LITTLE CUBE ALGEBRAS AND FACTORIZATION HOMOLOGY
COURSE NOTES

YONATAN HARPAZ

Contents

1. Introduction

In the field of algebra we often encounter objects described as sets, equipped with certain *operations*, which are required in turn to satisfy certain axioms. For example, a *monoid* is given by a set with one binary operations, which is associative and unital. When every element acts invertibly we say that the monoid is a *group*. In modern mathematics it is very useful to consider such algebraic structures on objects more general than sets. For example, if we replace sets by smooth manifolds we get the notion of a *Lie group*. If instead we consider algebraic varieties we get *algebraic groups*. In order to properly define such notions it is useful to describe the algebraic structure diagrammatically, that is, as certain arrows in the category we choose to work in. For example, a binary structure on a object X can be represented by an arrow $X \times X \longrightarrow X$. This makes sense whether X is a set, an algebraic variety, or something more exotic, as long as we know what the product of two objects are. This product can be something that behaves quite differently from the usual cartesian product of sets. For example, if we work in the category of abelian groups, and take the product to be the tensor product operation, then a monoid object will now be a *ring*.

To make this approach work we need a convenient way to encode algebraic structures that is independent of the specific type of objects in which we want to realize it. A successful framework for doing so is by using the notion of an *operad*. An operad encodes the information of an algebraic structure by specifying in a suitable way what are the operations, and what rules they are required to follow. For example, the operad that encodes the structure of a monoid is usually called the *associative operad*. The operad that encodes the structure of a commutative monoid is called the *commutative operad*.

In the realm of homotopy theory this approach encounters a new subtlety. This is because operads encode the algebraic structure using *strict* rules that the operations must obey to (such as associativity), but when dealing with objects with a homotopical nature (such as spaces, chain complexes, or even categories), we actually need to let our algebraic axioms hold only *up to homotopy*, in a suitable delicate sense where all homotopies need to be specified compatibly (this is usually called up to *coherent* homotopy). For example, consider a topological space X and let $x \in X$ be a point. The fundamental group of X is defined to be the group whose elements are homotopy classes of paths in X from x to itself, where the group operation is given by concatenation. This invariant is very important in algebraic topology. However, to some extent what is more interesting is the object that you get without identifying homotopical paths, that is, the topological space of all paths from x to x. We would like to say that this is a topological group with the operation of concatenation. Alas, a direct examination shows that concatenation is not, strictly speaking, associative. It turns out, however, that it is associative up to (coherent) homotopy.

In order to encode algebraic structures in which the rules only hold up to coherent homotopy one can use the homotopy theoretical avatar of operads, which are called ∞-*operads*. In the passage from operads to ∞-operads some new phenomena arises. One such phenomenon, which is the center of the course, is the fact that commutativity now comes in many flavors. More precisely, between the classical associative and commutative operads there is now an infinite tower of intermediate operads

$$\mathrm{Ass} = \mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, ...$$

which "converge" in a suitable sense, to the commutative operad $\mathbb{E}_\infty = \mathrm{Com}$. The operad $\mathbb{E}_n$ is known as the *little n-cube operad*. This hierarchy of commutativity levels can already be seen when considering monoid structures on *categories* (which have a certain limited level of homotopyness): between the notion of a monoidal category (which is an $\mathbb{E}_1$-structure) and a symmetric monoidal category (which is an $\mathbb{E}_\infty$-structure) there is an an intermediate option known as a *braided monoidal structure*, which corresponds to an $\mathbb{E}_2$-structure. In spaces, the typical examples of $\mathbb{E}_n$-monoids are the *n-fold loop spaces*. May's recognition principle (see also work of Boardman and Vogt) states that these are exactly the *group-like* $\mathbb{E}_n$-monoids in spaces.

We will start the course by recalling some homotopy theoretical preliminaries surrounding the notion of ∞-categories. We will then discuss symmetric monoidal ∞-categories and ∞-operads, setting up the stage for the study of algebraic structures in a homotopy theoretical setting. We will then introduce and study the main object of interest in this course, the little n-cube operad. We will prove a key result, known as *Dunn's additivity theorem*, which roughly states that specifying an $\mathbb{E}_n$-structure is equivalent to specifying n commuting $\mathbb{E}_1$-structures.

One of the interesting features of $\mathbb{E}_n$-algebras are their relations to the topology of n-manifolds. In particular, given an $\mathbb{E}_n$-algebra A with a suitable equivariance structure (taking values in a sufficiently nice ∞-category) and an n-manifold M with a corresponding tangential structure, we may integrate A along M to obtain an object $\int_M A$, known as the *factorization homology* of M with coefficients in A. We may think of these construction as either producing homological invariants of manifolds out of $\mathbb{E}_n$-algebras, or as producing homological invariants of $\mathbb{E}_n$-algebras out of manifolds: both point of views yield interesting insights. In addition, in the former point of view the resulting homology theories for manifolds can be characterized axiomatically, as we will describe towards the end of the course, following the approach of D. Ayala and J. Francis.

We note that any abelian group A can be considered as an $\mathbb{E}_n$-monoid in spaces. The factorization homology $\int_M A$ is then simply the chain complex $\mathcal{C}_*(M, A)$ with coefficients in A, whose homologies are $H_*(M, A)$. If we replace A by a general $\mathbb{E}_n$-monoid is spaces X then we get a *nonabelian generalization* of classical homology. When X is group-like, May's recognition principle gives that $X \simeq \Omega^n Y$ for some space Y. In this case, Lurie's *nonabelian Poincaré duality* states that $\int_M X$ is naturally equivalent to the mapping space from M to Y, that which can be considered as a generalization of *cohomology* to nonabelian coefficients. We will outline the mathematics needed to establish this theorem in the final parts of the course, following the ideas of Lurie.

The passage from X to its n-fold delooping Y can itself be considered as a variant of factorization homology: it is obtained by taking the *reduced* factorization homology of X along the n-sphere. This operation can be done for $\mathbb{E}_n$-algebras taking values in any nice ∞-category $\mathcal{C}$, as long as they admit an *augmentation* (which is what makes reduced homology make sense). This procedure takes an augmented $\mathbb{E}_n$-algebra and yields an *augmented $\mathbb{E}_n$-coalgebra*. On the other hand, given an $\mathbb{E}_n$-coalgebra, we may take its reduced factorization *cohomology* along the n-sphere and obtain again an $\mathbb{E}_n$-algebra. In spaces this reproduces the looping delooping procedure, where we note that every space is canonically an $\mathbb{E}_n$-comonoid for every n. In general, this delooping process is closely related to *Koszul* duality for $\mathbb{E}_n$-algebras. Given a closed n-manifold M one then obtains a canonical map from the factorization homology of A along M to the factorization cohomology of the "n-fold delooping" of A along M. In sufficiently nice cases this map is an equivalence, yielding *Poincaré-Koszul duality.* We will describe these ideas without proof in the last part of the course, following recent work of D. Ayala and J. Francis.

2. Preliminaries on ∞-categories

The point of departure of modern algebraic topology from traditional point-set topology is arguably the introduction of *homotopies* between maps. This has some remarkable consequences: it yields a notion of a homotopy equivalence between topological spaces, leads to the focus on invariants which respect this new equivalence relation, such as homotopy groups and homology groups, and demands modifying various non-homotopical construction to accommodate homotopy. This last point may seem surprising. If the notion of a homotopy between maps is natural from a topological point of view, how come many natural topological constructions do not respect homotopy equivalence? for example, if $f : X \longrightarrow Y$ is a continuous map and $y \in Y$ is a point, then the fiber $f^{-1}(y) \subset X$ is not a homotopy invariant notion: it is not stable under a continuous deformation of f or y, nor under replacing X or Y by homotopy equivalent spaces. A modern answer to this question is that the introduction of homotopies between maps means that we are no longer working in the category of topological spaces we thought we were working in. Indeed, this would explain why some constructions don't work anymore: these constructions are usually categorical, given by various limits and colimits, and hence should not be expected to make sense once the category we are working with changes.

But then in what category are we working in? One potential answer is that we should consider the *homotopy category* of spaces. This is the category whose objects are spaces and whose morphisms are homotopy classes of maps. This point of view is powerful, but has some serious drawbacks. Besides the general feeling that some crucial information got lost (what about homotopies between homotopies, homotopies between homotopies between homotopies, and so on?), working with the homotopy category does not explain the modifications that various categories constructions need in order to become homotopical. For example, the notion of the fiber $f^{-1}(y)$ of a map $f : X \longrightarrow Y$ is replaced in algebraic topology with the notion of the *homotopy fiber* of f, which is the space of pairs (x, η) where x is a point of X and η is a continuous path in Y from $f(x)$ to y. This is an example of switching from a limit to its corresponding *homotopy limit*, and yields a satisfyingly homotopy invariant replacement. It is *not*, however, the corresponding limit in the homotopy category of spaces. In fact, the homotopy category of spaces does not admit limits and colimits in general! If we choose the homotopy category as our categorical framework when studying spaces, then we will need to accept that the vast majority of our homotopically modified constructions, including all homotopy limits and colimits, are not determined by the underlying categorical structure, and thus need to be considered as additional structure. This is of course a valid approach, which in fact leads to Grothendieck's theory of *derivators*, though conceptually it is not very satisfying, at least if we wish to hold on to the idea that the homotopy versions of our categorical constructions should be justified by some manifest universality.

We are hence led to ask the following: is there a category in which we can consider topological spaces in such a way that homotopy limits and colimits will become limits and colimits with respect to this category? The answer to this question is yes - if we accept to expand what we mean by a category. Indeed, homotopies between maps are something that looks kind of familiar from a categorical point of view: it looks like morphisms between maps. Indeed, we can concatenate them, and there is a trivial homotopy from a map to itself. It hence seems that we have a kind of a categorical structure stacked on itself: we have objects (spaces), morphisms (continuous maps) and then morphisms between morphisms (homotopies). This seems to just almost work, but then fails at a surprising point: concatenation of homotopies is not, strictly speaking, associative (nor unital). But it *almost* is. It is associative and unital up to *homotopy between homotopies*. A traditional way to solve this problem would then be to replace homotopies by homotopy classes of homotopies. This indeed yields something that looks very categorical (it is in fact a 2-category in which all 2-arrows are invertible). Unfortunately, this "enhanced homotopy category", though keeping more of the information available than the homotopy category, suffers from the exact same problem: it does not provide a universal justification for homotopy limits and colimits. In fact, inspecting this solution we may see that it is very similar to the solution of taking the homotopy category. Indeed, up to some point-set topology subtleties we can describe the homotopy category as obtained by remembering from each mapping space its set of path components, while the extended version as obtained by remembering from each mapping space its *fundamental groupoid*.

At this point it is starting to be clear that in order to obtain a satisfying solution we would need to remember the information "all the way up". In other words, we need to have some kind of a categorical creature in which we have objects, morphisms, (invertible) morphisms between morphisms, (invertible) morphisms between morphisms between morphisms etc. The informal name for such a creature is an $(\infty, 1)$*-category*. Here the ∞ symbols stands for the fact that we have morphisms in all dimensions, and the 1 stands for the fact that the morphisms above dimension 1 are invertible (in a suitable sense). To make this idea precise is not trivial, and took the mathematical community a few decades to develop. Historically, several potential models for the notion of $(\infty, 1)$-category were suggested, and after a while they were all shown to be equivalent to one another. In this course we will mostly make use of the model developed by Joyal and Lurie, and which appears in the literature as either *quasi-category* or ∞*-category*. In these notes we will use the latter name.

2.1. **Two models for $(\infty, 1)$-categories.** Before we discuss ∞-categories let us consider a more basic idea. Instead of inventing a whole new notion of a higher category, why not use an already existing extension of category theory, the notion of an *enriched category*? In this setup we consider categorical structures in which the set morphisms is replaced by some other type of object. An example relevant to our story will be to take our enrichment in the category Set_{Δ} of *simplicial sets*. Recall that a *simplicial category* $\mathcal{C}$ consists of

- a set of objects $\mathrm{Ob}(\mathcal{C})$;
- for every two objects x, y a simplicial set $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathrm{Set}_{\Delta}$;

- for every three objects x, y, z a composition rule
$$\mathrm{Map}_{\mathcal{C}}(x,y) \times \mathrm{Map}_{\mathcal{C}}(y,z) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x,z)$$
which satisfies the associativity axiom.
- for every $x \in \mathrm{Ob}(\mathcal{C})$ a designated vertex $\mathrm{Id}_x \in \mathrm{Map}_{\mathcal{C}}(x,x)$, which is a two-sided unit with respect to composition.

If x is an object of $\mathcal{C}$ then we will usually write $x \in \mathcal{C}$ as a short for $x \in \mathrm{Ob}(\mathcal{C})$. We note that every simplicial category $\mathcal{C}$ has an *underlying* ordinary category $\overline{\mathcal{C}}$, whose objects are the same as those of $\mathcal{C}$ and such that $\mathrm{Hom}_{\overline{\mathcal{C}}}(x,y)$ is the set of vertices of $\mathrm{Map}_{\mathcal{C}}(x,y)$.

Definition 2.1.1. A *simplicial functor* $\mathcal{C} \longrightarrow \mathcal{D}$ between simplicial categories consists of a map of sets $\varphi : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D})$, together with maps $\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(\varphi(x),\varphi(y))$ for every $x, y \in \mathcal{C}$, which are compatible with the composition operation and preserve identity vertices.

Remark 2.1.2. There is also an associated notion of a *simplicial natural transformation* (though we will not make much use of it in this course): if $\varphi, \psi : \mathcal{C} \longrightarrow \mathcal{D}$ are two simplicial functors then a simplicial natural transformation $\tau : \varphi \Rightarrow \psi$ consists of a map $\tau_x : \varphi(x) \longrightarrow \psi(x)$ in the underlying category $\overline{\mathcal{C}}$ for every $x \in \mathcal{C}$ such that for every $x, y \in \mathcal{C}$ the composed map
$$\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(\varphi(x),\varphi(y)) \xrightarrow{(\tau_y)_*} \mathrm{Map}_{\mathcal{D}}(\varphi(x),\psi(y))$$
and the composed map
$$\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(\psi(x),\psi(y)) \xrightarrow{(\tau_X)^*} \mathrm{Map}_{\mathcal{D}}(\varphi(x),\psi(y))$$
coincide.

To see how this might work as a model for $(\infty,1)$-categories let us look again at our motivating example of spaces. Given two spaces X, Y and two maps $f, g : X \longrightarrow Y$, a homotopy from f to g is by definition a map of the form $[0,1] \times X \longrightarrow Y$ which restricts to f on $\{0\} \times X$ and to g on $\{1\} \times X$. Similarly, homotopy between homotopies can be expressed as a map $\mathbb{D}^2 \times X \longrightarrow Y$ (where $\mathbb{D}^2$ is the 2-disk) and so on for higher homotopies. Having in mind simplicial sets we may suggest the following manner to efficiently encode all this information at once. Given two topological spaces X, Y let us denote by $\mathrm{Map}(X,Y) \in \mathrm{Set}_\Delta$ the simplicial set whose n-simplices are given by
$$\mathrm{Map}(X,Y)_n := \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n| \times X, Y)$$
where $|\Delta^n|$ is the geometric realization of the n-simplex. We then see that the vertices of $\mathrm{Map}(X,Y)$ are the continuous maps $X \longrightarrow Y$ and the edges in $\mathrm{Map}(X,Y)$ are exactly the homotopies. Similarly, any homotopy between homotopies can be encoded by a suitable 2-simpelx $\Delta^2 \times X \longrightarrow Y$, and so on for higher homotopies. Generalizing from our example of interest we may consider the idea that *simplicial categories* can be used as a model for the notion of $(\infty,1)$-categories. In turns out that indeed any $(\infty,1)$-category can be represented by a simplicial category. This simplicial category will however not be unique, and in order to use this as a model we have to understand when two simplicial categories model the same $(\infty,1)$-category. Given a simplicial category $\mathcal{C}$, let us denote by $\mathrm{Ho}(\mathcal{C})$ the ordinary category whose objects are the objects of $\mathcal{C}$ and such that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) := \pi_0|\mathrm{Map}_{\mathcal{C}}(X,Y)|$.

Definition 2.1.3. Let $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ be a simplicial functor between simplicial categories. We will say that φ is a *Dwyer-Kan* equivalence if it satisfies the following two conditions:

(1) For every $x, y \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(\varphi(x), \varphi(y))$ is a weak equivalence of simplicial sets.
(2) The induced functor $\mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ is an equivalence of categories.

We will say that two simplcial categories $\mathcal{C}, \mathcal{D}$ are Dwyer-Kan equivalent if they can be connected by a zig-zag of Dwyer-Kan equivalences.

Two simplicial categories should be considered as modeling the same $(\infty, 1)$-category precisely when they are Dwyer-Kan equivalent. We can then say that $(\infty, 1)$-categories are modeled by simplicial categories *up to Dwyer-Kan equivalence.* This model has the advantage of behaving very much like ordinary categories, and so many categorical arguments pass through without change. However, while simplicial categories are useful to encode and construct individual $(\infty, 1)$-categories, they are not as convenient when it comes to encoding *functors* between $(\infty, 1)$-categories. Informally speaking, if $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor between $(\infty, 1)$-categories then we should be able to find simplicial categories $\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}}$ which model $\mathcal{C}$ and $\mathcal{D}$ such that φ can be represented as a simplicial functor $\widetilde{\varphi} : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{D}}$. However, such an $\widetilde{\varphi}$ will not necessarily exist for every choice of simplicial categories modelling $\mathcal{C}$ and $\mathcal{D}$. In particular, given a fixed pair of simplicial categories, it is not easy to understand directly what are the functors between the corresponding $(\infty, 1)$-categories, not to mention that we would like to have this collection of functors organized into an $(\infty, 1)$-category as well (much like for ordinary categories).

We shall now present another model for the theory of $(\infty, 1)$-categories, which has the advantage of being particularly amenable to the formation of functor categories. This amenability will also make this model very well suited for defining the notions of limits and colimits, which will finally give us a universal justification to the add-hoc constructions of homotopy limits and colimits in spaces. Recall that for $0 \leq i \leq n$ the i'th *horn* of Δ^n is the subsimplicial set $\Lambda^n_i \subseteq \Delta^n$ spanned by all the $(n-1)$-faces of Δ^n which contain the vertex i.

Definition 2.1.4. An ∞-*category* is a simplicial set $\mathcal{C}$ with the following property: for every $0 < i < n$ the dotted extension exists in any diagram of the form

$$
\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}
\tag{2.1}
$$

To obtain a preliminary intuition as to why Definition 2.1.4 makes sense let us explain how ordinary categories can be interpreted as ∞-categories via their *nerves.*

Definition 2.1.5. Let $\mathcal{I}$ be a small category. We define the *nerve* of $\mathcal{I}$ to be the simplicial set $\mathrm{N}(\mathcal{I})$ given by

$$\mathrm{N}(\mathcal{I})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{I}) \tag{2.2}$$

where $[n]$ is the category corresponding to the poset (partially ordered set) $\{0, ..., n\}$ with its usual linear order.

We invite the reader whose is not familiar with the following fact to prove it themselves:

Exercise 2.1.6. Prove that the nerve functor $\mathrm{N} : \mathrm{Cat} \longrightarrow \mathrm{Set}_\Delta$ is fully-faitful and its essential image consists of those simplicial sets for which every diagram as (2.1) admits a *unique* dotted extension.

The above exercise tells us that we can think of ordinary categories as a particular type of simplicial sets, and furthermore that this type of simplicial sets is characterized by a certain unique lifting condition. The notion of an ∞-category is then obtained by removing the uniqueness part of this condition.

Given an ∞-category $\mathcal{C}$ we will call the vertices of $\mathcal{C}$ its *objects*, and if we have an edge $e : \Delta^1 \longrightarrow \mathcal{C}$ then we will call it a *morphism* from $x := e|_{\Delta^{\{0\}}}$ to $y := e|_{\Delta^{\{1\}}}$, in which case we will also use diagramatic notation and write this edge as an arrow $x \longrightarrow y$. For a subset $S \subseteq \{0, ..., n\}$ we use the notation $\Delta^S \subseteq \Delta^n$ to denote the $|S|$-dimensional face of Δ^n whose vertices are S. Given an object $x \in \mathcal{C}$ we will consider the degenerate edge $s(x)$ on x as the *identity* $\mathrm{Id}_x : x \longrightarrow x$. Given a 2-simplex $\sigma : \Delta^2 \longrightarrow \mathcal{C}$ we will depict it diagramatically as

(2.3)
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z \end{tikzcd}$$

where x, y, z are the vertices obtained by restricting σ to $\Delta^{\{0\}}, \Delta^{\{1\}}$ and $\Delta^{\{2\}}$ respectively, and f, g, h are the morphisms obtained by restricting σ to $\Delta^{\{0,1\}}, \Delta^{\{1,2\}}$ and $\Delta^{\{0,2\}}$ respectively. We can think of σ as designating a *homotopy* from $g \circ f$ to h. However, it is important to note that in an ∞-category there isn't a specific edge which *is* $g \circ f$. Instead, we may consider any triangle of the form (2.3) to be *exhibiting h as the composition of f and g*. In particular, there could be two different triangles $\sigma, \sigma' : \Delta^2 \longrightarrow \mathcal{C}$ of the form (2.3), with different edges $\sigma|_{\Delta^{\{0,2\}}} = h \neq h' = \sigma'_{\Delta^{\{0,2\}}}$. However, in this case the edges h and h' will be *homotopic* in a suitable sense: indeed, let $\tau : \Delta^2 \longrightarrow \Delta^1 \longrightarrow \mathcal{C}$ be the degenerate triangle such that $\tau|_{\Delta^{\{1,2\}}} = \tau|_{\Delta^{\{0,2\}}} = f$ and $\tau|_{\Delta^{\{0,1\}}} = s(x)$ is degenerate on x. The triangles σ, σ' and τ then determine a map $\rho : \Lambda^2_3 \longrightarrow \mathcal{C}$ such that $\rho|_{\Delta^{\{0,1,2\}}} = \tau$, $\rho|_{\Delta^{\{0,2,3\}}} = \sigma$ and $\rho|_{\Delta^{\{1,2,3\}}} = \sigma'$. Since $\mathcal{C}$ is an ∞-category the map ρ extends to a map $\widetilde{\rho} : \Delta^3 \longrightarrow \mathcal{C}$. The 2-simplex $\widetilde{\rho}|_{\Delta^{\{0,1,3\}}}$ then determines a diagram in $\mathcal{C}$ of the form

(2.4)
$$\begin{tikzcd} & x \arrow[dr, "h'"] & \\ x \arrow[ur, equal, "\mathrm{Id}"] \arrow[rr, "h"] & & z \end{tikzcd}$$

which we will consider as a *homotopy* from h' to h. In this sense, while the composition of two morphisms in an ∞-category is not strictly speaking uniquely defined, it is uniquely defined up to homotopy. Elaborating on this argument one can show that the collection of compositions of f and g can be organized into a simplicial set which is a *contractible Kan complex* (see Remark 2.3.11 below). We may thus say that composition of two morphisms in an ∞-category is *essentially defined*. This essentially defined composition is associative in the following sense: if $f : x \longrightarrow y$, $g : y \longrightarrow z$ and $h : z \longrightarrow w$ are three arrows in $\mathcal{C}$ then we can compose the three of them in two different ways. We can either choose triangles of the form

(2.5)
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "t"] \arrow[dr, "s"'] & & z \arrow[dl, "h"] \\ & w & \end{tikzcd}$$

exhibiting t as the composition of f and g and s as the composition of t and h, or choose triangles

(2.6)
$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[dd, "t'"] & \\ x \arrow[ur, "f"] \arrow[dr, "s'"'] & & z \arrow[dl, "h"] \\ & w & \end{tikzcd}$$

exhibiting t' as the composition of g and h and s' as the composition of f and t'. We claim that in this case s is homotopic to s'. To see this, it will suffice by the argument before to show that there exists a triangle in $\mathcal{C}$ exhibiting s as the composition of f and t'. To find such a triangle we note that the two triangles in (2.5) together with the right triangle in (2.6) together determine a map $\rho : \Lambda^3_2 \longrightarrow \mathcal{C}$. Since $\mathcal{C}$ is an ∞-category we can extend ρ to $\overline{\rho} : \Delta^3 \longrightarrow \mathcal{C}$. The triangle $\rho|_{\Delta^{\{0,1,3\}}}$ then exhibits s as the composition of f and t'.

The above considerations concerning the essential well-definition and associativity up to homotopy of composition of arrows in an ∞-category have in particular the following outcome: we can define an ordinary category $\mathrm{Ho}(\mathcal{C})$ whose objects are the vertices of $\mathcal{C}$ and such that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x,y)$ is the set of *homotopy classes* of arrows from x to y. This category is known as the *homotopy category* of $\mathcal{C}$. There is an evident map $\mathcal{C} \longrightarrow \mathrm{N}(\mathrm{Ho}(\mathcal{C}))$ which sends each arrow to its homotopy class.

Definition 2.1.7. Let $\mathcal{C}$ be an ∞-category and $f : x \longrightarrow y$ an arrow in $\mathcal{C}$. We will say that f is *invertible* if there exists an arrow $g : y \longrightarrow x$ and triangles as in (2.3) exhibiting Id_x as the composition of f and g and Id_y as the composition of g and f. In this case we will also say that f is an *equivalence* in $\mathcal{C}$ from x to y. We will say that an object $x \in \mathcal{C}$ is *equivalent* to $y \in \mathcal{C}$ if there exists an invertible arrow from x to y. In this case we will write $x \sim y$. We note that, essentially by the definition of $\mathrm{Ho}(\mathcal{C})$, an arrow is invertible if and only if it maps to an isomorphism in $\mathrm{Ho}(\mathcal{C})$, and two objects are equivalent if and only if they become isomorphic in $\mathrm{Ho}(\mathcal{C})$.

Definition 2.1.8. Let X, Y be two simplicial sets. We will denote by Y^X the simplicial set such that

$$(Y^X)_n = \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times X, Y).$$

We will call Y^X the *mapping simplicial set* from X to Y.

Proposition 2.1.9 (Joyal)**.** *If $\mathcal{C}$ is an ∞-category then for every $K \in \mathrm{Set}_\Delta$ the mapping simplicial set $\mathcal{C}^K$ is again an ∞-category.*

We will think of $\mathcal{C}^K$ as the ∞-category of *K-indexed diagrams in $\mathcal{C}$*. If K is also an ∞-category then we will also call $\mathcal{C}^K$ the ∞-category of *functors from K to $\mathcal{C}$*. When we want to emphasize this point of view we will denote $\mathcal{C}^K$ also by $\mathrm{Fun}(K, \mathcal{C})$.

Definition 2.1.10. Let $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ be a map of ∞-categories. We will say that φ is an *equivalence* if there exists a $\psi : \mathcal{D} \longrightarrow \mathcal{C}$ such that $\psi \circ \varphi$ is equivalent to $\mathrm{Id}_{\mathcal{C}}$ in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $\varphi \circ \psi$ is equivalent to $\mathrm{Id}_{\mathcal{D}}$ in the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$.

Definition 2.1.11. We will say that an ∞-category $\mathcal{C}$ is an *∞-groupoid* if every arrow in $\mathcal{C}$ is invertible.

Theorem 2.1.12 (Joyal)**.** *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is an ∞-groupoid if and only if it is a Kan complex, that is, if and only if the dotted extension exists in any diagram of the form* (2.1) *where $0 \leq i \leq n$.*

We will give a proof of Theorem 2.1.12 in §2.3 below.

