

Lax limits of model categories

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Let \mathcal{J} be a small category and let $\mathcal{M} : \mathcal{J} \rightarrow \text{ModCat}$ be a diagram of model categories. We will denote by $\mathcal{M}^{\text{L}} : \mathcal{J} \rightarrow \text{ModCat}^{\text{L}}$ and $\mathcal{M}^{\text{R}} : \mathcal{J} \rightarrow \text{ModCat}^{\text{R}}$ the associated functors. Let

$$\text{Sec}(\mathcal{M}) = \left\{ s : \mathcal{J} \rightarrow \int_{\mathcal{J}} \mathcal{M} \mid \pi \circ s = \text{Id} \right\}$$

be the category of sections of π . More explicitly, the objects of $\text{Set}(\mathcal{M})$ are given by collections $X_i \in \mathcal{M}(i)$ for each $i \in \mathcal{J}$ together with maps $f_\alpha : \alpha_* X_i \rightarrow X_{i'}$ for every map $\alpha : i \rightarrow i'$ satisfying an obvious compatibility conditions for each commutative triangle

$$\begin{array}{ccc} & i' & \\ \alpha \swarrow & & \searrow \beta \\ i & \xrightarrow{\gamma} & i'' \end{array}$$

in \mathcal{J} . As shown in [2], when each $\mathcal{M}(i)$ is a **combinatorial model category** one can endow $\text{Sec}(\mathcal{M})$ with the **projective model structure** $\text{Sec}^{\text{proj}}(\mathcal{M})$ in which a map $T : s \rightarrow s'$ is a weak equivalence/fibration if and only if $T(i) : s(i) \rightarrow s'(i)$ is a weak equivalence/fibration for every i . Furthermore, it is not hard to verify that when each $\mathcal{M}(i)$ is endowed with a **simplicial structure** such that each $\alpha_* \dashv \alpha^*$ is a simplicial Quillen adjunction then $\text{Sec}^{\text{proj}}(\mathcal{M})$ inherits a simplicial structure which is given levelwise, i.e. $(K \otimes s)(i) = K \otimes s(i)$ for every section s and simplicial set K . We will henceforth fix these two assumptions. In particular, we will denote by ModCat_Δ the category of simplicial model categories and simplicial Quillen adjunctions, and restrict our attention to diagrams $\mathcal{M} : \mathcal{J} \rightarrow \text{ModCat}_\Delta$ of simplicial model categories in which each $\mathcal{M}(i)$ is combinatorial.

The model category $\text{Sec}^{\text{proj}}(\mathcal{M})$ can be considered as a model for the **lax limit** of \mathcal{M}^{R} (or, equivalently, of \mathcal{M}^{L}). As described in [2], if we want the **homotopy limits** of \mathcal{M}^{R} we must first **localize** $\text{Sec}^{\text{proj}}(\mathcal{M})$ so that the new fibrant objects will consist of the **Cartesian** fibrant sections, where we recall that a section $s \in \text{Sec}(\mathcal{M})$ is **Catesian** if the derived adjoint map

$$f_\alpha^{\text{ad}} : X_i \rightarrow \mathbb{R}\alpha^* X_{i'}$$

is a weak equivalence in $F(i)$ for every α . We will call the resulted model structure the **Cartesian model structure** and denote it by $\text{Sec}_r^{\text{proj}}(\mathcal{M})$. In general the desired localization needs not exist as a model category, except if $\text{Sec}^{\text{proj}}(\mathcal{M})$ is left proper. Since colimits in $\text{Sec}(\mathcal{M})$ are computed levelwise this condition is satisfied, for example, when each $\mathcal{M}(i)$ is left proper.

Definition 1. Given a simplicial model category \mathcal{C} we will denote by \mathcal{C}° the full subcategory of fibrant/cofibrant objects considered as a (fibrant) simplicial category. For the purpose of simplicity we will employ the notations

$$\text{Sec}^\circ(\mathcal{M}) \stackrel{\text{def}}{=} (\text{Sec}^{\text{proj}}(\mathcal{M}))^\circ \quad \text{and} \quad \text{Sec}_r^\circ(\mathcal{M}) \stackrel{\text{def}}{=} (\text{Sec}_r^{\text{proj}}(\mathcal{M}))^\circ$$

The purpose of this section is to relate the above picture to the ∞ -categorical one, and in particular to prove that the Cartesian model category is indeed a model for the corresponding limit of diagram of ∞ -categories.

Recall that we have fixed the assumption that each $\mathcal{M}(i)$ is a simplicial model category and each $\alpha_* \dashv \alpha^*$ is a simplicial Quillen adjunction. In this case the Grothendieck construction $\int_j \mathcal{M}$ inherits a natural enrichment over Set_Δ . Given two objects $i, j \in \mathcal{J}$ and two objects $X \in \mathcal{M}(i), Y \in \mathcal{M}(j)$ the simplicial mapping space $\text{Map}((i, X), (j, Y))$ is given by

$$\text{Map}((i, X), (j, Y)) = \coprod_{\alpha: i \rightarrow j} \text{Map}_{\mathcal{M}(j)}(\alpha_* X, Y)$$

where the coproduct is taken over all maps $\alpha : i \rightarrow j$ in \mathcal{J} . The resulting simplicial category is not fibrant in general, but we can obtain a good fibrant model for it by considering the natural full subcategory

$$\int_j^\circ \mathcal{M} \subseteq \int_j \mathcal{M}$$

consisting of all objects (i, X) such that X is fibrant and cofibrant in $\mathcal{M}(i)$. The simplicial category $\int_j^\circ \mathcal{M}$ is fibrant and we can take its coherent nerve

$$\mathfrak{G} = \text{N} \left(\int_j^\circ \mathcal{M} \right)$$

Then \mathfrak{G} is a (big) ∞ -category carrying a natural map

$$p : \mathfrak{G} \rightarrow \text{N}(\mathcal{J})$$

Proposition 2. *The map p is a Cartesian fibration.*

Proof. To prove the claim we must show that we have a sufficient supply of p -Cartesian edges. For this we will need to use a characterization of p -Cartesian edges, as provided by the following lemma:

