Higher semiadditivity, higher monoids, and the ∞-category of finite spans

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July 27, 2016

Let Pr^{L} be the ∞-category of presentable ∞-categories and left functors between them (equivalently, colimit preserving functors). The ∞-category Pr^{L} admits a symmetric monoidal structure such that if $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}$ are presentable ∞-categories then $\mathcal{C} \otimes \mathcal{D}$ is a presentable ∞-category equipped with a map $p : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C} \otimes \mathcal{D}$ satisfying the following property: for every presentable ∞-category $\mathcal{E}$, restriction along p induces a fully-faithful inclusion

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \xrightarrow{p^*} \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

whose essential image is spanned by those functors $\mathcal{C} \times \mathcal{D} \longrightarrow W$ which preserve colimits in each variable separately. By a **tensor** ∞-category we shall mean a commutative algebra object in $(\mathrm{Pr}^{\mathrm{L}}, \otimes)$. Alternatively, a tensor ∞-category is a symmetric monoidal ∞-category $\mathcal{C}$ such that $\mathcal{C}$ is presentable and the monoidal product preserves colimits in each variable separately. If $\mathcal{C}$ is a tensor ∞-category and $\mathcal{D}$ is a presentable ∞-category, we will say that $\mathcal{D}$ is **tensored** over $\mathcal{C}$ if it carries a structure of a module over $\mathcal{C}$ in $(\mathrm{Pr}^{\mathrm{L}}, \otimes)$. In this case we will also say that $\mathcal{D}$ carries a $\mathcal{C}$-tensor structure. This means that $\mathcal{D}$ carries an action $\mathcal{C} \otimes \mathcal{D} \longrightarrow \mathcal{D}$ which preserves colimits in each variable separately. Consequently, for every $c \in \mathcal{C}$ the functor $c \otimes (-) : \mathcal{D} \longrightarrow \mathcal{D}$ admits a right adjoint $(-)^c : \mathcal{D} \longrightarrow \mathcal{D}$ (the associated **cotensor structure**), and for each $d \in \mathcal{D}$ the functor $(-) \otimes d : \mathcal{C} \longrightarrow \mathcal{D}$ admits a right adjoint $\underline{\mathrm{Hom}}(d, (-)) : \mathcal{D} \longrightarrow \mathcal{C}$ (the associated **enriched mapping object**). In particular, when $\mathcal{D}$ is tensored over $\mathcal{C}$ it is also canonically enriched in $\mathcal{C}$.

Given a tensor ∞-category $\mathcal{C}$, any presentable ∞-category $\mathcal{D}$ can carry many non-equivalent $\mathcal{C}$-tensor structures. However, in some situations, a $\mathcal{C}$-tensor structure is unique if is exists. For example, if $\mathcal{C} = \mathcal{S}_*$ is the tensor category of pointed spaces (and smash product) then $\mathcal{D}$ is tensored over $\mathcal{C}$ if and only if it is **pointed** (i.e., the initial and terminal objects of $\mathcal{D}$ coincide), in which case the $\mathcal{C}$-tensor structure is essentially unique. Similarly, if $\mathcal{C} = \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ is the ∞-category of commutative (or $\mathbb{E}_\infty$) monoids in spaces, then being tensored over $\mathcal{C}$ is the same thing as being **semi-additive**, i.e., being pointed and having coproducts which are also products.

The following lemma suggests where one can find tensor ∞-categories which have this uniqueness-of-structure property:

Proposition 1. *Let $\mathcal{C}$ be a tensor ∞-category. The following conditions are equivalent:*

1. *The forgetful functor* $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}) \longrightarrow \mathrm{Pr}^{\mathrm{L}}$ *is fully-faithful.*

2. *For every $\mathcal{C}$-module $\mathcal{D}$, the structure map $\mathcal{C} \otimes \mathcal{D} \longrightarrow \mathcal{D}$ is an equivalence.*

3. *The monoidal product $\mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C}$ is an equivalence.*

4. *$\mathcal{C}$ is (-1)-cotruncated as a tensor ∞-category, i.e., if $\mathcal{D}$ is any other tensor ∞-category then the ∞-category* $\mathrm{Fun}^{\mathrm{ten}}(\mathcal{C}, \mathcal{D})$ *of monoidal left functors is either contractible or empty.*

Definition 2. Following T. Schlank we define a **mode** to be a tensor ∞-category satisfying the equivalent conditions of Proposition 1. If $\mathcal{C}$ is a mode then we will say that $\mathcal{C}$ **classifies** the property of being tensored over $\mathcal{C}$.