Definition 2.1.13. Given an ∞-category $\mathcal{C}$ we will denote by $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ the subsimplicial set such that an n-simplex $\sigma : \Delta^n \longrightarrow \mathcal{C}$ belongs to $\mathcal{C}^{\simeq}$ if and only if the edge $\sigma|_{\Delta^{\{i,j\}}}$ is invertible for every $i, j \in \{0, ..., n\}$. Then $\mathcal{C}^{\simeq}$ is also an ∞-category, and every morphism in $\mathcal{C}$ is invertible, i.e., $\mathcal{C}^{\simeq}$ is an ∞-groupoid. By construction, $\mathcal{C}^{\simeq}$ is the maximal sub-∞-groupoid of $\mathcal{C}$, that is, it contains any other subsimplicial set of $\mathcal{C}$ which is an ∞-groupoid.

Remark 2.1.14. By Theorem 2.1.12 the ∞-groupoid $\mathcal{C}^{\simeq}$ is a Kan complex and contains any other Kan subcomplex of $\mathcal{C}$.

Remark 2.1.15. Given a topological space X, its singular simplicial set $\mathrm{Sing}(X)$ is Kan, and is hence an ∞-groupoid by Theorem 2.1.12. We may thus call $\mathrm{Sing}(X)$ the *fundamental ∞-groupoid of X*, an invariant which refines the classical fundamental groupoid of X. The counit map $|\mathrm{Sing}(X)| \longrightarrow X$ is a weak homotopy equivalence of spaces, and so the fundamental groupoid captures all the information on X up to weak homotopy equivalence. On the other hand, if Z is a Kan complex then the unit map $Z \longrightarrow \mathrm{Sing}|Z|$ is an equivalence of ∞-categories (Definition 2.1.10) and so every Kan complex is (canonically) equivalent to the fundamental groupoid of a space. Elaborating on this argument we see that the notion of ∞-groupoids is essentially equivalent to that of topological spaces up to weak homotopy equivalence. Alternatively, if we restrict attention to CW complexes, then every weak homotopy equivalence is a homotopy equivalence, and so we may say that the notion of ∞-groupoid is equivalent to that of a CW-complex up to homotopy equivalence. The idea that spaces are essentially ∞-categories in which every morphism is invertible is known as *Grothendieck's homotopy hypothesis*, and is one of the conceptual pillars of higher category theory.

2.2. **Constructions of ∞-categories.** In this section we will discuss two ways to construct an ∞-category starting from from classical data. The first is via forming *coherent nerves* of simplicial categories. This operation can also be used to show the equivalence between the model of simplicial categories and that of ∞-categories.

Definition 2.2.1. For every $n \geq 0$ let us denote by $\mathfrak{C}(\Delta^n)$ the simplicial category whose objects are the numbers $0, ..., n$ and whose mapping simplicial sets are given by $\mathrm{Map}_{\mathfrak{C}(\Delta^n)}(i,j) = \mathrm{N}(\mathcal{P}(i,j))$ where $\mathcal{P}(i,j) \subseteq \mathrm{Sub}(\{0,...,n\})$ is the poset of subsets of $\{0,...,n\}$ whose minimal element is i and whose maximal element is j (in particular $\mathcal{P}(i,j) = \emptyset$ if $i > j$). The composition rule is induced by the poset map $\mathcal{P}(i,j) \times \mathcal{P}(j,k) \longrightarrow \mathcal{P}(i,k)$ which takes (A,B) to $A \cup B$.

The association $[n] \mapsto \mathfrak{C}(\Delta^n)$ determines a functor from the category Δ^{op} to the category Cat_Δ of simplicial categories. Since $\mathrm{Set}_\Delta = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$ is a category of presheaves (of sets), the functor $\mathfrak{C} : \Delta \longrightarrow \mathrm{Cat}_\Delta$ admits a unique colimit preserving extension

$$\mathfrak{C} : \mathrm{Set}_\Delta \longrightarrow \mathrm{Cat}_\Delta \qquad \mathfrak{C}(X) = \operatorname*{colim}_{\Delta^n \to X} \mathfrak{C}(\Delta^n)$$

where the colimit is taken of the category of simplices of X. By equally formal arguments the functor $\mathfrak{C}$ admits a *right adjoint*:

$$\mathrm{N} : \mathrm{Cat}_\Delta \longrightarrow \mathrm{Set}_\Delta \qquad \mathrm{N}(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}(\Delta^n), \mathcal{C}). \tag{2.7}$$

we note that we used the same notation in (2.7) as we did for the nerve functor of Definition 2.1.5: indeed if $\mathcal{C}$ is a simplicial category whose mapping simplicial sets are discrete (i.e., $\mathcal{C}$ is a actually an ordinary category) then (2.7) coincides with (2.2) and we may hence consider (2.7) as an extension of the nerve functor from ordinary to simplicial categories. The functor (2.7) is also known as the *coherent nerve* functor.

Definition 2.2.2. Let us say that a simplicial category $\mathcal{C}$ is *locally Kan* if for every $X, Y \in \mathcal{C}$ the mapping simplicial set $\mathrm{Map}_{\mathcal{C}}(X,Y)$ is Kan.

Remark 2.2.3. Recall that in simplicial sets we have a well-behaved Kan replacement functor

$$\mathrm{Ex}^\infty : \mathrm{Set}_\Delta \longrightarrow \mathrm{Set}_\Delta$$

which sends every simplicial set X to a Kan complex $\mathrm{Ex}^\infty(X)$ equipped with a natural weak equivalence $X \xrightarrow{\simeq} \mathrm{Ex}^\infty(X)$. One can then check that the functor Ex^∞ preserves cartesian products, and so if we have a simplicial category $\mathcal{C}$ then we can apply it to all mapping objects in $\mathcal{C}$. This yields a locally Kan simplicial category $\mathcal{C}_{\mathrm{Ex}^\infty}$ together with a Dwyer-Kan equivalence $\mathcal{C} \longrightarrow \mathcal{C}_{\mathrm{Ex}^\infty}$.

Theorem 2.2.4.

(1) If $\mathcal{C}$ is a locally Kan simplicial category then $\mathrm{N}(\mathcal{C})$ is an ∞-category.
(2) The functor N sends Dwyer-Kan equivalences between locally Kan simplicial categories to equivalence of ∞-categories, while the functor $\mathfrak{C}$ sends equivalences of ∞-categories to Dwyer-Kan equivalences.
(3) If $\mathcal{C}$ is a locally Kan simplicial category then the counit map $\mathfrak{C}(\mathrm{N}(\mathcal{C})) \longrightarrow \mathcal{C}$ is a Dwyer-Kan equivalence.
(4) If $\mathcal{D}$ is an ∞-category then the natural map $\mathcal{D} \longrightarrow \mathrm{N}(\mathfrak{C}(\mathcal{D})_{\mathrm{Ex}^\infty})$ is an equivalence of ∞-categories.

We may summarize Theorem 2.2.4 by saying that the functors $\mathcal{C} \mapsto \mathrm{N}(\mathcal{C})$ and $\mathcal{D} \mapsto \mathfrak{C}(\mathcal{D})_{\mathrm{Ex}^\infty}$ determine an equivalence between the notion of locally Kan simplicial categories (up to Dwyer-Kan equivalence) and that of ∞-categories (up to equivalence of ∞-categories). This equivalence can in fact be set up in the more powerful framework of *Quillen model categories*.

Definition 2.2.5. We will say that a map $K \longrightarrow L$ of simplicial sets is a *categorical equivalence* if for every ∞-category $\mathcal{C}$ the induced map

$$(\mathcal{C}^L)^{\simeq} \longrightarrow (\mathcal{C}^K)^{\simeq}$$

is an equivalence of Kan complexes.

Theorem 2.2.6.

[Bergner] *(1) There exists a model structure on* Cat_Δ*, which we will call the* Dwyer-Kan model structure*, whose weak equivalences are the Dwyer-Kan equivalences and whose fibrant objects are the locally Kan simplicial categories.*

[Joyal] *(2) There exists a model structure on* Set_Δ*, which we will call the* categorical model structure*, whose weak equivalences are the categorical equivalences, whose cofibrations are the levelwise injective maps and whose fibrant objects are the* ∞*-categories.*

[Joyal-Lurie] *(3) The adjunction* $\mathfrak{C} \dashv \mathrm{N}$ *is a Quillen equivalence between these two model structures.*

We remark that the fibrations in the categorical model structure, which are called *categorical fibrations* are not easy to describe in general, but become easy to describe when the codomain is an ∞-category.

Definition 2.2.7. Let $p : X \longrightarrow Y$ be a map of simplicial sets. We will say that p is an *inner fibration* if it has the right lifting property with respect to horn inclusions of the form $\Lambda^i_n \subseteq \Delta^n$ with $0 < i < n$, that is, if the dotted lift exists in any diagram of the form

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{2.8}$$

with $0 \leq i < n$.

Remark 2.2.8. The map $X \longrightarrow \Delta^0$ is an inner fibration if and only if X is an ∞-category.

Proposition 2.2.9 (Joyal)**.** *Let* $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ *be a map between* ∞*-categories. Then* φ *is a fibration in the categorical model structure if and only if it is an inner fibration and in addition for every* $x \in \mathcal{C}$ *and any invertible arrow* $f : \varphi(x) \longrightarrow y$ *in* $\mathcal{D}$ *there exists an invertible arrow* $\widetilde{f} : x \longrightarrow \widetilde{y}$ *in* $\mathcal{C}$ *such that* $\varphi(\widetilde{f}) = f$*.*

Let us now go back to our motivating question and show that we can describe *spaces* as forming a suitable ∞-category. Recall from earlier courses that we have two equivalent models for the homotopy theory of spaces (with homotopy equivalences). The first consists of a model structure on the category Top of spaces whose weak equivalences are the weak homotopy equivalences, whose cofibrant objects are the CW-complexes, and where all objects are fibrant. The second consists of the Kan-Quillen model structure on the category Set_Δ of simplicial sets whose weak equivalences are the maps which induce a weak equivalence on geometric realizations, whose fibrant objects are the Kan complexes, and where all objects are cofibrant. These two model categories are related via the *Quillen equivalence*

$$|-| : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Top} : \mathrm{Sing}\,.$$

We may hence choose any one of these models in order to construct our ∞-category of spaces. If we start from Top then we can do this by defining the simplicial category $\mathcal{CW}$ whose

- objects are the CW-complexes;
- for two CW-complexes X, Y the simplicial mapping set $\mathrm{Map}_{\mathcal{CW}}(X, Y)$ is given by $\mathrm{Map}_{\mathcal{CW}}(X, Y)_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n| \times X, Y)$.

On the other hand, if we start from the Kan-Quillen model structure on simplcial sets then we can consider instead the simplicial category $\mathcal{K}\mathrm{an}$ whose

- objects are the Kan complexes;
- for two Kan complexes X, Y the simplicial mapping set $\mathrm{Map}_{\mathcal{K}\mathrm{an}}(X, Y)$ is given by $\mathrm{Map}_{\mathcal{K}\mathrm{an}}(X, Y)_n = \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times X, Y)$.

Both $\mathcal{CW}$ and $\mathcal{K}\mathrm{an}$ are locally Kan simplicial categories. Furthermore, the functors Sing and $|-|$ are both *Dwyer-Kan equivalence*

$$\mathrm{Sing} : \mathcal{CW} \xrightarrow{\simeq} \mathcal{K}\mathrm{an} \qquad |-| : \mathcal{K}\mathrm{an} \xrightarrow{\simeq} \mathcal{CW},$$

and so it makes essentially no difference which option we choose for our model. In modern homotopy theory, it is often customary to take $\mathcal{K}\mathrm{an}$.

Definition 2.2.10. We define the ∞-category of spaces

$$\mathcal{S} := \mathrm{N}(\mathcal{K}\mathrm{an})$$

to be the coherent nerve of the simplicial category of Kan complexes.

Remark 2.2.11. Technically speaking, $\mathcal{S}$ is a *large* ∞-category, that is the set of n-simplices $\mathcal{S}_n$ is a proper class. However, this ∞-category is locally small, that is, its mapping spaces are all small.

In a similar fashion we can now define the ∞-category Cat_∞ of (small) ∞-categories. Let QC be the simplicial category whose objects are the ∞-categories and such that for each $\mathcal{C}, \mathcal{D} \in \mathrm{QC}$ we have $\mathrm{Map}_{\mathrm{QC}}(X, Y) = \mathrm{Fun}(X, Y)^{\simeq}$, the maximal ∞-groupoid of $\mathrm{Fun}(X, Y)$. Then QC is locally Kan by Theorem 2.1.12, and we define

$$\mathrm{Cat}_\infty := \mathrm{N}(\mathrm{QC})$$

to be its coherent nerve. We will refer to Cat_∞ as the ∞-category of ∞-categories. If $\mathcal{C}, \mathcal{D}$ are ∞-categories then we will usually denote $\mathrm{Map}_{\mathrm{QC}}(\mathcal{C}, \mathcal{D})$ simply by $\mathrm{Map}(\mathcal{C}, \mathcal{D})$.

We will now discuss a second major source of ∞-categories, which arise as localizations of ordinary categories by a collection of weak equivalences. More generally, one can localize ∞-categories by a collection of edges. Let us recall the definition.

Definition 2.2.12. Let $\mathcal{C}$ be an ∞-category and let W be a collection of arrows in $\mathcal{C}$. Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor which sends every arrow in W to an invertible edge in $\mathcal{D}$. We will say that f *exhibits* $\mathcal{D}$ *as the localization of* $\mathcal{C}$ *with respect to* W if for every ∞-category $\mathcal{E}$ the induced functor

$$\mathrm{Map}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Map}(\mathcal{C}, \mathcal{E})$$

identifies $\mathrm{Map}(\mathcal{D}, \mathcal{E})$ with the subspace of $\mathrm{Map}(\mathcal{C}, \mathcal{E})$ consisting of those functors $\mathcal{C} \longrightarrow \mathcal{E}$ which send every edge in W to an equivalence in $\mathcal{E}$. In this case we will also write $\mathcal{D} \simeq \mathcal{C}[W^{-1}]$.

The following variant of Definition 2.2.12 is also terminologically useful:

Definition 2.2.13. We will say that a map $\mathcal{C} \longrightarrow \mathcal{D}$ of ∞-categories is a *localization map* if it exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ with respect to the collection of all maps in $\mathcal{C}$ which are sent to equivalences in $\mathcal{D}$.

By general considerations we have that the localization of $\mathcal{C}$ with respect to W is unique as soon as it exists. One way to prove its existence is via the formalism of *marked simplicial sets*.

Definition 2.2.14. A *marked simplicial set* is a pair (X, E) where X is a simplicial set and E is a collection of edges in X which contains all the degenerate edges. We call the edges in E the *marked edges*.

We will denote by Set_Δ^+ the category of marked simplcial sets (and maps which send marked edges to marked edges). The forgetful functor $\mathrm{Set}_\Delta^+ \longrightarrow \mathrm{Set}_\Delta$ to simplicial sets has both a left adjoint and a right adjoint. The left adjoint sends $X \in \mathrm{Set}_\Delta$ to the marked simplicial set $X^\flat$ which is X with only the degenerate edges marked. The right adjoint sends $X \in \mathrm{Set}_\Delta$ to the marked simplicial set $X^\sharp$ which is X with all edges marked. If $\mathcal{C}$ is an ∞-category we will denote by $\mathcal{C}^\natural$ the marked simplicial set which is $\mathcal{C}$ with the marked edges being the invertible arrows.

Given two marked simplicial sets X, Y we will denote by $\mathrm{Map}^\sharp(X, Y)$ the simplicial set given by the formula $\mathrm{Map}^\sharp(X, Y)_n = \mathrm{Hom}_{\mathrm{Set}_\Delta^+}((\Delta^n)^\sharp \times X, Y)$. If $\mathcal{C}$ is an ∞-category and (X, E) is any marked simplicial set then $\mathrm{Map}^\sharp((X, E), \mathcal{C}^\natural)$ is by construction the subsimplicial set of $(\mathcal{C}^X)^\simeq$ spanned by those diagrams $X \longrightarrow \mathcal{C}$ which send E to invertible edges. Since $(\mathcal{C}^X)^\simeq$ is a Kan complex it follows that $\mathrm{Map}^\sharp((X, E), \mathcal{C}^\natural)$ is a Kan complex as well.

Definition 2.2.15. We will say that a map $X \longrightarrow Y$ of marked simplicial sets is a *marked categorical equivalence* if for every ∞-category $\mathcal{C}$ the induced map

$$\mathrm{Map}^\sharp(Y, \mathcal{C}^\natural) \longrightarrow \mathrm{Map}^\sharp(X, \mathcal{C}^\natural)$$

is an equivalence of Kan complexes.

Theorem 2.2.16 ([5, §3])**.** *There exists a model structure on* Set_Δ^+*, which we will call the* marked categorical model structure*, in which the weak equivalences are the marked categorical weak equivalences, the cofibrations are the injective maps and the fibrant objects are the marked simplicial sets of the form* $\mathcal{C}^\natural$ *for an* ∞*-category* $\mathcal{C}$*. Furthermore, the forgetful functor* $\overline{(-)} : \mathrm{Set}_\Delta^+ \longrightarrow \mathrm{Set}_\Delta$ *is a right Quillen equivalence to the categorical model structure on* Set_Δ*.*

Let $\mathcal{C}$ be an ∞-category and let W be a collection of edges in $\mathcal{C}$. Given an ∞-category $\mathcal{D}$, the data of a map $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ which sends W to equivalences in $\mathcal{D}$ can be encoded as a map of marked simplicial sets $\varphi_+ : (\mathcal{C}, W) \longrightarrow \mathcal{D}^\natural$. Comparing Definition 2.2.12 an Definition 2.2.15 we see that φ exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ by W if and only if φ_+ is a marked categorical weak equivalence.

Corollary 2.2.17. *An localization* $\mathcal{C}[W^{-1}]$ *exists for every* ∞*-category* $\mathcal{C}$ *and collection of arrows* W*.*

Proof. One can simply take a fibrant replacement with respect to the marked categorical model structure. □

Remark 2.2.18. The marked categorical model structure on Set_Δ^+ is *combinatorial* and in particular admits a functorial fibrant replacement. This means that the formation of localizations $(\mathcal{C}, W) \mapsto \mathcal{C}[W^{-1}]$ can be made functorial in the pair $(\mathcal{C}, W)$. Furthermore, one can show that the cartesian product of two marked categorical weak equivalences is again a marked categorical weak equivalence. It then follows that the functorial localization procedure preserves cartesian products up to equivalence, that is $(\mathcal{C} \times \mathcal{C}')[(W \times W')^{-1}] \simeq \mathcal{C}[W^{-1}] \times \mathcal{C}'[(W')^{-1}]$.

Notation 2.2.19. If $\mathcal{M}$ is a model category with class of weak equivalences W then we will denote the localization $\mathcal{M}[W^{-1}]$ also by $\mathcal{M}_\infty$, and call it the *underlying ∞-category* of $\mathcal{M}$.

Remark 2.2.20. If $\mathcal{M}$ is a model category then $\mathcal{M}$ would usually be a large category and hence technically not covered by Corollary 2.2.17. This issue can be bypassed by using suitable set theoretical machinery, such as Grothendieck universes. Of course, such an approach will only assure us that $\mathcal{M}_\infty$ exists as a large ∞-category, i.e., one in which the sets of simplices are a proper class. However, it could be shown that in the case of model categories $\mathcal{M}_\infty$ will always be locally small, e.g., it will be equivalent to the coherent nerve of a large simplicial category whose mapping spaces are small Kan complexes.

A situation in which the underlying ∞-category $\mathcal{M}_\infty$ of a model category $\mathcal{M}$ is especially accessible is when $\mathcal{M}$ is a *simplicial model category*.

Definition 2.2.21. A simplicial model category is a model category $\mathcal{M}$, equipped with the additional structure of a *simplicial category* such that the following two conditions hold:

(1) The enrichment of $\mathcal{M}$ in simplicial sets admits tensors and cotensors, that is, the functors $\mathrm{Map}_{\mathcal{M}}(-, X), \mathrm{Map}_{\mathcal{M}}(X, -) : \mathcal{M} \longrightarrow \mathrm{Set}_\Delta$ admit enriched left adjoints.
(2) If $i : A \longrightarrow B$ is a cofibration in $\mathcal{M}$ and $p : X \longrightarrow Y$ is a fibration in $\mathcal{M}$ then the map
$$\mathrm{Map}_{\mathcal{M}}(B, X) \longrightarrow \mathrm{Map}_{\mathcal{M}}(B, Y) \times_{\mathrm{Map}_{\mathcal{M}}(A,Y)} \mathrm{Map}_{\mathcal{M}}(A, X)$$
is a Kan fibration, which is furthermore trivial if i or p are trivial.

Examples 2.2.22. The Kan-Quillen model structure on Set_Δ and the marked categorical model structure on Set_Δ^+ are both simplicial model categories. On the other hand, the categorical model structure on Set_Δ and the Dwyer-Kan model structure on Cat_Δ are not simplicial.

Remark 2.2.23. Condition (2) of Definition 2.2.21 implies that when X is cofibrant and Y is fibrant the mapping simplicial set $\mathrm{Map}_{\mathcal{M}}(X, Y)$ is a Kan complex.

Definition 2.2.24. Let $\mathcal{M}$ be a simplicial model category. We define $\mathcal{M}^\circ \subseteq \mathcal{M}$ to be the full simplicial subcategory spanned by the fibrant-cofibrant objects. By Remark 2.2.23 we have that $\mathcal{M}^\circ$ is *locally Kan*.

Proposition 2.2.25. *Let $\mathcal{M}$ be a simplicial model category. Then there is a canonical equivalence of ∞-categories*
$$\mathrm{N}(\mathcal{M}^\circ) \simeq \mathcal{M}_\infty.$$

Remark 2.2.26. Applying Proposition 2.2.25 to the Kan-Quillen model structure on simplicial sets shows that the ∞-category $\mathcal{S}$ which we defined in §2.1 as the coherent nerve of $\mathcal{K}\mathrm{an}$ is equivalent to the localization of Set_Δ by weak homotopy equivalences. A similar claim holds for Cat_∞, that is, Cat_∞ is equivalent to the localization of Set_Δ by categorical equivalences. To see this, we can replace the categorical model structure on Set_Δ (which is not simplicial) by the (simplicial) *marked categorical* model structure on Set_Δ^+ in which the simplicial enrichment is given by $\mathrm{Map}^\sharp(X,Y)$ above. We then have that QC is isomorphic to the simplicial category of fibrant-cofibrant objects in Set_Δ^+, and so we can apply Proposition 2.2.25 to deduce that $\mathrm{Cat}_\infty \simeq (\mathrm{Set}_\Delta^+)_\infty$. The latter is also equivalent to the localization of the categorical model structure since the forgetful functor $\mathrm{Set}_\Delta^+ \longrightarrow \mathrm{Set}_\Delta$ is a right Quillen equivalence.

2.3. **Left and right fibrations.** Let X be a nice topological space. A classical result in algebraic topology asserts an equivalence between the category of *covering spaces* $Y \longrightarrow X$ and the category of functors $\Pi_1(X) \longrightarrow \mathrm{Set}$, where $\Pi_1(X)$ is the *fundamental groupoid* of X, i.e., the groupoid whose objects are the points in X and whose morphisms are homotopy classes of paths. A similar phenomenon happens in ordinary category theory. Recall that a functor $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ of ordinary categories is said to be *left fibered in sets* if for every $x \in \mathcal{C}$ and every morphism $f : \pi(x) \longrightarrow y$ in $\mathcal{D}$ there exists a *unique arrow* $\widetilde{f} : x \longrightarrow \widetilde{y}$ in $\mathcal{C}$ which maps to f. The notion of a functor right fibered in sets is similarly defined using lifts of morphisms $y \longrightarrow \pi(x)$. One then has a classification of functors left fibered in sets over a fixed category $\mathcal{D}$, analogous to the topological story of covering spaces: they correspond exactly to functors $\mathcal{D} \longrightarrow \mathrm{Set}$ from $\mathcal{D}$ to sets. In particular, given $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ we may construct the corresponding functor $\mathcal{D} \longrightarrow \mathrm{Set}$ by associating to each $y \in \mathcal{D}$ the set $\pi^{-1}(y)$ of objects of $\mathcal{C}$ lying above y. If $f : y \longrightarrow y'$ is map in $\mathcal{D}$ then for each $x \in \pi^{-1}(y)$ there exists a unique lift $\widetilde{f} : x \longrightarrow x'$ of f starting from x. We may hence associate to f the map of sets $\pi^{-1}(y) \longrightarrow \pi^{-1}(y')$ given by $x \mapsto x'$. This association preserves composition by the uniqueness of lifts. In the other direction, if we start from a functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}$ then we can associate to $\mathcal{F}$ a functor $\pi : \int_{\mathcal{D}} \mathcal{F} \longrightarrow \mathcal{D}$ left fibered in sets where $\int_{\mathcal{D}} \mathcal{F}$ is the category whose objects are pairs (y, a) where y is an object of $\mathcal{D}$ and a is an element of $\mathcal{F}(y)$. A morphism from (y, a) to (y', a') is a map $f : y \longrightarrow y'$ in $\mathcal{D}$ such that $f_!(a) = a'$ (where we denoted by $f_! : \mathcal{F}(y) \longrightarrow \mathcal{F}(y')$ the map associated to f by $\mathcal{F}$). The category $\int_{\mathcal{D}} \mathcal{F}$ is also known as the *category of elements* of $\mathcal{F}$, and in a more general context as the *Grothendieck construction* of $\mathcal{F}$. In a dual manner, the notion of a functor right fibered in sets will correspond to *contravariant functors* from $\mathcal{D}$ to Set.

In this section we will discuss the ∞-categorical avatar of this story, which are known as *left and right fibrations*. We first note that since we are replacing categories with ∞-categories, we will naturally want to replace the notion of sets with that of spaces, or more precisely, ∞-groupoids. We hence cannot expect to have a definition in which we require a lift to be unique, but only unique *up to equivalence*. Even more, we may want something such as unique up to a contractible space of choices. Let us see how such a definition can be made in the setting of ∞-categories.

Definition 2.3.1. Let $\pi : X \longrightarrow Y$ be a map of simplicial sets. We will say that π is a *left fibration* if it has the right lifting property with respect to horn inclusions of the form $\Lambda^i_n \subseteq \Delta^n$ with $0 \leq i < n$, that is, if the dotted lift exists in any diagram of the form

$$
\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
\tag{2.9}
$$

with $0 \le i < n$. Dually, we will say that π is a *right fibration* if is has the right lifting property with respect to the horn inclusions $\Lambda^i_n \subseteq \Delta^n$ such that $0 < i \le n$.

We would like to explain how one can think of the lifting condition in 2.9 with $i = 0$ as a kind of "unique lifting property" for arrows with a fixed domain. For this, we will need a construction of the ∞-category of arrows with a fixed domain.

