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ABSTRACT

Schinzel’s Hypothesis (H) was used by Colliot-Thélène and Sansuc, and later by Serre, Swinnerton-Dyer and others, to prove that the Brauer–Manin obstruction controls the Hasse principle and weak approximation on pencils of conics and similar varieties. We show that when the ground field is \mathbb{Q} and the degenerate geometric fibres of the pencil are all defined over \mathbb{Q} , one can use this method to obtain unconditional results by replacing Hypothesis (H) with the finite complexity case of the generalised Hardy–Littlewood conjecture recently established by Green, Tao and Ziegler.

Introduction

The finite complexity case of the generalised Hardy–Littlewood conjecture recently proved by Green and Tao [GT10, GT12] and Green–Tao–Ziegler [GTZ12] is of fundamental importance to number theory. The aim of this note is to explore some of its consequences for the Hasse principle and weak approximation on algebraic varieties over \mathbb{Q} .

Hasse used Dirichlet’s theorem on primes in an arithmetic progression to deduce what is now called the Hasse principle for quadratic forms in four variables from the global reciprocity law and the Hasse principle for quadratic forms in three variables, itself a corollary of global class field theory (see [Has75, pp. 16 and 87]). This ‘fibration method’ was taken up by Colliot-Thélène and Sansuc in [CS82]. They showed that Schinzel’s Hypothesis (H), a vast generalisation of Dirichlet’s theorem in which $at + b$ is replaced by a finite collection of irreducible polynomials in t of arbitrary degree [SS58], has strong consequences for the Hasse principle and weak approximation on conic bundles and some other pencils of varieties over \mathbb{Q} . The method of [CS82] was extended to number fields and generalised in various directions by Serre (see [Ser97, ch. II, Annexe]), Swinnerton-Dyer (see [Swi94], [Swi11] and references in that paper), Colliot-Thélène and others [CS94, CSS98, Wit07, Wei14]. In this note we show that when the ground field is \mathbb{Q} and the degenerate geometric fibres of the pencil are all defined over \mathbb{Q} , one can apply the methods of the aforementioned papers to obtain unconditional results by using the theorem of Green, Tao and Ziegler in place of Schinzel’s Hypothesis (H).

Additive combinatorics was recently applied to the study of rational points by Browning, Matthiesen and one of the authors in [BMS14]. The approach of [BMS14] uses the descent method of Colliot-Thélène and Sansuc; it crucially relies on the work of Matthiesen on representation of linear polynomials by binary quadratic forms [Mat12, Mat13] in order to prove the Hasse principle and weak approximation for certain varieties appearing after descent. In this paper we give a short proof of most of the results of [BMS14] as well as a generalisation to certain equations involving norms of cyclic (and some non-cyclic) extensions. Our approach is

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based directly on the Green–Tao–Ziegler theorem and on various generalisations of the method of Hasse, Colliot-Thélène–Sansuc and Swinnerton-Dyer mentioned above, thus avoiding the use of descent and the results of [Mat12, Mat13].

We recall the theorem of Green, Tao and Ziegler and give its first corollaries in §1. In §2 we compare Schinzel’s Hypothesis (H), in the form of Hypothesis (H₁), with Proposition 2.1, a consequence of the Green–Tao–Ziegler theorem. We prove our main results and deduce their first applications in §3. In §4 we consider applications to representation of norms (and products of norms) by products of linear polynomials.

1. A corollary of the generalised Hardy–Littlewood conjecture

In a series of papers Green and Tao [GT10, GT12] and Green *et al.* [GTZ12] proved the generalised Hardy–Littlewood conjecture in the finite complexity case. The following qualitative statement is [GT10, Corollary 1.9].

THEOREM 1.1 (Green, Tao, Ziegler). *Let $L_1(x, y), \dots, L_r(x, y) \in \mathbb{Z}[x, y]$ be pairwise non-proportional linear forms, and let $c_1, \dots, c_r \in \mathbb{Z}$. Assume that for each prime p , there exists $(m, n) \in \mathbb{Z}^2$ such that p does not divide $L_i(m, n) + c_i$ for any $i = 1, \dots, r$. Let $K \subset \mathbb{R}^2$ be an open convex cone containing a point $(m, n) \in \mathbb{Z}^2$ such that $L_i(m, n) > 0$ for $i = 1, \dots, r$. Then there exist infinitely many pairs $(m, n) \in K \cap \mathbb{Z}^2$ such that $L_i(m, n) + c_i$ are all prime.*

We shall use the following easy corollary of this result. For a finite set of rational primes S we write $\mathbb{Z}_S = \mathbb{Z}[S^{-1}]$.

PROPOSITION 1.2. *Suppose that we are given $(\lambda_p, \mu_p) \in \mathbb{Q}_p^2$ for p in a finite set of primes S , and a positive real constant C . Let e_1, \dots, e_r be elements of \mathbb{Z}_S . Then there exist infinitely many pairs $(\lambda, \mu) \in \mathbb{Z}_S^2$ such that:*

- (1) $\lambda > C\mu > 0$;
- (2) (λ, μ) is close to (λ_p, μ_p) in the p -adic topology for $p \in S$;
- (3) $\lambda - e_i\mu = p_i u_i$ with $u_i \in \mathbb{Z}_S^*$, for $i = 1, \dots, r$, where p_1, \dots, p_r are primes not in S such that $p_i = p_j$ if and only if $e_i = e_j$.

Proof. By eliminating repetitions we can assume that e_1, \dots, e_r are pairwise different. We can multiply λ, μ and all λ_p, μ_p by a product of powers of primes from S , and so assume without loss of generality that $(\lambda_p, \mu_p) \in \mathbb{Z}_p^2$ for $p \in S$. Replacing C with a larger constant we assume $C > e_i$, $i = 1, \dots, r$. Using the Chinese remainder theorem, we find $\lambda_0 \in \mathbb{Z}$ such that $\lambda_0 - \lambda_p$ is divisible by a sufficiently high power p^{n_p} for all $p \in S$, and similarly for $\mu_0 \in \mathbb{Z}$. In doing so we can assume that $\lambda_0 > C\mu_0 > 0$ and, in particular, $\lambda_0 - e_i\mu_0 > 0$ for all i .

Let d be a product of powers of primes from S such that $de_i \in \mathbb{Z}$ for $i = 1, \dots, r$. Let us write $d(\lambda_0 - e_i\mu_0) = M_i c_i$, where M_i is a product of powers of primes from S , and $c_i \in \mathbb{Z}$ is coprime to the primes in S . Let N be a product of primes in S such that $N > c_i - c_j$ for any i and j . Let $M = \prod_{p \in S} p^{m_p}$ where

$$m_p \geq \max\{n_p, \text{val}_p(N) + \text{val}_p(M_i)\}, \quad i = 1, \dots, r.$$

Then N divides M/M_i for each i . We now look for λ and μ of the form

$$\lambda = \lambda_0 + Mm, \quad \mu = \mu_0 + Mn, \quad (m, n) \in \mathbb{Z}^2. \tag{1.1}$$

Write $L_i(x, y) = M_i^{-1}Md(x - e_iy)$; then

$$\lambda - e_i\mu = d^{-1}M_i(L_i(m, n) + c_i) \tag{1.2}$$

for each $i = 1, \dots, r$. Let us check that the linear functions $L_i(x, y) + c_i$ satisfy the condition of Theorem 1.1. For $p \in S$, the integer $L_i(0, 0) + c_i$ is non-zero modulo p for each i . Now let p be a prime not in S . Since the determinant of the homogeneous part of the affine transformation (1.1) is in \mathbb{Z}_S^* and each $M_i^{-1}d(\lambda - e_i\mu)$ equals $M_i^{-1}d \in \mathbb{Z}_S^*$ at the point $(\lambda, \mu) = (1, 0)$, we see that there is $(m, n) \in \mathbb{Z}^2$ such that $\prod_{i=1}^r(L_i(m, n) + c_i)$ is not divisible by p .