Remark 3. The notion of a mode can be considered as a categorification of the notion of a **solid ring**. Indeed, a module structure over a solid ring is unique as soon as it exists.

Examples 4.

1. The tensor ∞-category $\mathcal{S}_*$ of pointed spaces is a mode, which classifies the properties of being **pointed**.

2. The tensor ∞-category $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ and commutative monoids is a mode, which classifies the property of being **semi-additive**.

3. For each n, the tensor ∞-category $\mathcal{S}_{\leq n}$ of n-truncated spaces is a mode, which classifies the property of being an $(n+1)$**-presentable category**. In particular, the category of sets is a mode which classifies the property of being an ordinary presentable category.

4. The tensor ∞-category of spectra is a mode, which classifies the property of being **stable**.

5. For every field k, the tensor category of k-vector spaces is a mode, which classifies the property of being k**-linear**.

6. If $\mathcal{C}$ is a mode and $\mathcal{D}$ is a symmetric monoidal accessible localization of $\mathcal{C}$ then $\mathcal{D}$ is a mode. This follows from characterization (4) above since the localization (tensor) functor $\mathcal{L} : \mathcal{C} \longrightarrow \mathcal{D}$ is (-1)-cotruncated.

By Proposition 1 the full subcategory of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ spanned by modes is a poset, i.e., the space of symmetric monoidal left functors between each two is either empty or contractible. We will say that a mode $\mathcal{C}$ is **minimal** if every symmetric monoidal left functor out of $\mathcal{C}$ is an equivalence. The problem of classifying all minimal modes is a very interesting open question in higher category theory. We may consider such modes as the closed points of the "spectrum" of Pr^{L}.

In this talk we will focus on the first two examples above, namely $\mathcal{S}_*$ and $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$. We claim that these form the first two steps in an infinite sequence of modes, which can be considered as **higher versions** of the notion of commutative monoids. Consider a space X. The structure of an $\mathbb{E}_\infty$-monoid on X can be informally described as a rule which associates to each finite collection of points $x_1, ..., x_n$ in X their **sum** $x_1 + ... + x_n \in X$. Using what may appear as an artificial analogy at this point, we can encode the collection $x_1, ..., x_n$ as a map $f : S \longrightarrow X$ where S is a finite (possibly empty) set, and write $\int_S f = x_1 + ... + x_n$ for the sum. The condition of being pointed can be considered as a similar "integral" operation, but defined only for sets S with $|S| \leq 1$. More precisely if (X, x_0) is a pointed space, then for $S = \{*\}$ a singleton the integral is given by $\int_S f = f(*)$, and if $S = \emptyset$ then the integral of the unique map $S \longrightarrow X$ is the base point x_0. We hence see that the structure of pointedness is naturally associated with the category of (-1)-truncated sets (or spaces), while the structure of a commutative monoid is associated with the category of finite sets (or finite 0-truncated spaces). We may hence contemplate a generalization of this idea when we replace S with, say, n-truncated spaces, satisfying suitable finiteness conditions. More precisely, we will consider the following types of spaces:

Definition 5. Let X be a space. For $n \geq 0$ we say that X is ***n*-truncated** if $\pi_i(X, x) = 0$ for every $i > n$ and every $x \in X$. We will say that X is (-1)-truncated if it is either empty of contractible and that X is (-2)-truncated if it is contractible. We will say that a map $f : X \longrightarrow Y$ is ***n*-truncated** if the homotopy fiber of f over every point of Y is n-truncated. Let X be a space. Then X is said to be **π-finite** if it is n-truncated for some n and all its homotopy groups/sets are finite. We will denote by $\mathcal{K}_n$ a set of representatives for the equivalence types of π-finite n-truncated spaces.