Lemma 3. *Let $(Y, j), (Z, k) \in \int_j^\circ \mathcal{M}$ be objects. Let $\beta : j \rightarrow k$ be a morphism in \mathcal{J} and $f : \beta_* Y \rightarrow Z$ a morphism in $\mathcal{M}(k)$. Then the edge of \mathfrak{G} corresponding to (β, f) is p -Cartesian if and only if the adjoint map*

$$f^{\text{ad}} : Y \rightarrow \beta^* Z$$

is a weak equivalence.

Proof. First assume that f^{ad} is a weak equivalence. Let $i \in \mathcal{J}$ be an object and $X \in \mathcal{M}(i)$ a fibrant-cofibrant object. We need to show that the natural commutative diagram

$$\begin{array}{ccc} \text{Hom}_{f_j \mathcal{M}}((i, X), (j, Y)) & \xrightarrow{(f, \beta)_*} & \text{Hom}_{f_j \mathcal{M}}((i, X), (k, Z)) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{J}}(i, j) & \xrightarrow{\beta_*} & \text{Hom}_{\mathcal{J}}(i, k) \end{array}$$

is homotopy Cartesian. For this, it will suffice to show that for every morphism $\alpha : i \rightarrow j$ in \mathcal{J} , the induced map

$$(\beta, f)_* : \text{Map}_{\mathcal{M}(j)}(\alpha_* X, Y) \rightarrow \text{Map}_{\mathcal{M}(k)}(\beta_* \alpha_* X, Z)$$

is a weak equivalence. We now observe that under the adjunction isomorphism

$$\text{Map}_{\mathcal{M}(k)}(\beta_* \alpha_* X, Z) \cong \text{Map}_{\mathcal{M}(j)}(\alpha_* X, \beta^* Z)$$

the map $(\beta, f)_*$ is given by post-composition with $f^{\text{ad}} : Y \rightarrow \beta^* Z$. Now since f^{ad} is a weak equivalence between fibrant objects and since $\alpha_* X$ is cofibrant we get the $(\beta, f)_*$ is indeed a weak equivalence.

In the other direction, assume that the edge associated to (β, j) is p -Cartesian. Then for every fibrant cofibrant $X \in \mathcal{M}(j)$ the square

$$\begin{array}{ccc} \text{Hom}_{f_j \mathcal{M}}((j, X), (j, Y)) & \xrightarrow{(f, \beta)_*} & \text{Hom}_{f_j \mathcal{M}}((i, X), (k, Z)) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{J}}(j, j) & \xrightarrow{\beta_*} & \text{Hom}_{\mathcal{J}}(j, k) \end{array}$$

is homotopy Cartesian. Using adjunction as above this means that the map

$$\text{Map}_{\mathcal{M}(j)}(X, Y) \rightarrow \text{Map}_{\mathcal{M}(j)}(X, \beta^* Z)$$

obtained by composition with f^{ad} is a weak equivalence. Since this is true for every fibrant cofibrant $X \in \mathcal{M}(j)$ we deduce f^{ad} is a weak equivalence as desired. \square

In order to complete the proof it will suffice to show that for every morphism $\beta : j \rightarrow k$ in \mathcal{J} and for every fibrant cofibrant object $Z \in \mathcal{M}(k)$ there exists a fibrant cofibrant object $Y \in \mathcal{M}(j)$ admitting a weak equivalence $Y \xrightarrow{\simeq} \beta^*Z$. But this is clear since we can choose a trivial fibration $Y \xrightarrow{\simeq} \beta^*Z$ such that Y is cofibrant. \square

Remark 4. Using a dual argument to the proof of Proposition 2 one can show that the map $p : \mathfrak{G} \rightarrow \mathbf{N}(\mathcal{J})$ is also a **coCartesian fibration**.

Notation 5. We will denote by \mathfrak{G}^Γ the marked simplicial set whose underlying simplicial set is \mathfrak{G} and whose marked edges are the p -**Cartesian** edges.

Let $(\mathrm{Set}_\Delta^+)_/\mathbf{N}(\mathcal{J})$ denote the category of marked simplicial sets over $\mathbf{N}(\mathcal{J})$ (whose objects consist of marked simplicial sets (X, M) equipped with an unmarked map $X \rightarrow \mathbf{N}(\mathcal{J})$). As explained in [1] one can endow $(\mathrm{Set}_\Delta^+)_/\mathbf{N}(\mathcal{J})$ with the **Cartesian model structure**, in which the fibrant objects are of the form X^\natural for some Cartesian fibration $X \rightarrow \mathbf{N}(\mathcal{J})$. In particular, we can view \mathfrak{G}^Γ as a fibrant object in the Cartesian model structure.