We now choose an open convex cone K . Choose $(m_0, n_0) \in \mathbb{Z}^2$, $m_0 > Cn_0 > 0$, for which the positive integers $L_i(m_0, n_0)$ are pairwise different. After re-ordering the subscripts, we see that the inequalities

$$x > Cy > 0, \quad L_1(x, y) > \dots > L_r(x, y) > 0$$

hold for $(x, y) = (m_0, n_0)$. Define $K \subset \mathbb{R}^2$ by these inequalities. We can apply Theorem 1.1 to the linear functions $L_i(x, y) + c_i$ and the cone K . Thus there exist infinitely many pairs $(m, n) \in K \cap \mathbb{Z}^2$ such that $L_i(m, n) + c_i = p_i$, where p_i is a prime not in S , for $i = 1, \dots, r$. The coefficients of each $L_i(x, y)$ are divisible by N , and hence

$$L_i(m, n) - L_{i+1}(m, n) \geq N > c_{i+1} - c_i.$$

Thus $p_i > p_{i+1}$ for each $i = 1, \dots, r - 1$, so all the primes p_i are pairwise different. Since $n > 0$ and $m > Cn$ we see that $\mu = \mu_0 + Mn > 0$ and $\lambda = \lambda_0 + Mm > C\mu$.

By (1.2) we see that $\lambda - e_i\mu$ differs from $L_i(x, y) + c_i$ by an element of \mathbb{Z}_S^* , so the proof is now complete. \square

Proposition 1.2 can be used to study the Hasse principle and weak approximation for rational points. In the proof of the following result, which is modelled on the original proof of Hasse (in the version of [CS82, Proposition 2]), it replaces Dirichlet’s theorem on primes in an arithmetic progression.

For a field extension K/\mathbb{Q} of degree n we denote by $N_{K/\mathbb{Q}}(\mathbf{x})$ the corresponding norm form of degree n in n variables $\mathbf{x} = (x_1, \dots, x_n)$, defined by choosing a basis of the \mathbb{Q} -vector space K .

THEOREM 1.3. *Let K_i be a cyclic extension of \mathbb{Q} of degree d_i and let $b_i \in \mathbb{Q}^*$, $e_i \in \mathbb{Q}$, for $i = 1, \dots, r$. Then the affine variety $V \subset \mathbb{A}^2 \times \mathbb{A}^{d_1} \times \dots \times \mathbb{A}^{d_r}$ over \mathbb{Q} defined by*

$$b_i(u - e_iv) = N_{K_i/\mathbb{Q}}(\mathbf{x}_i) \neq 0, \quad i = 1, \dots, r, \tag{1.3}$$

satisfies the Hasse principle and weak approximation.

Proof. We are given $M_p \in V(\mathbb{Q}_p)$ for each prime p , and $M_0 \in V(\mathbb{R})$. Let S be the set of places of \mathbb{Q} where we need to approximate. We include the real place in S . Note that the set of real points $(u, v, \mathbf{x}_1, \dots, \mathbf{x}_r) \in V(\mathbb{R})$ for which $(u, v) \in \mathbb{Q}^2$ is dense in $V(\mathbb{R})$, and so it will be enough to prove the claim in the case when the coordinates u and v of M_0 are in \mathbb{Q} . By a \mathbb{Q} -linear change of variables we can assume without loss of generality that M_0 has coordinates $(u, v) = (1, 0)$. Then we have $b_i > 0$ whenever K_i is totally imaginary.

We enlarge S so that $b_i \in \mathbb{Z}_S^*$, $e_i \in \mathbb{Z}_S$ and the field K_i is not ramified outside S , for all $i = 1, \dots, r$. Thus for each $p \in S$ we now have a pair $(\lambda_p, \mu_p) \in \mathbb{Q}_p^2$ such that

$$b_i(\lambda_p - e_i\mu_p) = N_{K_i/\mathbb{Q}}(\mathbf{x}_{i,p}) \neq 0, \quad i = 1, \dots, r,$$

for some $\mathbf{x}_{i,p} \in (\mathbb{Q}_p)^{d_i}$. Let C be a large positive constant to be specified later, such that $C > e_i$ for $i = 1, \dots, r$. An application of Proposition 1.2 produces $(\lambda, \mu) \in \mathbb{Z}_S^2$, $\lambda > C\mu > 0$, such that for each i the number $b_i(\lambda - e_i\mu)$ is a local norm for K_i/\mathbb{Q} at each finite place of S . This is also true for the real place because $b_i > 0$ whenever K_i is totally imaginary, and $\lambda - e_i\mu > 0$ for all i . Moreover, for each i we have $b_i(\lambda - e_i\mu) = p_i u_i$, where p_i is a prime not in S and $u_i \in \mathbb{Z}_S^*$. Recall that $p_i = p_j$ if and only if $e_i = e_j$.

Let $(K_i/\mathbb{Q}, b_i(\lambda - e_i\mu)) \in \text{Br}(\mathbb{Q})$ be the class of the corresponding cyclic algebra. By continuity we have $\text{inv}_p(K_i/\mathbb{Q}, b_i(\lambda - e_i\mu)) = 0$ for any $p \in S$, and also $\text{inv}_{\mathbb{R}}(K_i/\mathbb{Q}, b_i(\lambda - e_i\mu)) = 0$. Next, $b_i(\lambda - e_i\mu)$ is a unit at every prime $p \notin S \cup \{p_i\}$, and hence we obtain

$$\text{inv}_p(K_i/\mathbb{Q}, b_i(\lambda - e_i\mu)) = 0$$

for any $p \neq p_i$. The global reciprocity law now implies

$$\text{inv}_{p_i}(K_i/\mathbb{Q}, b_i(\lambda - e_i\mu)) = \text{inv}_{p_i}(K_i/\mathbb{Q}, p_i) = 0,$$

and since K_i/\mathbb{Q} is unramified outside S the prime p_i splits completely in K_i . In particular, $b_i(\lambda - e_i\mu)$ is a local norm at every place of \mathbb{Q} . By Hasse's theorem it is a global norm, so that

$$b_i(\lambda - e_i\mu) = N_{K_i/\mathbb{Q}}(\mathbf{x}_i) \neq 0$$

for some $\mathbf{x}_i \in \mathbb{Q}^{d_i}$. This proves the Hasse principle for V .

Let us now prove weak approximation. Write $d = d_1 \dots d_r$. Using weak approximation in \mathbb{Q} we find a positive rational number ρ that is p -adically close to 1 for each prime $p \in S$, and ρ^d is close to $\lambda > 0$ in the real topology. We now make the change of variables

$$\lambda = \rho^d \lambda', \quad \mu = \rho^d \mu', \quad \mathbf{x}_i = \rho^{d/d_i} \mathbf{x}'_i, \quad i = 1, \dots, r.$$

Then (λ', μ') is still close to (λ_p, μ_p) in the p -adic topology for each prime $p \in S$. In the real topology (λ', μ') is close to $(1, \mu/\lambda)$. Since $0 < \mu/\lambda < C^{-1}$, by choosing a large enough C we ensure that (λ', μ') is close to $(1, 0)$. We can conclude by using weak approximation in the norm tori $N_{K_i/\mathbb{Q}}(\mathbf{z}) = 1$. □

Remarks. (1) In the case when all the fields K_i are quadratic this result is Theorem 1.2 of [BMS14]. Eliminating u and v one sees that V is then isomorphic to an open subset of a complete intersection of $r - 2$ quadrics in \mathbb{A}^{2r} of a very special kind. These intersections of quadrics are important because of their relation to conic bundles, first pointed out by Salberger in [Sal86]. In [CS87b] Colliot-Thélène and Sansuc proved the main results of descent theory and gave a description of universal torsors by explicit equations. Given the conclusion of Theorem 1.3 their results imply that the Brauer–Manin obstruction is the only obstruction to weak approximation on conic bundles over $\mathbb{P}_{\mathbb{Q}}^1$ such that all the degenerate fibres are over \mathbb{Q} -points (this is [BMS14, Theorem 1.1]; see [CS87b, Theorem 2.6.4 and ch. III] or [Sko01, Proposition 4.4.8 and Corollary 6.1.3]).