Definition 2.3.2. Let X, Y be two simplicial sets. The *join* $X \star Y$ of X and Y is the simplicial set given by

$$(X \star Y)_n = \coprod_{i=-1,\dots,n} X_i \times Y_{n-1-i},$$

where by convention we set $X_{-1} = Y_{-1} = *$. We note that $X * Y$ comes equipped with two natural inclusions $X \hookrightarrow X * Y \hookleftarrow Y$. The special cases where one of X, Y is Δ^0 will be denoted as

$$X^{\triangleright} := X \star \Delta^0 \qquad Y^{\triangleleft} := \Delta^0 * Y.$$

Remark 2.3.3. The join operation $\star$ can be considered as a simplicial model for the join of topological spaces. However, in the simplicial case the join is not symmetric, and generally $X \star Y \neq Y \star X$. For example, the 1-simplices in $X \star Y$ which are not in X or Y all go from X to Y, and not the other way around.

Remark 2.3.4. If $\mathcal{C}, \mathcal{D}$ are ordinary categories then $\mathrm{N}(\mathcal{C}) \star \mathrm{N}(\mathcal{D})$ is the nerve of the ordinary category $\mathcal{E}$ which contains $\mathcal{C}$ and $\mathcal{D}$ as disjoint full subcategories and such that the hom sets from every object of $\mathcal{C}$ to every object of $\mathcal{D}$ are singletons, while the hom sets from every object of $\mathcal{D}$ to every object of $\mathcal{C}$ are empty. In particular, if $\mathcal{D} = *$ then $\mathrm{N}(\mathcal{C})^{\triangleright}$ is the nerve of the categorical right cone on $\mathcal{C}$. Similarly, $\mathrm{N}(\mathcal{C})^{\triangleleft}$ is the categorical left cone on $\mathcal{C}$.

Definition 2.3.5. Let $\mathcal{C}$ be an ∞-category and $p : K \longrightarrow \mathcal{C}$ a map from a simplicial set K. We will denote by $\mathcal{C}_{/p}$ the simplicial set given by

$$(\mathcal{C}_{/p})_n = \mathrm{Hom}_K(\Delta^n \star K, \mathcal{C}),$$

where $\mathrm{Hom}_K(-,-)$ refers to the set of morphisms which preserve the given map from K. Similarly, we will denote by $\mathcal{C}_{p/}$ the simplicial set given by

$$(\mathcal{C}_{p/})_n = \mathrm{Hom}_K(K \star \Delta^n, \mathcal{C}).$$

If $K = \Delta^0$ and $p : \Delta^0 \longrightarrow \mathcal{C}$ picks the object $x \in \mathcal{C}$ then we will also denote $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ by $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$ respectively. Similarly, if $K = \Delta^1$ and $p : \Delta^1 \longrightarrow \mathcal{C}$ picks the arrow $f : x \longrightarrow y$ then we will also denote $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ by $\mathcal{C}_{/f}$ and $\mathcal{C}_{f/}$ respectively.

If $\mathcal{C}$ is an ∞-category then we will see below that for every $p : K \longrightarrow \mathcal{C}$ the simplicial sets $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ are ∞-categories as well. Given a K-indexed diagram $p : K \longrightarrow \mathcal{C}$ in an ∞-category $\mathcal{C}$, the objects of $\mathcal{C}_{/p}$ are given by maps $\overline{p} : K^{\triangleleft} \longrightarrow \mathcal{C}$ extending p. Similarly, the objects of $\mathcal{C}_{p/}$ are given by extensions $\overline{p} : K^{\triangleright} \longrightarrow \mathcal{C}$. We will call $\mathcal{C}_{/p}$ the ∞-category of *left cones on* p, and $\mathcal{C}_{p/}$ as the ∞-category of *right cones on* p.

Example 2.3.6 (over ∞-categories). Given an object $x \in \mathcal{C}$, the ∞-category $\mathcal{C}_{/x}$ can be described as follows: the objects of $\mathcal{C}_{/x}$ are maps $f : z \longrightarrow x$ from some $z \in \mathcal{C}$ to x. A map in $\mathcal{C}_{/x}$ from $f : z \longrightarrow x$ to $f' : z' \longrightarrow x$ is given by a triangle $\sigma : \Delta^2 \longrightarrow \mathcal{C}$ of the form

$$\begin{tikzcd} z \arrow[rr, "g"] \arrow[dr, "f"'] & & z' \arrow[dl, "f'"] \\ & x & \end{tikzcd} \tag{2.10}$$

which we consider as a map $g : z \longrightarrow z'$ together with a homotopy exhibiting f as the composition of g and f'. We will call $\mathcal{C}_{/x}$ the ∞-category of *objects over* x. Similarly, we will call the ∞-category $\mathcal{C}_{x/}$ the ∞-category of *objects under* x.

Remark 2.3.7. If $\mathcal{C}$ is an ordinary category and $x \in \mathcal{C}$ is an object then $\mathrm{N}(\mathcal{C})_{/x}$ coincides with the nerve of the ordinary category $\mathcal{C}_{/x}$ of objects over x.

We are now ready to explain how Definition 2.3.1 can be considered as an essentially unique lifting property for arrows with a fixed domain. Let $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ be a map of ∞-categories. Identifying $(\partial\Delta^n)^{\triangleleft} \cong \Lambda^{n+1}_0$ and $(\Delta^n)^{\triangleleft} \cong \Delta^{n+1}$ we see that π satisfies the right lifting property with respect to $\Lambda^{n+1}_0 \subseteq \Delta^{n+1}$ if and only if for every $x \in \mathcal{C}$, the dotted lift exists in any diagram of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathcal{C}_{x/} \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}_{\pi(x)/} \end{tikzcd} \tag{2.11}$$

In other words, if and only if the map $\mathcal{C}_{x/} \longrightarrow \mathcal{D}_{\pi(x)/}$ is a trivial Kan fibration for every x. We interpret this as the ∞-categorical analogue of the unique arrow lifting property: it says in particular that the fiber $\mathcal{C}_{x/} \longrightarrow \mathcal{D}_{\pi(x)/}$ over a fixed arrow $f : \pi(x) \longrightarrow y$ is a *contractible Kan complex*.

We will see in §2.4 that, in analogy with the situation in ordinary category theory, left fibrations over a fixed ∞-category $\mathcal{D}$ essentially correspond to functors from $\mathcal{D}$ to ∞-groupoids, where the functor corresponding to a left fibration $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ is given informally by the "formula" $y \mapsto \pi^{-1}(y)$. Similarly, right fibrations correspond to contravariant functors from $\mathcal{D}$ to ∞-groupoids. For now, let us focus on concrete constructions and examples of left (and right) fibrations. We note that in ordinary categories, a canonical example of a functor $\mathcal{D} \longrightarrow \mathrm{Set}$ is given by the functor corepresented by an object $x \in \mathcal{D}$, that is, the functor $y \mapsto \mathrm{Hom}_{\mathcal{D}}(x,y)$. The category of elements of this functor is simply the category $\int_{\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(x,-) \cong \mathcal{D}_{x/}$ of objects under x, where the projection to $\mathcal{D}$ given by $[x \longrightarrow y] \mapsto y$. If $\mathcal{D}$ is an ∞-category then we have the ∞-categorical construction of the under category $\mathcal{D}_{x/}$ described in Example 2.3.6. We would like to show that this indeed results in a left fibration $\mathcal{D}_{x/} \longrightarrow \mathcal{D}$. More generally, we will show that for every diagram $p : K \longrightarrow \mathcal{C}$ the projection $\mathcal{C}_{p/} \longrightarrow \mathcal{C}$ is a left fibration (and similarly that $\mathcal{C}_{/p} \longrightarrow \mathcal{C}$ is a right fibration).

Recall the following terminology: we will say that a class of maps in Set_{Δ} is *weakly saturated* if it closed under pushouts (along any map), transfinite compositions and retracts.

Definition 2.3.8. We will say that a map of simplicial sets $X \longrightarrow Y$ is

(1) *inner anodyne* if it belongs to the smallest weakly saturated class of maps generated by $\Lambda^i_n \hookrightarrow \Delta^n$ for $0 < i < n$.
(2) *left anodyne* if it belongs to the smallest weakly saturated class of maps generated by $\Lambda^i_n \hookrightarrow \Delta^n$ for $0 \le i < n$.
(3) *right anodyne* if it belongs to the smallest weakly saturated class of maps generated by $\Lambda^i_n \hookrightarrow \Delta^n$ for $0 < i \le n$.

We note that the condition of being a left (resp. right) fibration is equivalent to the condition of having the right lifting property with respect to all left (resp. right) anodyne maps. Similarly, the condition of being an inner fibration is equivalent to having the right lifting property with respect to all inner anodyne maps.

Lemma 2.3.9. *Let $f : A_0 \hookrightarrow A$ and $g : B_0 \hookrightarrow B$ be inclusions of simplicial sets and consider the map*

$$h : A_0 \star B \coprod_{A_0 \star B_0} A \star B_0 \longrightarrow A \star B.$$

Then the following holds:

(1) If f is left anodyne then h is left anodyne.
(2) If g is right anodyne then h is right anodyne.
(3) If f is right anodyne or g is left anodyne then h is inner anodyne.

Proof. Let us begin with Claim (1). Let us fix g and consider the class of all f such that the h is left anodyne. Using the fact that the functor $(-) \star g$ preserves pushout squares and filtered colimits one can check that this class is weakly saturated. It will hence suffice to show that it contains all horn inclusions of the form $\Lambda^n_i \subseteq \Delta^n$ for $0 \le i < n$. On the other hand, if we fix f to be $\Lambda^n_i \subseteq \Delta^n$ then the collection of all g such that h is left anodyne is also weakly saturated. It will hence suffice to show it for f the inclusion $\Lambda^n_i \subseteq \Delta^n$ for $0 \le i < n$ and g the inclusion $\partial\Delta^m \subseteq \Delta^m$ (these maps generate the weakly saturated class of inclusions). In this case the map h identifies with the inclusion

$$\Lambda^{n+m+1}_i \subseteq \Delta^{n+m+1}$$

which is indeed left anodyne. The proof of (2) is completely dual to (1).

To prove (3) we argue in the same manner to reduce the claim to checking that the maps

$$\Lambda^n_i \star \Delta^m \coprod_{\Lambda^n_i \star \partial\Delta^m} \Delta^n \star \partial\Delta^m \longrightarrow \Delta^n \star \Delta^m \tag{2.12}$$

and

$$\Delta^m \star \Lambda^n_j \coprod_{\partial\Delta^m \star \Lambda^n_j} \partial\Delta^m \star \Delta^n \longrightarrow \Delta^m \star \Delta^n \tag{2.13}$$

are inner anodyne when $0 < i \le n$ and $0 \le j < n$. But (2.12) is just the horn inclusion $\Lambda^{n+m+1}_i \longrightarrow \Delta^{n+m+1}$ and (2.13) is the horn inclusion $\Lambda^{m+n+1}_{m+j+1} \longrightarrow \Delta^{m+n+1}$. Since both i and $m+j+1$ are strictly between 0 and $m+n+1$ we get that (2.12) and (2.13) are inner anodyne, as desired. □

Corollary 2.3.10. *Let $\mathcal{C}$ be an ∞-category and $p : K \longrightarrow \mathcal{C}$ a diagram. Let $K_0 \subseteq K$ be a subsimplicial set and write $p_0 = p|_{K_0} : K_0 \longrightarrow \mathcal{C}$. Then the projection*

$$\mathcal{C}_{p/} \longrightarrow \mathcal{C}_{p_0/}$$

is a left fibration which is furthermore a trivial Kan fibration if the inclusion $K_0 \subseteq K$ is right anodyne. More generally, if $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ is an inner fibration then the projection

$$\mathcal{C}_{p/} \longrightarrow \mathcal{C}_{p_0/} \times_{\mathcal{D}_{p_0/}} \mathcal{D}_{p/}$$

is a left fibration, which is furthermore a trivial Kan fibration if the inclusion is right anodyne. Dually, the projection

$$\mathcal{C}_{/p} \longrightarrow \mathcal{C}_{/p_0} \times_{\mathcal{D}_{/p_0}} \mathcal{D}_{/p}$$

is a right fibration which is furthermore a trivial Kan fibration if the inclusion $K_0 \subseteq K$ is left anodyne.

Remark 2.3.11. Applying Corollary 2.3.10 to the case where $K = \Delta^1$ and $K_0 = \Delta^{\{0\}}$ we may conclude that for every map $f : x \longrightarrow y$ in $\mathcal{C}$ the projection

$$\mathcal{C}_{f/} \longrightarrow \mathcal{C}_{y/} \tag{2.14}$$

is a trivial Kan fibration. In particular, for every $g : y \longrightarrow z$ the fiber $(\mathcal{C}_{f/})_g$ of (2.14) is a trivial Kan complex. We may interpret $(\mathcal{C}_{f/})_g$ as the space of compositions of f and g (its vertices consists indeed of maps $h : x \longrightarrow z$ together with a triangle exhibiting h as the compostion of f and g). We may then interpret the contractibility of $(\mathcal{C}_{f/})_g$ as the statement that composition is well-defined up to a contractible space of choices.

Our goal in this section is to prove Joyal's Theorem 2.1.12, which characterizes Kan complexes as those ∞-categories in which all morphisms are invertible. We begin with some auxiliary results.

Lemma 2.3.12 ([5, Proposition 2.1.1.5, 2.1.1.6])**.** *Let $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ be a left fibration of ∞-categories. Then the following holds:*

(1) The functor π detects invertible maps, that is, an arrow $f : x \longrightarrow y$ in $\mathcal{C}$ is invertible if and only if $\pi(f)$ is invertible in $\mathcal{D}$.
(2) For every object $y \in \mathcal{C}$ and every invertible arrow $f : x \longrightarrow \pi(y)$ in $\mathcal{D}$ there exists an arrow $\widetilde{f} : \widetilde{x} \longrightarrow y$ in $\mathcal{C}$ such that $\pi(\widetilde{f}) = f$.

Proof. We begin with Claim (1). Let g be a homotopy inverse to $\pi(f)$ in $\mathcal{D}$, so that there exists a triangle in $\mathcal{D}$ of the form

$$\begin{tikzcd} & \pi(y) \arrow[dr, "g"] & \\ \pi(x) \arrow[ur, "\pi(f)"] \arrow[rr, "\mathrm{Id}"] & & \pi(x) \end{tikzcd}$$

Since π is a left fibration we can lift this triangle to a triangle in $\mathcal{C}$ of the form

$$\begin{tikzcd} & y \arrow[dr, "\widetilde{g}"] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{Id}"] & & x \end{tikzcd}$$

for some $\widetilde{g}$. It follows that f admits a left homotopy inverse in $\mathcal{C}$. Since $\pi(\widetilde{g}) = g$ is an equivalence in $\mathcal{D}$ the same argument shows that $\widetilde{g}$ has a left homotopy inverse f' in $\mathcal{C}$. In particular, the image of $\widetilde{g}$ in $\mathrm{Ho}(\mathcal{C})$ has both a left and a right inverse and is hence an isomorphism there. It follows that the image of f in $\mathrm{Ho}(\mathcal{C})$ is an isomorphism and so f is invertible in $\mathcal{C}$.

Let us now prove (2). Let $g : \pi(y) \longrightarrow x$ be a homotopy inverse to f in $\mathcal{D}$, so that we have a triangle

(2.15)
$$\begin{tikzcd} & \pi(x) \arrow[dr, "f"] & \\ \pi(y) \arrow[ur, "g"] \arrow[rr, "\mathrm{Id}"] & & \pi(y) \end{tikzcd}$$

Since π is a left fibration there exists a morphism $\widetilde{g} : y \longrightarrow \widetilde{x}$ such that $\pi(\widetilde{g}) = g$. We may then lift (2.15) to a triangle

$$\begin{tikzcd} & \widetilde{x} \arrow[dr, "\widetilde{f}"] & \\ y \arrow[ur, "\widetilde{g}"] \arrow[rr, "\mathrm{Id}"] & & y \end{tikzcd}$$

in $\mathcal{C}$ for some $\widetilde{f} : \widetilde{x} \longrightarrow y$. Then $\pi(\widetilde{f}) = f$ and so $\widetilde{f}$ is the desired lift. □

Proposition 2.3.13. *Let $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ be an inner fibration and $f : x \longrightarrow y$ an invertible arrow in $\mathcal{C}$. Then the dotted lift exists in any diagram of the form*

(2.16)
$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

with $n \geq 2$ which maps $\Delta^{\{0,1\}} \subseteq \Lambda^n_0$ to the edge f.

Proof. The lifting problem (2.16) is equivalent by adjunction to the lifting problem

(2.17)
$$\begin{tikzcd} \Delta^{\{0\}} \arrow[r] \arrow[d] & \mathcal{C}_{/\Delta^{\{2,\dots,n\}}} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{/\partial\Delta^{\{2,\dots,n\}}} \times_{\mathcal{D}_{/\partial\Delta^{\{2,\dots,n\}}}} \mathcal{D}_{/\Delta^{\{2,\dots,n\}}} \end{tikzcd}$$

Since $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ is an inner fibration the right vertical map in (2.17) is a right fibration by Corollary 2.3.10. Now the composed arrow

$$\mathcal{C}_{/\partial\Delta^{\{2,\dots,n\}}} \times_{\mathcal{D}_{/\partial\Delta^{\{2,\dots,n\}}}} \mathcal{D}_{/\Delta^{\{2,\dots,n\}}} \longrightarrow \mathcal{C}_{/\partial\Delta^{\{2,\dots,n\}}} \longrightarrow \mathcal{C}$$

is a composition of right fibrations by Corollary 2.3.10 (and the fact that right fibrations are closed under base change) and the edge determined by the lower horizontal map in (2.17) is sent to an invertible edge in $\mathcal{C}$ by our assumption. It then follows from (the dual of) Lemma 2.3.12(1) that this edge is already invertible in $\mathcal{C}_{/\partial\Delta^{\{2,\dots,n\}}} \times_{\mathcal{D}_{/\partial\Delta^{\{2,\dots,n\}}}} \mathcal{D}_{/\Delta^{\{2,\dots,n\}}}$. The desired lift now follows from (the dual of) Lemma 2.3.12(2). □

Corollary 2.3.14 (Joyal)**.** *Let $\mathcal{C}$ be an ∞-category. Then $f : x \longrightarrow y$ is an invertible arrow if and only if the dotted lift exists in any diagram of the form*

(2.18)
$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

which maps $\Delta^{\{0,1\}} \subseteq \Lambda^n_0$ to the edge f.

Proof. The "only if" direction follows from Proposition 2.3.13. To show the "if" direction note that if the dotted lift exists in any diagram of the form (2.18) even just for $n = 2,3$ then in the homotopy category $\mathrm{Ho}(\mathcal{C})$ pre-composition with f induces a bijection $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(y,z) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x,z)$ for any $z \in \mathcal{C}$, and so f is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ (and hence invertible in $\mathcal{C}$) by the Yoneda lemma. □

Proof of Theorem 2.1.12. If $\mathcal{C}$ is a Kan complex then every extension problem of the form (2.18) has a solution and hence every arrow in $\mathcal{C}$ is invertible by Corollary 2.3.14. On the other hand, if every arrow in $\mathcal{C}$ is invertible then the same is true for $\mathcal{C}^{\mathrm{op}}$ and hence Corollary 2.3.14 applied to both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ implies that $\mathcal{C}$ has the extension property with respect to every horn extension $\Lambda^i_n \subseteq \Delta^n$ with $0 \leq i \leq n$, so that $\mathcal{C}$ is a Kan complex. □

We finish this section with another interesting corollary:

Corollary 2.3.15. *Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ an object. Then for every $y \in \mathcal{C}$, the fiber $(\mathcal{C}_{x/})_y$ of $\mathcal{C}_{x/} \longrightarrow \mathcal{C}$ over $y \in \mathcal{C}$ is a Kan complex.*

Proof. The map $(\mathcal{C}_{x/})_y \longrightarrow \Delta^0$ is a base change of a left fibration and hence a left fibration. It follows from Corollary 2.3.14 that every arrow in $(\mathcal{C}_{x/})_y$ is invertible, and hence $(\mathcal{C}_{x/})_y$ is a Kan complex by Theorem 2.1.12. □

Corollary 2.3.15 motivates the following definition

Definition 2.3.16. Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathcal{C}$. We will denote by

$$\mathrm{Map}^{\mathrm{L}}_{\mathcal{C}}(x,y) \coloneqq (\mathcal{C}_{x/})_y,$$

and refer to it as the *mapping space* in $\mathcal{C}$ from x to y.

Remark 2.3.17. Definition 2.3.16 is somewhat asymmetric. Indeed, we could instead first take the right fibration $\mathcal{C}_{/y}$ and then take its fiber $(\mathcal{C}_{/y})_x$ over $x \in \mathcal{C}$, which is usually denoted by $\mathrm{Map}^{\mathrm{R}}_{\mathcal{C}}(x,y)$. This results in a different, though canonically homotopy equivalent, Kan complex. Furthermore, if $\mathcal{C}$ is obtain as the coherent nerve of a locally Kan simplicial category $\mathcal{D}$ and $x, y \in \mathcal{D}$ are two objects then

$$\mathrm{Map}^{\mathrm{L}}_{\mathcal{C}}(x,y) \simeq \mathrm{Map}^{\mathrm{R}}_{\mathcal{C}}(x,y) \simeq \mathrm{Map}_{\mathcal{D}}(x,y).$$

We will hence often simplify notation and denote either one of the above spaces simply as $\mathrm{Map}_{\mathcal{C}}(x,y)$.

The definition of mapping spaces also allows for the following definition:

Definition 2.3.18. We will say that a map of ∞-categories $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ is *fully-faithful* if it induces an equivalence $\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(\varphi(x), \varphi(y))$ for every $x, y \in \mathcal{C}$.

Remark 2.3.19. The condition that $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence of ∞-categories (Definition 2.1.10) is in fact equivalent to the condition that φ is fully-faithful and essentially surjective (that is, every object in $\mathcal{D}$ is equivalent to an obejct in the image of φ). This is not evident from Definition 2.1.10, which is phrased in terms of the existence of an inverse functor. To show this one can use the Quillen equivalence between the categorical model structure and the Dwyer-Kan model structure on simplicial categories. In particular, φ is an equivalence in the sense of Definition 2.1.10 if and only if it is a categorical equivalence in the sense of Definition 2.2.5, and since every object in Set_Δ is cofibrant and $\mathfrak{C}$ is a left Quillen equivalence we have that φ is a categorical equivalence if and only if it is mapped to a Dwyer-Kan equivalence in Cat_Δ. One can then show $\mathrm{Map}_{\mathcal{C}}(x,y) \simeq \mathrm{Map}_{\mathfrak{C}(\mathcal{C})}(x,y)$ and so $\mathfrak{C}(\varphi)$ is a Dwyer-Kan equivalence if and only if φ is fully-faithful and essentially surjective.

2.4. **Cartesian and cocartesian fibrations.** In §2.3 we discussed left and right fibrations of ∞-categories. These are the ∞-categorical analogues of ordinary functors $\mathcal{C} \longrightarrow \mathcal{D}$ fibered in sets. In the classical setting, the association $\mathcal{F} \mapsto \int_{\mathcal{D}} \mathcal{F}$, which associates to a functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}$ its *category of elements*, determines an equivalence of categories

$$\mathrm{Fun}(\mathcal{D}, \mathrm{Set}) \xrightarrow{\simeq} \mathrm{Fib}^{\mathrm{set}}(\mathcal{D}),$$

where the right hand side refers to the full subcategory $\mathrm{Fib}^{\mathrm{set}}(\mathcal{D}) \subseteq \mathrm{Cat}_{/\mathcal{D}}$ spanned by those $\mathcal{C} \longrightarrow \mathcal{D}$ which are fibered in sets, that is, for which arrows admit a unique lift given a lift of their domain (see §2.3). It is natural to wonder what can be done if start with a functor not into sets but into *categories*. Here we encounter a small subtlety: the category Cat of (small) categories is actually a 2-category: in addition to objects (categories), and morphisms (functors), it also has morphisms between morphisms (natural transformations). Equivalently, it is a category enriched in categories (mapping sets are replaced by mapping categories). The notion of a diagram in Cat is therefore a bit subtle: the right notion to use here is that of a *pseudofunctor*

$$\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}\,.$$

This means that we associate to each object $x \in \mathcal{D}$ a category $\mathcal{F}(x) \in \mathrm{Cat}$, to each morphism $f : x \longrightarrow y$ in $\mathcal{D}$ a functor $f_! : \mathcal{F}(x) \longrightarrow \mathcal{F}(y)$, and to each composable pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{D}$ a natural isomorphism $\tau_{f,g} : g_! \circ f_! \overset{\cong}{\Rightarrow} (g \circ f)_!$. We then enforce a certain compatibility condition for each triple of composable arrows $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$. Here for simplicity we assume that $\mathcal{F}$ is strictly unital, that is, it sends the identity morphisms to the corresponding identity functors (though morally we should have also required here that this would only hold up to a specified natural isomorphism; there is however a very simple procedure that replaces every weakly unital pseudofunctor by a strictly unital one, and so we will ignore this point).

Now, given such a pseudofunctor $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}$, we may assemble the various categories $\{\mathcal{F}(x)\}_{x \in \mathcal{D}}$ into a global category $\int_{\mathcal{D}} \mathcal{F}$, known as the *Grothendieck construction* of $\mathcal{F}$. More precisely, if we denote by $f_! : \mathcal{F}(x) \longrightarrow \mathcal{F}(y)$ the functor associated to $f : x \longrightarrow y$ by $\mathcal{F}$, then $\int_{\mathcal{D}} \mathcal{F}$ is the category whose

- objects are pairs (x, a) where x is an object of $\mathcal{D}$ and a is an object of $\mathcal{F}(x)$;
- morphisms from (x,a) to (y,b) are pairs (f, φ) where $f : x \longrightarrow y$ is a map in $\mathcal{D}$ and $\varphi : f_! a \longrightarrow b$ is a map in $\mathcal{F}(y)$.

Composition of maps is defined in a straightforward way using the natural isomorphisms $\tau_{f,g} : g_! \circ f_! \overset{\cong}{\Rightarrow} (g \circ f)_!$. The category $\int_{\mathcal{D}} \mathcal{F}$ admits a natural projection $\pi : \int_{\mathcal{D}} \mathcal{F} \longrightarrow \mathcal{D}$ given by $(x,a) \mapsto x$. The somewhat surprising feature of the Grothendieck construction is that it actually does not forget any information: the original pseudofunctor $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}$ can be reconstructed from π. To see how this works let us introduce the following definition:

Definition 2.4.1. Let $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of ordinary categories and let $f : x \longrightarrow y$ be a morphism in $\mathcal{C}$. We will say that f is π-*cocartesian* if for every morphism $g : x \longrightarrow y$ in $\mathcal{C}$ and every factorization

$$\begin{tikzcd} \pi(x) \arrow[r, "\varphi(g)"] \arrow[d, "\varphi(f)"'] & \varphi(z) \\ \pi(y) \arrow[ur, dashed] & \end{tikzcd} \tag{2.19}$$

of $\pi(g)$ through $\pi(f)$, there exists a unique factorization

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "f"'] & z \\ y \arrow[ur, dashed] & \end{tikzcd} \tag{2.20}$$

of g through f whose image in $\mathcal{D}$ is (2.19).

Exercise 2.4.2. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}$ be an $\mathcal{D}$-indexed diagram of categories and let $\pi : \int_{\mathcal{D}} \mathcal{F} \longrightarrow \mathcal{D}$ be the projection from the Grothendieck construction as above. Show that an arrow $(f, \varphi) : (x, a) \longrightarrow (y, b)$ in $\int_{\mathcal{D}} \mathcal{F}$ is π-cocartesian if and only if $\varphi : f_! a \longrightarrow b$ is an isomorphism.