The Cartesian model category $(\mathrm{Set}_\Delta^+)_/\mathbf{N}(\mathcal{J})$ is tensored and cotensored over Set_Δ^+ with respect to the marked model structure, i.e., the Cartesian model structure over of a point. In particular, given two objects $X, Y \in (\mathrm{Set}_\Delta^+)_/\mathbf{N}(\mathcal{J})$ such that Y is fibrant we have a fibrant mapping object

$$\mathrm{Map}_{/\mathbf{N}(\mathcal{J})}^+(X, Y) \in \mathrm{Set}_\Delta^+$$

In this case, we will denote by

$$\mathrm{Map}_{/\mathbf{N}(\mathcal{J})}(X, Y) \in \mathrm{Set}_\Delta$$

the underlying (unmarked) ∞ -category of $\mathrm{Map}_{/\mathbf{N}(\mathcal{J})}^+(X, Y)$.

We now observe that if a section $s : \mathcal{J} \rightarrow \int_{\mathcal{J}} \mathcal{M}$ is fibrant and cofibrant in $\mathrm{Sec}^{\mathrm{proj}}(\mathcal{M})$, then it factors through a map

$$s^\circ : \mathcal{J} \rightarrow \int_{\mathcal{J}}^\circ \mathcal{M}$$

We hence obtain a diagram of simplicial categories

$$\begin{array}{ccc} \mathcal{J} \times \mathrm{Sec}^\circ(\mathcal{M}) & \xrightarrow{\quad} & \int_{\mathcal{J}}^\circ \mathcal{M} \\ & \searrow & \swarrow \\ & \mathcal{J} & \end{array}$$

inducing a diagram of simplicial sets

$$\begin{array}{ccc} \mathbf{N}(\mathcal{J}) \times \mathbf{N}(\mathrm{Sec}^\circ(\mathcal{M})) & \xrightarrow{\quad u \quad} & \mathfrak{G} \\ & \searrow & \swarrow \\ & \mathbf{N}(\mathcal{J}) & \end{array}$$

which can be considered as a map

$$u : \mathbb{N}^b(\mathcal{J}) \times \mathbb{N}^h(\text{Sec}^\circ(\mathcal{M})) \longrightarrow \mathfrak{G}^\Gamma \quad (0.1)$$

in $(\text{Set}_\Delta^+)_{/\mathbb{N}(\mathcal{J})}$. Our purpose in this section is to prove the following:

Theorem 6. *Let $\mathcal{M} : \mathcal{J} \longrightarrow \text{ModCat}$ be a diagram of simplicial model categories such that each $\mathcal{M}(i)$ is combinatorial. Then the map u of (0.1) induces an equivalence of ∞ -categories*

$$\mathbb{N}(\text{Sec}(\mathcal{M})^\circ) \longrightarrow \text{Map}_{/\mathbb{N}(\mathcal{J})} \left(\mathbb{N}^b(\mathcal{J}), \mathfrak{G}^\Gamma \right)$$

In order to attack theorem 6 it will be convenient to consider a slightly more general situation. Let \mathcal{J} be a simplicial category equipped with a functor $\varphi : \mathcal{J} \longrightarrow \mathcal{J}$ (where \mathcal{J} is considered as a discrete simplicial category). We may consider the category $\text{Fun}_{/\mathcal{J}}(\mathcal{J}, \mathcal{M})$ of functors $\mathcal{J} \longrightarrow \int_{\mathcal{J}} \mathcal{M}$ over \mathcal{J} . More explicitly, an object in $\text{Fun}_{/\mathcal{J}}(\mathcal{J}, \mathcal{M})$ is given by associating with each object $j \in \mathcal{J}$ an object $X_j \in \mathcal{M}(\varphi(j))$ and for each $j, j' \in \mathcal{J}$ and each $\alpha : \varphi(j) \longrightarrow \varphi(j')$ in \mathcal{J} a map in $\mathcal{M}(\varphi(j'))$ of the form

$$\text{Map}_{\mathcal{J}}^\alpha(j, j') \otimes \alpha_* X_j \longrightarrow X_{j'},$$

where $\text{Map}_{\mathcal{J}}^\alpha(j, j') \subseteq \text{Map}_{\mathcal{J}}(j, j')$ denotes the pre-image of $\alpha \in \text{Map}_{\mathcal{J}}(\varphi(j), \varphi(j'))$ in $\text{Map}_{\mathcal{J}}(j, j')$, which is a union of connected components.

Definition 7. Let $T : f \longrightarrow g$ be a map in $\text{Fun}_{/\mathcal{J}}(\mathcal{J}, \mathcal{M})$. We will say that a map T is a **levelwise** weak equivalence (resp. fibration, cofibration etc.) of $T(j) : f(j) \longrightarrow g(j)$ is a weak equivalence (resp. fibration, cofibration, etc.) in $\mathcal{M}(j)$ for every j .