In order to properly define our higher notion of a monoid we will need to consider suitable ∞-categories of **spans**. Let $\mathcal{S}$ denote the ∞-category of spaces and let $\mathrm{Span}(\mathcal{S})$ denote the ∞-category whose objects are spaces and such that the mapping space from X to Y is the classifying space of spans

$$\begin{array}{ccccc} & & Z & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\ X & & & & Y \end{array} \tag{0.1}$$

Composition of spans is given by the formation of pullbacks, see, e.g., [1], [2] for rigorous constructions. Given integers $-2 \leq m \leq n$ we let $\mathcal{S}_n^m \subseteq \mathrm{Span}(\mathcal{S})$ denote the subcategory spanned by those objects $X \in \mathrm{Span}(\mathcal{S})$ which are π-finite and n-truncated as spaces and those spans as in 0.1 for which p is m-truncated. When $m = -2$ we will also denote by $\mathcal{S}_n = \mathcal{S}_n^{-2}$, noting that $\mathcal{S}_n$ is just the ∞-category of n-truncated π-finite spaces and ordinary maps between them.

Definition 6. Let $m \geq -1$ be an integer and let $\mathcal{D}$ be an ∞-category admitting $\mathcal{K}_m$-indexed limits. An **m-commutative monoid** in $\mathcal{D}$ is functor $\mathcal{F} : \mathcal{S}_m^{m-1} \longrightarrow$

$\mathcal{D}$ with the following property: for every $X \in \mathcal{S}_m^{m-1}$ the collection of maps $\mathcal{F}(i_x^*) : \mathcal{F}(X) \longrightarrow \mathcal{F}(*)$ exhibits $\mathcal{F}(X)$ as the limit in $\mathcal{D}$ of the constant X-indexed diagram with value $\mathcal{F}(*)$. We will denote by $\mathrm{CMon}_m(\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ the full subcategory spanned by those functors which are m-commutative monoids.

Example 7. If $m = -1$ then $\mathcal{S}_m^{m-1} = \mathcal{S}_{-1}^{-2} = \mathcal{S}_{-1}$ is the ∞-category of (-1)-truncated spaces and ordinary maps between them. In particular, we may identify $\mathcal{S}_{-1}$ with the category consisting of two objects $\emptyset, *$ and a unique non-identity morphism $\emptyset \longrightarrow *$. An ∞-category $\mathcal{D}$ admits $\mathcal{K}_{-1}$-indexed limits exactly when it has a final object. A functor $\mathcal{S}_{-1} \longrightarrow \mathcal{D}$ is completely determined by the associated morphism $\mathcal{F}(\emptyset) \longrightarrow \mathcal{F}(*)$ in $\mathcal{D}$. By definition such a functor $\mathcal{F}$ is a (-1)-commutative monoid if and only if $\mathcal{F}(\emptyset)$ is a terminal object of $\mathcal{D}$. We may hence identify $\mathrm{CMon}_{-1}(\mathcal{D})$ with the full subcategory of the arrow category of $\mathcal{D}$ spanned by those arrows $A \longrightarrow B$ for which A is a final object. In particular, if we fix a particular final object $\star \in \mathcal{D}$ then we may form an equivalence $\mathrm{CMon}_{-1}(\mathcal{D}) \simeq \mathcal{D}_{\star/}$. In other words, we may identify $\mathrm{CMon}_{-1}(\mathcal{D})$ with the ∞-category of **pointed objects** $\mathcal{D}$.