Definition 2.4.3. Let $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of ordinary categories. We will say that π is a *cocartesian fibration* if for every object $x \in \mathcal{C}$ and morphism $f : \pi(x) \longrightarrow y$, there exists a π-cocartesian morphism $\widetilde{f} : x \longrightarrow \widetilde{y}$ lying above f.

Remark 2.4.4. The definitions of cocartesian edges and cocartesian fibrations can be dualized in an obvious manner. In particular, given a functor $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ with opposite $\pi^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$, an arrow $f : x \longrightarrow y$ in $\mathcal{C}$ is π-*cartesian* if and only if it is π^{op}-cocartesian when considered as an arrow in $\mathcal{C}^{\mathrm{op}}$, and π is a *cartesian fibration* exactly when π^{op} is a cocartesian fibration.

Example 2.4.5. If $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}$ is a $\mathcal{D}$-diagram of categories then the projection $\pi : \int_{\mathcal{D}} \mathcal{F} \longrightarrow \mathcal{D}$ is a cocartesian fibration: indeed, for every $(x, a) \in \int_D \mathcal{F}$ and $f : x \longrightarrow y$ we have the π-cocartesian lift $(f, \mathrm{Id}_{f_! a}) : (x, a) \longrightarrow (y, f_! a)$.

Definition 2.4.6. Let $\mathcal{D}$ be a small category. We define $\mathrm{Fib}^{\mathrm{coc}}(\mathcal{D})$ to be the 2-category whose objects are the cocartesian fibrations $\pi : \mathcal{C} \longrightarrow \mathcal{D}$, whose morphisms are functors

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr, "\pi"'] & & \mathcal{C}' \arrow[dl, "\pi'"] \\ & \mathcal{D} & \end{tikzcd}$$

over $\mathcal{D}$ which send π-cocartesian edges to π'-cocartesian edges, and whose 2-morphisms are the natural transformations which are compatible with the projection to $\mathcal{D}$.

The precise way in which the Grothendieck construction does not loose any information is summarized in the following folk theorem:

Theorem 2.4.7 (Grothendieck's correspondence, see, e.g, [4, Theorem 1.3.6])**.** *The Grothendieck construction determines an equivalence of 2-categories*

$$\int : \mathrm{PsFun}(\mathcal{D}, \mathrm{Cat}) \xrightarrow{\simeq} \mathrm{Fib}^{\mathrm{coc}}(\mathcal{D})$$

between the 2-category of pseudofunctors $\mathcal{D} \longrightarrow \mathrm{Cat}$ *and the 2-category of cocartesian fibrations over* $\mathcal{D}$*.*

The equivalence of Theorem 2.4.7 is of fundamental importance in ordinary category theory, but is also useful in practice, since cocartesian fibrations over $\mathcal{D}$ are more benign creatures than functors $\mathcal{D} \longrightarrow \mathrm{Cat}$. One problem with the latter is that it requires using the large category Cat, and so forces us to take into account some set theoretical issues. This issue will less concern us in this course. A more interesting problem is that the 2-categorical nature of Cat makes the notion of diagrams $\mathcal{D} \longrightarrow \mathrm{Cat}$ a rather intricate type of mathematical object (involving various coherence isomorphisms), while the notion of a cocartesian fibration $\int_{\mathcal{D}} \mathcal{F} \longrightarrow \mathcal{D}$ is much simpler in that respect. When passing from ordinary category to higher category theory, this second issue becomes considerably more important. In particular, if $\mathcal{D}$ is now an ∞-category, then a diagram of ∞-categories indexed by $\mathcal{D}$ is something that is very difficult to write down explicitly, while the (∞-categorical generalization of the) notion of a cocartesian fibration over $\mathcal{D}$ is a much more accessible type of structure. This makes the notion of cocartesian fibrations of ∞-categories, which we will discuss below, utterly indispensable in higher category theory.

In order to generalize the definition of a cocartesian edge to the ∞-categorical setting it will be useful to formulate the unique relative extension property in terms of the associated map of nerves $\pi : \mathrm{N}(\mathcal{D}) \longrightarrow \mathrm{N}(\mathcal{C})$ (which we also denote by π). Indeed, a pair of maps of the form $f : x \longrightarrow y, g : x \longrightarrow z$ in $\mathcal{D}$ can be encoded as a map of simplicial sets $f \vee g : \Lambda^2_0 \longrightarrow \mathrm{N}(\mathcal{D})$, and a factorization of $\pi(g)$ through $\pi(f)$ can be encoded as a map of simplicial sets $\Delta^2 \longrightarrow \mathrm{N}(\mathcal{C})$. If we drop the uniqueness condition, then the mere existence of a relative extension amounts to the existence of a dotted lift in the resulting square

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "f\vee g"] \arrow[d] & \mathrm{N}(\mathcal{D}) \arrow[d, "\pi"] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathrm{N}(\mathcal{C}) \end{tikzcd}$$

Somewhat surprisingly, the uniqueness can also be phrased as a similar lifting condition using the horn inclusion $\Lambda^3_0 \hookrightarrow \Delta^3$.

Exercise 2.4.8. Suppose that the morphism $f : x \longrightarrow y$ in $\mathcal{D}$ has the (non-unique) relative extension property. Then f has the unique extension property if and only if a dotted lift exists in any square of the form

$$\begin{tikzcd} \Lambda^3_0 \arrow[r, "\sigma"] \arrow[d] & \mathrm{N}(\mathcal{D}) \arrow[d, "\pi"] \\ \Delta^3 \arrow[r] \arrow[ur, dashed] & \mathrm{N}(\mathcal{C}) \end{tikzcd}$$

in which σ sends the edge $\Delta^{\{0,1\}} \subseteq \Lambda^3_0$ to f.

Definition 2.4.9. Let $\pi : X \longrightarrow S$ be a map of simplicial sets and let $f : x \longrightarrow y$ be an edge in X. We will say that f is π-*cocartesian* if for every $n \geq 2$ a dotted lift exists in every square of the form

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\sigma"] \arrow[d] & X \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{2.21}$$

in which σ maps $\Delta^{\{0,1\}} \subseteq \Lambda^n_0$ to f. Similarly, we will say that f is π-*cartesian* if the same holds when we replace $\sigma : \Lambda^n_0 \longrightarrow X$ in 2.21 by a map $\tau : \Lambda^n_n \longrightarrow X$ such that τ maps $\Delta^{\{n-1,n\}}$ to f.

Exercise 2.4.10. Let $\pi : X \longrightarrow S$ be an inner fibration.

(1) Show that $f : x \longrightarrow y$ in X is π-cocartesian if and only if the dotted lift exist in any diagram of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\rho"] \arrow[d] & X_{x/} \arrow[d] \\ \Delta^n \arrow[r] & S_{\pi(x)/} \end{tikzcd}$$

with $n \geq 1$ and such that $\rho(0) = f$.

(2) Let $x \in X$ be a vertex and $\overline{f} : \pi(x) \longrightarrow y$ an edge in S. Let $\mathcal{X} \subseteq X_{x/} \times_{S_{\pi(x)/}} \{\overline{f}\}$ be the subsimplicial set spanned by those $f : x \longrightarrow \widetilde{y}$ which are π-cocartesian edges in X. Show that $\mathcal{X}$ is either empty or a contractible Kan complex.

(3) Let $f : x \longrightarrow y$ and $f' : x \longrightarrow y'$ be two arrows in X which map to the same arrow $\overline{f}$ in S. Show that if f is equivalent to f' in $X_{x/} \times_{S_{\pi(x)/}} \{\overline{f}\}$ (the latter is an ∞-category since π is an inner fibration) then f is π-cocartesian if and only if f' is π-cocartesian.

Definition 2.4.11. Let $\pi : X \longrightarrow S$ be a map of simplicial sets. We will say that $\pi : X \longrightarrow S$ is a *cocartesian fibration* if the following conditions hold:

(1) π is an *inner fibration*, i.e., π satisfies the right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ with $0 < i < n$ (this condition is automatic when X and S are nerves of discrete categories).

(2) For every $x \in X$ and every edge $f : \pi(x) \longrightarrow y$ in S there exists a π-cocartesian edge $\widetilde{f} : x \longrightarrow \widetilde{y}$ such that $\pi(\widetilde{f}) = f$.

Example 2.4.12. The terminal map $\pi : X \longrightarrow *$ is a cocartesian fibration if and only if X is an ∞-category, in which case the π-cocartesian edges are exactly the equivalences by Corollary 2.3.14.

Remark 2.4.13. By definition the property of being a cocartesian fibration is invariant under base change. In other words, if $\pi : X \longrightarrow S$ is a cocartesian fibration and $T \longrightarrow S$ is any map then $X \times_S T \longrightarrow T$ is a cocartesian fibration. In particular, if $\pi : X \longrightarrow S$ is a cocartesian fibration then the fiber $\pi^{-1}(s)$ is an ∞-category for every $s \in S$.

Definition 2.4.14. Let S be a simplicial set. We define $\mathrm{Fib}^{\mathrm{coc}}_{\Delta}(S)$ to be the simplicial category whose objects are the cocartesian fibrations $\pi : X \longrightarrow S$ and such that for two cocartesian fibrations $\pi : X \longrightarrow S$ and $\pi' : Y \longrightarrow S$ the mapping simplicial set $\mathrm{Map}(X, Y)$ is the simplicial set of $\mathrm{Fun}_S(X, Y)^{\simeq}$ spanned by those maps $X \longrightarrow Y$ over S which send π-cocartesian edges to π'-cocartesian edges. It can then be shown that $\mathrm{Fib}^{\mathrm{coc}}_{\Delta}(S)$ is a *locally Kan* simplicial category, and we define $\mathrm{Fin}^{\mathrm{coc}}_{\Delta}(S) \coloneqq \mathrm{N}(\mathrm{Fin}^{\mathrm{coc}}_{\Delta}(S))$ to be its coherent nerve.

We then have the following higher categorical analogue of Theorem 2.4.7:

Theorem 2.4.15 (The Lurie-Grothendieck correspondence)**.** *For $S \in \mathrm{Set}_{\Delta}$ there is an equivalence of ∞-categories*

$$\mathrm{Un}^{\infty}_{S} : \mathrm{Fun}(S, \mathrm{Cat}_{\infty}) \xrightarrow{\simeq} \mathrm{Fib}^{\mathrm{coc}}(S) \tag{2.22}$$

such that

(1) Un^{∞}_{S} is compatible with base change along maps $T \longrightarrow S$.
(2) Un^{∞}_{S} reduces to the tautological identification

$$\mathrm{Fib}^{\mathrm{coc}}(*) = \mathrm{Cat}_{\infty} \xrightarrow{\mathrm{Id}} \mathrm{Cat}_{\infty} = \mathrm{Fun}(*, \mathrm{Cat}_{\infty})$$

for $S = *$.
(3) If $S = \mathrm{N}(\mathcal{D})$ is the nerve of a discrete category $\mathcal{D}$ and $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}_{\infty}$ factors through $\mathrm{Cat} \subseteq \mathrm{Cat}_{\infty}$, then $\mathrm{Un}^{\infty}_{S}(\mathcal{F})$ is naturally equivalent to $\int_{\mathcal{D}} \mathcal{F}$.

We will then say that the cocartesian fibration $\mathrm{Un}^{\infty}_{S}(\mathcal{F}) \longrightarrow S$ is classified *by $\mathcal{F}$, and will refer to Un^{∞}_{S} as the* ∞-unstraightening functor*.*

By *compatibility with base change* we mean that if $T \longrightarrow S$ is a map of simplicial sets then under the Lurie-Grothendieck correspondence the functor $\mathrm{Fib}^{\mathrm{coc}}(S) \longrightarrow \mathrm{Fib}^{\mathrm{coc}}(T)$ given by $(X \longrightarrow S) \mapsto (X \times_S T \longrightarrow T)$ corresponds to the restriction functor $\mathrm{Fun}(S, \mathrm{Cat}_{\infty}) \longrightarrow \mathrm{Fun}(T, \mathrm{Cat}_{\infty})$. Combined with the “normalization” condition for $S = *$ this means that if $\pi : X \longrightarrow S$ is a cocartesian fibration classified (up to equivalence) by a functor $\mathcal{F} : S \longrightarrow \mathrm{Cat}_{\infty}$, then for every $s \in S$ the ∞-category $\mathcal{F}(s)$ is equivalent to the fiber $\pi^{-1}(s)$.

Remark 2.4.16. The dual statement of Theorem 2.4.15 for *cartesian fibrations* yields a similar equivalence between the ∞-category $\mathrm{Fib}^{\mathrm{car}}(S)$ of cartesian fibrations (and functors which preserve cartesian edges) and $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Cat}_{\infty})$.

Given a cocartesian fibration $\pi : X \longrightarrow S$, the fiber $X_s \coloneqq X \times_S \{s\}$ over any vertex of S is an ∞-category by Remark 2.4.13 and Example 2.4.12. We would like to illustrate the idea encapsulated in Theorem 2.4.15 that X_s depends functorially on s. In particular, given an arrow $f : s \longrightarrow s'$ in S, we would like to construct the *transition functor* $f_! : X_s \longrightarrow X_{s'}$ associated to f. Informally speaking, we would like to do this by choosing for each $x \in X_s$ a cocartesian edge $\widetilde{f} : x \longrightarrow x'$ covering f in a way that is compatible along X_s. One way to describe this procedure is by a *lifting property* for natural transformations:

Proposition 2.4.17. *Let $\pi : X \longrightarrow S$ be an inner fibration and consider a lifting problem of the form*

$$\begin{tikzcd} K \times \Delta^{\{0\}} \arrow[r, "p"] \arrow[d] & X \arrow[d, "\pi"] \\ K \times \Delta^1 \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & S \end{tikzcd} \tag{2.23}$$

Assume that for every vertex $v \in K$ there exists a π-cocartesian edge $f : p(v) \longrightarrow x$ lifting $H(\{v\} \times \Delta^1)$. Then there exists a lift $\widetilde{H} : K \times \Delta^1 \longrightarrow X$ in (2.23) such that $\widetilde{H}(\{v\} \times \Delta^1)$ is π-cocartesian for every $v \in K$.

We note that given a cocartesian fibration $\pi : X \longrightarrow S$, solving the lifting problem (2.23) can be described as follows: if we are given a diagram $p : K \longrightarrow X$ and a natural transformation $H : K \times \Delta^1 \longrightarrow S$ from $\pi \circ p : K \longrightarrow S$ to some map $q : K \longrightarrow S$, then we can lift H to a natural transformation $\widetilde{H}$ in X starting from p. Furthermore, we can choose this natural transformation to be *pointwise π-cocartesian*.

We will prove Proposition 2.4.17 by working simplex by simplex. For this we will need the following lemma:

Lemma 2.4.18. *Let $\pi : X \longrightarrow S$ be an inner fibration and consider a lifting problem of the form*

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \coprod_{\partial\Delta^n \times \Delta^{\{0\}}} \Delta^n \times \Delta^{\{0\}} \arrow[r, "\rho"] \arrow[d] & X \arrow[d, "\pi"] \\ \Delta^n \times \Delta^1 \arrow[r, "\sigma"] \arrow[ur, dashed, "\widetilde{\rho}"] & S \end{tikzcd} \tag{2.24}$$

for $n \geq 1$. If $\rho(\Delta^{\{0\}} \times \Delta^1)$ is π-coCatesian then the dotted lift exists.

Proof. For $i = 0, ..., n$ let $\sigma_i : \Delta^{n+1} \longrightarrow \Delta^n \times \Delta^1$ be the $n+1$ simplex given on vertices by the formula

$$\sigma_i(j) = \begin{cases} (j, 0) & j \leq i \\ (j-1, 1) & j < i \end{cases}.$$

Consider the filtration $Z_{n+1} \subseteq Z_n \subseteq ... \subseteq Z_0 = \Delta^n \times \Delta^1$ such that Z_{n+1} is the top left corner of (2.24) and for $i = 0, ..., n$ the subsimplicial set Z_i is obtained from Z_{i+1} by adding the $(n+1)$-simplex σ_i. Let us construct the dotted lift $\widetilde{\rho}$ inductively on each Z_i. For this, we observe that the n-faces of σ_i are all contained in Z_{i+1} except the face across from the i'th vertex. It follows that we have a pushout square of simplicial sets

$$\begin{tikzcd} \Lambda^{n+1}_i \arrow[r] \arrow[d] & Z_{i+1} \arrow[d] \\ \Delta^{n+1} \arrow[r] & Z_i \end{tikzcd}$$

Since $0 \leq i < n+1$ we see that any partial extension $\widetilde{\rho}_{i+1} : Z_{i+1} \longrightarrow X$ can be extended to $\widetilde{\rho}_i : Z_i \longrightarrow X$, either because $0 < i < n+1$ or, when $i = 0$, by the assumption that ρ sends $\Delta^{\{0\}} \times \Delta^1$ to a π-cocartesian edge. □

Proof of Proposition 2.4.17. We argue inductively on the skeleton of K. We first define $\widetilde{\rho}$ on $\{v\} \times \Delta^1$ for every vertex $v \in K$ by choosing π-cocartesian lifts. Given $n \geq 1$ we then extend $\widetilde{\rho}$ from the $(n-1)^{\text{th}}$ skeleton to the n^{th} skeleton simplex by simplex using Lemma 2.4.18. □

Construction 2.4.19. Let $\pi : X \longrightarrow S$ be an inner fibration and $f : s \longrightarrow s'$ an edge in S such that for every $x \in X_s$ there exists a π-cocartesian edge of X lifting f. Consider the commutative diagram

$$\begin{tikzcd} X_s \arrow[r, "\iota"] \arrow[d] & X \arrow[d, "\pi"] \\ X_s \times \Delta^1 \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{2.25}$$

where the bottom horizontal map is the composition $X_s \times \Delta^1 \longrightarrow \Delta^1 \xrightarrow{[f]} S$. By Proposition 2.4.17 there exists a dotted lift in (2.25) which sends $\{x\} \times \Delta^1$ to a π-cocartesian edge for every $x \in X_s$. This can be interpreted as a natural transformation from the fiber inclusion $\iota : X_s \hookrightarrow X$ to some other functor $\iota' : X_s \longrightarrow X$ whose image is contained in the fiber over s'. In particular, ι' determines a map $f_! : X_s \longrightarrow X_{s'}$. It can be shown that this functor is equivalent to the functor $X_s \longrightarrow X_{s'}$ associated to f by the functor $\chi : S \longrightarrow \mathrm{Cat}_\infty$ classifying π, and in particular does not depend on the choice of lift (as long as it is "pointwise" π-cocartesian). We will refer to the $f_! : X_s \longrightarrow X_{s'}$ above as the *transition functor* associated to f.

An important particular case of cocartesian fibrations is the following:

Proposition 2.4.20. *Let $\pi : X \longrightarrow S$ be a cocartesian fibration. Then the following conditions are equivalent:*

(1) π is a left fibration.
(2) Every edge of X is π-cocartesian.
(3) For every $s \in S$ the ∞-category $\pi^{-1}(s)$ is an ∞-groupoid.
(4) The functor classifying π takes values in the full subcategory $\mathrm{Grp}_\infty \subseteq \mathrm{Cat}_\infty$ spanned by ∞-groupoids.

In particular, the Lurie-Grothendieck correspondence descends to an equivalence

$$\mathrm{Fun}(S, \mathcal{S}) \xrightarrow{\simeq} \mathrm{Fib}^{\mathrm{left}}(S)$$

between the ∞-category of functors $S \longrightarrow \mathcal{S}$ (resp. $S^{\mathrm{op}} \longrightarrow \mathcal{S}$) and the ∞-category of left (resp. right) fibrations over S.

Proof. The equivalence of (1) and (2) is by definition, and the equivalence of (3) and (4) is by the base change compatibility of the ∞-unstraightening functor (2.22). The implication (2) $\Rightarrow$ (3) is obtained by restricting attention to π-cocartesian edges contained in a fiber and using Example 2.4.12. To see that (3) $\Rightarrow$ (2) observe that every arrow in $\mathcal{C}$ factors as a composition of a π-cocartesian arrow followed by an arrow contained in a fiber, and so if (3) holds then every arrow is equivalent to a π-cocartesian arrow, and is therefore π-cocartesian (see Exercise 2.4.10(3)). ☐

Let us now say a few words about the proof of the Lurie-Grothendieck correspondence (see [5, §3]). The main idea consists of finding suitable model categories which model the ∞-categories on both sides and then constructing explicit Quillen equivalence between them. This is done by using the category Set^+_Δ of *marked simplicial sets*. Recall from Theorem 2.2.16 that there exists a model structure on Set^+_Δ whose underlying ∞-category is Cat_∞. The path to the proof of the Lurie-Grothendieck correspondence passes through the following steps:

(1) Let $\mathcal{C} = \mathfrak{C}[S]$ be the simplicial category generated from S. Then the model structure of Theorem 2.2.16 induces a model structure on the functor category $(\mathrm{Set}^+_\Delta)^{\mathcal{C}}$ whose underlying ∞-category is $\mathrm{Fun}(S, \mathrm{Cat}_\infty)$.
(2) The category $(\mathrm{Set}^+_\Delta)_{/S^\sharp}$ of marked simplicial set over $S^\sharp$ can be endowed with a model structure whose underlying ∞-category is $\mathrm{Fib}^{\mathrm{coc}}(S)$.
(3) There exists a Quillen equivalence

$$\mathrm{St}^+_S : (\mathrm{Set}^+_\Delta)_{/S^\sharp} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} (\mathrm{Set}^+_\Delta)^{\mathcal{C}} : \mathrm{Un}^+_S$$

which is suitably compatible with base change, and such that for $S = *$ the resulting Quillen equivalence $\mathrm{St}^+_* \dashv \mathrm{Un}^+_*$ is naturally equivalent to the identity. Furthermore, if $S = \mathrm{N}(\mathcal{D})$ is the nerve of a discrete category and $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}_\infty$ factors through $\mathrm{Cat} \subseteq \mathrm{Cat}_\infty$, then $\mathrm{Un}^\infty_S(\mathcal{F})$ is naturally equivalent to the nerve of $\int_{\mathcal{D}} \mathcal{F}$ (with the marked edges being the cocartesian ones).

The Quillen functors St^+_S and Un^+_S are known as the *straightening and unstraightening* functors. The ∞-unstraightening functor Un^∞_S of Theorem 2.4.15 is the one induced on underlying ∞-categories by Un^+_S.

2.5. **The relative nerve.** In this section we will describe an alternative and equivalent construction for the ∞-unstraightening which is valid when S is the nerve of an *ordinary* category. This construction is combinatorially simpler than that of St^+_S and reflects what happens in the ∞-categorical analogue of the Grothendieck construction more transparently. In [5, §3.2.5] this construction is described under the name *the relative nerve* construction.

Definition 2.5.1. Let $\mathcal{D}$ be an ordinary category and $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}_\Delta$ a functor. Define a simplicial set $\mathrm{N}_{\mathcal{F}}(\mathcal{D})$ as follows. An n-simplex of $\mathrm{N}_{\mathcal{F}}(\mathcal{D})$ consists of
(1) A functor $\sigma : [n] \longrightarrow \mathcal{D}$.
(2) A collection of simplices $\tau_S : \Delta^S \longrightarrow \mathcal{F}(\sigma(\max(S)))$ for every non-empty subset $S \subseteq [n]$ such that for every $S' \subseteq S \subseteq [n]$ the diagram

$$\begin{tikzcd} \Delta^{S'} \arrow[r, "\tau_S"] \arrow[d] & \mathcal{F}(\sigma(\max(S'))) \arrow[d] \\ \Delta^{S} \arrow[r, "\tau_S"] & \mathcal{F}(\sigma(\max(S))) \end{tikzcd} \tag{2.26}$$

commutes.

Forgetting the collection (τ_S) we obtain a natural map $\pi : \mathrm{N}_{\mathcal{F}}(\mathcal{D}) \longrightarrow \mathrm{N}(\mathcal{D})$.

Remark 2.5.2. The compatibility condition (2.26) implies in particular that the collection $\tau_S : \Delta^S \longrightarrow \mathcal{F}(\sigma(\max(S)))$ is completely determined by the collection $\tau_{\{0,..,i\}}$ for $i = 0, ...n$, and so we could have replaced in Definition 2.5.1 the compatible collection $\{\tau_S\}$ by a compatible collection $\{\tau_{\{0,...,i\}}\}$. However, choosing all the τ_S makes the simplicial structure on $\mathrm{N}_{\mathcal{F}}(\mathcal{D})$ more evident.

Example 2.5.3. If $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}_\Delta$ is such that $\mathcal{F}(x)$ is the nerve of an ordinary category $\mathcal{G}(x)$ for every $x \in \mathcal{D}$ then $\mathrm{N}_{\mathcal{F}}(\mathcal{D})$ is naturally isomorphic to the nerve of the Grothendieck construction of $\mathcal{G}$. In fact, a more general variant of this claim is true, see Proposition 2.5.9 below.

We note that the the fiber of $\mathrm{N}_{\mathcal{F}}(\mathcal{D}) \longrightarrow \mathrm{N}(\mathcal{D})$ over $x \in \mathcal{D}$ is canonically isomorphic to $\mathcal{F}(x)$. In particular, a vertex of $\mathrm{N}_{\mathcal{F}}(\mathcal{D})$ can be identifies with a pair (x, a) where x is an object of $\mathcal{D}$ and a is an object of $\mathcal{F}(x)$. Similarly, an edge from (x, a) to (y, b) is given by definition by a pair (f, α) where $f : x \longrightarrow y$ is an arrow in $\mathcal{D}$ and $\alpha : f_!a \longrightarrow b$ is an arrow in $\mathcal{F}(y)$, where we have denoted by $f_! : \mathcal{F}(x) \longrightarrow \mathcal{F}(y)$ the map associated to f by $\mathcal{F}$.

Proposition 2.5.4. *Let $\mathcal{D}$ be an ordinary category and $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}_\Delta$ a functor such that $\mathcal{F}(x)$ is an ∞-category for every $x \in \mathcal{D}$. Then $\pi : \mathrm{N}_{\mathcal{F}}(\mathcal{D}) \longrightarrow \mathrm{N}(\mathcal{D})$ is a cocartesian fibration. Furthermore, an edge $(f, \alpha) : (x, a) \longrightarrow (x, b)$ is π-cocartesian if and only if $\alpha : f_!a \longrightarrow b$ is an equivalence in $\mathcal{F}(y)$.*

Proof. Let us first show that π is an inner fibration. Consider a lifting problem of the form

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\rho"] \arrow[d] & \mathrm{N}_{\mathcal{F}}(\mathcal{D}) \arrow[d] \\ \Delta^n \arrow[r, "\sigma"] \arrow[ur, dashed] & \mathrm{N}(\mathcal{D}) \end{tikzcd} \tag{2.27}$$

with $0 < i < n$. For every $S \subseteq [n]$ other than $[n]$ and $[n] \setminus \{i\}$ the map ρ provides us with an S-simplex $\tau_S : \Delta^S \longrightarrow \mathcal{F}(\sigma(\max(S)))$ which satisfies the compatibility condition 2.26. The collection of composed maps

$$\Delta^S \xrightarrow{\tau_S} \mathcal{F}(\sigma(\max(S))) \longrightarrow \mathcal{F}(\sigma(n))$$

then determines a map $\rho' : \Lambda^n_i \longrightarrow \mathcal{F}(\sigma(n))$. The data of the dotted lift is than equivalent to an extension of ρ' to a map $\overline{\rho}' : \Delta^n \longrightarrow \mathcal{F}(\sigma(n))$. This extension exists by our assumption that $\mathcal{F}$ takes values in ∞-categories.