We now claim the following:

Proposition 8. *Let $\mathcal{M} : \mathcal{J} \longrightarrow \text{ModCat}_\Delta$ be a diagram of simplicial model categories such that each $\mathcal{M}(i)$ is combinatorial. Then there exists a combinatorial simplicial model structure $\text{Fun}_{/\mathcal{J}}^{\text{proj}}(\mathcal{J}, \mathcal{M})$, called the **projective model structure**, such that the weak equivalences/fibrations are the levelwise weak equivalences/fibrations, and cofibrations are the maps which satisfy the left lifting property with respect to levelwise trivial fibrations.*

Proof. The proof is completely standard, but we spell out the main details for the convenience of the reader. For each $j \in \mathcal{J}$ and $X \in \mathcal{M}(\varphi(j))$ let us denote by $\mathcal{F}_{j,X} \in \text{Fun}_{/\mathcal{J}}(\mathcal{J}, \mathcal{M})$ the functor given by

$$\mathcal{F}_{j,X}(j') = \coprod_{\alpha : \varphi(j) \longrightarrow \varphi(j')} \text{Map}_{\mathcal{J}}^\alpha(j, j') \otimes \alpha_* X$$

where the coproduct is taken over all maps $\alpha : \varphi(j) \longrightarrow \varphi(j')$ in \mathcal{J} and $\text{Map}_{\mathcal{J}}^\alpha(j, j')$ is as above. We note that for a fixed $j \in \mathcal{J}$ the functor $\mathcal{F}_{j,-} :$

$\mathcal{M}(j) \longrightarrow \text{Fun}_{/j}(\mathcal{J}, \mathcal{M})$ is left adjoint to the evaluation functor $\mathcal{G} \mapsto \mathcal{G}(j)$. Since evaluation functors preserve all colimits we have that $\text{Fun}_{/j}(\mathcal{J}, \mathcal{M})$ is monadic over $\prod_{j \in \text{Ob}(\mathcal{J})} \mathcal{M}(j)$ with monad $\mathcal{G} \mapsto \bigoplus_{j \in \mathcal{J}} \mathcal{F}_{j, \mathcal{G}(j)}$, and in particular presentable as an ordinary category.

Now note that if $\iota : X \hookrightarrow Y$ is a cofibration (resp. trivial cofibration) in $\mathcal{M}(\varphi(j))$ then the induced map

$$\mathcal{F}_{j, X} \longrightarrow \mathcal{F}_{j, Y}$$

is a levelwise cofibration (resp. trivial cofibration). For each $j \in \mathcal{J}$, let I_j, J_j be sets of generating cofibrations and trivial cofibrations respectively, and define $I = \bigcup_j I_j$ and $J = \bigcup_j J_j$. We then observe the following:

1. A map $T : f \longrightarrow g$ is an levelwise fibration if and only if it satisfies the right lifting property with respect to J .
2. A map $T : f \longrightarrow g$ is an levelwise trivial fibration if and only if it satisfies the right lifting property with respect to I .

A direct consequence of the above observation is that cofibrations in $\text{Fun}_{/j}^{\text{proj}}(\mathcal{J}, \mathcal{M})$ coincide with the weakly saturated class generated from I . The small object argument then gives the factorization of every map into a cofibration followed by a trivial fibration. We also have the lifting property of cofibrations against trivial fibrations by definition.

Let \bar{J} be the weakly saturated class of morphisms generated from J . Then \bar{J} is contained in the class of trivial cofibrations in $\text{Fun}_{/j}^{\text{proj}}(\mathcal{J}, \mathcal{M})$. Using the small object argumnet we can factor every map as a map in \bar{J} followed by a fibration. Similarly, maps in \bar{J} satisfy the left lifting property against fibration by observation (1) above. It will hence suffice to prove that \bar{J} coincides with the class of trivial cofibrations.

Let $T : f \longrightarrow g$ be a trivial cofibration. Then we can factor T as $f \xrightarrow{T'} h \xrightarrow{T''} g$ such that $T' \in \bar{J}$ and T'' is an levelwise fibration. Applying the 2-out-of-3 rule we see that T'' is a levelwise trivial fibration. Since T is a in particular a cofibration we obtain a lift in the square

$$\begin{array}{ccc} f & \xrightarrow{T'} & h \\ \downarrow T & \nearrow & \downarrow T'' \\ g & \xrightarrow{\text{Id}} & g \end{array}$$

This means that T is a retract of T' and so $T \in \bar{J}$. This establishes the existence of a combinatorial model structure as required. The existence of the levelwise simplicial structure is readily verified. \square

We shall now prove a few basic properties of the categories $\text{Fun}_{/j}(\mathcal{J}, \mathcal{M})$. We begin with some terminology.

Definition 9. Let \mathcal{C} be a simplicial category. We will denote the vertices $f \in (\text{Map}_{\mathcal{C}}(X, Y))_0$ simply as morphisms $f : X \rightarrow Y$. We will say that two morphisms $f, g : X \rightarrow Y$ in \mathcal{C} are **weakly homotopic** if they are in the same connected component of $\text{Map}_{\mathcal{C}}(X, Y)$. We will say that $f : X \rightarrow Y$ is a **weak homotopy equivalence** if there exists morphism $g : Y \rightarrow X$ such that $f \circ g$ and $g \circ f$ are weakly homotopic to the identity. This notion coincides with the notion of equivalence in the ∞ -category $\mathbf{N}(\mathcal{C}^{\text{fib}})$ where \mathcal{C}^{fib} denotes a fibrant replacement for \mathcal{C} and $\mathbf{N}(\bullet)$ is the coherent nerve functor.

Remark 10. Let \mathcal{C} be a **simplicial model category** and $f, g : X \rightarrow Y$ a pair of weakly homotopic maps. If **either** X is cofibrant **or** Y is fibrant then f, g will have the same image in $\text{Ho}(\mathcal{C})$. Furthermore, if

$$\mathcal{C} \begin{array}{c} \xrightarrow{\text{L}} \\ \xleftarrow{\text{R}} \end{array} \mathcal{D}$$

is a simplicial Quillen adjunction then L will preserve weak homotopies between maps whose domain is cofibrant and R will preserve weak homotopies between maps whose codomain is fibrant. Finally, since any model category is saturated as a relative category, we get that if $f : X \rightarrow Y$ is a weak homotopy equivalence between fibrant objects (or cofibrant objects) then f is a weak equivalence.