Example 8. If $m = 0$ then we may identify $\mathcal{S}_m^{m-1} = \mathcal{S}_0^{-1}$ with the category whose objects are finite sets, and such that a morphism from a finite set A to a finite set B is a pair (C, f) where C is a subset of A and $f : C \longrightarrow B$ is a map. In particular, $\mathcal{S}_0^{-1}$ is equivalent to the nerve of a discrete category. By sending a finite set A to the pointed set $A_+ = A \coprod \{*\}$ and sending a map (C, f) to the map $f' : A_+ \longrightarrow B_+$ which restricts to f on C and sends $A \setminus C$ to the base point of B_+ we obtain an equivalence $\mathcal{S}_0^{-1} \simeq \mathcal{F}\mathrm{in}_*$, where $\mathcal{F}\mathrm{in}_*$ is the category of finite pointed sets. To say that an ∞-category $\mathcal{D}$ has $\mathcal{K}_0$-indexed limits is to say that $\mathcal{D}$ admits finite products. Unwinding the definitions we see that a functor $\mathcal{S}_0^{-1} \longrightarrow \mathcal{D}$ is a 0-commutative monoid object if and only if the corresponding functor $\mathcal{F}\mathrm{in}_* \longrightarrow \mathcal{D}$ is a commutative monoid object in the sense of [4, Definition 2.4.2.1], also known as an $\mathbb{E}_\infty$**-monoid**. When $\mathcal{D}$ is the ∞-category of spaces this notion of commutative monoids was first developed by Segal under the name **special Γ-spaces**.

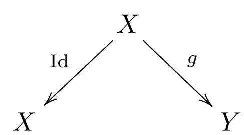
To get a feel for what these higher commutative monoids are, let us consider the example of the ∞-category $\mathcal{S}$ of spaces. Let $\mathcal{F} : \mathcal{S}_m^{m-1} \longrightarrow \mathcal{S}$ be an m-commutative monoid object and let us refer to $M = \mathcal{F}(*)$ as the **underlying space** of $\mathcal{F}$. We may then identify two types of morphisms in $\mathcal{S}_m^{m-1}$. The first type are morphisms of the form

$$\begin{array}{ccc} & X & \\ {}^{f}\swarrow & & \searrow^{\mathrm{Id}} \\ Y & & X \end{array}$$

where f is $(m-1)$-truncated, which we shall write as $\widehat{f} : Y \longrightarrow X$. These morphisms help us to identify the spaces $\mathcal{F}(X)$: by definition, the collection of maps $\widehat{i}_x : X \longrightarrow *$ exhibit $\mathcal{F}(X)$ as the limit of the constant X-indexed diagram with value $\mathcal{F}(*) = M$. In particular, we may identify $\mathcal{F}(X)$ with the mapping space $\mathrm{Map}_{\mathcal{S}}(X, M)$. Other morphisms of the form $\widehat{f} : Y \longrightarrow X$ don't really give more information: if $f : X \longrightarrow Y$ is an $(m-1)$-truncated map then for every $x \in X$ we have $\widehat{i}_x \circ \widehat{f} = \widehat{i}_{f(x)}$, and so the induced map

$$\widehat{f}_* : \mathrm{Map}_{\mathcal{S}}(Y, M) \simeq \mathcal{F}(Y) \longrightarrow \mathcal{F}(X) \simeq \mathrm{Map}_{\mathcal{S}}(X, M)$$

is forced to coincide with the restriction along f. The second type of morphisms in $\mathcal{S}_m^{m-1}$ are the spans of the form



where $g : X \longrightarrow Y$ is any map of π-finite m-truncated spaces. We can think of the associated map $g_* : \mathrm{Map}_{\mathcal{S}}(X, M) \longrightarrow \mathrm{Map}_{\mathcal{S}}(Y, M)$ as carrying the **structure** of M. Let X_y be homotopy fiber of g over $y \in Y$, equipped with its natural map $i_{X_y} : X_y \longrightarrow X$, and let $p_y : X_y \longrightarrow \{y\}$ be the constant map. Then $\widehat{i}_y \circ g = p_y \circ \widehat{i}_{X_y}$ and so for each $\varphi \in \mathrm{Map}_{\mathcal{S}}(X, M)$ the function $g_*(\varphi) \in \mathrm{Map}_{\mathcal{S}}(Y, M)$ maps the point y to the point $(p_y)_*(\varphi|_{X_y}) \in M$. Hence the core structure here is concentrated in the action of constant maps $p : X \longrightarrow *$, which is given by some map $p_* : \mathrm{Map}(X, M) \longrightarrow M$. In other words, the structure of being an m-commutative monoid means that for every m-truncated space X we can take an X-family $\{\varphi(x)\}_{x \in X}$ of points in M and "integrate" it to obtain a new point $p_*(\varphi) \in M$. The compatibility conditions that this integration satisfies is encoded in the fact that if we present X as a total space of a fibration $X \longrightarrow Y$ and we have a map $f : X \longrightarrow M$ then then we have a "twisted Fubini" path in M relating $\int_{x \in X} f(x)$ and $\int_{y \in Y} \int_{x \in F_y} f(x)$.

