Obstruction and Classification Theory for Stably Framed Smooth Manifolds

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1 Introduction

Let M be a smooth oriented n-manifold. By a theorem of Whitney M can be embedded in some Euclidian space $M \hookrightarrow \mathbb{R}^{n+k}$ and we can take the normal bundle $\nu \longrightarrow M$ to that inclusion.

It turns our that if we take two different embedding we would get vector bundles ν_1, ν_2 which are stably equivalent, i.e. $\nu_1 \oplus \mathbb{R}^a \cong \nu_2 \oplus \mathbb{R}^b$ for some $a, b \in \mathbb{N}$. Hence they define the same element in $\nu \in \widetilde{K}^0(M)$. This element is called the **stable normal bundle** of M. If one does not wish to describe ν by choosing some representing vector bundle one can also say it is a principle SO fibration, and then there is no choice.

We are interested in smooth closed oriented manifolds which are **stably framed** (these would be called stably framed manifolds from no on). This means in particular that the stable normal bundle is trivial $\nu = 0$ but we also add the information of a specific **trivialization**, also called a **stable framing**.

To be more specific we choose some representing vector bundle $\nu \longrightarrow M$ (obtained by as the normal bundle of some embedding $M \hookrightarrow \mathbb{R}^{n+k}$) and choose a specific isomorphism $\nu \cong \mathbb{R}^k$. If thinks of ν as a principle SO fibration then a trivialization is an isomorphism with the trivial SO fibration. Note that if M is a stably framed manifold then the various stable framings are classified by homotopy classes of maps $M \longrightarrow SO$.

We wish to study the classification theory of these objects. The theory we are about the present works for ordinary oriented manifolds as well but it is somewhat simpler in the stably framed case. Further more in many cases the passage from the stably framed case to the ordinary case is quite simple. In particular we will obtain information on the smooth category in concrete examples following Kervaire's paper.

Note that every smooth manifold has the homotopy type of a finite CW complex. Hence one can divide the classification process of stably framed n-manifolds into two steps:

1. Find which finite CW complexes are homotopy equivalent to stably framed manifolds.

2. Classify the isomorphism classes of stably framed manifolds with a prescribed homotopy type (isomorphisms are required to preserve the orientation and the stable framing).

If we complete these two steps then once the homotopy theorists are done classifying finite CW complexes then we would understand all stably framed manifolds...

In these notes we will talk mainly on the first part, but the tools described here can also be used to tackle the second. But first of all we need to work in right category.

Let X be a finite CW complex. We want to determine whether X is homotopy equivalent to a stably framed manifold. First of all if X is homotopy equivalent to an orientable manifold then it needs to satisfy Poincare duality. Hence we define:

Definition 1.1. An n-dimensional **Poincare complex** is a CW complex X together with a choice of an element $[X] \in H_n(X)$ (where by omitting the coefficients we mean integer coefficients) such that the cap product map

$$(-) \cap [X] : H^k(X) \longrightarrow H_{n-k}(X)$$

is an isomorphism.

Further more we will restrict attention to simply connected Poincare complexes, so from no on all Poincare complexes are assumed to be simply connected.

Now let X be an n-dimensional Poincare complex. Is that enough to make X homotopy equivalent to a stably framed manifold?

2 Stably Framed Cobordism and the First Obstruction

Now suppose that we have a Poincare complex X. In order to see X is homotopy equivalent to a stably equivalent manifold we first consider general maps $M \longrightarrow X$ where M is a stably framed manifold. Such maps can be classified if one introduces an appropriate notion of stably framed cobordism.

Definition 2.1. Let M_1, M_2 be two stably framed n-manifolds with chosen embeddings $M_1 \longrightarrow \mathbb{R}^{n+k}$ and $M_2 \longrightarrow \mathbb{R}^{n+k}$, normal bundles ν_1, ν_2 and trivialization $f_1 : \nu_1 \xrightarrow{\simeq} \mathbb{R}^k$, $f_2 : \nu_2 \xrightarrow{\simeq} \mathbb{R}^k$. A **stably framed cobordism** between M_1 and M_2 is an $(n+1)$-dimensional manifold with boundary $(W, \partial W)$ with an embedding $\iota : W \longrightarrow \mathbb{R}^{n+k} \times \mathbb{R}$, a normal bundle ν, a trivialization $F : \nu \longrightarrow \mathbb{R}^k$ and a diffeomorphism $F : M_1 \coprod M_2 \xrightarrow{\simeq} \partial W$ such that

1. The map $\iota \circ F$ embeds M_1 in $\mathbb{R}^{n+k} \times \{0\}$ nd M_2 in $\mathbb{R}^{n+k} \times \{1\}$ and coincides with the given embeddings of M_1 and M_2 under the obvious identification $\mathbb{R}^{n+k} \times \{0\} \cong \mathbb{R}^{n+k} \times \{0\} \cong \mathbb{R}^{n+k}$

2. The ν coincides with ν_1 and ν_2 when restricted to ∂W.

3. The trivialization F coincides with f_1 and f_2 when restricted to ∂W.

Note that we have chosen the same k for both manifolds, but this is not a restriction because we can always find a k in which we can represent both normal bundles.

The set stably framed cobordism classes of manifolds over X has a natural group structure obtained by disjoint union. This gives functor into groups which actually form a homology theory. The surprising work of Pontryagin, Thom and others shows that this homology theory is actually isomorphic to the homology theory of stable homotopy groups. In particular the stably framed cobordism group in dimension n over X is isomorphic to $\pi_n^s(X_+)$ (the plus there is because π_n^s is a reduced homology theory and stably framed cobordism is unreduced).

Let us try to explain how this works. We will explain it for the case of $X = *$, i.e. we will describe the correspondence between the n'th stably framed cobordism group and the n'th stable homotopy group of $*_+ = S^0$. Let $\alpha \in \pi_n^s(S^0)$ be an element represented by a map $f : S^{n+k} \longrightarrow S^k$. It can be shown that any such map is homotopic to a smooth map with regard to the standard smooth structures on the spheres, so we can assume f is smooth. One can then take a regular value $x \in S^k$ and consider the fiber $M = f^{-1}(x)$. Then M is a smooth n-manifold embedded in S^{n+k} or by removing a point in $\mathbb{R}^{n+k}$. The map f induces an isomorphism of vector bundles between the normal bundle of M in this embedding and the trivial vector bundle $f^*T_xS^k$. Hence we obtain a stably framed manifold.