We now have the following basic lemma:

Lemma 11. *Let $\varphi : \mathcal{J} \rightarrow \mathcal{I}$ be a map of simplicial categories and $f \in \text{Fun}_{/\mathcal{J}}(\mathcal{J}, \mathcal{M})$ a **fibrant** object. Let $\gamma : j \rightarrow j'$ be a weak homotopy equivalence in \mathcal{J} and let $\alpha = \varphi(\gamma)$ be the corresponding map in \mathcal{I} . Let $X_j \in \mathcal{M}(\varphi(j))$ and $X_{j'} \in \mathcal{M}(\varphi(j'))$ be the objects determined by f . Then the adjoint*

$$f_{\gamma}^{\text{ad}} : X_j \rightarrow \alpha^* X_{j'}$$

to the map determined by γ is a weak equivalence.

Proof. Since both X_j and $\alpha^* X_{j'}$ are fibrant it will be enough to prove that f_{γ}^{ad} is a weak homotopy equivalence (see Remark 10). Now since β is a weak homotopy equivalence there exists a $\delta : j' \rightarrow j$ such that $\delta \circ \gamma$ and $\gamma \circ \delta$ are weakly homotopic to the corresponding identity maps. In particular, $\beta = \varphi(\delta)$ is an inverse for α , which implies that $\alpha_* \dashv \alpha^*$ and $\beta_* \dashv \beta^*$ are Quillen equivalences such that both their compositions are (naturally isomorphic to) the identity Quillen equivalence. Now let

$$f_{\delta}^{\text{ad}} : X_{j'} \rightarrow \beta^* X_j$$

be the adjoint to the map associated to δ . Since $\gamma \circ \delta$ and $\delta \circ \gamma$ are weakly homotopic to the identity we get that the compositions

$$X_j \xrightarrow{f_{\gamma}^{\text{ad}}} \alpha^* X_{j'} \xrightarrow{\alpha^* f_{\delta}^{\text{ad}}} \alpha^* \beta^* X_j \cong X_j$$

and

$$X_{j'} \xrightarrow{f_{\delta}^{\text{ad}}} \beta^* X_j \xrightarrow{\beta^* f_{\gamma}^{\text{ad}}} \beta^* \alpha^* X_{j'} \cong X_{j'}$$

are also weakly homotopic to the identity. Applying α^* to the latter map we obtain that the composition

$$\alpha^* X_{j'} \xrightarrow{\alpha^* f_{\delta}^{\text{ad}}} X_j \xrightarrow{f_{\gamma}^{\text{ad}}} \alpha^* X_{j'}$$

is weakly homotopic to the identity map as well (see Remark 10). This means that f_{γ}^{ad} is a weak homotopy equivalence. \square

Definition 12. Consider a diagram of simplicial categories of the form

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{f} & \int_{\mathcal{J}} \mathcal{M} \\ \downarrow \psi & & \downarrow \\ \mathcal{J}' & \longrightarrow & \mathcal{J} \end{array}$$

The **enriched relative left Kan extension** $\psi_! f : \mathcal{J}' \rightarrow \int_{\mathcal{J}} \mathcal{M}$ of f is defined by

$$\psi_! f(j') = \text{coeq} \left(\coprod_{j_1, j_2 \in \mathcal{J}, \alpha: \varphi(j_1) \rightarrow \varphi(j_2)} [\text{Map}_{\mathcal{J}}^{\alpha}(j_1, j_2) \otimes \mathcal{F}_{\psi(j_2), \alpha_* f(j_1)}] \rightrightarrows \coprod_{j \in \mathcal{J}} \mathcal{F}_{\psi(j), f(j)}(j') \right)$$

It is straightforward to verify that for every map $g : \mathcal{J}' \rightarrow \int_{\mathcal{J}} \mathcal{M}$ over \mathcal{J} one has a canonical isomorphism

$$\text{Map}_{\text{Fun}/_{\mathcal{J}}(\mathcal{J}', \mathcal{M})}(\psi_! f, g) \cong \text{Map}_{\text{Fun}/_{\mathcal{J}}(\mathcal{J}, \mathcal{M})}(f, \psi^* g)$$

Furthermore, one has a canonical isomorphism

$$\psi_! \mathcal{F}_{j, X} \cong \mathcal{F}_{\psi(j), X}$$

The following proposition is a generalization of [1] A.3.3.8.

Proposition 13. *Let*

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{\psi} & \mathcal{J}' \\ & \searrow & \swarrow \\ & \mathcal{J} & \end{array}$$

be a diagram of simplicial categories. Then there exists a Quillen adjunction

$$\text{Fun}_{/\mathcal{J}}^{\text{proj}}(\mathcal{J}, \mathcal{M}) \xrightleftharpoons[\psi^*]{\psi_!} \text{Fun}_{/\mathcal{J}'}^{\text{proj}}(\mathcal{J}', \mathcal{M}) .$$

where ψ^ denotes restriction along ψ and $\psi_!$ denotes enriched relative left Kan extension. If ψ is an equivalence of simplicial categories then the adjunction above a **Quillen equivalence**.*