If $\mathcal{D}$ is a presentable ∞-category then $\mathrm{Mon}_m(\mathcal{D})$ is an accessible localization of $\mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ and in particular presentable. We note that $\mathcal{S}_m^{m-1}$ carries a natural symmetric monoidal structure inherited from the symmetric monoidal structure of $\mathrm{Span}(\mathcal{S})$ (see [2, Theorem 1.3(iv)]), which is given on the level of objects by $(X, Y) \mapsto X \times Y$, and on the level of morphisms by taking levelwise Cartesian products of spans. This structure induces a symmetric monoidal structure on $\mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ via Day convolution, and this structure descends to a symmetric monoidal structure on $\mathrm{Mon}_n(\mathcal{D})$, which we can think of as a **tensor product** of commutative monoids. In particular, $\mathrm{Mon}_n(\mathcal{D})$ is a tensor ∞-category.

The first result we wish to state is the following:

Proposition 9. *For every $m \geq -1$ the ∞-category $\mathrm{Mon}_m(\mathcal{S})$ is a mode.*

At this point one is led to wonder what is the property classified by $\mathrm{Mon}_m(\mathcal{S})$. This is closely related to the notion of **ambidexterity**, developed by Lurie and Hopkins. Ambidexterity is a category-theoretic phenomenon concerning diagrams $p : K \longrightarrow \mathcal{C}$ which have the property that their limits and colimit coincide. The simplest case where this can happen is when K is empty. In this case a colimit of K is simply an initial object of $\mathcal{C}$, and a limit of K is a final object of $\mathcal{C}$. Given that both $\emptyset$ and $*$ exist there is a canonical way to require that they coincide, namely, asserting that the unique map $\emptyset \longrightarrow *$ is an equivalence. In this case we say that $\mathcal{C}$ is **pointed**. An object $0 \in \mathcal{C}$ which is both initial and final is called a **zero object**.

Generalizing this property to cases where K is non-empty involves an immediate hic-up. In general, even if $p : K \longrightarrow \mathcal{C}$ can be extended to both a limit and a colimit diagram, there is apriori no natural choice of a map relating the limit and the colimit. Informally speaking, choosing a map $\mathrm{colim}(p) \longrightarrow \lim(p)$ is the same as choosing compatibly for every two objects $x, y \in K$ a map $p(x) \longrightarrow p(y)$ in $\mathcal{C}$. The diagram p, on its part, provides such a map $p(e) : p(x) \longrightarrow p(y)$ for every $e \in \mathrm{Map}_K(x, y)$. The problem is that now we have a whole space of maps $p(x) \longrightarrow p(y)$ in our disposal, but no way to pin-point any specific one (certainly not in a way that would be natural in x and y).

To see how this problem might be resolved assume for a moment that $\mathcal{C}$ is a pointed, and that $\mathrm{Map}_K(x, y)$ is either empty or contractible (i.e. K is equivalent to a partially ordered set, or a poset). Then for every $X, Y \in \mathcal{C}$ there is a distinguished point in $\mathrm{Map}_{\mathcal{C}}(X, Y)$, namely the essentially unique map which factors as $f : X \longrightarrow 0 \longrightarrow Y$, where $0 \in \mathcal{C}$ is a zero object. We may call this map $X \longrightarrow Y$ the **zero map**. Then we do have a choice of a map $f_{x,y} : p(x) \longrightarrow p(y)$ which is natural in x and y: if $\mathrm{Map}_K(x, y)$ is contractible then we just take $f_{x,y}$ to be $p(e)$ for the essentially unique map $e : x \longrightarrow y$, and if $\mathrm{Map}_K(x, y)$ is empty then we just take the zero map. It is then meaningful to ask whether the limits and colimit of a diagram $p : K \longrightarrow \mathcal{C}$ coincide: assuming both of them exist, we may ask whether the map $\mathrm{colim}(p) \longrightarrow \lim(p)$ we have just constructed is an equivalence.