Note that we have made several choices here: we could have changed f up to homotopy or change the point x. It can be shown that these changes will change M by a stably framed cobordism. We can also replace f by Σf. This will correspond to changing the embedding $M \hookrightarrow \mathbb{R}^{n+k}$ to $M \hookrightarrow \mathbb{R}^{n+k+1}$ by adding another coordinate to everything. This change the normal bundle to another representative of the stable normal bundle.

In the other direction let M be a stably framed manifold M with an embedding $\iota : M \longrightarrow \mathbb{R}^{n+k}$, a normal bundle ν and a framing $f : \nu \cong \mathbb{R}^k$. Let $M \subseteq N$ be a normal neighborhood of M. Then N can be identified with the total space of ν. Using the map f we obtain a map $N \longrightarrow \mathbb{R}^k$. This map extends to a map $S^{n+k} \longrightarrow \mathbb{R}^k \cup \{\infty\} = S^k$ by sending $S^{n+k} \setminus N$ to ∞ which gives us an element in $\pi_n^s(S^0)$. Again it can be showed that if we change M by a stably framed cobordism or change the embedding ι the resulting element in $\pi_n^s(S^0)$ will not change.

Returning to the stably framed cobordism group of X we see that there is a Hurewitz map $\pi_n^s(X_+) \longrightarrow \widetilde{H}_n(X_+) = H_n(X)$ which is realized geometrically by associating to a map $f : M \longrightarrow X$ the image $f_*[M] \in H_n(X)$. We are interested in finding a homotopy equivalence and so as a first step we need a map $f : M \longrightarrow X$ inducing an isomorphism on H_n (such maps will be called maps of degree 1 for the obvious reason). This will correspond to an element $\alpha \in \pi_n^s(X)$ which is sent to $[X]$ by the Hurewitz map.

Hence in order for X to be homotopy equivalent to a stably framed manifold it is necessay that the element $[X] \in H_n(X)$ will lift to $\pi_n^s(X)$. This can be considered as the first obstruction to X being homotopy equivalent to a stably framed manifold.

Comment: this obstruction can be generalized to answer the more difficult question - is X homotopy equivalent to any smooth manifold. The corresponding theory was developed mainly by Michael Spivak. See Spivak normal fibrations.

Now suppose that $[X]$ does lift to $\pi_n^s(X)$ so we have maps $f : M \longrightarrow X$ of degree 1. We wish to answer the question: can we change f by a stably framed cobordism such that it becomes a homotopy equivalence? It turns out that the answer is always yes if n is odd, but when n is even one gets an obstruction. This obstruction takes values in $\mathbb{Z}$ when n is divisible by 4 and $\mathbb{Z}/2$ when $n = 4k+2$. In these notes we will only discuss the case of $n = 4q + 2$.

Example:

Consider the 6-dimensional Poincare complex $X = (S^3 \vee S^3) \cup_\alpha e^6$ obtained via the gluing map $\alpha : S^5 \longrightarrow S^3 \vee S^3$ given by

$$\alpha = [\iota_1, \iota_2] + \iota_{1*}(\beta)$$

where $\iota_1, \iota_2 : S^3 \longrightarrow S^3 \vee S^3$ are the two natural embeddings and $\beta : S^5 \longrightarrow S^3$ is the composition

$$S^5 \xrightarrow{\Sigma^2 h} S^4 \xrightarrow{\Sigma h} S^3$$

of suspensions of the hopf map $h : S^3 \longrightarrow S^2$.

X is a Poincare complex because the $[\iota_1, \iota_2]$ part of the gluing map makes the cup product pairing non-degenerate on $H_3(X)$. The map $[\iota_1, \iota_2]$ is stably trivial but the map β is stably non-trivial, hence:

$$\Sigma^\infty X \simeq \Sigma^\infty((S^3 \cup_\beta e^6) \vee S^3) \not\simeq \Sigma^\infty X(S^6 \vee S^3 \vee S^3)$$

In particular the top class $[X] \in H_n(X)$ is not stably spherical so X is not homotopy equivalent to any stably framed manifold.

In fact, we claim that X is not homotopy equivalent to any smooth manifold! Since $\pi_n(BSO) = 0$ for $n = 3, 6$ we get that all vector bundles on X are stably trivial. Hence if X was homotopy equivalent to a smooth manifold then it would be equivalent to a stably framed one. It can even be showed that X doesn't have the homotopy type of any topological manifold, but this is more difficult.

3 Surgery Theory and Surgery Obstructions

On of the basic tools in constructing cobordisms is **surgery**. Let us first start with the simpler case of ordinary cobordism rather than stably framed cobordism. Let M_1 be a closed oriented n-manifold for $n \geq 5$ and let $f : S^p \hookrightarrow M$ be an embedded sphere with $2p \leq n$. We want to replace M_1 with a cobordant manifold in such a way that preserves the homotopy groups in dimension $< p$ but kills $[f] \in \pi_p(M)$.

Suppose the normal bundle to S^p in M is **trivial**. Then we can extend f to an embedding $\overline{f} : S^p \times D^q \longrightarrow M_1$ where $p + q = n$. Let M be the manifold obtained from M_1 by removing the image of f. Then M is a manifold with boundary $S^p \times S^{q-1}$. First of all it clear that M is still connected. From Van-Kampen's theorem we also see that $\pi_1(M) \cong \pi_1(M_1)$. Then from the long exact sequence in homology (with $\mathbb{Z}$ coefficients)

$$... \longrightarrow H_m(S^p \times S^{q-1}) \longrightarrow H_m(S^p \times D^q) \oplus H_m(M) \longrightarrow H_m(M_1) \longrightarrow H_{m-1}(S^p \times S^{q-1}) \longrightarrow ...$$

that $H_m(M) \cong H_m(M_1)$ for $m < q - 1$.

