mass (γ is Lorentz's factor and we are working in $c = 1$ coordinates, so $\overline{u} = \gamma(1, v_1, v_2, v_3)$ is the normalized 4-speed of the particle).

Hence we see that if we want to talk about density of mass we must also talk about density of momentum. Now 3-dimensional volume is also not Lorentz invariant - the space dimensions are mixed with time dimensions under Lorentz transformations. We need to talk about a 4-dimensional 3-forms in Minkowski space time. Such a 3-form will have four components: one space volume component and three surface-time components.

When we exchange the notion of 3-dimensional volume with surface-time volume the notion of density becomes the notion of **flux**. In particular mass density becomes mass flux and momentum density becomes momentum flux.

Suppose we have a relatively sparse rest-mass density $\rho_0 = \rho_0(x_0, x_1, x_2, x_3)$ where sparse means that the particles don't bump into each other and the gravitational field is negligible.

Let $\overline{u} = \overline{u}(x_0, x_1, x_2, x_3)$ be the average normalized 4-speed of particles at (x_0, x_1, x_2, x_3). Let $u^j = \sum_l g^{j,l} u_l$ be the covariant form of $\overline{u}$ which in the standard minkowsi metric just means $u^0 = -u_0$ and $u^i = u_i$ for $i = 1, 2, 3$. Then the information of density and flux of mass and momentum can be organized into a $(1, 1)$ tensor given by

$$T_i^j = \rho_0 u_i u^j$$

Let us try to understand how this measures density and flux. For $i = j = 0$ we have $T_0^0 = -\rho_0$ is minus rest-mass density. For $j = 0$ and $i > 0$ we have that $T_i^0 = -\rho_0 u_i$ is minus the density of momentum 4-vectors. For $i = 0$ and general $j > 0$ we have that $T_0^j = \rho_0 u^j$ is density of momentum covectors. This is supposed to be connected to mass flux somehow.

Let us go back to 3-dimensional newtonian mechanics for a second, so assume $\overline{u} \simeq (1, v_1, v_2, v_3) = (1, \overline{v})$ (because we're in $c = 1$ coordinates). If we have an infinitesimal area element dS with normal vector $\overline{n} = (n_1, n_2, n_3)$ then the amount of mass which flows through dS per unit time is equal to

$$\rho_0 \langle \overline{v}, \overline{n} \rangle \, dS = \left[\sum_{j=1}^{3} T_0^j n_j \right] dS$$

where $\overline{v}$ refers to the 3-speed vector. Hence we see that density field of momentum covectors is equivalent to the flux field of mass. Note that T_0^0 can be though of as flux of mass through time.

What about T_i^j for $i > 0, j > 0$. This measures **flux of momentum**. Going back again to newtonian 3-space, consider an open subset $V \subseteq \mathbb{R}^3$ with boundary S. Then (since there is no force changing momentum of particle) we get that the total x_i-momentum change in V is given by

$$(*) \quad \int_V \frac{\partial}{\partial t} \rho_0 v_i = \int_S \rho_0 v_i \langle \overline{v}, \overline{n} \rangle \, dS = \int_S \left[\sum_j T_i^j n_j \right] dS$$

Note in particular that the divergence of the vector (T_i^1, T_i^2, T_i^3) is equal the time derivative of $-T_i^0$, which means that the 4-divergence of $(T_i^0, T_i^1, T_i^2, T_i^3)$ vanishes:

$$\sum_{j=0}^{4} \frac{\partial T_i^j}{\partial x_j} = 0$$

This is also true for $i = 0$.

Now if the particles do exert force on each other then we need to modify T_i^j for $i, j > 0$ in order to make equation $(*)$ true, i.e. we need to add to T_i^j the effect of the total force applied to V by external particles which contributes to the total change of momentum inside V. We assume that in every physical situation there is a well-defined $(1,1)$-tensor T_i^j measuring the correct mass-momentum density-flux and in particular satisfying

$$\sum_{j=0}^{4} \frac{\partial T_i^j}{\partial x_j} = 0$$

We refer to this property as "the divergence of the $(1,1)$-tensor T_i^j is 0".