For general posets and general pointed ∞-categories $\mathcal{C}$ it turns out that the map $\mathrm{colim}(p) \longrightarrow \lim(p)$ is rarely an equivalence. For example, if $K = [1]$ then our map $p(1) \simeq \mathrm{colim}(p) \longrightarrow \lim(p) \simeq p(0)$ is the 0-map, and is hence an equivalence if and only if both $p(0)$ and $p(1)$ are zero objects. However, there is a class of posets for which this property turns out to yield something more interesting, namely, the class of **discrete finite posets**, i.e., finite posets for which the order relation is the equality. In this case we may consider K as a **finite set** and identify $\mathrm{colim}(p) \simeq \coprod_{x \in K} p(x)$ and $\lim(p) \simeq \prod_{x \in K} p(x)$. The map

$$\mathcal{F}_p : \coprod_{x \in K} p(x) \longrightarrow \prod_{x \in K} p(x)$$

we constructed above is then given by the "matrix" of maps $[f_{x,y}]_{x,y \in K}$, where $f_{x,y} : p(x) \longrightarrow p(y)$ is the identity if $x = y$ and the zero map if $x \neq y$. When a pointed ∞-category satisfies the property that $\mathcal{F}_p$ is an equivalence for every finite set K and every diagram $p : K \longrightarrow \mathcal{C}$ we say that $\mathcal{C}$ is **semiadditive**. In the world of discrete categories examples of semiadditive categories include all abelian categories. Similarly, every stable ∞-category is semiadditive. However, semiadditivity is a strictly weaker property. For example, the category of commutative monoids is semiadditive but not abelian. Similarly, if $\mathcal{C}$ is any ∞-category with finite products then the ∞-category $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ of $\mathbb{E}_\infty$-monoids in $\mathcal{C}$ is semiadditive.

In their paper [6], Lurie and Hopkins develop the theory of ambidexterity further by observing that the passage from pointed ∞-categories to semiadditive ones is just a first step in a more general process. Suppose, for example, that $\mathcal{C}$ is a semiadditive ∞-category. Then for every $X, Y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ carries a natural structure of an $\mathbb{E}_\infty$-monoid, where the sum of two maps $f, g : X \longrightarrow Y$ is given by the composition

$$X \stackrel{\mathrm{diag}}{\longrightarrow} X \times X \stackrel{f \times g}{\longrightarrow} Y \times Y \simeq Y \coprod Y \stackrel{\mathrm{codiag}}{\longrightarrow} Y.$$

Now suppose that K is an ∞-category whose mapping spaces are finite and discrete and that $p : K \longrightarrow \mathcal{C}$ is a diagram which admits both a limit and colimit. Then we may construct a natural map $\mathrm{colim}(p) \longrightarrow \lim(p)$ by choosing, for every $x, y \in K$, the map $\sum_{e \in \mathrm{Map}_K(x,y)} p(e) : p(x) \longrightarrow p(y)$, where the sum is taken with respect to the natural $\mathbb{E}_\infty$-monoid structure on $\mathrm{Map}_{\mathcal{C}}(X, Y)$. The resulting map

$$\mathrm{Nr}_p : \mathrm{colim}(p) \longrightarrow \lim(p)$$

is known as the **norm map**. We may now ask when the norm map is an equivalence. Again, it turns out that this rarely happens in general. However, if K is assumed in addition to be ∞-groupoid (and hence a finite discrete groupoid by our assumption) then this map is an equivalence in many interesting examples. For example, this holds when $\mathcal{C}$ is the category of vector spaces (or chain complexes) of a field of characteristic 0. When the norm map is an equivalence for every finite grouopoid K we say that $\mathcal{C}$ is **1-semiadditive**.