Now define

$$M_2 = M \coprod_{S^p \times S^{q-1}} \overline{D^{p+1} \times S^{q-1}}$$

The reverse orientation is needed to make the orientation of M_2 consistent. Then the same considerations show that $\pi_1(M_2) \cong \pi_1(M)$ and that $H_m(M_2) \cong H_m(M)$ for $m < p$. From Hurewitz's theorem we get that $\pi_i(M_2) = \pi_i(M_1)$ for $i < \min(p, q - 1)$.

Now suppose for a moment that $p < q - 1$. Then $H_p(M) \cong H_p(M_1)$ and $H_p(M_2)$ is the quotient of $H_p(M)$ by the element which is the hurewitz image of $[f]$. This is why this process can be used to make M more and more connected, at least until we get the middle dimensions.

We then say that M_2 is obtained form M_1 by a k-dimensional **surgery step**. The point is that M_2 is actually cobordant to M_1. The appropriate W can be constructed as follows: Define

$$W_1 = M_1 \times I$$

$$W_2 = M_2 \times I$$

$$D = D^{p+1} \times D^q$$

Then

$$\partial W_1 = M_1 \coprod \overline{M_1} = M_1 \coprod \left[\overline{M} \coprod_{S^p \times S^{q-1}} \overline{S^p \times D^q} \right]$$

$$\partial W_2 = M_2 \coprod \overline{M_2} = \left[M \coprod_{S^p \times S^{q-1}} \overline{D^{p+1} \times S^{q-1}} \right] \coprod \overline{M_2}$$

$$\partial D = [S^p \times D^q] \coprod_{S^p \times S^{q-1}} \left[D^{p+1} \times S^{q-1} \right]$$

Thus we can consistently glue these three manifolds together:

$$W = W_1 \coprod_M W_2 \coprod_{\partial D} D$$

and obtain an oriented manifold with boundary $\partial W = M \coprod \overline{N}$, so M and N are cobordant. Note that the new manifold

By using Morse theory, one can show that each cobordism can be partitioned into a finite composition of cobordisms of the form above, hence two manifolds are cobordant if and only if one can be obtained from the other by a finite sequence of surgery steps.

Now suppose that we are working with cobordisms over a given space X, i.e. we have a map $f : M \longrightarrow X$. We can then use the surgery operations in order to make f more and more connected, i.e. more and more close to a homotopy equivalence.

In order to make surgery work in the relative case we want to kill elements in the relative homotopy groups, rather then the homotopy groups them selves. To be precise, we want to kill elements in the homotopy fiber of F of f. Such elements can be represented by a diagrams of the form

$$\begin{array}{ccc} S^p & \longrightarrow & M \\ \downarrow & & \downarrow \\ D^{p+1} & \longrightarrow & X \end{array}$$

up to a homotopy preserving the commutativity of the diagram. Then if we can extends this diagram to a diagram

$$\begin{array}{ccc} S^p \times D^q & \longrightarrow & M \\ \downarrow & & \downarrow \\ D^{p+1} \times D^q & \longrightarrow & X \end{array}$$

such that the top horizontal map is an embedding, then we can do the surgery construction on the embedding, only now we have enough information to extend f along W, making into a cobordism over X. Let F' be the new fiber. Then similarly to before, if $p < q-1$ then $\pi_i(F')$ will be isomorphic to $\pi_i(F)$ for $i < p$ and at $i = p$ will be a quotient by the element that we wanted to kill. Hence as long as we can represent elements in $\pi_i(F)$ by embeddings with trivial normal bundles we can kill them using surgery.

3.1 Surgery Steps in Stably Framed Cobordism

Now suppose that M_1 is a stably framed manifold. If the cobordism W we've constructed supports a structure of stably framed cobordism we say that we can do a stable surgery on f. The stably framed case has a very important advantage which makes surgery actually effective in killing homotopy groups.

Consider for a moment the non-relative case (or equivalently the over a sphere case). We have a manifold M and we want to surger it into a homotopy sphere. Suppose M is $m-1$ connected and we are trying to kill an element $[\alpha] \in \pi_m(M)$. The first problem is that we need to the represent $[\alpha]$ by an **embedding** $\alpha : S^m \longrightarrow M$. For this we use an important embedding theorem of Whitney:

Theorem 3.1. *Let $f : N^m \longrightarrow M^n$ be a map of connected manifolds with $n - m > 2$. Then if either $2m < n$ or $2m \leq n$ and N is simply connected then f is homotopic to embedding.*

Note that since we have Poincare duality we only need to make an n-manifold $\lfloor \frac{n}{2} \lceil$-connected in order for it to be a homotopy sphere. This is basically why we need to assume that $\dim(M) \geq 5$: in that case can assume that $n - m > 2$ and that by the time $2m = n$, M is already simply connected.

The second problem is that we don't want just any embedding $\alpha : S^m \longrightarrow M$, we need one with a trivial normal bundle. Here comes handy the fact that M is stably framed. Let ν_α be the normal bundle to S^m in M and let ν_M be a (trivialized) bundle representing the stable normal bundle of M. Then $\nu_M \oplus S^m$ represents the stable normal bundle of S^m, which is trivial. But ν_M is already trivial, which means that ν_α is stably trivial. If $2m < n$ then the rank $n - m$ of ν_α is bigger the dimension of S^m and then the theory of vector bundles tells us that it is in the stable range, i.e. if it is stably trivial then it is trivial. Hence we only encounter a problem when $2m = n$ (which in particular only happen when n is odd).