Let us now consider a lifting problem as in (2.27) with $i = 0$ and let $\rho' : \Lambda^n_i \longrightarrow \mathcal{F}(\sigma(n))$ be as above. By Corollary 2.3.14 the desired extension exists if $\rho'|_{\Delta^{\{0,1\}}}$ is an invertible edge of $\mathcal{F}(\sigma(n))$. This will indeed by the case if the edge $\tau_{\{0,1\}} : \Delta^{\{0,1\}} \longrightarrow \mathcal{F}(\sigma(1))$ is invertible. We may hence conclude that a general edge $(f, \alpha) : (x, a) \longrightarrow (y, b)$ as above is π-cocartesian if the edge $\alpha : f_!a \longrightarrow b$ is an equivalence in $\mathcal{F}(y)$. Conversely, if (f, α) is π-cocartesian then we can deduce that $\alpha : f_!x \longrightarrow y$ is an equivalence from Corollary 2.3.14 by considering the lifting problem (2.27) as above when $\sigma : \Delta^n \longrightarrow \mathrm{N}(\mathcal{D})$ factors through the surjective map $\Delta^n \longrightarrow \Delta^1$ which sends $\Delta^{\{0,1\}}$ isomorphicaly to Δ^1. To show that π is a cocartesian fibration it will hence suffice to show that if $f : x \longrightarrow y$ is an arrow in $\mathcal{D}$ and $a \in \mathcal{F}(x)$ is an object then there exist $b \in \mathcal{F}(y)$ and an equivalence $\alpha : f_!a \longrightarrow b$. But this is clear: just take $b = f_!a$ and α the identity on $f_!a$. $\square$

Remark 2.5.5. Proposition 2.5.4 implies in particular that when $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}_\Delta$ takes values in ∞-categories the relative nerve $\mathrm{N}_{\mathcal{F}}(\mathcal{D})$ is an ∞-category. The mapping spaces in this ∞-category can be explicitly described. Specifically, if $x, y \in \mathcal{D}$ are two objects and a is an object of $\mathcal{F}(x)$ then there is a natural isomorphism of simplicial sets

$$\mathrm{N}_{\mathcal{F}}(\mathcal{D})_{(x,a)/} \times_{\mathrm{N}_{\mathcal{F}}(\mathcal{D})} \mathcal{F}(y) \cong \coprod_{f : x \to y} \mathcal{F}(y)_{f_!a/}, \tag{2.28}$$

where the coproduct ranges over all maps $f : x \longrightarrow y$ in $\mathcal{D}$. Taking the fibers over a particular $b \in \mathcal{F}(y)$ we may conclude that

$$\mathrm{Map}_{\mathrm{N}_{\mathcal{F}}(\mathcal{D})}((x, a), (y, b)) \simeq \coprod_{f : x \to y} \mathrm{Map}_{\mathcal{F}(y)}(f_!a, b).$$

Remark 2.5.6. If $\tau : \mathcal{F} \longrightarrow \mathcal{F}'$ is a natural transformation such that $\tau_x : \mathcal{F}(x) \longrightarrow \mathcal{F}'(x)$ is an equivalence of ∞-categories for every $x \in \mathcal{D}$ then the induced map $\tau_* : \mathrm{N}_{\mathcal{F}}(\mathcal{D}) \longrightarrow \mathrm{N}_{\mathcal{F}}(\mathcal{D})$ is an equivalence of ∞-categories. Indeed, the discussion above shows that if τ is levelwise essentially surjective then τ_* is essentially surjective and the formula in (2.28) shows that if τ_* is levelwise fully-faithful then τ_* is fully-faithful (see Remark 2.3.19).

Remark 2.5.7. In Definition 2.5.1, if we assume instead that $\mathcal{F} : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}$ is a *contravariant functor* form $\mathcal{D}$ to Set_{Δ} then we can define the cartesian version of the relative nerve $\mathcal{N}^{\mathcal{F}}(\mathcal{D}) \longrightarrow \mathrm{N}(\mathcal{D})$ whose simplices are given by pairs $(\sigma, \{\tau_S\})$ as in Definition 2.5.1 where τ_S is now a simplex of $\mathcal{F}(\sigma(\min(s)))$ instead of $\mathcal{F}(\sigma(\max(s)))$. The projection $\mathcal{N}^{\mathcal{F}}(\mathcal{D}) \longrightarrow \mathrm{N}(\mathcal{D})$ is then a *cartesian fibration* which is the one corresponding to $\mathcal{F}$ by the Lurie-Grothendieck correspondence for cartesian fibrations.

Remark 2.5.8. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}_{\Delta}$ be a diagram of ∞-categories indexed by an ordinary category $\mathcal{D}$. Then the fiber of $\pi : \mathrm{N}_{\mathcal{F}}(\mathcal{D}) \longrightarrow \mathrm{N}(\mathcal{D})$ over an object $x \in \mathrm{N}(\mathcal{D})$ is canonically isomorphic to $\mathcal{F}(x)$. Let $f : x \longrightarrow y$ be a map in $\mathcal{D}$ corresponding to a map of simplicial sets $\sigma_f : \Delta^1 \longrightarrow \mathrm{N}(\mathcal{D})$. Consider the lifting problem

$$\begin{tikzcd} \mathcal{F}(x) \arrow[r] \arrow[d] & \mathrm{N}_{\mathcal{F}}(\mathcal{D}) \arrow[d] \\ \mathcal{F}(x) \times \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathrm{N}(\mathcal{D}) \end{tikzcd} \tag{2.29}$$

where the bottom horizontal map is the composition $\mathcal{F}(x) \times \Delta^1 \longrightarrow \Delta^1 \xrightarrow{\sigma_f} \mathrm{N}(\mathcal{D})$. Unwinding the definitions we see that we can construct a lift in (2.29) by sending an n-simplex in $\mathcal{F}(x) \times \Delta^1$ given by a pair $(\eta, \rho) \in \mathcal{F}(x)_n \times (\Delta^1)_n$ to the n-simplex of $\mathrm{N}_{\mathcal{F}}(\mathcal{D})$ given by the data $(\sigma, \{\tau_S\}_{S \subseteq [n]})$ where $\sigma = \sigma_f \circ \rho : \Delta^n \longrightarrow \mathrm{N}(\mathcal{D})$ and $\tau_S : \Delta^S \longrightarrow \mathcal{F}(\sigma(\max(S))$ is given by the restriction of $\rho : \Delta^n \longrightarrow \mathcal{F}(x)$ to Δ^S if $\rho(\max(S)) = 0$ and the composition $\rho|_{\Delta^S} : \Delta^S \longrightarrow \mathcal{F}(x) \xrightarrow{f_!} \mathcal{F}(y)$ if $\rho(\max(S)) = 1$. We then see that the resulting natural transformation $H : \mathcal{F}(x) \times \Delta^1 \longrightarrow \mathrm{N}_{\mathcal{F}}(\mathcal{D})$ is point-wise π-cocartesian by the description of π-cocartesian edges in Proposition 2.5.4, and that $H|_{\mathcal{F}(x) \times \Delta^{\{1\}}}$ maps $\mathcal{F}(x)$ to the fiber $\mathcal{F}(y)$ of π over y via $f_!$. We may hence conclude that the transition functor $\mathcal{F}(x) \longrightarrow \mathcal{F}(y)$ associated to π (see Construction 2.4.19) is exactly $f_!$.

We finish this section with another version of the Grothendieck construction, this time for a family of *simplicial categories* indexed by an *ordinary category* $\mathcal{D}$. Given a functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}_{\Delta}$, let $\int_{\mathcal{D}} \mathcal{F}$ be the simplicial category whose

- objects are pairs (x, a) where x is an object of $\mathcal{D}$ and a is an object of $\mathcal{F}(x)$;
- The mapping simplicial set from (x, a) to (y, b) is given by

$$\mathrm{Map}_{\int_{\mathcal{D}} \mathcal{F}}((x,a),(y,b)) = \coprod_{f : x \to y} \mathrm{Map}_{\mathcal{F}(y)}(f_! x, y)$$

  where the coproduct is taken over all maps $f : x \longrightarrow y$ in $\mathcal{D}$, and $f_! : \mathcal{F}(x) \longrightarrow \mathcal{F}(y)$ is the simplcial functor associated to f by $\mathcal{F}$.

We then have the following observation:

Proposition 2.5.9. *Let $\mathcal{D}$ be an ordinary category and $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Cat}_{\Delta}$ a diagram of simplicial categories. Let $\mathrm{N}\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}_{\Delta}$ be the composition of $\mathcal{F}$ with the coherent nerve functor. Then there is a natural* isomorphism *of simplicial sets*

$$\mathrm{N}\Big(\int_{\mathcal{D}} \mathcal{F}\Big) \cong \mathrm{N}_{\mathrm{N}\mathcal{F}}(\mathcal{D})$$

which is compatible with the projection to $\mathrm{N}(\mathcal{D})$.

Proof. Let $\Theta : [n] \longrightarrow \mathrm{Set}_\Delta$ be the functor $i \mapsto \Delta^{\{0,...,i\}}$. In view of Remark 2.5.2 we see that given an n-simplex $\sigma : \Delta^n \longrightarrow \mathrm{N}(\mathcal{D})$, the lifts $\widetilde{\sigma} : \Delta^n \longrightarrow \mathrm{N}_{\mathrm{N}\,\mathcal{F}}(\mathcal{D})$ of σ are in bijection with natural transformations

$$\Theta \Rightarrow \sigma^* \mathrm{N}\,\mathcal{F} \tag{2.30}$$

of functors from $[n]$ to Set_Δ, where we have identified σ with a functor $[n] \longrightarrow \mathcal{D}$. By adjunction, a natural transformation as in (2.30) is the same as a natural transformation

$$\mathfrak{C}\Theta \Rightarrow \sigma^*\mathcal{F}, \tag{2.31}$$

where we denote by $\mathfrak{C}\Theta : [n] \longrightarrow \mathrm{Cat}_\Delta$ the composition of Θ and $\mathfrak{C} : \mathrm{Set}_\Delta \longrightarrow \mathrm{Cat}_\Delta$. Explicitly, a natural transformation as in (2.31) is given by a collection of simplicial functors $\rho_i : \mathfrak{C}(\Delta^{\{0,...,i\}}) \longrightarrow \mathcal{F}(i)$ for $i = 0, ..., n$, such that for every $i \leq j$ the diagram

$$\begin{tikzcd}
\mathfrak{C}(\Delta^{\{0,...,i\}}) \arrow[r] \arrow[d] & \mathcal{F}(i) \arrow[d] \\
\mathfrak{C}(\Delta^{\{0,...,j\}}) \arrow[r] & \mathcal{F}(j)
\end{tikzcd}$$

commutes. We claim that this is the same data as a map $\eta : \mathfrak{C}(\Delta^n) \longrightarrow \int_{[n]} \sigma^*\mathcal{F}$ over $[n]$ (and hence the same data as an n-simplex of $\mathrm{N}(\int_{\mathcal{D}} \mathcal{F})$ lying above σ). Indeed, for each $i = 0, ..., n$ let $\mathcal{F}_i : [i] \longrightarrow \mathrm{Cat}_\Delta$ be the restriction of $\sigma^*\mathcal{F}$ to $[i] = \{0, ..., i\} \subseteq [n]$. Given an η as above we may restrict it to a map

$$\eta_i : \mathfrak{C}\big(\Delta^{\{0,...,i\}}\big) \longrightarrow \int_{[i]} \mathcal{F}_i \cong [i] \times_{[n]} \int_{[n]} \sigma^*\mathcal{F}$$

over $[i]$ for every $i = 0, ..., n$. We now observe that since i is terminal in $[i]$ we have a natural functor $\int_{[i]} \mathcal{F}_i \longrightarrow \mathcal{F}(i)$ (which maps each $\mathcal{F}(i') \subseteq \int_{[i]} \mathcal{F}_i$ for $i' \leq i$ to $\mathcal{F}(i)$ via the map $\mathcal{F}(i') \longrightarrow \mathcal{F}(i)$ determined by the functor $\mathcal{F}$). Composing these functors with η_i we get a compatible collection of functors

$$\overline{\eta}_i : \mathfrak{C}\big(\Delta^{\{0,...,i\}}\big) \longrightarrow \mathcal{F}(i).$$

The association $\eta \mapsto \{\overline{\eta}_i\}$ then gives the desired bijection. □

2.6. **Limits and colimits in ∞-categories.** A major advantage of the model of ∞-categories over that of simplicial categories is that it allows for a straightforward definition of limits and colimits.

Definition 2.6.1. Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ an object. We will say that x is *final* if $\mathrm{Map}_{\mathcal{C}}(y, x)$ is contractible for every $y \in \mathcal{C}$. Dually, we will say that x is *initial* if $\mathrm{Map}_{\mathcal{C}}(x, y) \simeq *$ for every $y \in \mathcal{C}$.

Remark 2.6.2. It follows from Remark 2.3.17 that if $\mathcal{C}$ is an ordinary category and $x \in \mathcal{C}$ is an object then x is final (resp. initial) in $\mathcal{C}$ in the usual sense if and only if x is final (resp. initial) in $\mathrm{N}(\mathcal{C})$ in the sense of Definition 2.6.1. More generally, if $\mathcal{C}$ is a locally Kan simplicial category then x is final in $\mathrm{N}(\mathcal{C})$ if and only if it is homotopy final in $\mathcal{C}$ in the sense that $\mathrm{Map}_{\mathcal{C}}(y, x)$ is weakly contractible for every $y \in \mathcal{C}$.

Definition 2.6.3. Let $\mathcal{C}$ be an ∞-category and $p : K \longrightarrow \mathcal{C}$ a diagram in $\mathcal{C}$. We will say that a left cone $\overline{p} : K^{\triangleleft} \longrightarrow \mathcal{C}$ is a *limit cone* if it is terminal as an object of the ∞-category $\mathcal{C}_{/p}$. Similarly, we will say that a right cone $\overline{p} : K^{\triangleleft} \longrightarrow \mathcal{C}$ is a *colimit cone* if it is initial as an object of $\mathcal{C}_{/p}$.

Remark 2.6.4. Given a simplicial set K, the embedding of the cone point $\{*\} \subseteq K^{\triangleleft}$ is left anodyne. It then follows from Proposition 2.6.8 below and Corollary 2.3.10 that if $\overline{p} : K^{\triangleleft} \longrightarrow \mathcal{C}$ is a limit cone with $p := \overline{p}|_K$ and $x := \overline{p}(*)$ then in the diagram

$$\begin{tikzcd} & \mathcal{C}_{/\overline{p}} \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ \mathcal{C}_{/p} \arrow[dr] & & \mathcal{C}_{/x} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

both top projections are trivial Kan fibrations. This expresses the idea that if $\overline{p}$ is a limit cone then the object $x = p(*)$ *represents* the functor classifying the right fibration $\mathcal{C}_{/p} \longrightarrow \mathcal{C}$.

Theorem 2.6.5. *The classical notion of homotopy limits and colimits in spaces, coincides with the notion of limits and colimits in the ∞-category $\mathcal{S}$. More generally, in any model category $\mathcal{M}$ the construction of homotopy limits and colimits, as given, for example, by the explicit construction of Bousfield and Kan, coincides with limits and colimit in the underlying ∞-category $\mathcal{M}_\infty$.*

The cocartesian fibration modelling a diagram of ∞-categories can also be used to express its limits and colimit without making any reference to model categorical homotopy limits and colimits. More precisely, we quote the following assertion, which is a combinatorion of [5, Corollary 3.3.3.2, Corollary 3.3.3.4, Corollary 3.3.4.3, Corollary 3.3.4.6].

Theorem 2.6.6. *Let $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ be a cocartesian fibration of ∞-categories classified by a diagram $\chi : \mathcal{D} \longrightarrow \mathrm{Cat}_\infty$. Then the* colimit *of π in Cat_∞ is equivalent to the localization of $\mathcal{C}$ (Definition 2.2.12) by the collection of π-cocartesian edges, while the* limit *of π in Cat_∞ is equivalent to the ∞-category of sections $\mathcal{D} \longrightarrow \mathcal{C}$ of π which send every edge to a π-cocartesian edge. The analogous statemet holds if we replace Cat_∞ by the ∞-category $\mathcal{S}$ of spaces and π by the corresponding left fibration.*

Our next goal is to show that limits and colimits are essentially unique once they exist. Given their definition via initial and final objects we may formulate this uniqueness as follows:

Proposition 2.6.7. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by the final vertices. Then $\mathcal{C}_0$ is either empty or a contractible Kan complex.*

Proposition 2.6.7 will follow rather directly from the following:

Proposition 2.6.8. *Let $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ is a right fibration of ∞-categories. Then the following conditions are equivalent:*

(1) The fibers of π are trivial Kan complexes.
(2) π is a trivial Kan fibration.

Proof. We first note that (2) ⇒ (1) since the base change of a trivial Kan fibration is a trivial Kan fibration. Now assume that the fibers of π are contractible Kan complexes. We need to show that a dotted lift exists in any square of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\sigma"] \arrow[d] & \mathcal{C} \arrow[d, "\pi"] \\ \Delta^n \arrow[r, "\tau"] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} \tag{2.32}$$

If $n = 0$ then this follows from the fact that the fibers of π are not empty. We may hence suppose that $n \geq 1$. Consider the composed map $H : \Delta^n \times \Delta^1 \longrightarrow \Delta^n \longrightarrow \mathcal{C}$ in which the first map is given on vertices by the rule $(i, 1) \mapsto i$ and $(i, 0) \mapsto 0$. Consider the restriction $H' \coloneqq H|_{\partial\Delta^n\times\Delta^1}$, which we can consider as a natural transformation from the constant map $\partial\Delta^n \longrightarrow \mathcal{D}$ with value $\tau(0)$ to $\pi\sigma : \partial\Delta^n \longrightarrow \mathcal{D}$. Since π is a right fibration we can use the (dual version of the) lifting property for natural transformations given in Proposition 2.4.17 to lift H' to a natural transformation $\overline{H}' : \partial\Delta^n \times \Delta^1 \longrightarrow \mathcal{C}$ from some $\sigma' : \partial\Delta^n \longrightarrow \mathcal{C}$ to σ, and such that $\pi\sigma' : \partial\Delta^n \longrightarrow \mathcal{D}$ is constant with value $\tau(0)$. In particular the image of σ' is concentrated in the fiber over $\tau(0)$. Since this fiber is a contractible Kan complex by assumption we can extend σ' to a map $\tau' : \Delta^n \longrightarrow \mathcal{C}$ such that $\pi\tau' : \Delta^n \longrightarrow \mathcal{D}$ is constant with value $\tau(0)$. The maps $\overline{H}'$ and τ' together then determine a diagram of the form

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \coprod_{\partial\Delta^n\times\Delta^{\{0\}}} \Delta^n \times \Delta^{\{0\}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "\pi"] \\ \Delta^n \times \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} \tag{2.33}$$

Note that we assumed that $n \geq 1$. The top horizontal map in (2.33) then sends the edge $\Delta^{\{0\}} \times \Delta^1$ to an edge which lies over the identity $\tau(0) \longrightarrow \tau(0)$ in $\mathcal{D}$, and which is hence invertible by Lemma 2.3.12(1). Since any invertible edge is π-cocartesian (Example 2.4.12) we can apply Lemma 2.4.18 to deduce that the dotted lift exists in (2.33). Restricting this lift to $\Delta^n \times \Delta^{\{1\}}$ then yields a lift in the original diagram (2.32). □

Corollary 2.6.9. *Let $\mathcal{C}$ be an ∞-category and $\rho : \partial\Delta^n \longrightarrow \mathcal{C}$ a map. If $\rho(n)$ is final in $\mathcal{C}$ then ρ extends to $\overline{\rho} : \Delta^n \longrightarrow \mathcal{C}$.*

Proof. By adjunction we can transform this extension problem into a lifting problem of the form

$$\begin{tikzcd} \partial\Delta^{n-1} \arrow[r, "\sigma"] \arrow[d] & \mathcal{C}_{/\sigma(n)} \arrow[d, "\pi"] \\ \Delta^{n-1} \arrow[r, "\tau"] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd} \tag{2.34}$$

where the desired lift exists by Proposition 2.6.8 since the projection $\mathcal{C}_{/\sigma(n)} \longrightarrow \mathcal{C}$ is a right fibrations whose fibers are contractible Kan complexes. □

Proof of Proposition 2.6.7. This follows immediately from Corollary 2.6.9. □

Exercise 2.6.10. Let $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ be a cartesian fibration of ∞-categories. Assume that for every $x \in \mathcal{D}$ the fiber $\mathcal{C}_x := \mathcal{C} \times_{\mathcal{D}} \{x\}$ over x has an initial object. Show that any lifting problem of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\rho"] \arrow[d] & \mathcal{C} \arrow[d, "\pi"] \\ \Delta^n \arrow[r, "\sigma"] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

with $n \geq 1$ has a solution as soon as $\rho(\Delta^{\{0\}})$ is initial in $\mathcal{C}_{\sigma(0)}$. Deduce that if we denote by $\mathcal{C}_0 \subseteq \mathcal{C}$ the collection of all objects which are initial in the fiber then the restriction $\pi|_{\mathcal{C}_0} : \mathcal{C}_0 \longrightarrow \mathcal{D}$ is a trivial Kan fibration. Hint: argue as in the proof of Proposition 2.6.8 and use (the dual of) Corollary 2.6.9.

Definition 2.6.11. Let $q : X \longrightarrow Y$ be a map of simplicial sets. We will say that q is *cofinal* if for every ∞-category $\mathcal{C}$ and every diagram $p : Y \longrightarrow \mathcal{C}$ the induced map

$$\mathcal{C}_{p/} \longrightarrow \mathcal{C}_{pq/}$$

is an equivalence of ∞-categories. Dually, we will say that q is *coinitial* if for every ∞-category $\mathcal{C}$ and every diagram $p : Y \longrightarrow \mathcal{C}$ the induced map

$$\mathcal{C}_{/p} \longrightarrow \mathcal{C}_{/pq}$$

is an equivalence of ∞-categories.

Example 2.6.12. It follows from Corollary 2.3.10 that any left anodyne map $X \hookrightarrow Y$ is coinitial and any right anodyne map is cofinal. In fact, it can be shown that any cofinal map arises in this way up to a categorical equivalence (see [5, Corollary 4.1.1.12]).

It follows directly from the definition that if $q : X \longrightarrow Y$ is a cofinal map then for every diagram $p : Y \longrightarrow \mathcal{C}$ in an ∞-category $\mathcal{C}$ we have that p admits a colimit if and only if $pq : X \longrightarrow \mathcal{C}$ admits a colimit and that a given cone $\overline{p} : Y^{\triangleright} \longrightarrow \mathcal{C}$ is a colimit cone if and only if $\overline{p}q : X^{\triangleright} \longrightarrow \mathcal{C}$ is a colimit cone. Similarly, if q is coinitial then the same claim holds for limits instead of colimits. The operation of restricting along cofinal/coinitial map is a very common process in the computation of limits and colimits. It is hence important to be able to identify when a given map of simplicial sets is cofinal/coinitial. One of the principal criteria in practice is given by the following fundamental theorem:

Theorem 2.6.13 (Quillen's theorem A for ∞-categories)**.** *Let $q : X \longrightarrow \mathcal{C}$ be a map of simplicial sets whose codomain is an ∞-category. Then q is cofinal if and only if for every $y \in \mathcal{C}$ the simplicial set $X \times_{\mathcal{C}} \mathcal{C}_{y/}$ is weakly contractible. Dually, q is coinitial if and only if for $X \times_{\mathcal{C}} \mathcal{C}_{/y}$ is weakly contractible.*

For reasons of scope we will not describe the proof of Theorem 2.6.13. We refer the interested reader to [5, §4.1.3].

2.7. **Kan extensions.** Let $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of ∞-categories and let $\mathcal{E}$ be a third ∞-category. Then we may associate with φ the restriction functor

$$\varphi^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E}).$$

In many cases the need arises to go the other way, namely, starting from a functor $\psi : \mathcal{C} \longrightarrow \mathcal{D}$, to *extend* it to a functor $\psi' : \mathcal{D} \longrightarrow \mathcal{E}$. To be more precise, by a *left* *extension* of ψ along φ we will mean a pair (ψ', τ) where $\psi' : \mathcal{D} \longrightarrow \mathcal{E}$ is a functor and $\tau : \psi \Rightarrow \varphi^*\psi'$ is a natural transformation (that is, an arrow in $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ from ψ to $\varphi^*\psi'$). The collection of such extensions can be organized into an ∞-category, which we call the ∞-category of left extensions of ψ. To see how this is done, consider the left mapping cone

$$\mathrm{Cone}^{\mathrm{L}}_{\varphi} := [\mathcal{C} \times \Delta^1] \coprod_{\mathcal{C}\times\Delta^{\{1\}}} \mathcal{D} \times \Delta^{\{1\}}.$$

Then the data of a triple (ψ, ψ', τ) as above is literally the same as the data of a map of simplicial sets $\mathrm{Cone}^{\mathrm{L}}_{\varphi} \longrightarrow \mathcal{E}$. Restriction along $\mathcal{C} \times \Delta^{\{0\}} \subseteq \mathrm{Cone}^{\mathrm{L}}_{\varphi}$ then determines a categorical fibration

$$\mathrm{Fun}(\mathrm{Cone}^{\mathrm{L}}_{\varphi}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \tag{2.35}$$

whose fiber over $\psi \in \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is the desired ∞-category of left extensions of ψ. Dually, we define a *right extension* of ψ along φ to be a pair (ψ', τ) where $\psi' : \mathcal{D} \longrightarrow \mathcal{E}$ is a functor and $\tau : \varphi^*\psi' \Rightarrow \psi$ is a natural transformation. These can be organized into an ∞-category by considering the right mapping cone

$$\mathrm{Cone}^{\mathrm{R}}_{\varphi} := [\mathcal{C} \times \Delta^1] \coprod_{\mathcal{C}\times\Delta^{\{0\}}} \mathcal{D} \times \Delta^{\{0\}},$$

in which case we can identify right extensions with objects in the fiber of the categorical fibration

$$\mathrm{Fun}(\mathrm{Cone}^{\mathrm{R}}_{\varphi}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \tag{2.36}$$

over ψ.

Definition 2.7.1. We will say that $\tau : \psi \Rightarrow \varphi^*\psi'$ *exhibits ψ' as a left Kan extension of ψ* if (ψ, ψ', τ) is initial in the fiber of (2.35) over ψ. Dually, we will say that $\tau : \varphi^*\psi' \Rightarrow \psi$ *exhibits ψ' as a right Kan extension of ψ* if (ψ, ψ', τ) is final in the fiber of (2.36) over ψ.