The definition of n-semiaddivity for higher n is a bit subtle: to define what it means for $\mathcal{C}$ to be n-semiadditive we need to assume that it is already $(n-1)$-semiadditive. Instead of trying to reproduce the definition of Lurie and Hopkins, let us try to continue along the same route as above. We saw that in order to define the notion of 1-semiadditivity, we needed to assume that $\mathcal{C}$ is already 0-semiadditive, and we needed to use the fact that in this case $\mathcal{C}$ is naturally enriched in commutative monoids. The passage from $(n-1)$-semiadditivity to n-semiadditivity can be similarly described using a suitable enrichment in n-commutative monoids. Informally speaking, we can define inductively the notion of n-semiadditivity using the following logic:

1. Being (-1)-semiadditive is to be pointed, and to be 0-semiadditive is to be semiadditive in the usual sense.

2. To define the notion of n-semiadditive, one must first have a notion of $(n-1)$-semiadditive. This notion should have the following property: if $\mathcal{C}$ is $(n-1)$-semiadditive then $\mathcal{C}$ should admit $\mathcal{K}_{n-1}$-indexed limits and colimits (which coincide) and should be naturally enriched in $(n-1)$-commutative monoids.

3. Let $\mathcal{C}$ be an $(n-1)$-semiadditive ∞-category. Suppose we have a diagram $p : K \longrightarrow \mathcal{C}$ where K is an ∞-category whose mapping spaces are all in $\mathcal{K}_{n-1}$, and suppose that p admits both a limit and a colimit. Then we may use the enrichment in $\mathrm{CMon}_n(\mathcal{S})$ to construct a natural **norm map**

$$\mathrm{Nr}_p : \mathrm{colim}(p) \longrightarrow \lim(p).$$

The data of such a map is equivalent to a compatible collection of maps $\mathrm{Nr}_{x,y} : p(x) \longrightarrow p(y)$ for $(x, y) \in K^{\mathrm{op}} \times K$. In this case $\mathrm{Nr}_{x,y}$ will be given by the "integral"

$$\int_{\alpha \in \mathrm{Map}_{\mathcal{C}}(x,y)} p(\alpha) \in \mathrm{Map}_{\mathcal{C}}(p(x), p(y))$$

which is defined using the $(n-1)$-commutative monoid structure of $\mathrm{Map}_{\mathcal{C}}(p(x), p(y))$ and the fact that $\mathrm{Map}_K(x, y)$ is in $\mathcal{K}_{n-1}$.

4. If $\mathcal{C}$ is $(n-1)$-semiadditive then we say that it is n-semiadditive if it admits $\mathcal{K}_n$-indexed limits and colimits and such that for every $K \in \mathcal{K}_n$ and every diagram $p : K \longrightarrow \mathcal{C}$ the norm map $\mathrm{Nr}_p : \mathrm{colim}(p) \longrightarrow \lim(p)$ is an equivalence.

We may now describe our main results:

Theorem 10. *The ∞-category $\mathrm{CMon}_n(\mathcal{S})$ is the free n-semiadditive presentable ∞-category generated by a single object. If $\mathcal{D}$ is any other presentable ∞-category, then $\mathcal{D}$ is tensored over $\mathrm{CMon}_n(\mathcal{S})$ if and only if $\mathcal{D}$ is n-semiadditive, in which case the $\mathrm{CMon}_n(\mathcal{S})$-tensor structure is unique. In other words, the mode $\mathrm{CMon}_n(\mathcal{S})$ classifies the property of being n-semiadditive.*

Corollary 11. *The ∞-category of K_n-local spectra is tensored over n-commutative monoids.*