Theorem 3.2. *The group* $\ker(\pi_{m-1}(SO(m)) \longrightarrow \pi_{m-1}(SO))$ *of stably trivial vector bundles of degree n is cyclic and is generated by the tangent bundle. If* $n = 1, 3, 7$ *the tangent bundle is trivial so this group is* 0. *If n is odd and not* $1, 3, 7$ *then the order of this group is* 2. *If m is even it is infinitely cyclic.*

Proof. From the stable range concept above we see that

$$\ker(\pi_{m-1}(SO(m)) \longrightarrow \pi_{m-1}(SO)) = \ker(\pi_{m-1}(SO(m)) \longrightarrow \pi_{m-1}(SO(m+1)))$$

Consider the fibration sequence $SO(m) \longrightarrow SO(m+1) \longrightarrow S^m$. Then we get that

$$\longrightarrow \pi_m(SO(m+1)) \longrightarrow \pi_m(S^m) \longrightarrow \pi_{m-1}(SO(m)) \longrightarrow \pi_{m-1}(SO(m+1)) \longrightarrow ...$$

and since $\pi_m(S^m) = \mathbb{Z}$ we get that the desired kernel is cyclic. It is a nice geometric exercise to show that the image of the identity $1 \in \pi_m(S^m)$ is actually the tangent bundle (which is nice because it gives the tangent bundle some homotopic naturality).

When $m = 1, 3, 7$ S^m is an H-space and so the tangent bundle is trivial. For the other cases we need to understand the map $f : \pi_m(SO(m+1)) \longrightarrow \pi_m(S^m) = \mathbb{Z}$. This map takes an $m+1$ bundle on S^{m+1} and returns its Euler class (times the identity).

When m is odd but not $1, 3, 7$ then the Hopf invariant one problem shows us that the image of f is contained in the even numbers. Since the tangent bundle then has Euler class 2 we get the desired result.

When m is even $m + 1$ and so we can use this theorem for the odd case and so $\pi_m(SO(m+1))$ is mapped to $\pi_m(SO)$ with kernel $\mathbb{Z}/2$. But since m is even $\pi_m(SO)$ is either 0 or $\mathbb{Z}/2$ so $\pi_m(SO(m+1))$ is a 2-torsion group and can't carry any non-zero map into $\mathbb{Z}$. □

Hence when $n = 2m$ and we try to surger an element $[\alpha] \in \pi_m(M)$ represented by an embedding $\alpha : S^m \hookrightarrow M$ we get an obstruction which is that the normal bundle to S^m in M is required to be trivial. If m is even this obstruction leaves in $\mathbb{Z}$. If m is odd and not $1, 3, 7$ then this obstruction lies in $\mathbb{Z}/2$. If $m = 1, 3, 7$ then we don't get an obstruction at this point.

3.2 Making The Surgery Step Stably Framed

So now suppose that this obstruction vanishes and we can represent $[\alpha]$ by an embedding $\alpha : S^m \hookrightarrow M$ with a trivial normal bundle. We still need to make sure that we can do the surgery in a stably framed manner, i.e. do s stably framed cobordism.

By general theorems on can make the cobordism W embed in $\mathbb{R}^{n+k+1}$ in a way such a way that the first two properties are satisfied. The problem will be with extending the trivialization.

Let $D^{p+1} = D^{p+1} \times \{0\} \subseteq D \subseteq W$ and its boundary $S^p = f(S^p \times \{0\}) \subseteq M_1$. Let η be the normal bundle to D^{p+1} in $\mathbb{R}^{n+k+1}$. Then η is trivial (and has only one trivialization) because D^{p+1} is contractible. This induces a trivialization of $\eta|_{S^p}$ which is also the normal bundle to S^p in $\mathbb{R}^{n+k} \times \{0\}$.

The bundle $\eta|_{S^p}$ has a sub-bundle which is $\nu_1|_{S^k}$ and has a trivialization obtained by restricting the framing $f_1 : \nu_1 \xrightarrow{\simeq} \mathbb{R}^k$. Hence we a trivialization of a sub k-bundle inside a trivialized $k + q$ bundle. This defines an element in $[\alpha] \in \pi_p(V_{k,q})$ where $V_{k,q} = SO(k + q)/SO(q)$ is the space of orthonormal k-frames in $\mathbb{R}^{k+q}$. If the framing f_1 extends to W then in particular it extends to D^{p+1} which means that the map $\alpha : S^p \longrightarrow V_{k,q}$ extends to D^{p+1} and so $[\alpha] = 0$.

It turns out that $[\alpha]$ can be constructed only from the embedding of S^p itself and in fact it somehow encompasses two obstructions. First of all given only and embedding $S^p \hookrightarrow M$ we need to extend it to an embedding of $S^p \times D^q$. This is equivalent to saying that the normal bundle to S^p in M is trivial. Consider the exact sequence

$$... \longrightarrow \pi_p(SO(q)) \longrightarrow \pi_p(SO(k + q)) \longrightarrow \pi_p(V_{k,q}) \longrightarrow \pi_{p-1}(SO(q)) \longrightarrow ...$$

It turns out that the image of $[\alpha]$ in $\pi_{p-1}(SO(q))$ is exactly the class representing the normal bundle to S^p in M. If this normal bundle is trivial then α actually lies in the cokernel $\pi_p(SO(k + q))/\pi_p(SO(q))$.

We have the following theorem (see [Br])

Theorem 3.3. *If $p < q$ then $\pi_p(V_{q,k}) = 0$. If p is even then $\pi_p(V_{p,k}) = \mathbb{Z}$ and if p is odd then $\pi_p(V_{p,k})$ is even. Further more, if $p \neq 1, 3, 7$ then the boundary map $\pi_p(V_{p,k}) \longrightarrow \pi_{p-1}(SO(p))$ is a monomorphism.*

What we learn from this is that there isn't any problem when p is less then half the dimension. Further more unless the dimension of M is $2, 6$ or 14, the middle dimension obstruction is simply that an embedded sphere $S^k \hookrightarrow M$ has a trivial normal bundle.

3.3 Surgery in the Middle Dimension

We now describe what actually happens in the middle dimension for the case $n = 2m$. Let $f : M \longrightarrow X$ be a map of degree 1 where M is a stably framed n-manifold and X an n-dimensional Poincare complex.

By all the theory we've explained so far we see that we can serger f to be $m-1$-connected. Let F be fiber of f. Since X is simply connected by the relative Hurewitz formula we see that the relative homology groups $H_i(C_f)$ vanish for $i \leq m$ and $H_{m+1}(C_f) \cong \pi_n(F)$.