Proof. The fact that ψ^* is a right Quillen functor is immediate from the definition. We will say that ψ is a **local trivial cofibration over \mathcal{J}** if for every $j, k \in \mathcal{J}$ and $\alpha : \varphi(j) \rightarrow \varphi(j')$ the map ψ induces a trivial cofibration

$$\mathrm{Map}_{\mathcal{J}}^{\alpha}(j, k) \hookrightarrow \mathrm{Map}_{\mathcal{J}'}^{\alpha}(\psi(j), \psi(k))$$

As in the proof of [1] A.3.3.8, we begin by reducing to the case where ψ is a local trivial cofibration over \mathcal{J} . To perform this reduction, factor the induced map $\mathcal{J} \amalg \mathcal{J}' \rightarrow \mathcal{J}'$ as a cofibration $\kappa \amalg \sigma : \mathcal{J} \amalg \mathcal{J}' \rightarrow \mathcal{J}''$ followed by a trivial fibration $\pi : \mathcal{J}'' \rightarrow \mathcal{J}'$. Note that by construction the map π is equipped with a section $\sigma : \mathcal{J}' \rightarrow \mathcal{J}''$.

We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{\kappa} & \mathcal{J}'' \\ & \searrow \psi & \nearrow \pi \\ & & \mathcal{J}' \end{array} \quad \begin{array}{c} \\ \\ \sigma \end{array}$$

of simplicial categories over \mathcal{J} , and by the 2-out-of-3 rule one can deduce that all maps appearing in this diagram are weak equivalences. Since Quillen equivalences are closed under 2-out-of-3, it will be enough to prove the theorem for κ and σ . But κ and σ are both local trivial cofibrations over \mathcal{J} . Hence we can assume without loss of generality that ψ is a local trivial cofibration over \mathcal{J} .

Our next step is to observe that the functor

$$\psi^* : \mathrm{Fun}_{/\mathcal{J}}^{\mathrm{proj}}(\mathcal{J}', \mathcal{M}) \rightarrow \mathrm{Fun}_{/\mathcal{J}}^{\mathrm{proj}}(\mathcal{J}, \mathcal{M})$$

preserves all weak equivalences, and, in view of Lemma 11, also detects weak equivalences between fibrant objects. It is hence enough to show that for every cofibrant object $f \in \mathrm{Fun}_{/\mathcal{J}}^{\mathrm{proj}}(\mathcal{J}, \mathcal{M})$ the unit map

$$f \rightarrow \psi^* \psi_! f$$

is a weak equivalence. Proceeding as in the proof of Proposition A.3.3.8 of [1], we will say that a map $T : f \rightarrow f'$ in $\mathrm{Fun}_{/\mathcal{J}}^{\mathrm{proj}}(\mathcal{J}, \mathcal{M})$ is **good** if the induced map

$$f' \amalg_f \psi^* \psi_! f \rightarrow \psi^* \psi_! f'$$

is a trivial cofibration. It will be enough to prove that every cofibration is good. We then observe that the collection of all good maps is weakly saturated and so it will suffice to prove that every generating cofibration is good. Let $j \in \mathcal{J}$ be an object and $X \hookrightarrow Y$ a generating cofibration of $\mathcal{M}(\varphi(j))$. We need to show that the map $\mathcal{F}_{j,X} \hookrightarrow \mathcal{F}_{j,Y}$ in $\mathrm{Fun}_{/\mathcal{J}}(\mathcal{J}, \mathcal{M})$ is good. Unwinding the definitions, it will be enough to show that for every $j' \in \mathcal{J}$ and every $\alpha : \varphi(j) \rightarrow \varphi(j')$ the induced map

$$[\mathrm{Map}_{\mathcal{J}}^{\alpha}(j, j') \otimes \alpha_* Y] \amalg_{\mathrm{Map}_{\mathcal{J}}^{\alpha}(j, j') \otimes \alpha_* X} [\mathrm{Map}_{\mathcal{J}}^{\alpha}(\psi(j), \psi(j')) \otimes \alpha_* X] \rightarrow \mathrm{Map}_{\mathcal{J}}^{\alpha}(\psi(j), \psi(j')) \otimes \alpha_* Y$$

is a trivial cofibration. But this follows from the fact that $\alpha_* X \hookrightarrow \alpha_* Y$ is a cofibration and ψ is a local trivial cofibration over \mathcal{J} . \square

Now let $X \rightarrow N(\mathcal{J})$ be a map of simplicial sets, let $\mathcal{J} \rightarrow \mathcal{J}$ be a map of simplicial categories and let

$$\begin{array}{ccc} \mathfrak{C}(X) & \xrightarrow{\varphi} & \mathcal{J} \\ & \searrow & \swarrow \\ & \mathcal{J} & \end{array}$$

be a commutative diagram of simplicial categories such that φ is a weak equivalence. We obtain a diagram of simplicial categories

$$\begin{array}{ccccc} \mathfrak{C}(X) \times \text{Fun}_{/\mathcal{J}}^{\circ}(\mathcal{J}, \mathcal{M}) & \longrightarrow & \mathcal{J} \times \text{Fun}_{/\mathcal{J}}^{\circ}(\mathcal{J}, \mathcal{M}) & \longrightarrow & \int_{\mathcal{J}}^{\circ} \mathcal{M} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{J} & & \end{array}$$

inducing a diagram of simplicial sets

$$\begin{array}{ccc} X \times N\left(\text{Fun}_{/\mathcal{J}}^{\circ}(\mathcal{J}, \mathcal{M})\right) & \xrightarrow{u} & \mathfrak{G} \\ & \searrow & \swarrow \\ & & N(\mathcal{J}) \end{array}$$

which can be considered as a map

$$u : X^{\flat} \times N\left(\text{Fun}_{/\mathcal{J}}^{\circ}(\mathcal{J}, \mathcal{M})\right) \longrightarrow \mathfrak{G}^{\Gamma}$$

in $(\text{Set}_{\Delta}^{\dagger})_{/N(\mathcal{J})}$. One can then formulate the following generalization of Theorem 6 above:

Theorem 14. *In the notation above, the map u induces an equivalence of ∞ -categories*

$$N\left(\text{Fun}_{/\mathcal{J}}^{\circ}(\mathcal{J}, \mathcal{M})\right) \simeq \text{Map}_{/N(\mathcal{J})}\left(X^{\flat}, \mathfrak{G}^{\Gamma}\right)$$

Proof. We want to establish an isomorphism in the homotopy category hSet_{Δ} associated to the Joyal model structure on simplicial. Hence it will suffice to show that for every simplicial set Y composition with u induces a bijection of sets

$$\text{Hom}_{\text{hSet}_{\Delta}}\left(Y, N\left(\text{Fun}_{/\mathcal{J}}^{\circ}(\mathcal{J}, \mathcal{M})\right)\right) \simeq \text{Hom}_{\text{hSet}_{\Delta}}\left(Y, \text{Map}_{/N(\mathcal{J})}\left(X^{\flat}, \mathfrak{G}^{\Gamma}\right)\right)$$

We begin by observing the right hand side can be identified with

$$\text{Hom}_{\text{hSet}_{\Delta}^{\dagger}}\left(Y^{\flat}, \text{Map}_{/N(\mathcal{J})}^{\dagger}\left(X^{\flat}, \mathfrak{G}^{\Gamma}\right)\right) \cong \text{Hom}_{\text{h}\left(\text{Set}_{\Delta}^{\dagger}/N(\mathcal{J})\right)}\left(Y^{\flat} \times X^{\flat}, \mathfrak{G}^{\Gamma}\right)$$

As for the left hand side, note first that we have an isomorphism of categories

$$\mathrm{Fun}(\mathfrak{C}(Y), \mathrm{Fun}_{/\mathcal{J}}(\mathcal{J}, \mathcal{M})) \cong \mathrm{Fun}_{/\mathcal{J}}(\mathfrak{C}(Y) \times \mathcal{J}, \mathcal{M})$$

Furthermore, the projective model structure on the right hand side coincides with the twice nested projective model structure on the left hand side. Hence by [1] Proposition 4.2.4.4 we get an isomorphism of sets

$$\mathrm{Hom}_{\mathrm{hSet}_{\Delta}}\left(Y, \mathrm{N}\left(\mathrm{Fun}_{/\mathcal{J}}^{\circ}(\mathcal{J}, \mathcal{M})\right)\right) \cong \pi_0\left(\mathrm{N}\left(\mathrm{Fun}_{/\mathcal{J}}^{\circ}(\mathfrak{C}(Y) \times \mathcal{J}, \mathcal{M})\right)\right)$$

This means that by replacing X with $Y \times X$ and \mathcal{J} with $\mathfrak{C}(Y) \times \mathcal{J}$ we may assume that $Y = \Delta^0$. It is hence left to show that the induced map

$$u_* : \pi_0\left(\mathrm{N}\left(\mathrm{Fun}_{/\mathcal{J}}^{\circ}(\mathcal{J}, \mathcal{M})\right)\right) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{h}(\mathrm{Set}_{\Delta^+/\mathrm{N}(\mathcal{J})})}\left(X^{\flat}, \mathfrak{G}^{\Gamma}\right)$$

is a bijection of sets. Furthermore, in light of Lemma 13 we may assume that $\mathcal{J} = \mathfrak{C}(X)$ and φ is the identity. In this case the map u_* admits a particularly simple description. Every fibrant/cofibrant functor $f : \mathfrak{C}(X) \rightarrow \int_{\mathcal{J}} \mathcal{M}$ factors through

$$\mathfrak{C}(X) \xrightarrow{f} \int_{\mathcal{J}}^{\circ} \mathcal{M} \hookrightarrow \int_{\mathcal{J}} \mathcal{M}$$

and one can identify $u_*([f])$ with the homotopy class of the adjoint map $f^{\mathrm{ad}} : X^{\flat} \rightarrow \mathfrak{G}^{\Gamma}$ (using the fact that marked maps $X^{\flat} \rightarrow \mathfrak{G}^{\Gamma}$ are the same as unmarked maps $X \rightarrow \mathfrak{G}$).

We start by showing that u_* is surjective. Let $g : X^{\flat} \rightarrow \mathfrak{G}^{\Gamma}$ be a map over $\mathrm{N}(\mathcal{J})$. By adjunction we obtain a map

$$g_{\mathrm{ad}} : \mathfrak{C}(X) \rightarrow \int_{\mathcal{J}}^{\circ} \mathcal{M}$$

over \mathcal{J} determining a fibrant object in $\mathrm{Fun}_{/\mathcal{J}}^{\mathrm{proj}}(\mathfrak{C}(X), \mathcal{M})$. Let $g'_{\mathrm{ad}} \rightarrow g_{\mathrm{ad}}$ be a cofibrant replacement for g_{ad} such that $g'_{\mathrm{ad}} \in \mathrm{Fun}_{/\mathcal{J}}^{\circ}(\mathfrak{C}(X), \mathcal{M})$. The map g'_{ad} is in turn adjoint to some map $g' : X^{\flat} \rightarrow \mathfrak{G}^{\Gamma}$ over $\mathrm{N}(\mathcal{J})$. The equivalence $g'_{\mathrm{ad}} \rightarrow g_{\mathrm{ad}}$ then determines a homotopy from g' to g in $(\mathrm{Set}_{\Delta^+})_{/\mathrm{N}(\mathcal{J})}$ and so