(2) In §3.3 below we give a somewhat different proof of Theorem 1.3 deducing it from the main result of this paper, Theorem 3.1.

2. Comparison with Schinzel's Hypothesis (H)

Our aim in this section is to show that in the study of rational points on pencils of varieties Proposition 1.2 can replace Schinzel's Hypothesis (H). In such applications it is often more

convenient to use a consequence of (H), stated in [CS94, § 4] under the name of Hypothesis (H₁). We reproduce it below in the case of linear polynomials over \mathbb{Q} . In this case it was already used in [CS82, § 5].

HYPOTHESIS (H₁). *Let e_1, \dots, e_r be pairwise different rational numbers. Let S be a finite set of primes containing the prime factors of the denominators of e_1, \dots, e_r and the primes $p \leq r$. Suppose that we are given $\tau_p \in \mathbb{Q}_p$ for $p \in S$ and a positive real constant C . Then there exist $\tau \in \mathbb{Q}$ and primes p_1, \dots, p_r not in S such that:*

- (1) τ is arbitrarily close to τ_p in the p -adic topology, for $p \in S$;
- (2) $\tau > C$;
- (3) $\text{val}_p(\tau - e_i) = 0$ for any $p \notin S \cup \{p_i\}$, $i = 1, \dots, r$;
- (4) $\text{val}_{p_i}(\tau - e_i) = 1$ for any $i = 1, \dots, r$.

Hypothesis (H₁) is usually supplemented with the following statement. Let K/\mathbb{Q} be a cyclic extension unramified outside S . Assuming the conclusion of (H₁) we have the following implication.

If $\sum_{p \in S} \text{inv}_p(K/\mathbb{Q}, \tau_p - e_i) = 0$ for some i , then p_i splits completely in K/\mathbb{Q} .

Hypothesis (H₁) and its supplement can be compared to the following consequence of Proposition 1.2.

PROPOSITION 2.1. *Let e_1, \dots, e_r be rational numbers. Let S be a finite set of primes containing the prime factors of the denominators of e_1, \dots, e_r . Suppose that we are given $\tau_p \in \mathbb{Q}_p$ for $p \in S$ and a positive real constant C . Then there exist $\tau \in \mathbb{Q}$ and primes p_1, \dots, p_r not in S , with $p_i = p_j$ if and only if $e_i = e_j$, such that:*

- (1) τ is arbitrarily close to τ_p in the p -adic topology, for $p \in S$;
- (2) $\tau > C$;
- (3) $\text{val}_p(\tau - e_i) \leq 0$ for any $p \notin S \cup \{p_i\}$, $i = 1, \dots, r$;
- (4) $\text{val}_{p_i}(\tau - e_i) = 1$ for any $i = 1, \dots, r$;
- (5) for any cyclic extension K/\mathbb{Q} unramified outside S and such that

$$\sum_{p \in S} \text{inv}_p(K/\mathbb{Q}, \tau_p - e_i) = c \in \mathbb{Q}/\mathbb{Z}$$

for some i , we have $\text{inv}_{p_i}(K/\mathbb{Q}, \tau - e_i) = -c$ and, in particular, if $c = 0$, then p_i splits completely in K/\mathbb{Q} .

Proof. By increasing the list of rational numbers e_i we may assume that $r \geq 2$ and $e_i \neq e_j$ for some $i \neq j$. Let us also assume $C > e_i$ for $i = 1, \dots, r$. We then apply Proposition 1.2 to $(\lambda_p, \mu_p) = (\tau_p, 1)$ for $p \in S$. This produces $(\lambda, \mu) \in \mathbb{Z}_S^2$ such that $\tau = \lambda/\mu$ satisfies all the properties in the proposition. Indeed, properties (1) and (2) are clear. For $p \notin S$ we have $\text{val}_p(\mu) \geq 0$. For $p \notin S \cup \{p_i\}$ we have $\text{val}_p(\lambda - e_i\mu) = 0$, so that

$$\text{val}_p(\tau - e_i) = \text{val}_p(\lambda - e_i\mu) - \text{val}_p(\mu) \leq 0,$$

which proves property (3). We claim that $\text{val}_{p_i}(\mu) = 0$ for $i = 1, \dots, r$. Indeed, $\text{val}_{p_i}(\mu) > 0$ implies $\text{val}_{p_i}(\lambda) > 0$; taking j such that $e_i \neq e_j$, we obtain $\text{val}_{p_i}(\lambda - e_j\mu) > 0$, thus contradicting property (3) of Proposition 1.2. This proves property (4). Since (λ, μ) is close to $(\tau_p, 1)$ in the p -adic topology for $p \in S$, by continuity we have

$$\sum_{p \in S} \text{inv}_p(K/\mathbb{Q}, \lambda - e_i\mu) = c.$$

We also have $\lambda - e_i\mu > 0$, and hence $\text{inv}_{\mathbb{R}}(K/\mathbb{Q}, \lambda - e_i\mu) = 0$. By the global reciprocity law of class field theory this implies

$$\sum_{p \notin S} \text{inv}_p(K/\mathbb{Q}, \lambda - e_i\mu) = -c.$$

Since K/\mathbb{Q} is unramified outside S , we have $\text{inv}_p(K/\mathbb{Q}, \lambda - e_i\mu) = 0$ for any prime $p \notin S \cup \{p_i\}$, because in this case $\text{val}_p(\lambda - e_i\mu) = 0$. Thus

$$\text{inv}_{p_i}(K/\mathbb{Q}, \lambda - e_i\mu) = -c.$$

In the case when $c = 0$, we deduce from $\text{val}_{p_i}(\lambda - e_i\mu) = 1$ that p_i splits completely in K , so that property (5) is proved. \square

Remark. One can give a stronger variant of this proposition by proving that τ , in addition to properties (1) to (5), can be chosen in a given Hilbertian subset of \mathbb{Q} . The proof uses Ekedahl's effective version of Hilbert's irreducibility theorem and the fact that any Hilbertian subset of \mathbb{Q} is open in the topology induced by the product topology of $\prod_p \mathbb{Q}_p$, where the product is over all primes. We do not give a detailed proof, because this variant will not be used in this paper.

3. Varieties fibred over the projective line

3.1 Main results

Recall that for a variety X over \mathbb{Q} one denotes the image of $\text{Br}(\mathbb{Q})$ in $\text{Br}(X)$ by $\text{Br}_0(X)$. If $\pi : X \rightarrow \mathbb{P}^1$ is a dominant morphism of integral varieties over \mathbb{Q} , then the corresponding vertical Brauer group is defined as follows:

$$\text{Br}_{\text{vert}}(X) = \text{Br}(X) \cap \pi^* \text{Br}(\mathbb{Q}(\mathbb{P}^1)) \subset \text{Br}(\mathbb{Q}(X)).$$

By a \mathbb{Q} -fibre of $\pi : X \rightarrow \mathbb{P}^1$ we understand a fibre above a \mathbb{Q} -point of \mathbb{P}^1 .

We denote the completions of \mathbb{Q} by \mathbb{Q}_v , and the ring of adèles of \mathbb{Q} by $\mathbf{A}_{\mathbb{Q}}$. Let $\overline{\mathbb{Q}}$ be an algebraic closure of \mathbb{Q} . For a subfield $K \subset \overline{\mathbb{Q}}$ we write Γ_K for the Galois group $\text{Gal}(\overline{\mathbb{Q}}/K)$.

THEOREM 3.1. *Let X be a geometrically integral variety over \mathbb{Q} with a smooth and surjective morphism $\pi : X \rightarrow \mathbb{P}^1$ such that:*

- (a) *with the exception of finitely many \mathbb{Q} -fibres, denoted by X_1, \dots, X_r , each fibre of π contains a geometrically integral irreducible component;*
- (b) *for each i , the fibre X_i contains an irreducible component such that the algebraic closure of \mathbb{Q} in its field of functions is an abelian extension of \mathbb{Q} .*

Then $\mathbb{P}^1(\mathbb{Q}) \cap \pi(X(\mathbf{A}_{\mathbb{Q}}))$ is dense in $\pi(X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}}) \subset \mathbb{P}^1(\mathbf{A}_{\mathbb{Q}}) = \prod_v \mathbb{P}^1(\mathbb{Q}_v)$.