In particular the map $f_* : H_m(M) \longrightarrow H_m(X)$ is surjective. Since f preserves the top class it preserves Poincare duality (in the appropriate sense) and hence the map $f_* : H_{m+1}(M) \longrightarrow H_{m+1}(X)$ is an isomorphism. This means that

$$\pi_m(F) \cong H_{m+1}(C_f) \cong \ker(H_m(f))$$

First we want to show that we can ignore torsion. From the universal coefficients theorem and Poincare duality we get

$$\mathrm{Tor}(H_m(M)) \cong \mathrm{Tor}(H^{m+1}(M)) \cong \mathrm{Tor}(H_{m-1}(M))$$

and similarly

$$\mathrm{Tor}(H_m(X)) \cong \mathrm{Tor}(H^{m+1}(X)) \cong \mathrm{Tor}(H_{m-1}(X))$$

Since f induces an isomorphism $H_{m-1}(M) \xrightarrow{\simeq} H_{m-1}(X)$ we see that f induces an isomorphism

$$\mathrm{Tor}(H_m(M)) \xrightarrow{\simeq} \mathrm{Tor}(H_m(X))$$

This means that $\ker(H_m(f))$ is torsion free. Similarly $\mathrm{coker}\,(H^m(f))$ is torsion free and we can embed it in $H^m(M)$ inside the subgroup of elements which are orthogonal (with respect to cup product) to the image of $H^m(f)$. Call this embedding $K^n \subseteq H^n(M)$. Then Poincare duality $H^n(M) \cong H_m(M)$ identifies K^n with $\ker(H_n(f))$.

Since K^n was a space orthogonal to a subspace on which the cup product is non-degenerate (as it is isomorphic to the cup product on $H^m(X)$) we see that the cup product on K^n is non-degenerate as well. Hence we can find a simplectic basis $x_1, ..., x_r, y_1, ..., y_r \in K^n$. For each element in this basis there is a Poincare dual element in $\ker(H_n(f)) = \pi_n(F)$. This element is represented by a mapping

$$\begin{array}{ccc} S^p & \longrightarrow & M \\ \downarrow & & \downarrow \\ D^{p+1} & \longrightarrow & X \end{array}$$

This mapping carries a $\mathbb{Z}/2$ obstruction which has to vanish in order for us to be able to do surgery on it. This defines a mapping $\phi : K^n \longrightarrow \mathbb{Z}/2$. It can be shown that this mapping actually factors through a map

$$K^n/2K^n \longrightarrow \mathbb{Z}/2$$

which is a quadratic form compatible with the cup product. Such quadratic forms are classified by a unique invariant, called the **Arf** invariant which is defined to be

$$\mathrm{Arf}(\phi) = \sum_i \phi(x_i)\phi(y_i)$$

which turns out to be independent of the choice of simplectic basis. We claim that we can surger f into a homotopy equivalence if and only if $\mathrm{Arf}(f) = 0$. Let us sketch the reason.

For each i, if we look at $\phi(x_i), \phi(y_i), \phi(x_i + y_i)$ then either exactly one of them is 1 or all 3 are 1. What the Arf invariant measures is the parity of the numbers of pairs x_i, y_i for which it is 3. We will now show that any pair with only one 1 can be surgered out.

If exactly one of $\phi(x_i), \phi(y_i), \phi(x_i + y_i)$ then is 1 then we can assume that $\phi(x_i) = \phi(y_i) = 0$. This means in particular that we can embed the dual elements $\widetilde{x_i}, \widetilde{y_i}$ as two spheres with trivial normal bundles and transverse intersection at a unique point. We need to show that the surgery won;t affect $\pi_{m-1}(F)$ (recall that for a $S^p \times D^q$ type surgery we could only guaranty that it would not affect the homotopy groups in dimensions strictly smaller then $q-1$. Here $p = q = m$ so we need some how to make sure that $\pi_{m-1}(F)$ is not affected.

At the intersection point $p \in \widetilde{x_i} \cap \widetilde{y_i}$ we can identify the normal disc to $\widetilde{x_i}$ in M with a small neighborhood of p in $\widetilde{y_i}$. This means that when I fatten $\widetilde{x}$ to an embedded $S^m \times D^m$ and remove it, the removed copies of D^m won't create new elements in $\pi_{m-1}(F)$ because the boundary $p \times S^{m-1}$ of $p \times D^k$ can be contracted through the other side of $\widetilde{y_i}$. Then it can be showed that the surgery process won't change $\pi_{m-1}(F)$ but will replace $\pi_m(F)$ by $\pi_m(F)/ < x_i, y_i >$. Note that this process won't change the Arf invariant.

Now suppose that all the pairs are 3's but we have more then one pair. Then we can take two pairs x_1, y_1, x_2, y_2 and preform the following change of variables:

$$x'_1 = x_1 + x_2$$
$$y'_1 = y_1$$
$$x'_2 = x_2$$
$$y'_2 = y_2 + y_1$$

Then we see that we have switched both pairs to pairs of type 1 and we can surger away. But if the Arf invariant is 1 then we will eventually get stuck with a single pair of type 3 and we could not surger it out. In particular one shows that the Arf invariant is a stably framed cobordism invariant and so it has to vanish in order for us to be able to surger f into a homotopy equivalence.

comment: For $n = 2m + 1$ odd we don't have any surgery obstruction but we still need use the Poincare dual of an element $x \in \pi_m(F)$ in order to do surgery on it. We will not spell out the details here. Note that this means in particular that every stably framed manifold of odd dimension is cobordant to a homotopy sphere.

4 Kervaire's Example

In his paper Kervaire constructs a homotopical invariant Φ of 10-dimensional 4-connected Poincare complexes which takes values in $\mathbb{Z}/2$. Since $\pi_4^s(S^0) = 0$ every such complex has a stably spherical top class and so we can find degree 1 4-connected maps $f : M \longrightarrow X$ where M is a stably framed manifold.

We will show that the surgery obstruction to make this map a homotopy equivalence is exactly $\Phi(M) - \Phi(X)$. Hence X is equivalent to a stably framed manifold if and only if there exists a manifold in its stably framed cobordism group with the same Φ. Following Kervaire, we will then show that $\Phi(M) = 0$ for every stably framed manifold but there exists a 4-connected 10-dimensional Poincare complex X with $\Phi(X) = 1$. We will then show that X is in fact a topological manifold.