Remark 2.7.2. One can show that the restriction

$$\mathrm{Fun}(\mathrm{Cone}^{\mathrm{L}}_{\varphi}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is a *cartesian fibration*. It then follows from (the dual of) Exercise 2.6.10 that if every $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ admits a left Kan extension and we denote by $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Cone}^{\mathrm{L}}_{\varphi}, \mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Cone}^{\mathrm{L}}_{\varphi}, \mathcal{E})$ the full subcategory spanned by those triples (ψ, ψ', δ) which correspond to left Kan extensions of ψ along φ then the projection

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Cone}^{\mathrm{L}}_{\varphi}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is a trivial Kan fibration. More generally, if we do not assume that every ψ admits a left Kan extension but instead we take some full subcategory $\mathcal{X} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ such that every $\psi \in \mathcal{X}$ admits a left Kan extension then the projection

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Cone}^{\mathrm{L}}_{\varphi}, \mathcal{E}) \times_{\mathrm{Fun}(\mathcal{C},\mathcal{E})} \mathcal{X} \longrightarrow \mathcal{X}$$

is a trivial Kan fibration.

The main result that we will need about left and right Kan extensions is their relation to the notions of limits and colimits described in 2.6. We will describe this result without giving the proof, and refer the interested reader to [5, §4.3]. To phrase this result, we first note that the cones $\mathrm{Cone}^{\mathrm{L}}_{\varphi}$ and $\mathrm{Cone}^{\mathrm{R}}_{\varphi}$ are simplicial sets which are generally not ∞-categories. To fix this, let us define

$$\mathcal{M}^{\mathrm{L}}_{\varphi} \coloneqq \mathrm{N}_{[\varphi]}([1]) \longrightarrow \Delta^1$$

to be the cocartesian fibration obtained by applying the relative nerve construction of Definition 2.5.1 to the diagram $[\varphi] : [1] \longrightarrow \mathrm{Set}_{\Delta}$ determined by the arrow $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ in Set_{Δ}. We have inclusions

$$\mathcal{C} \hookrightarrow \mathcal{M}^{\mathrm{L}}_{\varphi} \hookleftarrow \mathcal{D}$$

identifying $\mathcal{C}$ with $\mathcal{M}^{\mathrm{L}}_{\varphi} \times_{\Delta^1} \Delta^{\{0\}}$ and $\mathcal{D}$ with $\mathcal{M}^{\mathrm{L}}_{\varphi} \times_{\Delta^1} \Delta^{\{1\}}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C} \times \Delta^{\{1\}} \arrow[r] \arrow[d] & \mathcal{C} \times \Delta^1 \arrow[d] \\
\mathcal{D} \times \Delta^{\{1\}} \arrow[r] & \mathcal{M}^{\mathrm{L}}_{\varphi}
\end{tikzcd} \tag{2.37}$$

where the right vertical map is constructed as in Remark 2.5.8.

Lemma 2.7.3. *The map*

$$\mathrm{Cone}^{\mathrm{L}}_{\varphi} \longrightarrow \mathcal{M}^{\mathrm{L}}_{\varphi}$$

determined by the square (2.37) *is a categorical equivalence.*

Remark 2.7.4. Since the map $\mathcal{C} \times \Delta^{\{1\}} \longrightarrow \mathcal{C} \times \Delta^1$ is a cofibration and the categorical model structure is left proper it follows that the pushout defining $\mathrm{Cone}^{\mathrm{L}}_{\varphi}$ is also a homotopy pushout. The claim of Lemma 2.7.3 is hence equivalent to the claim that the square (2.37) is a homotopy pushout square in the categorical model structure.

Proof of Lemma 2.7.3. Since the coherent nerve functor is a right Quillen equivalence there exists a map $\widetilde{\varphi} : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{D}}$ of fibrant simplicial categories such that the arrow $\mathrm{N}(\widetilde{\varphi}) : \mathrm{N}(\widetilde{\mathcal{C}}) \longrightarrow \mathrm{N}(\widetilde{\mathcal{D}})$ is levelwise categorically equivalent to $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$. In addition, we may choose $\widetilde{\varphi}$ to be a cofibration in Cat_{Δ}. By Remark 2.5.6 and Remark 2.7.4 we may then prove the claim for $\mathrm{N}(\widetilde{\varphi})$ instead of φ. By Proposition 2.5.9 the square (2.37) for $\mathrm{N}(\widetilde{\varphi})$ is the image under N of the square of simplicial categories

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \times \{1\} \arrow[r] \arrow[d, "\widetilde{\varphi}"] & \widetilde{\mathcal{C}} \times [1] \arrow[d] \\
\widetilde{\mathcal{D}} \times \{1\} \arrow[r] & \textstyle\int_{[1]}[\widetilde{\varphi}]
\end{tikzcd} \tag{2.38}$$

where we have denoted by $[\widetilde{\varphi}] : [1] \longrightarrow \mathrm{Cat}_{\Delta}$ the functor corresponding to the arrow $\widetilde{\varphi} : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{D}}$. Inspecting the square 2.38 we now see that it is an actual pushout square in Cat_{Δ}. Since $\widetilde{\varphi}$ was chosen to be a cofibration in Cat_{Δ} and Cat_{Δ} is left proper the square (2.38) is also a homotopy pushout square. The claim now follows from the fact that N is a right Quillen equivalence and hence preserves homotopy pushout squares consisting of fibrant objects. □

By Lemma 2.7.3 we have that $\mathrm{Fun}(\mathcal{M}^{\mathrm{L}}_{\varphi}, \mathcal{E}) \simeq \mathrm{Fun}(\mathrm{Cone}^{\mathrm{L}}_{\varphi}, \mathcal{E})$ is equivalent to the category of triples (ψ, ψ', τ) as above. We will then say that a functor $\overline{\psi} : \mathcal{M}^{\mathrm{L}}_{\varphi} \longrightarrow \mathcal{E}$ is a *left Kan extension* if in the corresponding triple τ exhibits ψ' as the left Kan extension of ψ along φ. In other words, $\overline{\psi}$ is a left Kan extension if it is initial in the fiber of $\mathrm{Fun}(\mathcal{M}^{\mathrm{L}}_{\varphi}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ above $\overline{\psi}|_{\mathcal{C}}$.

In the next theorem we will employ the following terminology. For each $y \in \mathcal{D}$, let us denote by $\mathcal{I}_y \coloneqq \mathcal{C} \times_{\mathcal{M}} \mathcal{M}_{/y}$. Inspecting the relative nerve construction we see that there is a natural *isomorphism* of simplicial sets $\mathcal{I}_y \cong \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/y}$. The natural map $\mathcal{I}_y \longrightarrow \mathcal{C}$ canonically extends to a map $\mathcal{I}_y^{\triangleright} \longrightarrow \mathcal{M}$ which sends the cone point to y. We then have the following key statement:

Theorem 2.7.5 ([5, §4.3.3])**.** *Let $\psi : \mathcal{E}$ be a functor and suppose that for every $y \in \mathcal{D}$ the composed functor $\mathcal{I}_y \longrightarrow \mathcal{C} \xrightarrow{\psi} \mathcal{E}$ admits a colimit in $\mathcal{E}$. Then*

(1) There exists a left Kan extension $\overline{\psi} : \mathcal{M}^{\mathrm{L}}_{\varphi} \longrightarrow \mathcal{E}$ such that $\overline{\psi}|_{\mathcal{C}} = \psi$.

(2) An arbitrary functor $\overline{\psi} : \mathcal{M}^{\mathrm{L}}_{\varphi} \longrightarrow \mathcal{E}$ with $\overline{\psi}|_{\mathcal{C}} = \psi$ is a left Kan extension if and only if for every $y \in \mathcal{D}$ the composed map $\mathcal{I}_y^{\triangleright} \longrightarrow \mathcal{M} \longrightarrow \mathcal{E}$ is a colimit cone.

Remark 2.7.6. It follows from Theorem 2.7.5 that if $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor and $\mathcal{D}_0$ is a full subcategory of $\mathcal{D}$ such that φ factors through $\varphi_0 : \mathcal{C} \longrightarrow \mathcal{D}_0$, then if $\delta : \psi \Rightarrow \varphi^*\psi'$ exhibits $\psi' : \mathcal{D} \longrightarrow \mathcal{E}$ as a left Kan extension of ψ along φ then δ also exhibits $\psi'|_{\mathcal{D}_0}$ as a left Kan extension of ψ along φ_0.

A useful property of left Kan extensions which we will need is the following, also known as the *pasting lemma* for left Kan extensions:

Proposition 2.7.7. *Let $\mathcal{B} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\rho} \mathcal{D}$ be two composable functors and $\mathcal{E}$ another ∞-category. Suppose we are given three functors*

$$\begin{tikzcd} \mathcal{B} \arrow[dr, "\psi_{\mathcal{B}}"'] & \mathcal{C} \arrow[d, "\psi_{\mathcal{C}}"] & \mathcal{D} \arrow[dl, "\psi_{\mathcal{D}}"] \\ & \mathcal{E} & \end{tikzcd}$$

and two natural transformations $\delta : \psi_{\mathcal{B}} \Rightarrow \varphi^*\psi_{\mathcal{C}}$ and $\delta' : \psi_{\mathcal{C}} \Rightarrow \rho^*\psi_{\mathcal{D}}$. Suppose that δ exhibits $\psi_{\mathcal{C}}$ as a left Kan extension of $\psi_{\mathcal{B}}$ along φ. Then δ' exhibits $\psi_{\mathcal{D}}$ as a left Kan extension of $\psi_{\mathcal{C}}$ along ρ if and only if the composed transformation $\psi_{\mathcal{B}} \Rightarrow \varphi^*\psi_{\mathcal{C}} \Rightarrow \varphi^*\rho^*\psi_{\mathcal{D}}$ exhibits $\psi_{\mathcal{D}}$ as a left Kan extension of $\psi_{\mathcal{B}}$ along $\rho \circ \varphi$.

A dual story exists for right Kan extensions by using the dual version of the relative nerve construction (see Remark 2.5.7) to define a cartesian fibration

$$\mathcal{M}^{\mathrm{R}}_{\varphi} = \mathrm{N}^{[\varphi]}([1]) \longrightarrow \Delta^1 \tag{2.39}$$

classified by the diagram $(\Delta^1)^{\mathrm{op}} \longrightarrow \mathrm{Cat}_{\infty}$ corresponding to $[\mathcal{D} \xleftarrow{\varphi} \mathcal{C}]$. We then have the dual statement of Theorem 2.7.5 phrased using limit cones instead of colimits cones.

3. Symmetric monoidal ∞-categories

3.1. **Introduction.** Recall that a *monoid* is set M together with an associative product $\cdot : M \times M \longrightarrow \mathcal{M}$ which admits a unit $1 \in M$. We say that M is commutative if $x \cdot y = y \cdot x \in M$ for every $x, y \in M$. In many natural situations we are faced with *categories* $\mathcal{C}$ which posses a similar structure, namely, we have a product operation $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ which obeys similar axioms. Alas, in the case of categories, we do not expect to have a product which is associative on the nose, but only up to a *natural* *isomorphism*. As a result, the definition of the notion of a monoidal product on a category becomes slightly more elaborate:

Definition 3.1.1. Let $\mathcal{C}$ be a category. A *monoidal structure* on $\mathcal{C}$ consists of

(1) a product functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$;
(2) an object $1 \in \mathcal{C}$, known as the *unit object*;
(3) a natural isomorphism

$$a_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (z \otimes y),$$

known as the *associator*;
(4) natural isomorphisms

$$\lambda_x : 1 \otimes x \xrightarrow{\cong} x \quad \text{and} \quad \rho_x : x \otimes 1 \xrightarrow{\cong} x,$$

known as the *left and right unitors*.

We then require that the associator obeys the *pentagon identity*, which says that the diagram
(3.1)

$$\begin{tikzcd}
 & & (w \otimes x) \otimes (y \otimes z) \arrow[drr, "a_{w,x,y\otimes z}"] & & \\
((w \otimes x) \otimes y) \otimes z \arrow[urr, "a_{w\otimes x,y,z}"] \arrow[dr, "a_{w,x,y}\otimes \mathrm{Id}"] & & & & w \otimes (x \otimes (y \otimes z)) \\
 & (w \otimes (x \otimes y)) \otimes z \arrow[rr, "a_{w,x\otimes y,z}"] & & w \otimes ((x \otimes y) \otimes z) \arrow[ur, "\mathrm{Id} \otimes a_{x,y,z}"'] &
\end{tikzcd}$$

commutes, and that the associator and the unitors satisfy the triangle identity, which says that the diagram

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "a_{x,1,y}"] \arrow[dr, "\rho_x \otimes \mathrm{Id}"'] & & x \otimes (1 \otimes y) \arrow[dl, "\mathrm{Id} \otimes \lambda_y"] \\
 & x \otimes y &
\end{tikzcd}$$

commutes.

The situation becomes even more elaborate if we want a monoidal structure which is *commutative*, or as it is usally called in this context, symmetric.

Definition 3.1.2. Let $\mathcal{C}$ be a category. A *symmetric monoidal structure* on $\mathcal{C}$ is a monoidal structure $(\otimes, 1, a_{x,y,z}, \lambda_x, \rho_x)$ together with a natural isomorphism

$B_{x,y} : x \otimes y \xrightarrow{\cong} y \otimes x$ such that $B_{x,y} \circ B_{y,x} = \mathrm{Id}_{x \otimes y}$ and such that the diagram

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{x,y,z}"] \arrow[d, "B_{x,y} \otimes \mathrm{Id}"'] & x \otimes (y \otimes z) \arrow[r, "B_{x,y \otimes z}"] & (y \otimes z) \otimes x \arrow[d, "a_{y,z,x}"] \\
(y \otimes x) \otimes z \arrow[r, "a_{y,x,z}"] & y \otimes (x \otimes z) \arrow[r, "\mathrm{Id} \otimes B_{x,z}"] & y \otimes (z \otimes x)
\end{tikzcd}$$

commutes.

Examples of symmetric monoidal categories in nature are abundant: sets with cartesian product (more generally, any category in which cartesian products exist), sets with disjoint union (more generally, any category in which cocartesian products exist), abelian groups with tensor product etc. Similarly, in higher category theory we will often wish to consider ∞-categories equipped with a symmetric monoidal structure, such as spaces with cartesian products, or chain-complexes with (derived) tensor products. To make higher category theory useful in such situations we need to be able to generalize Definition 3.1.2 to the realm of ∞-categories. This is not a trivial task: indeed, the fact that in categories we had to replace the associativity, unitality and symmetry axioms by their analogue "up to a natural isomorphism" (and then require suitable coherence conditions) is just the tip of the iceberg of what we can expect if we replace $\mathcal{C}$ with an ∞-category. For example, the pentagon axiom (3.1) should now be replaced with a homotopy making the diagram commute, and this homotopy should be part of the structure, thus prompting higher coherences and so on. This approach becomes unfeasible at some point (especially when the symmetric structure is introduced). Instead, we need to find a way to encode all these coherences implicitly, similar to the way in which cocartesian fibrations encode the coherence conditions of diagrams. In order to do this it will be convenient to introduce the category of *finite pointed sets*. For $n \geq 0$ we will denote by $\langle n \rangle$ the object corresponding to the finite pointed set $\{0, ..., n\}$ with 0 as its base point. We will denote by Fin_* the skeleton of the category of finite pointed sets spanned by the objects $\langle n \rangle$ for $n \geq 0$. We will denote by $\langle n \rangle^\circ := \langle n \rangle \smallsetminus \{0\} = \{1, ..., n\}$ the complement of the base point in $\langle n \rangle$.

Construction 3.1.3. Let $\mathcal{C}$ be a symmetric monoidal category. We construct a category $\mathcal{C}^\otimes$ whose

- objects are pairs $(n, (x_1, ..., x_n))$ consisting of a non-negative integer n and a family of objects in $\mathcal{C}$ parameterized by $\langle n \rangle^\circ$.
- maps from $(n, (x_1, ..., x_n))$ to $(m, (y_1, ..., y_m))$ are given by maps $\alpha : \langle n \rangle \longrightarrow \langle m \rangle$ in Fin_*, together with, for every $j = 1, ..., m$, a map in $\mathcal{C}$ of the form

$$f_j : \bigotimes_{i \in \alpha^{-1}(j)} x_i \longrightarrow y_j.$$

We will denote by $\pi : \mathcal{C}^\otimes \longrightarrow \mathrm{Fin}_*$ the natural projection $(n, (x_1, ..., x_n)) \mapsto \langle n \rangle$.

We now claim that, similarly to the case of the Grothendieck construction described in §2.4, the entire symmetric monoidal structure on $\mathcal{C}$ can be completely reconstructed from $\mathcal{C}^\otimes$ together with the functors $\pi : \mathcal{C}^\otimes \longrightarrow \mathrm{Fin}_*$ and the identification of the fiber $\mathcal{C}^\otimes_{\langle 1 \rangle}$ with $\mathcal{C}$. To see this, we first observe the following:

Lemma 3.1.4. *An edge $(\alpha, \{f_j\}) : (n, (x_1, ..., x_n)) \longrightarrow (m, (y_1, ..., y_m))$ in $\mathcal{C}^\otimes$ is π-cocartesian if and only if each $f_j : \otimes_{i \in \alpha^{-1}(j)} x_i \longrightarrow y_j$ is an isomorphism. In particular, π is a cocartesian fibration.*

Exercise 3.1.5. Prove Lemma 3.1.4.

By the Grothendieck correspondence (Theorem 2.4.7) we may associate with $\pi : \mathcal{C}^{\otimes} \longrightarrow \mathrm{Fin}_*$ a pseudofunctor $\mathrm{Fin}_* \longrightarrow \mathrm{Cat}$, which sends a finite pointed set $\langle n\rangle$ to the category $\mathcal{C}^n$ and a map of finite pointed sets $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ to the functor $\alpha_! : \mathcal{C}^n \longrightarrow \mathcal{C}^m$ given by $\alpha_!(x_1, ..., x_n) = (y_1, ..., y_m)$ with $y_j = \otimes_{i\in\alpha^{-1}(j)} x_i$. In particular, the monoidal product itself is encoded in the functor $\alpha_! : \mathcal{C}^2 \longrightarrow \mathcal{C}$ associated to the map $\alpha : \langle 2\rangle \longrightarrow \langle 1\rangle$ given by $\alpha(0) = 0, \alpha(1) = \alpha(2) = 1$, and the associator can be reconstructed from the natural isomorphisms $\alpha_! \circ (\beta_1)_! \cong \gamma_! \cong (\beta_2)_! \circ \alpha_!$ associated with the commutative square of finite pointed sets

$$\begin{tikzcd}
\langle 3\rangle \arrow[r, "\beta_1"] \arrow[d, "\beta_2"'] \arrow[dr, "\gamma"] & \langle 2\rangle \arrow[d, "\alpha"] \\
\langle 2\rangle \arrow[r, "\alpha"] & \langle 1\rangle
\end{tikzcd}$$

where for $j = 1, 2$ we denote by $\beta_j : \langle 3\rangle \longrightarrow \langle 2\rangle$ is the unique pointed surjective map such that $\beta_j^{-1}(j) = \{j, j + 1\}$.

We may consider the association $\mathcal{C} \mapsto \mathcal{C}^{\otimes}$ as an operation which transforms a symmetric monoidal category to a cocartesian fibrations over Fin_*. However, not every cocartesian fibration over Fin_* can be obtained in this way: indeed, the fiber $\mathcal{C}^{\otimes}_{\langle n\rangle}$ over $\langle n\rangle$ (equivalently, the category associated to $\langle n\rangle$ by the corresponding pseudofunctor) is always the n-fold cartesian product of the fiber $\mathcal{C}^{\otimes}_{\langle 1\rangle}$. To phrase this condition more precisely let us introduce some terminology.

Definition 3.1.6. We will say that a morphism $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ is *active* if $\alpha^{-1}(0) = \{0\}$ and that it is *inert* if $\alpha^{-1}(j) \subseteq \langle n\rangle$ contains exactly one element for every $j \in \{1, ...n\}$.

Remark 3.1.7. Every morphism in Fin_* can be factored as an inert morphism followed by an active morphism in an essentially unique way. In addition, this factorization is initial in the category of factorizations whose second map is active and terminal in the category of factorizations whose first map is inert.

Notation 3.1.8. For an $\langle n\rangle \in \mathrm{Fin}_*$ and an $i = 1, ..., n$ let us denote by $\rho^i : \langle n\rangle \longrightarrow \langle 1\rangle$ the inert morphism such that $(\rho^i)^{-1}(1) = \{i\}$.

We then have the following folk theorem:

Theorem 3.1.9. *The association $\mathcal{C} \mapsto \mathcal{C}^{\otimes}$ induces an equivalence between the 2-category of symmetric monoidal categories, symmetric monoidal functors and symmetric monoidal natural transformations and the full sub-2-category of* $\mathrm{Fun}^{\mathrm{coc}}(\mathrm{Fin}_*)$ *spanned by the cocartesian fibrations $\mathcal{D} \longrightarrow \mathrm{Fin}_*$ with the following property: for every $\langle n\rangle \in \mathrm{Fin}_*$ the functor*

$$(\rho^1_!, ..., \rho^n_!) : \mathcal{D}_{\langle n\rangle} \longrightarrow \prod_{i=1}^{n} \mathcal{D}_{\langle 1\rangle}$$

determined by the maps $\rho^1, ..., \rho^n : \langle n\rangle \longrightarrow \langle 1\rangle$ is an equivalence of categories.

Remark 3.1.10. We have not defined terms “symmetric monoidal functor” appearing in Theorem 3.1.9. It consists of a functor $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ between symmetric monoidal categories together with a specified isomorphism $1_{\mathcal{D}} \xrightarrow{\cong} \mathcal{F}(1_{\mathcal{C}})$ and a specified natural isomorphism $\mu_{x,y} : \mathcal{F}(x) \otimes \mathcal{F}(y) \longrightarrow \mathcal{F}(x \otimes y)$ which satisfy several compatibility conditions with respect to the associators, unitors and symmetry isomorphisms of $\mathcal{C}$ and $\mathcal{D}$. Theorem 3.1.9 tells us that this data is essentially the same as the data of a functor $\mathcal{C}^{\otimes} \longrightarrow \mathcal{D}^{\otimes}$ which preserves cocartesian edges, a notion which is considerably simpler in comparison. This reflects the fact that the "fibration" picture encodes homotopy coherences in a much more efficient manner than the "algebraic" picture. When passing to symmetric monoidal ∞-categories this improvement makes a critical difference.

Theorem 3.1.9 suggests the following way to define the notion of a symmetric monoidal ∞-category:

Definition 3.1.11. A *symmetric monoidal ∞-category* is a cocartesian fibration $\pi : \mathcal{C}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ such that for every n the map

$$(\rho^1_!, ..., \rho^n_!) : \mathcal{C}^{\otimes}_{\langle n \rangle} \longrightarrow \prod_{i=1}^{n} \mathcal{C}^{\otimes}_{\langle 1 \rangle}$$

determined by the transition functors of $\rho^1, ..., \rho^n : \langle n \rangle \longrightarrow \langle 1 \rangle$ (see Construction 2.4.19) is an equivalence of ∞-categories. We will denote by $\mathcal{C} := \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ the fiber over 1, and will refer to it as the *underlying ∞-category of* $\mathcal{C}^{\otimes}$. In this situation we will also say that $\pi : \mathcal{C}^{\otimes} \longrightarrow \mathrm{Fin}_*$ *exhibits $\mathcal{C}$ as a symmetric monoidal ∞-category.*

If $\pi : \mathcal{C}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ and $\pi' : \mathcal{D}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ are two symmetric monoidal ∞-categories then we define a *symmetric monoidal functor* from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ to be a functor $\mathcal{C}^{\otimes} \longrightarrow \mathcal{D}^{\otimes}$ over $\mathrm{N}(\mathrm{Fin}_*)$ which sends π-cocartesian edges to π'-cocartesian edges.

Remark 3.1.12. The notation $\mathcal{C}^{\otimes}$ in Definition 3.1.11 is merely suggestive in light of the discussion in §3.1, we do not mean that $\mathcal{C}^{\otimes}$ is obtained by applying some construction to $\mathcal{C}$.

Remark 3.1.13. Let $\mathcal{C}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1 \rangle}$. Using Construction 2.4.19 we see that the active morphisms $\langle 0 \rangle \longrightarrow \langle 1 \rangle$ and $\langle 2 \rangle \longrightarrow \langle 1 \rangle$ determine functors

$$\Delta^0 \longrightarrow \mathcal{C} \quad \text{and} \quad \otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

which are well-defined up to a contractible space of choices. The first of these functors determines an object of $\mathcal{C}$ which we will denote by 1 and refer to as the *unit object* of $\mathcal{C}$. One can then check that the unit object 1 and the product $\otimes$ satisfy the axioms of a symmetric monoidal category up to homotopy. In particular, the homotopy category $\mathrm{Ho}(\mathcal{C})$ inherits a canonical symmetric monoidal structure.

3.2. **Examples and constructions.**

In this section we will review some examples and constructions of symmetric monoidal ∞-categories. We first note the following immediate example:

Example 3.2.1. If $\mathcal{C}$ is an ordinary category then the nerve of the category $\mathcal{C}^{\otimes}$ constructed in §3.1 is a symmetric monoidal ∞-category.

The following proposition provides a large source of symmetric monoidal ∞-categories:

Proposition 3.2.2 ([6, Proposition 4.1.7.4]). *Let $\mathcal{C}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ and let W be a collection of morphisms in $\mathcal{C}$ which contains all equivalences and is closed under 2-out-of-3. Suppose that for every $f : x \longrightarrow y, f' : x' \longrightarrow y'$ in W the map $f \otimes f' : x \otimes x' \longrightarrow y \otimes y'$ is again in W. Then there exists a symmetric monoidal ∞-category $\mathcal{D}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ and a symmetric monoidal functor $f : \mathcal{C}^{\otimes} \longrightarrow \mathcal{D}^{\otimes}$ such that*

(1) for every symmetric monoidal ∞-category $\mathcal{E}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ the induced map

$$\mathrm{Fun}^{\otimes}(\mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}) \longrightarrow \mathrm{Fun}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{E}^{\otimes})$$

is fully-faithful and its essential image consists of those symmetric monoidal functors $\mathcal{C}^{\otimes} \longrightarrow \mathcal{E}^{\otimes}$ such that the underlying functor $\mathcal{C} \longrightarrow \mathcal{E}$ sends W to invertible edges;

(2) the induced functor $f_{\langle 1 \rangle} : \mathcal{C} \longrightarrow \mathcal{D}$ on underlying ∞-categories exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ with respect to W.

Remark 3.2.3. In the situation of Proposition 3.2.2, the existence of $\mathcal{D}$ such that (2) holds essentially follows from Remark 2.2.18, which says that the association $(\mathcal{C}, W) \mapsto \mathcal{C}[W^{-1}]$ can be made functorial and products preserving. Indeed, by the Lurie-Grothendieck correspondence the cocartesian fibration $\mathcal{C}^{\otimes} \longrightarrow \mathrm{Fin}_*$ corresponds to a functor $\chi : \mathrm{Fin}_* \longrightarrow \mathrm{Cat}_{\infty}$ given by the formula $\langle n \rangle \mapsto \mathcal{C}^n$. The hypothesis in Proposition 3.2.2 implies that χ can be considered as a diagram of *relative ∞-categories*, that is, a diagram in the ∞-category of ∞-categories equipped with a collection of arrows. Performing localization levelwise we obtain a functor $\chi' : \mathrm{Fun}_* \longrightarrow \mathrm{Cat}_{\infty}$ which by the Lurie-Grothendieck correspondence can be encoded again as a cocartesian fibration $\mathcal{D}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$. Since localization preserves products $\mathcal{D}^{\otimes}$ is a symmetric monoidal ∞-category, which satisfies (2) by construction. The main non-trivial point of Proposition 3.2.2 is that $\mathcal{D}^{\otimes}$ will actually also satisfy (1).