Note that the assumptions of Theorem 3.1 imply that the generic fibre of $\pi : X \rightarrow \mathbb{P}^1$ is geometrically integral. Thus all but finitely many \mathbb{Q} -fibres of π are geometrically integral. The cokernel of the natural map $\text{Br}(\mathbb{Q}) \rightarrow \text{Br}_{\text{vert}}(X)$ is finite by [CS00, Lemma 3.1]. Note finally that when $r = 1$ the statement of Theorem 3.1 is well known; see [CSS98, § 2.1].

Proof of Theorem 3.1. Without loss of generality we can assume that X_i is the fibre above a point $e_i \in \mathbb{A}^1(\mathbb{Q})$, for $i = 1, \dots, r$. Let K_i be the abelian extension of \mathbb{Q} as in assumption (b).

We follow the proof of [CSS98, Theorem 1.1] which uses the same method as [CS94, Theorem 4.2]. Steps 1 and 2 of the proof below repeat the proof of [CSS98, Theorem 1.1] verbatim. Step 3 contains modifications necessary to replace the use of Hypothesis (H₁) by Proposition 2.1.

Step 1. Let us recall a well-known description of $\text{Br}_{\text{vert}}(X)$. Write K_i as a compositum of cyclic extensions K_{ij}/\mathbb{Q} , and let $\chi_{ij} : \Gamma_{\mathbb{Q}} \rightarrow \mathbb{Q}/\mathbb{Z}$ be a character such that K_{ij} is isomorphic to the invariant subfield of $\text{Ker}(\chi_{ij})$. Let t be a coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$ so that $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(t)$. The class

$$A_{ij} = (K_{ij}/\mathbb{Q}, t - e_i) \in \text{Br}(\mathbb{Q}(t))$$

of the corresponding cyclic algebra is ramified on \mathbb{P}^1 only at e_i and ∞ with residues χ_{ij} and $-\chi_{ij}$, respectively. Let $A \in \text{Br}(\mathbb{Q}(t))$ be such that $\pi^*A \in \text{Br}(X)$. Assumptions (a) and (b), together with [CS94, Proposition 1.1.1], imply that A on \mathbb{P}^1 is unramified away from e_1, \dots, e_r , and that the residue of A at e_i belongs to

$$\text{Ker}[\text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\Gamma_{K_i}, \mathbb{Q}/\mathbb{Z})].$$

This group is generated by the characters χ_{ij} . Hence there exist $n_{ij} \in \mathbb{Z}$ such that $A - \sum n_{ij}A_{ij}$ is unramified on \mathbb{A}^1 . Since $\text{Br}(\mathbb{A}^1) = \text{Br}(\mathbb{Q})$ we conclude that $A = \sum n_{ij}A_{ij} + A_0$ for some $A_0 \in \text{Br}(\mathbb{Q})$, and this implies, by considering residues at ∞ , that

$$\sum n_{ij}\chi_{ij} = 0 \in \text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z}). \tag{3.1}$$

Therefore, every element of $\text{Br}_{\text{vert}}(X)$ is of the form $\sum n_{ij}\pi^*A_{ij} + A_0$ for some n_{ij} satisfying (3.1) and some $A_0 \in \text{Br}(\mathbb{Q})$.

Step 2. We can clearly assume that $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}} \neq \emptyset$, otherwise there is nothing to prove. Pick any $(M_p) \in X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}}$, where M_0 is a point in $X(\mathbb{R})$. By a small deformation we can assume that M_p does not belong to any of the fibres X_1, \dots, X_r .

We include the real place in the finite set of places S where we need to approximate. The set of real points $M_0 \in V(\mathbb{R})$ for which $\pi(M_0) \in \mathbb{P}^1(\mathbb{Q})$ is dense in $V(\mathbb{R})$, and so it is enough to approximate adelic points (M_p) such that $\pi(M_0) \in \mathbb{P}^1(\mathbb{Q})$. By a change of variables we then assume that $\pi(M_0) = \infty$. By another small deformation of the points M_p for each prime p we can further assume that $\pi(M_p) \neq \infty$ when $p \neq 0$.

We include in S the primes of bad reduction for X . We ensure that $e_i \in \mathbb{Z}_S$ for all i , that $e_i - e_j \in \mathbb{Z}_S^*$ for all $i \neq j$, and that no prime outside of S is ramified in any of the fields K_i . Furthermore, we increase S so that if K_i has a place of degree 1 over $p \notin S$, then the corresponding \mathbb{F}_p -component of the degenerate fibre of π over the reduction of e_i has an \mathbb{F}_p -point. This is achieved by using the Lang–Weil estimate (see [CSS98, Lemma 1.2]). By a similar argument we assume that on the reduction of X modulo $p \notin S$ any geometrically integral component of a fibre over an \mathbb{F}_p -point contains an \mathbb{F}_p -point. All these \mathbb{F}_p -points are smooth, because π is a smooth morphism.

Since $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}} \neq \emptyset$, by the result of Step 1 we can use Harari’s ‘formal lemma’ [Har94, Corollary 2.6.1] to increase $S \subset S_1$ and choose $M_p \in X(\mathbb{Q}_p)$ for $p \in S_1 \setminus S$ away from the fibres X_1, \dots, X_r so that for all i, j we have

$$\sum_{p \in S_1} \text{inv}_p(A_{ij}(\pi(M_p))) = 0. \tag{3.2}$$

Step 3. Let τ_p be the coordinate of $\pi(M_p)$, where p is a prime in S_1 . An application of Proposition 2.1 produces $\tau \in \mathbb{Q}$ which is an arbitrarily large positive real number, and is close to τ_p in the p -adic topology for the primes $p \in S_1$.

Let us prove that $X_{\tau}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$. By the inverse function theorem we have $X_{\tau}(\mathbb{R}) \neq \emptyset$ and $X_{\tau}(\mathbb{Q}_p) \neq \emptyset$ for $p \in S_1$. Thus it remains to consider the following two cases.

Case 1: $\mathbb{Q}_v = \mathbb{Q}_p$, where $p = p_i$, $i = 1, \dots, r$. Since $\text{val}_{p_i}(\tau - e_i) = 1$, the reduction of τ modulo p_i equals the reduction of e_i . In view of (3.2), property (5) of Proposition 2.1 implies that for each given value of i all the cyclic fields K_{ij} are split at p_i , and thus K_i is also split. Hence there is a geometrically integral irreducible component of the \mathbb{F}_{p_i} -fibre over the reduction of e_i modulo p_i . We arranged that it has an \mathbb{F}_{p_i} -point. By Hensel's lemma it gives rise to a \mathbb{Q}_{p_i} -point in X_τ .

Case 2: $\mathbb{Q}_v = \mathbb{Q}_p$, where $p \notin S_1 \cup \{p_1, \dots, p_r\}$. We have $\text{val}_p(\tau - e_i) \leq 0$ for each $i = 1, \dots, r$, and hence the reduction of τ modulo p is a point of $\mathbb{P}^1(\mathbb{F}_p)$ other than the reduction of any of e_1, \dots, e_r . Thus any \mathbb{F}_p -point on a geometrically integral irreducible component of the fibre at $\tau \bmod p$ gives rise to a \mathbb{Q}_p -point on X_τ , by Hensel's lemma.