4.1 The invariant

Let us first explain why the invariant exists. Let X be a 4-connected 10 dimensional Poincare complex. Choosing a basis for $H_5(X)$ one can construct a cell structure of the form

$$\left(\bigvee_i S_i^5\right) \cup_{\gamma_X} e^{10}$$

and so all of the homotopical information is encoded in one gluing element $\gamma_X \in \pi_9(\bigvee_i S_i^5)$. But it turns out that $\pi_9(S^5) = \mathbb{Z}/2$ and generator is the special element $\alpha_5 = [\iota, \iota]$ coming from the tangent bundle. Further more from a general theorem we know that

$$\pi_9\left(\bigvee_i S_i^5\right) = \oplus_i \pi_9\left(S_i^5\right) \oplus_{i,j} [\iota_i, \iota_j]_* \pi_9\left(S^9\right) \cong \oplus_i \mathbb{Z}/2 \oplus_{i,j} \mathbb{Z}$$

It is known that the $\mathbb{Z}$'s encode the cup product on the dual basis in $H^5(X)$. What the Φ invariant captures is the $\mathbb{Z}/2$ part of the map. In order to capture it we do the following. Let $Y = S^5 \cup_2 e^6$. It can be shown that the image of α_5 in $\pi_9(Y)$ is non-trivial. By abuse we will call its image α_5 as well. Consider

$$Y^* = Y \cup_{\alpha_5} e^1 0$$

Let $e_1 \in H^5(Y, \mathbb{Z}/2), e^2 \in H^{10}(Y, \mathbb{Z}/2)$ be generators. Then for every element $x \in H_5(X, \mathbb{Z})$ we get a unique map f_x of the 5-skeleton of X into Y (which can be taken to have its image in the 5-skeleton of Y) such that $f_x^*(e_1) = x$.

The obstruction to extending this map to X lies in $H^{10}(X, \pi_9(Y^*)) = \pi_9(Y^*)$. It is obtained in the following way: we push the gluing element γ_X using f_x into the 5-skeleton S^5 of Y^*. This element has to die inside Y^* in order for us to be able to extend our map. But $\pi_9(S^5) = \mathbb{Z}/2$ and the non-trivial element indeed dies in Y^*.

Note that the extension of f_x to $\widetilde{f_x} : X \longrightarrow Y$ is not unique. We can modify a chosen extension f_x by an element $H^{10}(X, \pi_{10}(Y)) = \pi_{10}(Y)$ in the following way: given a map $g : S^{10} \longrightarrow Y$ we can take the composition

$$X \longrightarrow X \vee S^{10} \overset{\widetilde{f_x} \vee g}{\longrightarrow} Y^*$$

where the map $X \longrightarrow X \vee S^{10}$ is obtained by collapsing a small sphere inside the top cell of X to a point. We claim that any map from S^{10} to Y^* sends the top class of S^{10} to an even class in $H_{10}(Y^*)$ (or in other word that the image of the Hurewitz map in $H_{10}(Y^*) = \mathbb{Z}$ contains only even elements).

We give the following argument: the image of the Hurewitz map in $H_{10}(Y^*)$ contains 2. This is given by the map induced from the commutative diagram

$$\begin{array}{ccc} S^9 & \xrightarrow{2} & S^9 \\ \downarrow & & \downarrow{\scriptstyle \alpha_5} \\ D^{10} & \longrightarrow & Y \end{array}$$

Now if the Hurewitz image contained any odd element then it would contain 1. Let $g : S^{10} \longrightarrow Y^*$ be such an element. Then we get a map

$$S^{10} \vee S^5 \overset{g \vee Id}{\longrightarrow} Y^*$$

which is a homology equivalence of simply connected spaces and hence a homotopy equivalence. But this map can't have an inverse map because any inverse on the 5-skeleton would encounter a non-trivial obstruction to extending to Y^*.

The conclusion from this discussion is that the parity of the element $\widetilde{f_x}[X] \in H_n(Y^*)$ depends only on f_x. In fact we claim that it is simply determined by whether $f_{x*}(\gamma_X) \in \pi_9(S^5)$ is trivial or not. If it is trivial then $\widetilde{f}$ factors through the collapse $X \longrightarrow S^{10}$ of the 5-skeleton of X and then as we saw that $\widetilde{f_{x*}}[X]$ would have to be even. On the other hand If $f_{x*}(\gamma_X) = \alpha_5$ then we can choose the extension obtained from the commutative diagram

$$\begin{array}{ccc} S^9 & \xrightarrow{Id} & S^9 \\ \downarrow{\scriptstyle \gamma_X} & & \downarrow{\scriptstyle \alpha_5} \\ \bigvee_i S^5_i & \xrightarrow{f_x} & S^5 \end{array}$$

which sends $[X]$ to $[Y^*]$. This motivates the definition

$$\phi(x) = \widetilde{f_x}^*(e_2)[X] \in \mathbb{Z}/2$$

which we see depends only on x and can also be identified with $f_{x*}\gamma_x \in \pi_9(S^5) = \mathbb{Z}/2$.

Write

$$\gamma_X = \sum_i a_i[\iota_i, \iota_i] + \sum_{i \neq j} b_{i,j}[\iota_i, \iota_j]$$

where $\iota_i : S^5_i \in \bigvee_i S^5_i$ is the inclusion of the i'th component, $a_i \in \mathbb{Z}/2$ and $b_i \in \mathbb{Z}$. Let $\{t_i\} \in H^5(X)$ be the dual basis to the homology basis given by the 5-skeleton. Then note that Then if $x = \sum_i c_i t_i$ then

$$f_{x*}\gamma_x = \left(\sum_i a_i c_i + \sum_{i \neq j} b_{i,j} c_i c_j \right) \alpha_5$$