A prominent source of examples of Proposition 3.2.2 comes from *symmetric monoidal model categories*. These are model categories equipped with a closed symmetric monoidal structure (that is, the tensor product has left adjoints in each variable separately) which is compatible with the model structure in following sense:

(1) For every pairs of cofibrations $f : A \longrightarrow B, g : X \longrightarrow Y$ the pushout-product

$$A \otimes Y \coprod_{A \otimes X} B \otimes X \longrightarrow B \otimes Y$$

is a cofibration, which is furthermore a trivial cofibration if either f or g is trivial.

(2) The unit $1 \in \mathcal{M}$ is cofibrant.

We note that in a symmetric monoidal model category the tensor product $\otimes$ does not necessarily preserve weak equivalence in each variable separately, and so $(\mathcal{M}, W)$ will not usually satisfy the hypothesis of Proposition 3.2.2. However, the axioms above do imply that the full subcategory $\mathcal{M}^c \subseteq \mathcal{M}$ of cofibrant objects is closed under tensor products (and contains the unit) and hence inherits a symmetric monoidal structure. Furthermore, the pushout-product axiom (1) implies that tensoring with a cofibrant object $X \otimes (-) : \mathcal{M} \longrightarrow \mathcal{M}$ is a left Quillen functor and hence preserves weak equivalences between cofibrant objects. This means that $(\mathcal{M}^c, W^c)$ satisfies the hypothesis of Proposition 3.2.2 and so the ∞-category $\mathcal{M}^c[(W^c)^{-1}]$ inherits a canonical symmetric monoidal structure (here we denotes by W^c the collection of weak equivalences between cofibrant objects). Fortunately, it follows from the general machinery of model categories that the natural map

$$\mathcal{M}^c[(W^c)^{-1}] \longrightarrow \mathcal{M}[W^{-1}] = \mathcal{M}_\infty$$

induced by the inclusion $\mathcal{M}^c \subseteq \mathcal{M}$ is an equivalence of ∞-categories. We may hence conclude that if $\mathcal{M}$ is a symmetric monoidal model category then the underlying ∞-category $\mathcal{M}_\infty$ inherits a canonical symmetric monoidal structure.

Example 3.2.4. The Kan-Quillen model structure and the categorical model structure are both symmetric monoidal with respect to cartesian product. It then follows that $\mathcal{S}$ and Cat_∞ inherit canonical symmetric monoidal structures (see Remark 2.2.26). These symmetric monoidal structures are the corresponding *cartesian* symmetric monoidal structures, see §3.3.

Example 3.2.5. Let R be a ring. Then the category $\mathrm{Ch}(R)$ of unbounded chain-complexes over R can be endowed with the *projective* model structure in which the weak equivalences are the quasi-isomorphisms and the fibrations are the levelwise surjective maps. Furthermore, this model structure is compatible with tensor products of chain-complexes, and so $\mathrm{Ch}(R)$ is a symmetric monoidal model category. The underlying ∞-category $\mathrm{Ch}(R)_\infty$ then inherits a canonical symmetric monoidal structure.

Another plentiful source of examples comes from *symmetric monoidal simplicial categories*:

Definition 3.2.6. A symmetric monoidal simplicial category is a simplicial category $\mathcal{C}$ equipped with the same type of structure $(\otimes, 1, a, \lambda, \mu)$ as in the definition of an ordinary symmetric monoidal category (see Definition 3.1.2) only that $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a simplicial functor and a, λ and μ are simplicial natural transformations (see Remark 2.1.2).

Construction 3.2.7. Let $\mathcal{C}$ be a symmetric monoidal simplicial category. We construct a simplicial category $\mathcal{C}^\otimes$ whose

- objects are pairs $(n, (x_1, ..., x_n))$ consisting of a non-negative integer n and a family of objects in $\mathcal{C}$ parameterized by $\langle n \rangle^\circ$;
- the simplicial set of maps from $(n, (x_1, ..., x_n))$ to $(m, (y_1, ..., y_m))$ is given by

$$\coprod_{\alpha : \langle n \rangle \to \langle m \rangle} \prod_{j=1}^{n} \mathrm{Map}_{\mathcal{C}}(\otimes_{i \in \alpha^{-1}(j)} x_i, y_j),$$

  where α ranges over all maps from $\langle n \rangle$ to $\langle m \rangle$ in Fin_*.

We will denote by $\pi : \mathcal{C}^\otimes \longrightarrow \mathrm{Fin}_*$ the natural projection $(n, (x_1, ..., x_n)) \mapsto \langle n \rangle$, which is a simplicial functor (where we consider Fin_* as a simplicial category whose mapping objects are discrete simplicial sets).

Proposition 3.2.8. *Let $\mathcal{C}$ be a locally Kan symmetric monoidal simplicial category. Then the map*

$$\mathrm{N}(\mathcal{C}^\otimes) \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$$

is a cocartesian fibration which exhibits $\mathrm{N}(\mathcal{C})$ as a symmetric monoidal ∞-category.

Example 3.2.9. The simplicial categories $\mathcal{K}\mathrm{an}$ and QC of ∞-groupoids and ∞-categories respectively satisfy the assumptions of Proposition 3.2.8 with respect to cartesian products. We hence obtain an induced symmetric monoidal structures on $\mathcal{S} = \mathrm{N}(\mathcal{K}\mathrm{an})$ and $\mathrm{Cat}_\infty = \mathrm{N}(\mathrm{QC})$. These structures agree with those of Example 3.2.4 (see [6, Corollary 4.1.7.16]) and both coincide with the respective cartesian symmetric monoidal structures (see §3.3).

A large supply of symmetric monoidal simplicial categories comes from *symmetric monoidal simplicial model categories*. These are the symmetric monoidal model categories $\mathcal{M}$ which admit a symmetric monoidal left Quillen functor $\iota : \mathrm{Set}_\Delta \longrightarrow \mathcal{M}$, in which case they automatically acquire the structure of a simplicial model category (see Definition 2.2.21). In this case the simplicial enrichment associated to the simplicial structure will endow $\mathcal{M}$ with the structure of a (symmetric monoida) simplicial category. However, the resulting simplicial category will not be locally Kan in general, and so we cannot apply Proposition 3.2.8 directly. On the other hand, we have the full simplicial subcategory $\mathcal{M}^\circ \subseteq \mathcal{M}$ spanned by fibrant-cofibrant objects, which is locally Kan and satisfies further that $\mathrm{N}(\mathcal{M}^\circ) \simeq \mathcal{M}_\infty$. Unfortunately, $\mathcal{M}^\circ$ will generally not be closed under $\otimes$: the tensor product of two fibrant-cofibrant objects will generally not be fibrant anymore. The solution to this problem is however not complicated:

Proposition 3.2.10. *Let $\mathcal{M}$ be a simplicial symmetric monoidal model category, and let $\mathcal{M}^{\otimes,\circ} \subseteq \mathcal{M}^\otimes$ denote the full subcategory spanned by those $(n,(x_1,...,x_n))$ such that $x_1,...,x_n \in \mathcal{M}^\circ$. Then the map*

$$\mathrm{N}(\mathcal{M}^{\otimes,\circ}) \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$$

is a cocartesian fibration which exhibits $\mathrm{N}(\mathcal{M}^\circ)$ as a symmetric monoidal ∞-category.

Remark 3.2.11. If $\mathcal{M}$ is a simplicial symmetric monoidal model category then $\mathrm{N}(\mathcal{M}^\circ) \simeq \mathcal{M}_\infty \simeq \mathcal{M}^c[(W^c)^{-1}]$ can be endowed with a symmetric monoidal structure by either Proposition 3.2.2 or by Proposition 3.2.10. These two symmetric monoidal structure are however equivalent to each other, see [6, Corollary 4.1.7.16].

3.3. **cartesian and cocartesian symmetric monoidal structures.** Let $\mathcal{C}$ be an ∞-category which admits *finite coproducts*, that is colimits for diagrams indexed by finite sets. Then we can expect that $\mathcal{C}$ can be endowed with a symmetric monoidal structure in which the monoidal product is given by the categorical coproduct. Similarly, if $\mathcal{C}$ admits *finite products* then we can expect to have a symmetric monoidal structure with the operation of cartesian products. In this section we will see how to express these symmetric monoidal structures explicitly in the formalism of symmetric monoidal ∞-categories.

We begin with the case of coproducts. Define a category Γ^* as follows:

- The objects of Γ^* are pairs $(\langle n\rangle, i)$ where $\langle n\rangle \in \mathrm{Fin}_*$ and $i \in \langle n\rangle^\circ$.
- A morphism in Γ^* from $(\langle n\rangle, i)$ to $(\langle m\rangle, j)$ is a map of pointed sets $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ such that $\alpha(i) = j$.

The category Γ^* admits an obvious forgetful functor $\Gamma^* \longrightarrow \mathrm{Fin}_*$ sending $(\langle n\rangle, i)$ to $\langle n\rangle$.

Definition 3.3.1. Let $\mathcal{C}$ be an ∞-category. We define $\mathcal{C}^\amalg$ to be the simplicial set whose n-simplices are pairs (σ,ρ) where $\sigma : \Delta^n \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ is an n-simplex of $\mathrm{N}(\mathrm{Fin}_*)$ and $\rho : \Delta^n \times_{\mathrm{N}(\mathrm{Fin}_*)} \mathrm{N}(\Gamma^*) \longrightarrow \mathcal{C}$ is a map of simplicial sets, where the fiber product is taken with respect to σ. By construction the simplicial set $\mathcal{C}^\amalg$ comes equipped with a map $\mathcal{C}^\amalg \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ sending (σ,ρ) to σ.

We note that the vertices of $\mathcal{C}^{\amalg}$ can be identified with pairs $(\langle n\rangle, \overline{x})$ where $\langle n\rangle \in \mathrm{Fin}_*$ and $\overline{x} : \{\langle n\rangle\} \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^* \cong \langle n\rangle^{\circ} \longrightarrow \mathcal{C}$ is a map, which we can identify with a tuple $\overline{x} = (x_1, ..., x_n)$ of objects in $\mathcal{C}$. Similarly, an arrow of $\mathcal{C}^{\amalg}$ consists of a tuple $(\alpha, \overline{f})$ where $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ is an arrow in $\mathrm{N}(\mathrm{Fin}_*)$ and

$$\overline{f} : \Delta^1 \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^* = \mathrm{N}(\mathcal{I}_{\alpha}) \longrightarrow \mathcal{C} \tag{3.2}$$

is a map, a data which we may identify with s triple $(\overline{x}, \overline{y}, \{f_i\})$ where $\overline{x} = (x_1, ..., x_n)$ is an n-tuple of objects in $\mathcal{C}$, $\overline{y} = (y_1, ..., y_m)$ is an m-tuple of objects in $\mathcal{C}$ and $f_i : x_i \longrightarrow y_{\alpha(i)}$ is a map for each $i \in \alpha^{-1}(\langle m\rangle^{\circ}) \subseteq \langle n\rangle^{\circ}$.

Exercise 3.3.2. Suppose that $\mathcal{C}$ is a locally Kan simplicial category. Define a simplicial category $\mathcal{C}^{\amalg}$ as follows: the objects of $\mathcal{C}^{\amalg}$ are pairs $(n, (x_1, ..., x_n))$ as in Construction 3.2.7 and the simplicial mapping sets are given by

$$\mathrm{Map}((n, (x_1, ..., x_n)), (m, (y_1, ..., y_m)) = \coprod_{\alpha:\langle n\rangle \to \langle m\rangle} \prod_{j=1}^{m} \prod_{i \in \alpha^{-1}(j)} \mathrm{Map}_{\mathcal{C}}(x_i, y_j),$$

where α ranges over all maps from $\langle n\rangle$ to $\langle m\rangle$ in Fin_*. Show that there is a canonical isomorphism of simplicial sets

$$\mathrm{N}(\mathcal{C}^{\amalg}) \cong \mathrm{N}(\mathcal{C})^{\amalg}$$

where the left hand side is constructed as in Definition 3.3.1.

Proposition 3.3.3 ([6, Proposition 2.4.3.3])**.** *Let $\mathcal{C}$ be an ∞-category. Then $\pi : \mathcal{C}^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ is an inner fibration and an arrow*

$$(\alpha, \{f_i\}) : (\langle n\rangle, (x_1, ..., x_n)) \longrightarrow (\langle m\rangle, (y_1, ..., y_m))$$

as above is π-cocartesian if and only if for every $j = 1, ..., m$ the collection of maps $f_i : x_i \longrightarrow y_j$ for $i \in \alpha^{-1}(j)$ exhibit y_j as the coproduct of $\{x_i\}_{i \in \alpha^{-1}(j)}$. In addition, if $\mathcal{C}$ admits finite coproducts then π is a symmetric monoidal structure on $\mathcal{C}$, which we call the cocartesian symmetric monoidal structure*.*

Proof. Let us show that π is an inner fibration. Consider a lifting problem of the form

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\rho"] \arrow[d] & \mathcal{C}^{\amalg} \arrow[d] \\ \Delta^n \arrow[r, "\sigma"] \arrow[ur, dashed] & \mathrm{N}(\mathrm{Fin}_*) \end{tikzcd} \tag{3.3}$$

With $0 < i < n$. The n-simplex σ determines a sequence of composable arrows

$$\langle k_0\rangle \xrightarrow{\alpha_0} \langle k_1\rangle \xrightarrow{\alpha_1} ... \xrightarrow{\alpha_{n-1}} \langle k_n\rangle$$

in Fin_*. For $j, j' \in \{0, ..., n\}$ let us denote $\alpha_{j,j'} : \langle k_j\rangle \longrightarrow \langle k_{j'}\rangle$ the composition of $\alpha_j, ..., \alpha_{j'-1}$. By the construction of $\mathcal{C}^{\amalg}$ this lifting problem is equivalent to an extension problem of the form

$$\begin{tikzcd} \Lambda^n_i \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^* \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^* \arrow[ur, dashed] & \end{tikzcd} \tag{3.4}$$

We note that $\Delta^n \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^*$ can be identified with the nerve of a poset $\mathcal{I}_\sigma$ whose objects are pairs (j,a) with $j \in \{0,...,n\}$ and $a \in \langle k_j \rangle^\circ$ and such that $(j,a) \le (j',a')$ if and only if $j \le j'$ and $\alpha_{j,j'}(a) = a'$. We then observe that m-simplices $\Delta^m \longrightarrow \mathrm{N}(\mathcal{I}_\sigma)$ are in bijections with pairs (τ, a) where $\tau : \Delta^m \longrightarrow \Delta^n$ is an m-simplex in Δ^n and $a \in \langle k_{\tau(0)} \rangle^\circ$ is such that $\alpha_{\tau(0),\tau(m)}(a) \in \langle k_{\tau(m)} \rangle^\circ$. Furthermore, such a simplex is non-degenerate if and only if τ is a non-degenerate m-simplex of Δ^n.

Let $A \subseteq \langle k_0 \rangle^\circ$ be the subset containing those elements $a \in \langle k_0 \rangle$ such that $\alpha_{0,n}(a) \in \langle k_n \rangle^\circ$. Using this explcit description we see that the non-degenerate simplices of $\mathrm{N}(\mathcal{I}_\sigma)$ that are not in $\Lambda^n_i \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^*$ are either of the form $(\Delta^{\{0,...,n\}}, a)$ for $a \in A$ or of the form $(\Delta^{\{0,...,\hat{i},...,n\}}, a)$ for $a \in A$, for a fixed a the latter simplex is the i'th face of the former, while all the other faces of the former are contained in $\Lambda^n_i \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^*$. It follows that the simplicial set $\mathrm{N}(\mathcal{I}_n)$ is obtained from $\Lambda^n_i \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^*$ by performing a pushout along $\Lambda^i_n \subseteq \Delta^n$ for each $a \in A$. Since $\mathcal{C}$ is an ∞-category it follows that the dotted extension exists in (3.4), and so the dotted lift exists in (3.3). We may hence conclude that $\mathcal{C}^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ is an inner fibration.

Let us now consider a lifting problem as in (3.3) with $i = 0$ and such that ρ sends the edge $\Delta^{\{0,1\}}$ to an arrow given by $(\alpha, \overline{f}) : (\langle n \rangle, \overline{x}) \longrightarrow (\langle m \rangle, \overline{y})$ as above, where $\overline{x} : \langle n \rangle^\circ \longrightarrow \mathcal{C}$ can be identified with a tuple $(x_1,...,x_n)$, $\overline{y} : \langle m \rangle^\circ \longrightarrow \mathcal{C}$ can be idetified with a tuple $(y_1,...,y_m)$ and

$$\overline{f} : \Delta^1 \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^* = \mathrm{N}(\mathcal{I}_\alpha) \longrightarrow \mathcal{C} \tag{3.5}$$

can be identified with a collection of maps $f_i : x_i \longrightarrow y_{\alpha(i)}$ for each $i \in \alpha^{-1}(\langle m \rangle^\circ) \subseteq \langle n \rangle^\circ$. Such a lifting problem is then equivalent to an extension problem as in (3.4) with $i = 0$. Let $B \subseteq \langle k_1 \rangle^\circ$ be the subset containing those elements $b \in \langle k_1 \rangle$ such that $\alpha_{1,n}(b) \in \langle k_n \rangle^\circ$. The non-degenerate simplices of $\mathrm{N}(\mathcal{I}_\sigma)$ that are not in $\Lambda^n_0 \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^*$ are either of the form $(\Delta^{\{0,...,n\}}, a)$ for $a \in A$ or of the form $(\Delta^{\{1,...,n\}}, b)$ for $b \in B$. In addition, all the faces of $(\Delta^{\{0,...,n\}}, a)$ are contained in $\Lambda^n_0 \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^*$ except the 0'th face which is $(\Delta^{\{1,...,n\}}, \alpha_0(a))$. We then see that $\mathrm{N}(\mathcal{I}_\sigma)$ is obtained from $\Lambda^n_0 \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^*$ by performing, for each $b \in B$, a pushout along a map of the form

$$S_b \star \partial\Delta^{\{1,...,n\}} \longrightarrow S_b \star \Delta^{\{1,...,n\}} \tag{3.6}$$

where $S_b \subseteq A$ is the preimage of b. Hence to solve the extension problem (3.4) we need to solve a series of extension problems of the form

$$\begin{tikzcd} S_b \star \partial\Delta^{\{1,...,n\}} \arrow[r] \arrow[d] & \mathcal{C} \\ S_b \star \Delta^{\{1,...,n\}} \arrow[ur, dashed] & \end{tikzcd} \tag{3.7}$$

Let $\overline{x}_b : S_b \longrightarrow \mathcal{C}$ be the restriction of $\overline{x} : \langle n \rangle \longrightarrow \mathcal{C}$. The extension problem 3.7 is equivalent to the extension problem

$$\begin{tikzcd} \partial\Delta^{\{1,...,n\}} \arrow[r] \arrow[d] & \mathcal{C}_{\overline{x}_b/} \\ \Delta^{\{1,...,n\}} \arrow[ur, dashed] & \end{tikzcd} \tag{3.8}$$

Let $\overline{f}_b : S_b \star \Delta^{\{1\}} \longrightarrow \mathcal{C}$ be the restriction of (3.2) to $S_b \star \Delta^{\{1\}} \subseteq \Delta^1 \times_{\mathrm{N}(\mathrm{Fin}_*)} \Gamma^*$. By Corollary 2.6.9 we see that the extension problem 3.8 is solvable as soon as $\overline{f}_b$ is initial in $\mathcal{C}_{\overline{x}_b}$, that is, as soon as $\overline{f}_b$ is a colimit diagram. We may thus conclude that the edge $(\overline{x}, \overline{y}, \{f_i\})$ is π-cocartesian if (and in fact only if) for each $b \in \langle m \rangle^{\circ}$ the collection of maps $f_a : x_a \longrightarrow y_b$ for $a \in S_b$ exhibit y_b is the coproduct of $\{x_a\}_{a \in S_b}$.

Now assume that $\mathcal{C}$ has coproducts. In this case we have by the above that for every $(\langle n \rangle, (x_1, ..., x_n))$ in $\mathcal{C}^{\amalg}$ and every $\alpha : \langle n \rangle \longrightarrow \langle m \rangle$ there exists a cocartesian edge $(\langle n \rangle, (x_1, ..., x_n)) \longrightarrow (\langle m \rangle, (y_1, ..., y_m))$ lying above α, and so π is a cocartesian fibration. To finish the proof we need to check that the transition functors along the inert maps $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$ induce an equivalence

$$\mathcal{C}^{\amalg}_{\langle n \rangle} \simeq \prod_i \mathcal{C}^{\amalg}_{\langle 1 \rangle}. \tag{3.9}$$

Inspecting the definition 3.3.1 we now observe that base change

$$\mathcal{C}^{\amalg} \times_{\mathrm{N}(\mathrm{Fin}_*)} \mathrm{N}(\mathrm{Fin}_*)_{\mathrm{in}} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)_{\mathrm{in}}$$

is canonically isomorphic to the relative nerve construction (see Definition 2.5.1) of the functor $(\mathrm{Fin}_*)_{\mathrm{in}} \longrightarrow \mathrm{Set}_{\Delta}$ which sends $\langle n \rangle$ to $\mathcal{C}^{\langle n \rangle^{\circ}}$ and each inert map $\alpha : \langle n \rangle \longrightarrow \langle m \rangle$ corresponding to an injection $\alpha' : \langle m \rangle^{\circ} \longrightarrow \langle n \rangle^{\circ}$ to the restriction

$$(\alpha')^* : \mathcal{C}^{\langle n \rangle^{\circ}} \longrightarrow \mathcal{C}^{\langle m \rangle^{\circ}}$$

along α'. In particular, there is a canonical isomorphism of simplicial sets $\mathcal{C}^{\amalg}_{\langle n \rangle} \cong \mathcal{C}^{\langle n \rangle^{\circ}}$ and the transition functor $\rho^i_! : \mathcal{C}^{\amalg}_{\langle n \rangle} \longrightarrow \mathcal{C}^{\amalg}_{\langle 1 \rangle}$ is equivalent to the projection to the i^{th} factor $\mathcal{C}^{\langle n \rangle^{\circ}} \longrightarrow \mathcal{C}$, hence the equivalence (3.9). $\square$

Remark 3.3.4. Proposition 3.3.3 implies that $\mathcal{C}^{\amalg}$ is an ∞-category as soon as $\mathcal{C}$ is an ∞-category (even if $\mathcal{C}$ does not admit finite coproducts). The mapping spaces in $\mathcal{C}^{\amalg}$ can then be explicitly identified: given $\overline{x} = (x_1, ..., x_n) \in \mathcal{C}^{\amalg}_{\langle n \rangle}$ and $\overline{y} = (y_1, ..., y_m) \in \mathcal{C}^{\amalg}_{\langle m \rangle}$ we have a canonical isomorphism of simplicial sets

$$\mathcal{C}^{\amalg}_{\overline{x}/} \times_{\mathrm{N}(\mathrm{Fin}_*)} \mathcal{C}^{\amalg}_{\langle m \rangle} \cong \coprod_{\alpha : \langle n \rangle \to \langle m \rangle} \prod_{j \in \langle m \rangle^{\circ}} \mathcal{C}_{\overline{x}_j /},$$

where $\overline{x}_j := \overline{x}|_{\alpha^{-1}(j)} : \alpha^{-1}(j) \longrightarrow \mathcal{C}$ is the restriction of $\overline{x}$ to the preimage of j. We hence obtain a canonical equivalence of spaces

$$\mathrm{Map}_{\mathcal{C}^{\amalg}}((\langle n \rangle, \overline{x}), (\langle m \rangle, \overline{y})) \simeq \coprod_{\alpha : \langle n \rangle \to \langle m \rangle} \prod_{j \in \langle m \rangle^{\circ}} \prod_{i \in \alpha^{-1}(j)} \mathrm{Map}_{\mathcal{C}}(x_i, y_j).$$

We now turn our attention to the case of *products*. Define a category $\Gamma^{\times}$ as follows:

- The objects of $\Gamma^{\times}$ are pairs $(\langle n \rangle, I)$ where $\langle n \rangle \in \mathrm{Fin}_*$ and $I \subseteq \langle n \rangle^{\circ}$.
- A morphism in $\Gamma^{\times}$ from $(\langle n \rangle, I)$ to $(\langle m \rangle, J)$ is a map of pointed sets $\alpha : \langle n \rangle \longrightarrow \langle m \rangle$ such that $\alpha^{-1}(J) \subseteq I$.

The category $\Gamma^{\times}$ admits an obvious forgetful functor $\Gamma^{\times} \longrightarrow \mathrm{Fin}_*$ sending $(\langle n \rangle, I)$ to $\langle n \rangle$.

Definition 3.3.5. Let $\mathcal{C}$ be an ∞-category. We define $\widetilde{\mathcal{C}}^{\times}$ to be the simplicial set whose n-simplices are pairs (σ, ρ) where $\sigma : \Delta^n \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ is an n-simplex of $\mathrm{N}(\mathrm{Fin}_*)$ and $\rho : \Delta^n \times_{\mathrm{N}(\mathrm{Fin}_*)} \mathrm{N}(\Gamma^{\times}) \longrightarrow \mathcal{C}$ is a map of simplicial sets, where the fiber product is taken with respect to σ. By construction the simplicial set $\widetilde{\mathcal{C}}^{\times}$ comes equipped with a map $\widetilde{\mathcal{C}}^{\times} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ sending (σ, ρ) to σ. We note that for $\langle n \rangle \in \mathrm{Fin}_*$ the fiber $\widetilde{\mathcal{C}}^{\times}_{\langle n \rangle}$ is canonically isomorphic to the simplicial set of functors

$\mathrm{N}(\mathcal{P}_n)^{\mathrm{op}} \longrightarrow \mathcal{C}$ where $\mathcal{P}_n$ denote the poset of subsets of $\langle n\rangle^{\circ}$. We then define $\mathcal{C}^{\times} \subseteq \widetilde{\mathcal{C}}^{\times}$ be the subsimplicial set spanned by those vertices whose corresponding functors $\varphi : \mathrm{N}(\mathcal{P}_n)^{\mathrm{op}} \longrightarrow \mathcal{C}$ satisfy the following property: for every $I \subseteq \langle n\rangle^{\circ}$ the maps $\varphi(I) \longrightarrow \varphi(\{i\})$ for $i \in I$ exhibit $\varphi(I)$ as the product in $\mathcal{C}$ of the objects $\{\varphi(\{i\})\}$.

Proposition 3.3.6 ([6, Proposition 2.4.1.5])**.** *Let $\mathcal{C}$ be an ∞-category which admits products. Then the projection $\mathcal{C}^{\times} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ exhibits $\mathcal{C}^{\times}$ as a symmetric monoidal ∞-category whose underlying ∞-category is $\mathcal{C}^{\times}_{\langle 1\rangle} = \mathcal{C}$. We will refer to $\mathcal{C}^{\times}$ as the* cartesian symmetric monoidal structure*.*

4. ∞-Operads and their algebras

4.1. **From colored operads to ∞-operads.** Our goal in this section is to define and study the notion of ∞-operad, which is a central object of interest in this course. The notion of an ∞-operad is essentially an ∞-categorical version of the classical notion of a *colored symmetric operad*. In this course all operads will be colored and symmetric, and so we will generally omit these adjectives from our discussion. The notion of an operad allows one to encode algebraic structures abstractly in a way that is independent of the specific type of objects in which we will wish to realize them. Similarly, an ∞-operad allows one to do so in a higher categorical manner, so that the axioms of the algebraic structure can be imposed only up to coherent homotopy. We begin with the basic definitions pertaining to the case of ordinary categories.