In both cases we constructed a \mathbb{Q}_p -point that comes from a \mathbb{Z}_p -point on an integral model of X_τ , and therefore $X_\tau(\mathbf{A}_\mathbb{Q}) \neq \emptyset$. The theorem is proved. \square

COROLLARY 3.2. *In the situation of Theorem 3.1, let us further assume that all but finitely many \mathbb{Q} -fibres of $\pi : X \rightarrow \mathbb{P}^1$ satisfy the Hasse principle. Then $\pi(X(\mathbb{Q}))$ is dense in $\pi(X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}})$. If, in addition, these \mathbb{Q} -fibres X_τ are such that $X_\tau(\mathbb{Q})$ is dense in $X_\tau(\mathbf{A}_\mathbb{Q})$, then $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}}$.*

Remark. If the generic fibre of $\pi : X \rightarrow \mathbb{P}^1$ is proper, then all but finitely many fibres of π are proper. For proper \mathbb{Q} -fibres X_τ the approximation assumption in Corollary 3.2 is that of *weak approximation*, since in this case $X_\tau(\mathbf{A}_\mathbb{Q}) = \prod_v X_\tau(\mathbb{Q}_v)$. By Hironaka's theorem one can always replace $\pi : X \rightarrow \mathbb{P}^1$ by a partial compactification $\pi' : X' \rightarrow \mathbb{P}^1$ such that X is a dense open subset of X' and the morphism π' is smooth with proper generic fibre.

We now give a statement for a smooth and proper variety X , to be compared with [CSS98, Theorem 1.1].

THEOREM 3.3. *Let X be a smooth, proper and geometrically integral variety over \mathbb{Q} with a surjective morphism $\pi : X \rightarrow \mathbb{P}^1$ such that:*

- (a) *with the exception of finitely many \mathbb{Q} -fibres, denoted by X_1, \dots, X_r , each fibre of π contains a geometrically integral irreducible component of multiplicity one;*
- (b) *for each i , the fibre X_i contains an irreducible component of multiplicity one such that the algebraic closure of \mathbb{Q} in its field of functions is an abelian extension of \mathbb{Q} .*

Then $\mathbb{P}^1(\mathbb{Q}) \cap \pi(X(\mathbf{A}_\mathbb{Q}))$ is dense in $\pi(X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}}) \subset \mathbb{P}^1(\mathbf{A}_\mathbb{Q}) = \prod_v \mathbb{P}^1(\mathbb{Q}_v)$. If, moreover, all but finitely many \mathbb{Q} -fibres of π satisfy the Hasse principle and weak approximation, then $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}}$.

Proof. Let $Y \subset X$ denote the smooth locus of π . By assumptions (a) and (b) each fibre of $\pi : X \rightarrow \mathbb{P}^1$ contains a multiplicity one irreducible component, and hence $\pi(Y) = \mathbb{P}^1$. Thus Theorem 3.1 can be applied to $\pi : Y \rightarrow \mathbb{P}^1$. We have $\text{Br}_{\text{vert}}(X) = \text{Br}(X) \cap \text{Br}_{\text{vert}}(Y)$, and by [CS00, Lemma 3.1], the set $Y(\mathbf{A}_\mathbb{Q})^{\text{Brvert}(Y)}$ is dense in $X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}(X)}$. This proves the first statement. The second one now follows from Corollary 3.2. \square

Remarks. (1) We note that assumptions (a) and (b) will hold for any smooth, proper and geometrically integral variety X' over \mathbb{Q} with a surjective morphism $\pi' : X' \rightarrow \mathbb{P}^1$ such that the generic fibres of π and π' are isomorphic. This follows from [Sko96, Lemma 1.1]; see also [Wit07, Lemme 3.8].

(2) Using the remark after Proposition 2.1 one can prove a stronger variant of Theorem 3.3 with ‘all but finitely many \mathbb{Q} -fibres of π ’ replaced by ‘the \mathbb{Q} -fibres of π over a Hilbertian subset of \mathbb{Q} ’. The details are left to the interested reader.

3.2 Application to pencils of Severi–Brauer and similar varieties

Theorem 3.3 can be applied to the varieties considered by Colliot-Thélène and Swinnerton-Dyer in [CS94], see also [CSS98].

COROLLARY 3.4. *Let X be a smooth, proper and geometrically integral variety over \mathbb{Q} with a morphism $\pi : X \rightarrow \mathbb{P}^1$. Suppose that the generic fibre of π is a Severi–Brauer variety (for example, a conic), a two-dimensional quadric, or a product of such. If all the fibres of π that are not geometrically integral are \mathbb{Q} -fibres, then $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}}$.*

Proof. The assumptions of Theorem 3.3 are satisfied for X and π by well-known results of class field theory and by the structure of fibres of regular models of quadric bundles [Sko90] and Artin models of Severi–Brauer varieties [Fro97] (or see [Wit07, pp. 117–118] for an alternative argument). \square

In particular, this result gives a uniform approach to Theorems 1.1, 1.3 and 1.4 of [BMS14] that does not use descent and leads to shorter and more natural proofs. See [BMS14] for a survey of known results on conic and quadric bundles, most of which are established over an arbitrary number field in place of \mathbb{Q} but under a strong restriction on the number of degenerate geometric fibres.

3.3 Theorem 1.3 as a consequence of Theorem 3.1

Here is one way to deduce Theorem 1.3 from Corollary 3.2 that keeps the Brauer group calculations to the minimum.

Let W be the quasi-affine subvariety of $\mathbb{A}^2 \times \mathbb{A}^{d_1} \times \cdots \times \mathbb{A}^{d_r}$ given by

$$b_i(u - e_i v) = N_{K_i/\mathbb{Q}}(\mathbf{x}_i), \quad i = 1, \dots, r, \quad (u, v) \neq (0, 0).$$

The variety V defined in (1.3) is a dense open subset of W . The projection to the coordinates (u, v) defines a morphism $W \rightarrow \mathbb{A}^2 \setminus (0, 0)$. Let $\pi : W \rightarrow \mathbb{P}^1$ be the composed morphism $W \rightarrow \mathbb{A}^2 \setminus (0, 0) \rightarrow \mathbb{P}^1$, and let $X \subset W$ be the smooth locus of π . It is easy to see that $\pi(X) = \mathbb{P}^1$. Let $\pi' : Y \rightarrow \mathbb{P}^1$ be a partial compactification of $\pi : X \rightarrow \mathbb{P}^1$ as in the remark after Corollary 3.2, so that π' is smooth with proper generic fibre.

Let $t = u/v$ be a coordinate on \mathbb{P}^1 . It is straightforward to see that conditions (a) and (b) of Theorem 3.1 hold. In order to deduce Theorem 1.3 from Corollary 3.2 we need to prove that:

- (1) geometrically integral, proper \mathbb{Q} -fibres of π' satisfy the Hasse principle and weak approximation;
- (2) $\text{Br}_{\text{vert}}(Y) = \text{Br}_0(Y)$.

The fibre of π at $\tau \in \mathbb{Q}$, $\tau \neq e_i$, is the affine variety

$$\frac{N_{K_1/\mathbb{Q}}(\mathbf{x}_1)}{b_1(\tau - e_1)} = \cdots = \frac{N_{K_r/\mathbb{Q}}(\mathbf{x}_r)}{b_r(\tau - e_r)} \neq 0.$$

This is a principal homogeneous space of the torus T defined by

$$N_{K_1/\mathbb{Q}}(\mathbf{t}_1) = \cdots = N_{K_r/\mathbb{Q}}(\mathbf{t}_r) \neq 0.$$

The Hasse principle and weak approximation hold for smooth compactifications of principal homogeneous spaces of T if $\text{III}_\omega^2(\mathbb{Q}, \hat{T}) = 0$; see [San81, ch. 8] or [Sko01, Theorem 6.3.1]. Let $T_i = R_{K_i/\mathbb{Q}}^1(\mathbb{G}_m)$ be the norm torus attached to K_i/\mathbb{Q} , and let G_i be the cyclic group $\text{Gal}(K_i/\mathbb{Q})$. Then T is an extension of \mathbb{G}_m by the product of the tori T_i , so that we have the exact sequence of $\Gamma_{\mathbb{Q}}$ -modules of characters

$$0 \rightarrow \mathbb{Z} \rightarrow \hat{T} \rightarrow \prod_{i=1}^r \hat{T}_i \rightarrow 0. \tag{3.3}$$

The long exact sequence of Galois cohomology gives rise to the exact sequence

$$\prod_{i=1}^r \text{Hom}(G_i, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(\mathbb{Q}, \hat{T}) \rightarrow \prod_{i=1}^r H^2(\mathbb{Q}, \hat{T}_i).$$

Let K be the compositum of the fields K_i , and let $G = \text{Gal}(K/\mathbb{Q})$. Since $\text{III}_\omega^2(\mathbb{Q}, \hat{T}_i) = 0$ it is enough to show that if $\alpha \in \text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z})$ goes to $\text{III}_\omega^2(\mathbb{Q}, \hat{T})$ in $H^2(\mathbb{Q}, \hat{T})$, then α belongs to $\text{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Now, the tori T_i and T are split by K , and hence (3.3) is split as an extension of Γ_K -modules. It follows that the restriction of α to $H^2(K, \mathbb{Z}) = \text{Hom}(\Gamma_K, \mathbb{Q}/\mathbb{Z})$ is in $\text{III}_\omega^2(K, \mathbb{Z}) = 0$, and thus $\alpha \in \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$. This proves assertion (1).