This implies that ϕ satisfies

$$\phi(x+y) = \phi(x) + \phi(y) + x \cup y$$

From this we conclude that ϕ induces a quadratic form $H^5(X, \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ which is compatible with the cup product. Such objects are classified by their so called **Arf invariant**, which is defined as follows: Since the cup product pairing on $H^5(X, \mathbb{Z}/2)$ is anti-symmetric (in the since the the cup of an element with it self is zero) and non-degenerate it has to have an even dimension $2m$ and must admit a basis $\{x_1, ..., x_m, y_1, ..., y_m\}$ such that

$$x_i \cup x_j = 0$$

$$y_i \cup y_j = 0$$

$$x_i \cup y_j = \delta_{i,j}$$

Then define

$$\mathrm{Arf}(\phi) = \sum_i \phi(x_i)\phi(y_i) \mod 2$$

Then it can be showed that this does not depend on the choice of basis. We now define

$$\Phi(X) = \mathrm{Arf}(\phi)$$

Theorem 4.1. *If M is a 4-connected stably framed manifold then $\Phi(M)$ is invariant under stably framed cobordism.*

Proof. It is enough to show that if M is a stably framed boundary of a manifold W then $\Phi(M) = 0$. Using surgery theory as before we can alter W so that it becomes 4-connected.

Since $H^i(W, \mathbb{Z}/2) = H^i(M, \mathbb{Z}/2) = 0$ for $i = 1, ..., 4$ we get from the long exact sequence in cohomology that $H^i(W, M, \mathbb{Z}/2) = 0$ for $i = 1, ..., 4$. The Lefschetz Poincare duality with $\mathbb{Z}/2$ coefficients gives us a non-degenerate pairing between $H^i(W, \mathbb{Z}/2)$ and $H^{11-i}(W, \mathbb{Z}/2)$. This means that $H^i(W) = H^i(W, M) = 0$ for $i = 7, 8, 9, 10$ as well. In the middle dimensional we get 5-term exact sequence

$$0 \longrightarrow H^5(W, M, \mathbb{Z}/2) \longrightarrow H^5(W, \mathbb{Z}/2) \xrightarrow{\iota_*} H^5(M, \mathbb{Z}/2) \xrightarrow{\partial} H^6(W, M, \mathbb{Z}/2) \longrightarrow H^6(W, \mathbb{Z}/2) \longrightarrow 0$$

Again from Poincare duality we get that the dimension of the image of ι_* is exactly half the dimension of $H^5(M, \mathbb{Z}, 2)$. Further more the Poincare pairing is compatible with ι_* and ∂ in the since that if $x \in H^5(M,\mathbb{Z}/2)$ and $y \in H^5(W,\mathbb{Z}/2)$ then

$$(\partial x \cup y)[W,M] = x \cup \iota_* y$$

In particular the image of ι_* is a lagrangian subspace of $H^5(M,\mathbb{Z}/2)$: if two elements in it have a trivial cup and its dimension is half the dimension of $H^5(M,\mathbb{Z}/2)$. Note that since W is 4 connected, for each element $u \in H^5(V,\mathbb{Z}/2)$ there is a (non-unique) map f_u from the 6-skeleton of W to Y^* such that $f_u^* e_1 = u$. Since $H^i(W,\mathbb{Z}/2) = 0$ for $i = 7, 8, 9, 10$ and the homotopy groups of $Y \subseteq Y^*$ are 2-torsion we see that we can always extend f_u to the 10'th skeleton. Since f_u is a manifold the 10'th skeleton is a deformation retract of W with a small 11-ball removed. Removing this ball results in a cobordism from M to the sphere. This means that the image of $(f_u|_M)_*[M]$ is in the image of the Hurewitz map and so is an even multiple of $[Y^*]$. This means that $\phi(x) = 0$ for every x which is in the image of $H^5(W,\mathbb{Z}/2) \longrightarrow H^5(M,\mathbb{Z}/2)$. Let $x_1, ..., x_m$ be a basis for that subspace. Since it is a lagrangian subspace we can extend this basis to $x_1, ..., x_m, y_1, ..., y_m$ such that

$$x_i \cup x_j = 0$$

$$y_i \cup y_j = 0$$

$$x_i \cup y_j = \delta_{i,j}$$

Since $\phi(x_i) = 0$ we get that

$$\Phi(M) = \mathrm{Arf}(\phi) = \sum_i \phi(x_i)\phi(y_i) = 0$$

and we are done. □

We will now show that $\Phi(M) = 0$ for every stably famed 10-manifold. The 10'th stably framed cobordism group $\pi_1^s 0(S^0) = \mathbb{Z}/6$. Since Φ is a homomorphism into $\mathbb{Z}/2$ we just need to show that it vanishes on the two torsion part of $\pi_1^s 0(S^0)$ which generated by a single element α.

It can shown using calculations in the homotopy groups of spheres that this element is actually a product of the hopf element $\eta \in \pi_1^s(S^0)$ and an element $\beta \in \pi_9^s(S^0)$. But from the surgery discussion above we know that β can be represented in the stably framed cobordism group by a 9-homotopy sphere. Since η is represented by a circle we see that we can represent α by a product of a 9-homotopy sphere and a circle. This is a manifold without homotopy groups from the middle dimension down except for π_1. Hence a simple 1-dimensional surgery step will make it into a 10-homotopy sphere, and so $\Phi(M)$ of every stably framed 10-manifold is 0.

4.2 The connection between $\Phi(X)$ and the Surgery Obstruction

Let M be a 4-connected stably framed manifold. Say we want to preform surgery on the middle dimension. Then the obstruction to doing surgery on an element $\alpha : S^5 \longrightarrow M$ is the non-triviality its normal bundle. We claim that if $x \in H^5(M)$ is Poincare dual to $[\alpha]$ then this normal bundle is trivial if and only if the normal bundle is trivial. The key step is the following:

Theorem 4.2. *Let m be odd and let $\eta \longrightarrow S^m$ be the tangent bundle. Then the Thom space $T(\eta)$ is homotopy equivalent to*

$$S^m \cup_{\alpha_m} e^{2m}$$

where α_m is the image of the tangent bundle under the map $\pi_{m-1}(SO(m)) \longrightarrow \pi_{m-1}(\Omega^m S^m) = \pi_{2m-1}(S^m)$.