One would then guess that Einstein's equation should be

$$R_i^j = CT_i^j$$

where $R_i^j = \sum_l g^{j,l} R_{i,l}$ is the $(1,1)$ version of the Rici tensor $R_{i,j}$ and C is some constant (which in $c = 1$ coordinates is $C = 8\pi\kappa$ where κ is the universal gravity constant). This is also what Einstein guessed in the beginning. But there is something too strong in this equation, because it would imply that the divergence of R_i^j is 0. Since this is apparently too strong, we modify this equation by replacing R_i^j with the Einstein tensor

$$G_i^j = R_i^j - \frac{1}{2}R$$

where $R = \sum_i R_i^i$ is the trace of the Rici tensor. It can be shown (see `http://en.wikipedia.org/wiki/Einstein_field_equations` under the title "derivation of local energy momentum conservation") that the divergence the Einstien tensor always vanishes, so the equation

$$G_i^j = CT_i^j$$

makes more sense. Apparently it has also been verified by experiments, so we're cool.

3 Description of the Einstein Tensor through Sectional Curvatures

Let M be a semi-Riemannian manifold semi-Riemmanian metric $g(v, u) = \langle v, u \rangle$, Rici tensor $\mathrm{Ric}(v)$, scalar curvature $R = \mathrm{Tr}(\mathrm{Ric})$ and Einstein tensor

$$G(v) = \mathrm{Ric}(v) - \frac{1}{2}Rv$$

Let $p \in M$ be a point and $e_0, e_1, e_2, e_3 \in T_pM$ a g-orthonormal basis, i.e. $\langle e_0, e_0 \rangle = -1$, $\langle e_i, e_i \rangle = 1$ for $i > 0$ and $\langle e_i, e_j \rangle = 0$ for $i \neq j$. Then we have by definition

$$\langle \mathrm{Ric}(e_i), e_i \rangle = \sum_l \varepsilon_l \langle R(e_l, e_i)e_i, e_l \rangle = \varepsilon_i \sum_{l \neq i} K(e_i \wedge e_l)$$

where for any two vectors $v, u \in T_p$ we denote by

$$K(v \wedge u) = \frac{\langle R(u, v)v, u \rangle}{\langle v, v \rangle \langle u, u \rangle - \langle v, u \rangle^2}$$

the sectional curvature of the plain spanned by v, u. This means that the scalar curvature is equal to

$$R = \sum_i \varepsilon_i \langle \mathrm{Ric}(e_i), e_i \rangle = \sum_{l \neq i} K(e_i \wedge e_l) = \sum_{i<l} 2K(e_i \wedge e_l)$$

So we can think of the scalar curvature as a sort of average section curvature.

This gives us

$$\langle G(e_i), e_i \rangle = \langle \mathrm{Ric}(e_i), e_i \rangle - \frac{1}{2} R \langle e_i, e_i \rangle =$$

$$\varepsilon_i \left[\sum_{l \neq i} K(e_i \wedge e_l) - \sum_{k<l} K(e_k \wedge e_l) \right] = -\varepsilon_i \sum_{k<l, k,l \neq i} K(e_k \wedge e_l)$$

In particular we see that the quantity $\sum_{k<l,k,l \neq i} K(e_k \wedge e_l)$ depends only on e_i and not on $\{e_j\}_{j \neq i}$ as long as they complete e_i to an orthonormal basis. It is there for reasonable that this quantity will have a geometric description in terms of e_i. Indeed, one can show the following:

Theorem 3.1. *Let $V \subseteq M$ be a totally geodesic submanifold containing p and orthogonal to $v \in T_p$. Then g induces either a Riemannian or semi-Riemannian metric on V. Let R_V be the scalar curvature of V under this metric. Then*

$$(R_V)_p = 2 \left[K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + K(e_2 \wedge e_3) \right]$$

Where e_1, e_2, e_3 is an orthonormal basis for the orthogonal complement of v in T_pM.

This means that we can write Einstein's equation as

$$\frac{1}{2} R_V = - \langle v, v \rangle 8\pi\kappa T(v, v)$$

where V is any totally geodesic 3-manifold orthogonal to v.