Since $\pi : X \rightarrow \mathbb{P}^1$ factors through the inclusion of X into Y , to prove assertion (2) it is enough to prove $\text{Br}_{\text{vert}}(X) = \text{Br}_0(X)$. Write $\chi_i \in \text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z})$ for the image of a generator of $\text{Hom}(G_i, \mathbb{Q}/\mathbb{Z})$. Recall from Step 1 of the proof of Theorem 3.1 that any $A \in \text{Br}(\mathbb{Q}(t))$ such that $\pi^* A \in \text{Br}(X)$ can be written as

$$A = \sum_{i=1}^r n_i(\chi_i, t - e_i) + A_0 \quad \text{where} \quad \sum_{i=1}^r n_i \chi_i = 0,$$

for some $A_0 \in \text{Br}(\mathbb{Q})$. In $\text{Br}(X)$ we have

$$\pi^* A = \sum_{i=1}^r n_i(\chi_i, u - e_i v) - \sum_{i=1}^r n_i(\chi_i, v) + A_0 = - \sum_{i=1}^r n_i(\chi_i, b_i) + A_0 \in \text{Br}_0(X),$$

since $(\chi_i, N_{K_i/\mathbb{Q}}(\mathbf{x}_i)) = 0$ in $\text{Br}(\mathbb{Q}(X))$. This finishes the proof of assertions (1) and (2).

4. Norms and their products as products of linear polynomials

4.1 Cyclic extensions

Consider the following system of Diophantine equations:

$$N_{K_i/\mathbb{Q}}(\mathbf{x}_i) = P_i(t), \quad i = 1, \dots, r, \tag{4.1}$$

where K_i/\mathbb{Q} are cyclic extensions and the polynomials $P_i(t)$ are products of (possibly repeated) linear factors over \mathbb{Q} .

COROLLARY 4.1. *Let X be a smooth, proper and geometrically integral variety over \mathbb{Q} with a morphism $\pi : X \rightarrow \mathbb{P}^1$ such that the generic fibre of π is birationally equivalent to the affine variety (4.1) over $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(t)$. Then $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}}$.*

Proof. Each fibre of π outside infinity and the zero set of $P_1(t) \dots P_r(t) = 0$ contains a geometrically integral irreducible component of multiplicity one. Since π has a section over

the compositum $K_1 \dots K_r$, which is an abelian extension of \mathbb{Q} , assumptions (a) and (b) of Theorem 3.3 hold for $\pi : X \rightarrow \mathbb{P}^1$. By Hasse’s norm theorem the varieties $N_{K/\mathbb{Q}}(\mathbf{z}) = c$, where K/\mathbb{Q} is cyclic and $c \in \mathbb{Q}^*$, satisfy the Hasse principle. Moreover, smooth and proper models of principal homogeneous spaces of cyclic norm tori satisfy the Hasse principle and weak approximation, by [San81, ch. 8]. We conclude by Theorem 3.3. \square

For $r = 1$ the statement of Corollary 4.1 was previously known for any finite extension K/\mathbb{Q} in the case when P_1 has at most two roots. See [HS02], [CHS03]; see also [Swa13], where \mathbb{Q} was replaced by an arbitrary number field.

For any cyclic extension of fields K/k the affine variety $N_{K/k}(\mathbf{x}) = c$, where $c \in k^*$, is well known to be birationally equivalent to the Severi–Brauer variety defined by the cyclic algebra $(K/k, c)$. Thus Corollary 4.1 can be seen as a particular case of Corollary 3.4.

When each polynomial $P_i(t)$ is linear we have the following consequence of Corollary 4.1.

COROLLARY 4.2. *Let K_i be a cyclic extension of \mathbb{Q} of degree d_i , $i = 1, \dots, r$. Let $b_i \in \mathbb{Q}^*$ and $e_i \in \mathbb{Q}$, $i = 1, \dots, r$. Then the variety over \mathbb{Q} defined by*

$$b_i(t - e_i) = N_{K_i/\mathbb{Q}}(\mathbf{x}_i) \neq 0, \quad i = 1, \dots, r,$$

satisfies the Hasse principle and weak approximation.

Proof. An easy calculation shows that this variety X is smooth. By Corollary 4.1 it is enough to prove that $\text{Br}_{\text{vert}}(X) = \text{Br}_0(X)$. In Step 1 of the proof of Theorem 3.1 we saw that for any $A \in \text{Br}(\mathbb{Q}(t))$ such that $\pi^*A \in \text{Br}(X) \subset \text{Br}(\mathbb{Q}(X))$ there exists $A_0 \in \text{Br}(\mathbb{Q})$ for which we can write

$$A = \sum_{i=1}^r n_i(K_i/\mathbb{Q}, t - e_i) + A_0.$$

Since $(K_i/\mathbb{Q}, N_{K_i/\mathbb{Q}}(\mathbf{x}_i)) = 0$ in $\text{Br}(\mathbb{Q}(X))$, the element $\pi^*A \in \text{Br}(X)$ can be written as

$$\pi^*A = - \sum_{i=1}^r n_i(K_i/\mathbb{Q}, b_i) + A_0 \in \text{Br}_0(X). \quad \square$$

The following statement is deduced from Corollary 4.1 by an easy application of the fibration method in the form of [Har97, Theorem 3.2.1].

COROLLARY 4.3. *Let X be a smooth and proper model of the variety over \mathbb{Q} defined by the system of equations*

$$N_{K_i/\mathbb{Q}}(\mathbf{x}_i) = P_i(t_1, \dots, t_n), \quad i = 1, \dots, r, \quad (4.2)$$

where each K_i is a cyclic extension of \mathbb{Q} and each $P_i(t_1, \dots, t_n)$ is a product of polynomials of degree 1 over \mathbb{Q} . Then $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$.

In [SS14], under a mild general position condition, this was proved for $r = 1$ and $\deg P_1 \leq 2n$, but with the cyclic extension K of \mathbb{Q} replaced by any finite extension of an arbitrary number field.

4.2 Products of norms

Instead of a norm form of a cyclic extension of \mathbb{Q} we can consider a product of norm forms associated to field extensions of \mathbb{Q} satisfying certain conditions. We start with one more application of Theorem 3.3.

COROLLARY 4.4. *Let $P(t)$ be a product of (possibly repeated) linear factors over \mathbb{Q} . Let L_1, \dots, L_n be $n \geq 2$ finite field extensions of \mathbb{Q} such that L_1/\mathbb{Q} is abelian and linearly disjoint from the compositum $L_2 \dots L_n$. Let X be a smooth, proper and geometrically integral variety over \mathbb{Q} with a morphism $\pi : X \rightarrow \mathbb{P}^1$ such that the generic fibre of π is birationally equivalent to the affine variety*

$$N_{L_1/\mathbb{Q}}(\mathbf{x}_1) \dots N_{L_n/\mathbb{Q}}(\mathbf{x}_n) = P(t) \tag{4.3}$$

over $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(t)$. Then X satisfies the Hasse principle and weak approximation.