Proof. Compactifying the fibers of η from $\mathbb{R}^m$ to S^m we get a sphere fibration $C(\eta) \longrightarrow S^m$ with structure loop group $\Omega^m S^m$. Let $\beta \in \pi_{m-1}(\Omega^m S^m)$ be its classifying element. Another way to put this is to say that we have a natural map $SO(m) \longrightarrow \Omega^m S^m$ and β is the image of α under the induced map on π_m.

Let $Y = T(\eta)$ be the Thom construction on η. We claim that Y is homotopy equivalent to the space $S^m \cup_\beta e^{2m}$. To show this first consider the unit $m-1$-sphere bundle $p : S(\eta) \longrightarrow S^m$. Then $T(\eta)$ can be constructed as the cone of the map $S(\eta) \longrightarrow S^m$.

The non-zero section of η induces a section of $S(\eta)$ which reduces its structure loop group from unpointed to pointed maps $S^{m-1} \longrightarrow S^{m-1}$, or simply $\Omega^{m-1} S^{m-1}$. Hence it has a classifying element $\widetilde{\beta} \in \pi_{m-1}(\Omega^{m-1} S^{m-1}) = \pi_{2m-2}(S^{m-1})$. It is not hard to see that if we suspend the fibers of $S(\eta)$ we get $C(\eta)$ and so the image of $\widetilde{\beta}$ under suspension is exactly β.

Since $S(\eta)$ fibers over the sphere with spherical fibers and admits a section it can be given a fairly simple cell structure, reading:

$$S(\eta) \simeq (S^m \vee S^{m-1}) \cup_f e^{2m-1}$$

where $f \in \pi_{2m-1}(S^m \vee S^{m-1})$ has e property that when composed with the retraction $S^m \vee S^{m-1} \longrightarrow S^{m-1}$ it gives the map $\widetilde{\beta} \in \pi_{2m-2}(S^{m-1})$. The S^m cell is constructed using a section and so it is mapped with homotopy equivalence to S^m under the fibration map p.

This means that the cone on p can be given the cell structure

$$S^m \cup_\beta e^{2m}$$

and we are done. □

The idea now is basically the following. If the normal bundle of α is non-trivial then we can map M to the Thom space on the normal bundle in a way that maps $[M]$ to the generator of $H_{10}(T(\eta))$. Since $T(\eta)$ is more or less the space Y^* used by Kervaire and the map we get corresponds to x we get that $\phi(x) = 1$. If the normal bundle is trivial then we Thom space is just $S^5 \vee S^{10}$ and since the map corresponds to x we can use it to construct a map $M \longrightarrow Y^*$ that sends $[M]$ to twice the top class of Y^*.

4.3 Constructing the Example

It is simple to construct the example as a Poincare space. Simply put

$$X = \left(S^5 \vee S^5\right) \cup_\gamma e^1 0$$

where

$$\gamma = [\iota_1, \iota_1] + [\iota_1, \iota_2] + [\iota_2, \iota_2]$$

The trick is to show that this can be realized as a topological manifold. This is done as following. Let $D \longrightarrow S^5$ be the disc bundle of the tangent bundle. This is a 10 manifold with boundary. We will start by gluing two copies D_1, D_2 of D along open subsets as follows. Since D_1, D_2 are fiber bundles they trivialize on small discs. Pick a small $D^5 \subseteq S^5$ and let U_1, U_2 be the restriction of the fiber bundles D_1, D_2 to D^5. Then $U_1 \cong U_2 \cong D^5 \times D^5$. Now let W be the gluing of D_1 and D_2 via the diffeomorphism $U_1 \cong U_2$ which switches the two D^5 components (i.e. the fiber components in U_1 is identified with the base component in U_2 and vice versa).

We claim that W is a smooth manifold with boundary ∂W. If you follow the construction you will see that the boundary is actually the result of one surgery step (with $p = 5$) on the boundary of D_1. We claim ∂W is actually a homotopy sphere.

First of all it easy to see that both W and ∂W are simply connected. Further more W deformation retracts to $S^5 \vee S^5$ which can be taken to be smoothly embedded spheres intersecting transversely. The Lefschetz Poincare duality gives us an isomorphism $H_i(W, \partial W) \cong H^{10-i}(W)$ which means that $H_i(W, \partial W) = \mathbb{Z} \oplus \mathbb{Z}$ if $i = 5$ and 0 if $i = 1, 2, 3, 4, 6, 7, 8, 9$.

In particular we see that neither $H_*(W)$ nor $H_*(W, \partial W)$ has torsion, which means that we can realize the Lefschetz Poincare duality by means of a non-degenerate intersection pairing

$$H_5(W) \otimes H_5(W, \partial W) \longrightarrow \mathbb{Z}$$

But $H_5(W)$ is generated by two embedded spheres which intersect transversely at a single point. Hence the intersection pairing is non-degenerate on the level of $H_5(W)$. This implies that the map

$$H_5(W) \longrightarrow H_5(W, \partial W)$$

has to be an isomorphism. This means that ∂W must be a homotopy sphere. Since all manifolds which are homotopy spheres are actually homeomorphic to spheres we get that ∂W is a topological sphere. Gluing on it a D^5 we obtain a topological manifold M. This M will be homotopy equivalent to X above and so will have $\Phi(M) = 1$. This implies that M cannot be homotopy equivalent (let alone homeomorphic) to any stably framed manifold.

We claim that M can't be homotopy equivalent to any smooth manifold at all. Suppose M was homotopic to a manifold. Since M is 4-connected and since $\pi_5(BSO) = 0$ we get that we can trivialize it's stable normal bundle on the

9-skeleton, or more geometrically, on the complement of some small embedded 10-ball. Looking at the boundary of that ball we find a sphere with two different trivializations of its stable normal bundle. If we would show that these two trivializations must agree then we would get that the stable normal bundle is trivial on all of M.

The two different trivializations both give S^9 a structure of a stably framed manifold which is stably framed null cobordant. Hence the gap between this two trivializations is in the kernel of the J-homomorphism

$$\pi_9(SO) \longrightarrow \pi_9^s(S^0)$$

But the J-homomorphism is injective at dimension 9 and so these trivializations must agree, i.e. we get that M must be stably framed.