The proof of Theorem 10 is obtained by first proving that $\mathcal{S}^n_n$ itself is the free n-semiadditive ∞-category generated by a single object. One then shows that $\mathrm{CMon}_n(\mathcal{S})$ is the presentable completion of $\mathcal{S}^n_n$, considered as a small ∞-category with $\mathcal{K}_n$-indexed colimits, yielding the same universal property for $\mathrm{CMon}_n(\mathcal{S})$. More generally, if $\mathcal{D}$ is any presentable ∞-category, then $\mathrm{CMon}_n(\mathcal{D}) \simeq \mathrm{CMon}(\mathcal{S}) \otimes \mathcal{D}$ is the free presentable n-semiadditive ∞-category generated from $\mathcal{D}$. This relationship between $\mathcal{S}^n_n$ and $\mathrm{CMon}(\mathcal{S})$ means in particular that there is a natural yoneda type functor $\mathcal{S}^n_n \longrightarrow \mathrm{CMon}(\mathcal{S})$ which can be considered as sending $X \in \mathcal{S}^n_n$ to the free n-commutative monoid generated from X. We note that this functor is also symmetric monoidal. One motivation for this circle of ideas is their relation to **topological field theories**. Indeed, if X is a π-finite n-truncated space then there is a 1-dimensional TFT $\mathrm{Bord}^{\mathrm{un}}_1 \longrightarrow \mathcal{S}^n_n$ which sends a 0-manifold M to $\mathrm{Map}(M, X)$ and sends a cobordism W from M_1 to M_2 to $\mathrm{Map}(W, X)$, considered as a span from $\mathrm{Map}(M_1, X)$ to $\mathrm{Map}(M_2, X)$. Composing this TFT with the yoneda map $\mathcal{S}^n_n \longrightarrow \mathrm{CMon}_n(\mathcal{S})$ one obtains a TFT valued in n-commutative monoids. By the universal property of $\mathrm{CMon}_n(\mathcal{S})$ one may transport this TFT to any other n-semiadditive presentable tensor ∞-category $\mathcal{D}$, such as the ∞-category of K_n-local spectra. More generally, one may obtain a TFT starting from a π-finite n-truncated space X and an X-indexed family of dualizable objects in $\mathcal{D}$. It is quite desirable to obtain higher dimensional generalizations of this result, which would allow one to construct higher dimensional TFT's. This would be related to the ∞-category $2-\mathcal{S}_n^n$ of spans of spans, whose objects are π-finite n-truncated spaces and such that the ∞-category of morphisms from Y to Z is the ∞-category of spans in $(\mathcal{S}_n^n)_{/Y\times Z}$. Fixing a π-finite n-truncated space X one obtains a 2-dimensional TFT $\mathrm{Bord}_2^{\mathrm{un}} \longrightarrow 2-\mathcal{S}_n^n$ which sends a 0-manifold M to $\mathrm{Map}(M,X)$, a 1-cobordism W from M_1 to M_2 to $\mathrm{Map}(W,X)$ considered as a span from $\mathrm{Map}(M_1,X)$ to $\mathrm{Map}(M_2,X)$ and a 2-cobordism Z from W_1 to W_2 to $\mathrm{Map}(Z,X)$ considered as a bordism from $\mathrm{Map}(W_1,X)$ to $\mathrm{Map}(W_2,X)$. One should then be able to transport this TFT to a Pr^{L}-valued TFT by composing with a natural 2-functor which associates to X the tensor ∞-category $\mathrm{Fun}(X,\mathrm{CMon}(\mathcal{S})) \simeq \mathrm{CMon}(\mathrm{Fun}(X,\mathcal{S}))$ of local systems of m-commutative monoids on X. To understand this 2-functor, observe that $\mathrm{Map}_{2-\mathcal{S}_n^n}(X,Y)$ is the free n-semiadditive ∞-category generated from the ∞-groupoid $X \times Y$, while, sense $\mathrm{CMon}(\mathrm{Fun}(X,\mathcal{S}))$ is the free presentable n-semiadditive ∞-category generated from the ∞-groupoid X we have that

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{CMon}(\mathrm{Fun}(X,\mathcal{S})),\mathrm{CMon}(\mathrm{Fun}(Y,\mathcal{S}))) \simeq \mathrm{Fun}(X,\mathrm{CMon}(\mathrm{Fun}(Y,\mathcal{S}))) \simeq$$
$$\simeq \mathrm{Fun}(X\times Y,\mathrm{CMon}(\mathcal{S}))$$

is the free presentable n-semiadditive ∞-category generated from $X \times Y$. Under suitable conditions, one should be able to transport this from $\mathrm{CMon}(\mathcal{S})$ to other n-semiadditive presentable ∞-categories, such as $\mathcal{K}_n$-local spectra.

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