Proof. This proof is similar to that of Corollary 4.1. Assumptions (a) and (b) of Theorem 3.3 are satisfied since L_1/\mathbb{Q} is abelian. To prove that almost all \mathbb{Q} -fibres satisfy the Hasse principle and weak approximation it is enough, by [San81, ch. 8] (see also [Sko01, Theorem 6.3.1]), to verify that $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T}) = 0$, where T is the multinorm torus over \mathbb{Q} attached to the fields L_1, \dots, L_n . This was proved by Demarche and Wei [DW14, Theorem 1]. To finish the proof we note that $\text{Br}_{\text{vert}}(X) = \text{Br}_0(X)$. Indeed, let $A \in \text{Br}(\mathbb{Q}(t))$ be such that $\pi^*A \in \text{Br}(X)$. The morphism π has a section defined over L_i , for each $i = 1, \dots, n$. By restricting A to this section we see that the restriction of A to $\text{Br}(L_i(t))$ comes from $\text{Br}(\mathbb{P}_{L_i}^1) = \text{Br}(L_i)$. In particular, the residues of A at the roots of $P(t)$ are in the kernel of the map $H^1(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(L_i, \mathbb{Q}/\mathbb{Z})$. Since $L_1 \cap L_2 \dots L_n = \mathbb{Q}$, there is no non-trivial cyclic extension of \mathbb{Q} contained in all of the L_i . This implies that A is not ramified at the zero set of $P(t)$. A well-known argument [CS94, Proposition 1.1.1] shows that A is unramified away from the zero set of $P(t)$. Hence $A \in \text{Br}(\mathbb{A}^1) = \text{Br}(\mathbb{Q})$. \square

For more cases when $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T}) = 0$ for the multinorm torus T see [DW14, Theorem 1 and Corollary 8]. One can extend Corollary 4.4 to systems of equations (4.3) and replace $P(t)$ by a product of linear polynomials in several variables. We leave the details to the interested reader.

Following Wei [Wei14, Theorem 3.5] we now consider a case where the \mathbb{Q} -fibres do not satisfy the Hasse principle nor weak approximation.

PROPOSITION 4.5. *Let $P(t)$ be a product of (possibly repeated) linear factors over \mathbb{Q} , and let $a, b \in \mathbb{Q}^*$. Let X be a smooth, proper and geometrically integral variety over \mathbb{Q} with a morphism $\pi : X \rightarrow \mathbb{P}^1$ such that the generic fibre of π is birationally equivalent to the affine variety*

$$N_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\mathbf{x})N_{\mathbb{Q}(\sqrt{b})/\mathbb{Q}}(\mathbf{y})N_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}}(\mathbf{z}) = P(t) \tag{4.4}$$

over $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(t)$. Then $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$.

Proof. We can assume that $\mathbb{Q}(\sqrt{a})$, $\mathbb{Q}(\sqrt{b})$ and $\mathbb{Q}(\sqrt{ab})$ are quadratic fields, otherwise the variety X is rational and the statement is clear. Let V be the smooth locus of the affine variety (4.4), and let U be the image of V by the projection to the coordinate t . It is clear that $\mathbb{P}^1 \setminus U$ is a finite union of \mathbb{Q} -points. The fibres of $V \rightarrow U$ are principal homogeneous spaces of the torus T that is given by

$$N_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\mathbf{x})N_{\mathbb{Q}(\sqrt{b})/\mathbb{Q}}(\mathbf{y})N_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}}(\mathbf{z}) = 1.$$

Let E be a smooth equivariant compactification of T (which exists by [CHS05]), and let $V^c = V \times^T E$ be the contracted product. Then $V^c \rightarrow U$ is a fibre-wise smooth compactification of

$V \rightarrow U$. We take $\pi : X \rightarrow \mathbb{P}^1$ such that $X \times_{\mathbb{P}^1} U = V^c$. We compose π with an automorphism of \mathbb{P}^1 to ensure that the fibre at infinity is smooth and is close to the real point that we need to approximate; in particular, the fibre at infinity contains a real point. An obvious change of variables shows that X contains an open set which is the smooth locus of the affine variety given by

$$N_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\mathbf{x})N_{\mathbb{Q}(\sqrt{b})/\mathbb{Q}}(\mathbf{y})N_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}}(\mathbf{z}) = Q(t),$$

where $Q(t)$ is a polynomial with rational roots e_1, \dots, e_r such that U is the complement to $\{e_1, \dots, e_r\}$ in \mathbb{P}^1 . Note that for any $\tau \in U(\mathbb{Q})$ we have $X_\tau(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ by [Col14, Proposition 5.1].

The quaternion algebra $A = (N_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\mathbf{x}), b)$ defines an element of $\text{Br}(\pi^{-1}(U))$.

We are given points $M_p \in X(\mathbb{Q}_p)$ for all primes p and $M_0 \in X(\mathbb{R})$ such that $(M_p) \in X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$. Since $\text{Br}(X)/\text{Br}_0(X)$ is finite, by a small deformation we can assume that $\pi(M_p)$ is a point in $U \cap \mathbb{A}^1$ where t equals $\tau_p \in \mathbb{Q}_p$.

Let S_0 be the finite set of places of \mathbb{Q} where we need to approximate. We can find a finite set S of places containing S_0 and the real place, such that $\pi : X \rightarrow \mathbb{P}^1$ extends to a proper morphism $\pi : \mathcal{X} \rightarrow \mathbb{P}_{\mathbb{Z}_S}^1$ with \mathcal{X} regular. By doing so we can ensure that S contains the primes where at least one of our quadratic fields is ramified, and that we have $a, b \in \mathbb{Z}_S, Q(t) \in \mathbb{Z}_S[t]$, and $e_i \in \mathbb{Z}_S$ for $i = 1, \dots, r$. By Harari’s ‘formal lemma’ [Har94, Corollary 2.6.1] we can further enlarge S so that

$$\sum_{p \in S} \text{inv}_p(A(M_p)) = 0, \quad \sum_{p \in S} \text{inv}_p(b, \tau_p - e_i) = 0, \quad i = 1, \dots, r.$$

(For this we may need to modify the points M_p for $p \in S \setminus S_0$.) Let \mathcal{U} be the complement to the Zariski closure of $e_1 \cup \dots \cup e_r$ in $\mathbb{P}_{\mathbb{Z}_S}^1$. The same algebra A defines a class in $\text{Br}(\pi^{-1}(\mathcal{U}))$. An application of Proposition 2.1 gives a \mathbb{Q} -point τ in $U \cap \mathbb{A}^1$ that is as large as we want in the real topology and is close to τ_p in the p -adic topology for the primes $p \in S$. For $p \notin S \cup \{p_1, \dots, p_r\}$ we see from property (3) of Proposition 2.1 that the Zariski closure of τ in $\mathbb{P}_{\mathbb{Z}_p}^1$ is contained in $\mathcal{U} \times_{\mathbb{Z}_S} \mathbb{Z}_p$. This implies that for any $N_p \in X_\tau(\mathbb{Q}_p)$ the value $A(N_p) \in \text{Br}(\mathbb{Q}_p)$ comes from $\text{Br}(\mathbb{Z}_p) = 0$. From property (5) we see that for each $i = 1, \dots, r$ the prime p_i splits in $\mathbb{Q}(\sqrt{b})$, and hence $A(N_{p_i}) = 0$ for any $N_{p_i} \in X_\tau(\mathbb{Q}_{p_i})$. By continuity and the inverse function theorem we can find $N_p \in X_\tau(\mathbb{Q}_p)$ close enough to M_p , for $p \in S$ (including the real place), so that $\sum_{p \in S} \text{inv}_p(A(N_p)) = 0$. Summing over all places of \mathbb{Q} we now have $\sum_p \text{inv}_p(A(N_p)) = 0$, for any choice of $N_p, p \notin S$. By [Col14, Theorem 4.1] the algebra A generates $\text{Br}(X_\tau)$ modulo the image of $\text{Br}(\mathbb{Q})$. By [San81, ch. 8] or [Sko01, Theorem 6.3.1] the set $X_\tau(\mathbb{Q})$ is dense in $X_\tau(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$, so we can find a \mathbb{Q} -point in X_τ close to M_p , for $p \in S$. \square

4.3 Non-cyclic extensions of prime degree

The method of Colliot-Thélène and Wei [Wei14, Theorem 3.6] can be used to prove the following result.