$$[g] = [g'] = u_*[f]$$

is in the image of u_* . It is left to show that u_* is injective. Let $f, g : \mathcal{J} \rightarrow \int_{\mathcal{J}} \mathcal{M}$ be fibrant cofibrant objects such that $f^{\mathrm{ad}}, g^{\mathrm{ad}} : X^{\flat} \rightarrow \mathfrak{G}^{\Gamma}$ are homotopic with respect to the Cartesian model structure. Since \mathfrak{G}^{Γ} is fibrant there exists a direct homotopy

$$H : (\Delta^1)^{\sharp} \times X^{\flat} \rightarrow \mathfrak{G}^{\Gamma}$$

from f^{ad} to g^{ad} . By adjunction we obtain a map

$$H_{\mathrm{ad}} : \mathfrak{C}(\Delta^1 \times X^{\flat}) \rightarrow \int_{\mathcal{J}}^{\circ} \mathcal{M}$$

whose restriction to $\mathfrak{C}(\{0\} \times X)$ is f and whose restriction to $\mathfrak{C}(\{1\} \times X)$ is g . Furthermore, since the marked edges in the fibers of \mathfrak{G} are equivalences we get that for every vertex x of X (i.e. every object of $\mathfrak{C}(X)$) the composed map

$$\mathfrak{C}(\Delta^1 \times \{x\}) \longrightarrow \mathfrak{C}(\Delta^1 \times X^b) \xrightarrow{H_{\text{ad}}} \int_{\mathcal{J}}^{\circ} \mathcal{M}$$

determines a weak equivalence from $f(x)$ to $g(x)$ in $\mathcal{M}(\varphi(x))$. Note that the map H_{ad} is not yet an honest natural equivalence from f to g but only a **homotopy coherent** one. In order to strictify it we will need to employ Lemma 13 again.

We can consider the map H_{ad} as a fibrant object in $\text{Fun}_{/\mathcal{J}}(\mathfrak{C}(\Delta^1 \times X^b), \mathcal{M})$. We have a natural map

$$\phi : \mathfrak{C}(\Delta^1 \times X^b) \longrightarrow \mathfrak{C}(\Delta^1) \times \mathfrak{C}(X^b)$$

which is a weak equivalence of simplicial categories. From Lemma 13 it follows that there exists a fibrant cofibrant object $H'_{\text{ad}} \in \text{Fun}_{/\mathcal{J}}(\mathfrak{C}(\Delta^1) \times \mathfrak{C}(X^b), \mathcal{M})$ such that $\phi^* H'_{\text{ad}}$ is weakly equivalent to H_{ad} . This implies, in particular, that the restriction f' of H'_{ad} to $\{0\} \times \mathfrak{C}(X)$ is weakly equivalent to f and the restriction g' of H'_{ad} to $\{1\} \times \mathfrak{C}(X)$ is weakly equivalent to g . However, the map H'_{ad} now determines an honest weak equivalence from f' to g' in $\text{Fun}_{/\mathcal{J}}(\mathfrak{C}(X), \mathcal{M})$, which means that

$$u_*[f] = u_*[f'] = u_*[g'] = u_*[g]$$

and the proof is complete. \square

Corollary 15. *In the notation above, there is a natural equivalence of ∞ -categories*

$$\mathbb{N}^{\natural}(\text{Sec}_r^{\circ}(\mathcal{M})) \simeq \text{Map}_{/\mathbb{N}(\mathcal{J})}(\mathbb{N}^{\natural}(\mathcal{J}), \mathfrak{G}^r)$$

Proof. We have a diagram of ∞ -categories

$$\begin{array}{ccc} \mathbb{N}^{\natural}(\text{Sec}_r^{\circ}(\mathcal{M})) & \longrightarrow & \text{Map}_{/\mathbb{N}(\mathcal{J})}(\mathbb{N}^{\natural}(\mathcal{J}), \mathfrak{G}^r) \\ \downarrow & & \downarrow \\ \mathbb{N}^{\natural}(\text{Sec}^{\circ}(\mathcal{M})) & \longrightarrow & \text{Map}_{/\mathbb{N}(\mathcal{J})}(\mathbb{N}^b(\mathcal{J}), \mathfrak{G}^r) \end{array}$$

where the vertical maps are the respective fully-faithful inclusions and the lower horizontal map is an equivalence by Theorem 6. It will hence be enough to verify that this diagram is homotopy Cartesian. For this, it will suffice to show that a fibrant cofibrant section $s : \mathcal{J} \longrightarrow \int_{\mathcal{J}}^{\circ} \mathcal{M}$ is Cartesian if and only if the corresponding map $\mathbb{N}(\mathcal{J}) \longrightarrow \mathfrak{G}$ sends every edge to a p -Cartesian edge. But this is a direct consequence of Lemma 3. \square

References

- [1] Lurie J., *Higher Topos Theory*, Annals of Mathematics Studies, 170, Princeton University Press, 2009, <http://www.math.harvard.edu/~lurie/papers/highertopoi.pdf>.
- [2] Barwick C. On left and right model categories and left and right Bousfield localizations, *Homology, Homotopy and Applications*, 12.2, 2010, p. 245-320