THEOREM 4.6. *Let $P(t)$ be a product of (possibly repeated) linear factors over \mathbb{Q} . Let K be a non-cyclic extension of \mathbb{Q} of prime degree such that the Galois group of the normal closure of K over \mathbb{Q} has a non-trivial abelian quotient. Let X be a smooth, proper and geometrically integral variety over \mathbb{Q} with a morphism $\pi : X \rightarrow \mathbb{P}^1$ such that the generic fibre of π is birationally equivalent to the affine variety $N_{K/\mathbb{Q}}(\mathbf{x}) = P(t)$ over $\mathbb{Q}(t)$. Then X satisfies the Hasse principle and weak approximation.*

This result covers the ‘generic’ case of the field $K = \mathbb{Q}[t]/(f(t))$, for which $f(t)$ is a polynomial of prime degree $\ell > 2$ such that the Galois group of $f(t)$ is the symmetric group S_ℓ (or the dihedral group $D_{2\ell}$). In particular, the assumption on K in Theorem 4.6 is automatically satisfied when $[K : \mathbb{Q}] = 3$.

Proof of Theorem 4.6. We start in the same way as in the proof of Proposition 4.5. We can assume that X contains an open set which is the smooth locus of the affine variety $N_{K/\mathbb{Q}}(\mathbf{x}) = Q(t)$, where $Q(t)$ is a product of powers of $t - e_i$, $i = 1, \dots, r$, with the additional assumption that the fibre at infinity is smooth and contains a real point close to the real point that we want to approximate.

LEMMA 4.7. *We have $\text{Br}(X) = \text{Br}_0(X)$.*

Proof. Let T be the norm torus $N_{K/\mathbb{Q}}(\mathbf{x}) = 1$. Since $\ell = [K : \mathbb{Q}]$ is prime, [CS87a, Propositions 9.1 and 9.5] gives $H^1(F, \text{Pic}(Z \times_F \bar{F})) = \text{III}_\omega^2(F, \hat{T}) = 0$ for any smooth and proper variety Z over a field F such that a dense open subset of Z is a principal homogeneous space of T . Applying this to the generic fibre of $\pi : X \rightarrow \mathbb{P}^1$ we see that $\text{Br}(X) = \text{Br}_{\text{vert}}(X)$.

Now let $A \in \text{Br}(\mathbb{Q}(t))$ be such that $\pi^*A \in \text{Br}(X)$. The morphism π has a section defined over K . By restricting to it we see that the image of A in $\text{Br}(K(t))$ belongs to the injective image of $\text{Br}(\mathbb{P}_K^1) = \text{Br}(K)$. In particular, the residue of A at e_i lies in the kernel of the map

$$H^1(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(K, \mathbb{Q}/\mathbb{Z}).$$

Since K contains no cyclic extension of \mathbb{Q} , this kernel is zero. Thus A is not ramified at the zero set of $Q(t)$. Since A is also unramified outside of the zero set of $Q(t)$, we see that $A \in \text{Br}(\mathbb{Q})$. \square

Let L be the normal closure of K/\mathbb{Q} . By assumption, there exists a cyclic extension k/\mathbb{Q} of prime degree such that $k \subset L$. Let $\ell = [K : \mathbb{Q}]$, $q = [k : \mathbb{Q}]$. Since $\text{Gal}(L/\mathbb{Q}) \subset S_\ell$ and $k \neq K$ we see that $q < \ell$.

LEMMA 4.8. *Let $a \in \mathbb{Q}^*$. If p is a prime unramified in L and inert in k , then the equation $N_{K/\mathbb{Q}}(\mathbf{x}) = a$ is solvable in \mathbb{Q}_p .*

Proof. Write $K \otimes_{\mathbb{Q}} \mathbb{Q}_p = K_{v_1} \oplus \dots \oplus K_{v_s}$, and let $d_i = [K_{v_i} : \mathbb{Q}_p]$.

If $s > 1$, then since $\ell = d_1 + \dots + d_s$ is a prime number, there exist integers n_1, \dots, n_s such that $1 = n_1 d_1 + \dots + n_s d_s$. It follows that

$$a = \prod_{i=1}^s N_{K_{v_i}/\mathbb{Q}_p}(a^{n_i}) \in N_{K/\mathbb{Q}}(K \otimes_{\mathbb{Q}} \mathbb{Q}_p),$$

and so we are done.

If $s = 1$, then $K \otimes_{\mathbb{Q}} \mathbb{Q}_p = K_v$ is a field extension of \mathbb{Q}_p of degree ℓ . By assumption, p is inert in k , so that $k \otimes_{\mathbb{Q}} \mathbb{Q}_p = k_w$ is a field. Since $[k_w : \mathbb{Q}_p] = q$ is a prime less than ℓ , the fields k_w and K_w are linearly disjoint over \mathbb{Q}_p , so that $k_w K_v$ is a field. Thus p is inert in $kK \subset L$, which implies that the Frobenius at p in $\text{Gal}(L/\mathbb{Q})$ is an element of order divisible by ℓq . However, S_ℓ contains no such elements, so the case $s = 1$ is impossible. \square

End of proof of Theorem 4.6. We are given points $M_p \in X(\mathbb{Q}_p)$ for all primes p and $M_0 \in X(\mathbb{R})$. By a small deformation we can assume that $\pi(M_p)$ is a point in $U \cap \mathbb{A}^1$ where t equals $\tau_p \in \mathbb{Q}_p$. Let S be the finite set of places of \mathbb{Q} where we need to approximate, containing the real place

and the primes of bad reduction for X . We also assume that L is unramified away from S . Consider the cyclic algebras

$$A_i = (k/\mathbb{Q}, t - e_i) \in \text{Br}(\mathbb{Q}(X)), \quad i = 1, \dots, r.$$

Harari’s ‘formal lemma’ [Har94, Corollary 2.6.1] and Lemma 4.7 imply that we can introduce new primes into S and choose the corresponding points M_p so that

$$\sum_{p \in S} \text{inv}_p(A_i(\tau_p)) \neq 0, \quad i = 1, \dots, r.$$

An application of Proposition 2.1 gives a \mathbb{Q} -point τ in $U \cap \mathbb{A}^1$ that is as large as we want in the real topology and is close to τ_p in the p -adic topology for the primes $p \in S$. This ensures that $X_\tau(\mathbb{R}) \neq \emptyset$ and $X_\tau(\mathbb{Q}_p) \neq \emptyset$ for all $p \in S$. For $p \notin S \cup \{p_1, \dots, p_r\}$ we see from property (3) of Proposition 2.1 that τ reduces modulo p to a point of $\mathbb{P}^1(\mathbb{F}_p)$ other than the reduction of any of e_1, \dots, e_r . The corresponding fibre over \mathbb{F}_p contains a principal homogeneous space of a torus over a finite field, and hence an \mathbb{F}_p -point, by Lang’s theorem. By Hensel’s lemma it gives rise to a \mathbb{Q}_p -point in X_τ . Finally, property (5) of Proposition 2.1 gives $\text{inv}_{p_i}(A_i(\tau)) \neq 0$. By property (4) this implies that p_i is inert in k . Now an application of Lemma 4.8 shows that $X_\tau(\mathbb{Q}_{p_i}) \neq \emptyset$. This holds for all $i = 1, \dots, r$, and so we conclude that $X_\tau(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$. To finish the proof we note that $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T}) = 0$ implies that the principal homogeneous spaces of T over \mathbb{Q} satisfy the Hasse principle and weak approximation [San81, ch. 8]. \square

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