Definition 4.1.1. A (colored, symmetric) *operad* $\mathcal{O}$ consists in

- a set $\mathrm{Ob}(\mathcal{O})$ of objects (sometimes called colors);
- for every finite set I, every I-indexed collection of objects $\{x_i\}_{i\in I}$ and every object $y \in \mathrm{Ob}(\mathcal{O})$ a set $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}, y)$, which we call the set of *multimaps* from $\{x_i\}_{i\in I}$ to y. We will generally denote such multimaps as arrows $\{x_i\}_{i\in I} \longrightarrow y$.
- for every finite collections $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$, every map of finite sets $\alpha : I \longrightarrow J$, and every object $z \in \mathrm{Ob}(\mathcal{O})$ a composition map

$$\Big[\prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in\alpha^{-1}(j)}, y_j)\Big] \times \mathrm{Mul}_{\mathcal{O}}(\{y_j\}_{j\in J}, z) \longrightarrow \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}, z). \tag{4.1}$$

- for every object $x \in \mathrm{Ob}(\mathcal{O})$ a designated identity multimap $\mathrm{Id}_x \in \mathrm{Mul}_{\mathcal{O}}(\{x\}, x)$ which is (left and right) neutral with respect to the composition map above.

If I has cardinality n then we will say that $\{x_i\}_{i\in I} \longrightarrow y$ is a multimap of *arity* n. The composition rule is required to be associative in the following sense: for every sequence of maps $I \xrightarrow{\alpha} J \xrightarrow{\beta} K$ of finite sets with $\gamma = \beta \circ \alpha$, every triple of collections $\{x_i\}_{i\in I}, \{y_j\}_{j\in J}, \{z_k\}_{k\in K}$ and every object $w \in \mathrm{Ob}(\mathcal{O})$, the diagram

$$\begin{tikzcd}
 & \prod_{j\in J}\mathrm{Mul}(\{x_i\}_{i\in\alpha^{-1}(j)}, y_j) \times \mathrm{Mul}(\{y_j\}_{j\in J}, w) \arrow[dr] & \\
 & \prod_{j\in J}\mathrm{Mul}(\{x_i\}_{i\in\alpha^{-1}(j)}, y_j) \times \prod_{k\in K}\mathrm{Mul}(\{y_j\}_{j\in\beta^{-1}(k)}, z_k) \times \mathrm{Mul}(\{z_k\}_{k\in K}, w) \arrow[u] \arrow[d] & \mathrm{Mul}(\{x_i\}_{i\in I}, w) \\
 & \prod_{k\in K}\mathrm{Mul}(\{x_i\}_{i\in\gamma^{-1}(k)}, z_k) \times \mathrm{Mul}(\{z_k\}_{k\in K}, w) \arrow[ur] &
\end{tikzcd}$$

commutes.

A *map of operads* $\varphi : \mathcal{O} \longrightarrow \mathcal{P}$ consists of a map of sets $\mathrm{Ob}(\mathcal{O}) \longrightarrow \mathrm{Ob}(\mathcal{P})$ together with maps $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}, y) \longrightarrow \mathrm{Mul}_{\mathcal{P}}(\{\varphi(x_i)\}_{i\in I}, \varphi(y))$ for every finite collection $\{x_i\}_{i\in I}$ and object y, which are compatible with the composition operation and the identity multimaps.

Remark 4.1.2. Every operad $\mathcal{O}$ has an *underlying category* whose objects are the objects of $\mathcal{O}$ and whose morphism sets are given by $\mathrm{Mul}(\{x\}, y)$. We may hence consider an operad as a category with additional structure, namely, the multimaps $\{x_i\}_{i\in I} \longrightarrow y$. Some authors hence use the term *multicategory* to talk about these kinds of operads.

Example 4.1.3. Let $\mathcal{C}$ be a symmetric monoidal category. Then we may associate to $\mathcal{C}$ an operad as follows: the objects of $\mathcal{O}$ are the objects of $\mathcal{C}$ and for every finite collection $\{x_i\}_{i\in I}$ and object $y \in \mathcal{C}$ we set

$$\mathrm{Mul}_{\mathcal{C}}(\{x_i\}_{i\in I}, y) = \mathrm{Hom}_{\mathcal{C}}(\otimes_i x_i, y).$$

We will refer to this operad as the *underlying operad* of $\mathcal{C}$.

Warning 4.1.4. If $\mathcal{C}$ is a symmetric monoidal category then $\mathcal{C}$ can be completely reconstructed from its underlying operad by the Yoneda lemma. However, if $\mathcal{C}$ and $\mathcal{D}$ are two symmetric monoidal categories then a map of operads $\mathcal{C} \longrightarrow \mathcal{D}$ is generally *not* the same as a map of symmetric monoidal categories. The latter is given by a symmetric monoidal functor while the former corresponds to a *lax* symmetric monoidal functor from $\mathcal{C}$ to $\mathcal{D}$.

The main motivation of the notion of an operad is that it can be used to encode the information of an algebraic structure.

Definition 4.1.5. Let $\mathcal{O}$ be an operad and $\mathcal{C}$ a symmetric monoidal category. An $\mathcal{O}$-*algebra* in $\mathcal{C}$ is a map of operads $\mathcal{O} \longrightarrow \mathcal{C}$, where $\mathcal{C}$ is considered as an operad by Example 4.1.3.

Example 4.1.6. If $\mathcal{O}$ has only multimaps of arity 1 then $\mathcal{O}$ is simply a category. In this case an $\mathcal{O}$-algebra in a symmetric monoidal category $\mathcal{C}$ is the same as a functor of ordinary categories $\mathcal{O} \longrightarrow \mathcal{C}$.

Examples 4.1.7.

(1) Let Com be the operad with a single object $*$ and such that the set of multimaps $\mathrm{Mul}_{\mathrm{Com}}(\{*\}_{i\in I}, *)$ contains a single element for every finite set I (in particular, Com is terminal in the category of operads). If $\mathcal{C}$ is a symmetric monoidal category then the data of a Com-algebra in $\mathcal{C}$ consists of an object $A \in \mathcal{C}$ (the image of $*$) together with maps $\otimes_{i\in I} A \longrightarrow A$ for every finite set I, which satisfy a compatibility condition for every $I \longrightarrow J$. Unwinding the definition we see that this is exactly the data of a commutative algebra structure on A.
(2) Let Triv be the operad with a single object $*$ and such that the only multimap is the identity. Then a Triv-operad in a symmetric monoidal category $\mathcal{C}$ is simply an object of $\mathcal{C}$ with no additional structure. We will refer to Triv as the *trivial operad*.
(3) Let Poi be the operad with a single object $*$ and such that the set of multimaps $\mathrm{Mul}_{\mathrm{Poi}}(\{*\}_{i\in I}, *)$ contains a single element if $|I| \leq 1$ and is empty otherwise. Then the data of a Poi-algebra object in a symmetric monoidal ∞-category $\mathcal{C}$ consists of an object $A \in \mathcal{C}$ together with a map $1 \longrightarrow A$ from the unit of $\mathcal{C}$. We will refer to Poi as the operad of *pointed objects*.
(4) Let Ass be the operad with a single object $*$ and such that for a finite set I the multimaps $\{*\}_{i\in I} \longrightarrow *$ are given by *linear orderings* on I (composition is defined by concatenating linear orders). If $\mathcal{C}$ is a symmetric monoidal category then the data of an Ass-algebra in $\mathcal{C}$ consists of an object $A \in \mathcal{C}$ (the image of $*$) together with maps $\otimes_{i\in I} A \longrightarrow A$ for every finite *linearly ordered* set I, which satisfy a compatibility condition for every order preserving map $I \longrightarrow J$. Unwinding the definition we see that this is exactly the data of an associative algebra structure on A.
(5) Let AssInv be the operad with a single object $*$ and for a finite set I, the multimaps $\{*\}_{i\in I} \longrightarrow *$ are given by pairs $(\leq, \varepsilon)$ where $\leq$ is a linear order on I and $\varepsilon : I \longrightarrow \{-1, 1\}$ is an assignment of signs to each $i \in I$. Composition of multimaps is given by the concatenation of linear orders, reversal of linear orders according to signs, and multiplication of signs. For example, the composition of the multimaps
$$\{*\}_{i\in\{1,2\}} \xrightarrow{1<2,\varepsilon_1=\varepsilon_2=1} \{*\}_{i\in\{1\}} \xrightarrow{\varepsilon_1=-1} *$$
is the multimap
$$\{*\}_{i\in\{1,2\}} \xrightarrow{2<1,\varepsilon_1=\varepsilon_2=-1} * \,.$$
Algebras over AssInv in a symmetric monoidal category $\mathcal{C}$ are known as *algebras with involution*. They are given by an associative algebra object A equipped with an isomorphism $\tau : A \longrightarrow A^{\mathrm{op}}$, where A^{op} denotes the same algebra with multiplication reversed.

The examples above all have a single object. Here are some examples with several objects:

Examples 4.1.8.

(1) Given an operad $\mathcal{O}$ we can produce another operad $\mathrm{M}\mathcal{O}$ such that $\mathrm{Ob}(\mathrm{M}\mathcal{O}) = \mathrm{Ob}(\mathcal{O})\times\{a, m\}$, and such that the set of multimaps $\mathrm{Mul}_{\mathrm{M}\mathcal{O}}(\{(x_i, y_i)\}_{i\in I}, (z, w))$ is equal to the set $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}, z)$ if $w = y_i = a$ for all i or if $w = m$ and $y_i = m$ for exactly one i, and is empty otherwise. The data of an $\mathrm{M}\mathcal{O}$-algebra in $\mathcal{C}$ is given by a pair A, M where A is an $\mathcal{O}$-algebra and M is a functor from the

underlying category of $\mathcal{O}$ to $\mathcal{C}$, equipped with an *action* of A. We will refer to M as an A-*module*, though the exact meaning of this notion depends on $\mathcal{O}$. For example, in the case of $\mathcal{O} = \mathrm{Com}$ we get that A is a commutative algebra object and M is an A-module in the usual sense. When $\mathcal{O} = \mathrm{Ass}$ we have that A is an associative algebra object and M is an A-*bimodule*.

(2) Let $\mathrm{LAss} \subseteq \mathrm{MAss}$ be the sub operad with the same objects where we only take those multimaps $\{(*, y_i)\}_{i\in I} \longrightarrow (*, w)$ in MAss such that either $w = y_i = a$ or such that $w = m$ and in the corresponding linear ordering on I the unique i_0 such that $y_{i_0} = m$ is *last*. Then the data of an LAss-algebra in $\mathcal{C}$ is given by a pair A, M where A is an associative algebra object and M is a *left module* over A. If instead of last we required *first* then we would get a suboperad $\mathrm{RAss} \subseteq \mathrm{MAss}$ whose algebras are pairs of associative algebras and *right modules*.

(3) Given an operad $\mathcal{O}$, let $\mathrm{P}\mathcal{O}$ be the operad with the same objects as $\mathrm{M}\mathcal{O}$ and such that the set of multimaps $\mathrm{Mul}_{\mathrm{P}\mathcal{O}}(\{(x_i, y_i)\}_{i\in I}, (z, w))$ is equal to $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}, z)$ if $w = y_i = a$ or if $w = m$ and $y_i = m$ for *at most* one i, and is empty otherwise. Then the data of a $\mathrm{P}\mathcal{O}$-algebra in $\mathcal{C}$ is given by a triple (A, M, f) where A is an $\mathcal{O}$-algebra, M is an A-module (in the sense of (1) above) and $f : A \longrightarrow M$ is a map of A-modules. For example, a PAss-algebra consists of an associative algebra A, a bimodule M and a map of bimodules $f : A \longrightarrow M$.

(4) As above, the operad PAss contains two suboperads $\mathrm{PLAss}, \mathrm{PRAss} \subseteq \mathrm{PAss}$ whose algebras are given by pairs (A, M, f) where A is an associative algebra object, M is a left (resp. right) module and $f : A \longrightarrow M$ is a map of left (resp. right) modules. In this case, the data of f is equivalent to that of a map in $\mathcal{C}$ of the form $1 \longrightarrow M$, and we may consider M as a left (resp. right) module object in the symmetric monoidal category $\mathcal{C}_{1/}$ of pointed objects. We will refer to such (left or right) modules as *pointed modules*.

When passing to a higher categorical setting, the notion of an algebraic structure (say, on an object in a symmetric monoidal ∞-category $\mathcal{C}$), is more subtle: we need to let the axioms of our algebraic theory hold only up to a (specified) homotopy, and these homotopies need to be compatible up to higher homotopies, etc. As with previous types of constructions discussed in this course, trying to keep explicit tabs on all coherence homotopies becomes unfeasible in general. We hence need a formalism in which the notion of an algebraic structure could be defined in an ∞-categorical setting, such that all the required coherence homotopy will be taken into account implicitly. The formalism of ∞-operads is meant to do exactly that.

The idea of how to encode ∞-operads can be traced back to Example 4.1.3: if symmetric monoidal categories are a particular kind of operads, then it makes sense to try to define ∞-operads by weakening the definition of a symmetric monoidal ∞-category. Recall that if $\mathcal{C}$ is an ordinary symmetric monoidal category then we can view it as a symmetric monoidal ∞-category by constructing the category $\mathcal{C}^{\otimes}$ over Fin_* (see Construction 3.1.3) and then taking the nerve. A key observation is that if we start with an operad $\mathcal{O}$ instead of a symmetric monoidal category then we can still construct an analogous category $\mathcal{O}^{\otimes}$ over Fin_*, only that in general it will not be a cocartesian fibration.

Construction 4.1.9. Let $\mathcal{O}$ be an operad. The category $\mathcal{O}^{\otimes}$ is defined as follows:

- its objects are pairs $(n, (x_1, ..., x_n))$ consisting of a non-negative integer n and a family of objects in $\mathcal{O}$ parameterized by $\{1, ..., n\}$.

- maps from $(n,(x_1,...,x_n))$ to $(m,(y_1,...,y_m))$ are given by maps $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ in Fin_*, together with, for every $j = 1,...,n$, a multimap in $\mathcal{O}$ of the form

$$f_j : \{x_i\}_{i\in\alpha^{-1}(j)} \longrightarrow y_j.$$

We will denote by $\pi : \mathcal{O}^{\otimes} \longrightarrow \mathrm{Fin}_*$ the natural projection $(n,(x_1,...,x_n)) \mapsto \langle n\rangle$.

We emphasize that this time π is not a cocartesian fibration, i.e., not every edge in Fin_* admits cocartesian lifts. However, some edges in Fin_* do have cocartesian lifts: if $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ is an *inert map* and $(n,(x_1,...,x_n))$ is an object of $\mathcal{O}^{\otimes}$ lying over $\langle n\rangle$ then maps $f : (n,(x_1,...,x_n)) \longrightarrow (m,(y_1,...,y_m))$ lying above α are given by specifying, for each $j = 1,...,m$, a map $f_j : x_i \longrightarrow y_j$ in the underlying category of $\mathcal{O}$, where $i \in \{1,...,n\}$ is the unique element mapping to j by α. It is then not difficult to check that f is π-cocartesian if and only if each f_j is an isomorphism in the underlying category of $\mathcal{O}$. We then see that each inert map $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ admits a cocartesian lift starting at an arbitrary object over $\langle n\rangle$. This means that π becomes a cocartesian fibration when restricted to the subcategory $\mathrm{Fin}_*^{\mathrm{in}} \subseteq \mathrm{Fin}$ consisting of all objects and all inert maps.

Similarly to the case of symmetric monoidal category, the operad $\mathcal{O}$ can be completely reconstructed from the category $\mathcal{O}^{\otimes}$ together with the forgetful functor $\pi : \mathcal{O}^{\otimes} \longrightarrow \mathrm{Fin}_*$. For example, the underlying category of $\mathcal{O}$ can be identified with the fiber $\mathcal{O}^{\otimes}_{\langle 1\rangle} = \pi^{-1}(\langle 1\rangle)$. More generally, suppose that $n \geq 0$ and for $1 \leq i \leq n$ let $\rho^i : \langle n\rangle \longrightarrow \langle 1\rangle$ be as in Notation 3.1.8. Since ρ^i is inert it has by the discussion above an associated transition functor $\rho^i_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \longrightarrow \mathcal{O}^{\otimes}_{\langle 1\rangle}$ (see Construction 2.4.19), and the collection of transition functors $\rho^i_!$ determines an equivalence $\mathcal{O}^{\otimes}_{\langle n\rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1\rangle})^n$. Given an object in $\mathcal{O}^{\otimes}_{\langle n\rangle}$ we may hence identify it with a sequence $x_1,...,x_n \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$ *intrinsically*. Furthermore, if $\overline{x}$ is an object of $\mathcal{O}^{\otimes}_{\langle n\rangle}$ which corresponds to the tuple $(x_1,...,x_n)$ and $y \in \mathcal{O}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{O}$ then maps from $\overline{x}$ to y in $\mathcal{O}^{\otimes}$ which lie above the unique active map $\langle n\rangle \longrightarrow \langle 1\rangle$ are in bijection with multimaps $\{x_i\}_{i=1}^n \longrightarrow y$ in $\mathcal{O}$. Elaborating further along these lines one can also reconstruct the composition of multimaps in $\mathcal{O}$.

The discussion above, together with the definition of a symmetric monoidal ∞-category (see Definition 3.1.11), suggest that we might attempt to define an ∞-operad as an ∞-category $\mathcal{O}^{\otimes}$ over Fin_* which satisfies certain conditions. As far as we know defining ∞-operads in this way was first done in by Lurie (see [6, §2.1.1]).

Definition 4.1.10. An ∞-*operad* is a map of ∞-categories $\pi : \mathcal{O}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ such that the following conditions hold:

(1) For every $\overline{x} \in \mathcal{O}^{\otimes}$ lying above $\langle n\rangle$ and every inert map $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ there exists a π-cocartesian edge $f : \overline{x} \longrightarrow \overline{y}$ lying above α.

(2) For every n the transition functors $\rho^1_!,...,\rho^n_!$ (whose existence is guaranteed by (1)) induce an equivalence

$$\mathcal{O}^{\otimes}_{\langle n\rangle} \xrightarrow{\simeq} \left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^n.$$

(3) Let $\overline{y} \in \mathcal{O}^{\otimes}$ be an object lying above $\langle m\rangle$ and for each $j = 1,...,m$ let $f_j : \overline{y} \longrightarrow y_i$ be a π-cocartesian edge lying above ρ^j. Then the maps $\{f_j\}$ exhibit $\overline{y}$ as the *relative product* of $y_1,...,y_m$ in $\mathcal{O}^{\otimes}$ over $\mathrm{N}(\mathrm{Fin}_*)$ in the following sense: for

every $\overline{x} \in \mathcal{O}^{\otimes}$ the square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{O}^{\otimes}}(\overline{x},\overline{y}) \arrow[r] \arrow[d] & \prod_j \mathrm{Map}_{\mathcal{O}^{\otimes}}(\overline{x},y_i) \arrow[d] \\ \mathrm{Map}_{\mathrm{Fin}_*}(\langle n\rangle,\langle m\rangle) \arrow[r] & \mathrm{Map}_{\mathrm{Fin}_*}(\langle n\rangle,\langle 1\rangle) \end{tikzcd}$$

is a cartesian square in $\mathcal{S}$.

In this case we will also say that π *exhibits* $\mathcal{O}^{\otimes}$ *as an* ∞*-operad*.

Example 4.1.11. If $\mathcal{O}$ is an ordinary operad and $\mathcal{O}^{\otimes}$ is obtained from $\mathcal{O}$ via construction 4.1.9 then the map $\mathrm{N}(\mathcal{O}^{\otimes}) \longrightarrow \mathrm{N}(\mathrm{Fin}))$ exhibits $\mathrm{N}(\mathcal{O}^{\otimes})$ as an ∞-operad. In particular, all the examples of (4.1.7) also yield examples of ∞-operads. We note that $\mathrm{Com}^{\otimes}$ is equivalent to Fin_* itself and $\mathrm{Poi}^{\otimes}$ is equivalent to the subcategory of Fin_* consisting of all objects and all maps $\alpha : \langle n\rangle \longrightarrow \langle m\rangle$ such that $|\alpha^{-1}(i)| \leq 1$ for $i = 1, ..., n$. Similarly, $\mathrm{Triv}^{\otimes}$ is simply the subcategory of Fin_* consisting of all inert maps.

Example 4.1.12. Let $\mathcal{C}$ be an ∞-category and let $\pi : \mathcal{C}^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ be as in Definition 3.3.1. Then Proposition 3.3.3 implies that $\mathcal{C}^{\amalg}$ satisfies Condition (1) of Definition 4.1.10. Arguing as in the final part of the proof of Proposition 3.3.3 we see that $\pi : \mathcal{C}^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin}_a *)$ is in fact an ∞-operad (even if $\mathcal{C}$ does not have coproducts). We call $\mathcal{C}^{\amalg}$ the *cocartesian* ∞*-operad* of $\mathcal{C}$.

Generalizing Example 4.1.11 we may also construct ∞-operads starting from *simplicial operads*. Recall that a simplicial operad $\mathcal{O}$ is given by the simplicially enriched version of Definition 4.1.1: we have a set of objects $\mathrm{Ob}(\mathcal{O})$ and for every collection of objects $\{x_i\}_{i\in I}$ indexed by a finite set I and every object $y \in \mathrm{Ob}(\mathcal{O})$, a *simplicial set* $\mathrm{Map}_{\mathcal{O}}(\{x_i\}, y)$ of multimaps from $\{x_i\}_{i\in I}$ to y. The composition of multimaps is then given as in (4.1) by replacing sets with simplicial sets. Given a simplicial operad $\mathcal{O}$ we may perform the analogue of Construction 4.1.9 to obtain a simplicial category $\mathcal{O}^{\otimes}$ whose

- objects are pairs $(n, (x_1, ..., x_n))$ consisting of a non-negative integer n and a family of objects in $\mathcal{O}$ parameterized by $\{1, ..., n\}$.
- the simplicial set of maps from $(n, (x_1, ..., x_n))$ to $(m, (y_1, ..., y_m))$ is given by

$$\coprod_{\alpha:\langle n\rangle\to\langle m\rangle} \prod_{j=1}^{n} \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in\alpha^{-1}(j)}, y_j)$$

where α ranges over all maps from $\langle n\rangle$ to $\langle m\rangle$ in Fin_*.

As before we have the forgetful functor $\pi : \mathcal{O}^{\otimes} \longrightarrow \mathrm{Fin}_*$ sending $(n, (x_1, ..., x_n))$ to $\langle n\rangle$. In what follows we say that a simplicial operad $\mathcal{O}$ is *locally Kan* if the simplical sets $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}, y)$ are all Kan.

Proposition 4.1.13. *Let* $\mathcal{O}$ *be a locally Kan simplicial operad. Then the map* $\mathrm{N}(\mathcal{O}^{\otimes}) \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ *exhibits* $\mathrm{N}(\mathcal{O}^{\otimes})$ *as an* ∞*-operad. We will refer to the* ∞*-operad* $\mathrm{N}(\mathcal{O}^{\otimes})$ *as the* operadic nerve *of* $\mathcal{O}$.

In §5 below we will use the operadic nerve construction in order to define the *little* n*-cube* ∞*-operads* $\mathbb{E}_n$, which serve as one of the main objects of interest in this course.

Definition 4.1.14. Let $\pi : \mathcal{O}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ be an ∞-operad. We will say that a map $f : \overline{x} \longrightarrow \overline{y}$ in $\mathcal{O}^{\otimes}$ is *inert* if it is π-cocartesian and $\pi(f)$ is inert.

Definition 4.1.15. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be two ∞-operads. A *map of ∞-operads* is a map of ∞-categories $\mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ over $\mathrm{N}(\mathrm{Fin}_*)$ which sends inert maps to inert maps. We will refer to ∞-operad maps from $\mathcal{O}$ to $\mathcal{P}$ as *$\mathcal{O}$-algebras in $\mathcal{P}$*. We will denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \subseteq \mathrm{Fun}_{\mathrm{N}(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes})$ the full subcategory spanned by the ∞-operad maps.

Remark 4.1.16. The terminology of Definition 4.1.15 is motivated by the example where $\mathcal{P}^{\otimes} = \mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, in which case we should think of ∞-operad maps $\mathcal{O}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ are the homotopy coherent analogue of the classical notion of an algebra over an operad in a symmetric monoidal category.

Definition 4.1.17. Let $\mathcal{O}\mathrm{p}$ denote the simplicial category whose objects are the ∞-operads and such that $\mathrm{Map}_{\mathcal{O}\mathrm{p}}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}) = \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq}$. Then $\mathcal{O}\mathrm{p}$ is locally Kan and we define the *∞-category of ∞-operads* to be its coherent nerve

$$\mathcal{O}\mathrm{p}_{\infty} := \mathrm{N}(\mathcal{O}\mathrm{p}).$$

In [6] Lurie constructs a simplicial model category whose underlying ∞-category is $\mathcal{O}\mathrm{p}_{\infty}$. Let us denote by E_{in} the collection of edges of $\mathrm{N}(\mathrm{Fin}_*)$ which correspond to *inert maps*. Given an ∞-operad $\pi : \mathcal{O}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ we will denote by $\mathcal{O}^{\otimes,\natural}$ the marked simplicial set whose underlying simplicial set is $\mathcal{O}^{\otimes}$ and whose marked edges are exactly the inert maps. We note that the map π then becomes a map of marked simplicial sets $\pi : \mathcal{O}^{\otimes,\natural} \longrightarrow (\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}})$.

Definition 4.1.18. Let $p : (X, E) \longrightarrow (Y, F)$ be a map of marked simplicial sets over $(\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}})$. We will say that f is an *operadic equivalence* if for every ∞-operad $\mathcal{O}^{\otimes}$ the induced map

$$\mathrm{Map}^{\sharp}_{\mathrm{N}(\mathrm{Fin}_*)}((Y, F), (\mathcal{O}^{\otimes})^{\sharp}) \longrightarrow \mathrm{Map}^{\sharp}_{\mathrm{N}(\mathrm{Fin}_*)}((Y, F), (\mathcal{O}^{\otimes})^{\sharp})$$

is an equivalence of Kan complexes, where $\mathrm{Map}^{\sharp}_{\mathrm{N}(\mathrm{Fin}_*)}(-,-)$ denotes the subsimplicial set of $\mathrm{Map}^{\sharp}(-,-)$ spanned by the maps which preserve the projection to $\mathrm{N}(\mathrm{Fin}_*)$ (see §**??**).

Theorem 4.1.19 ([6, §2.1.4])**.** *There exists a simplicial model structure on the category $(\mathrm{Set}^+_{\Delta})_{/(\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}})}$ of marked simplicial sets over $(\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}})$ whose weak equivalences are the operadic equivalences, whose cofibrations are the monomorphisms and whose fibrant objects are those of the form $\mathcal{O}^{\otimes,\natural}$ for $\mathcal{O}^{\otimes}$ an ∞-operad. The underlying ∞-category of this model structure is naturally equivalent to $\mathcal{O}\mathrm{p}_{\infty}$. We will refer to this model structure as the* operadic model structure*.*

Recall that if $\mathcal{C}$ is an ∞-category which admits finite products then $\mathcal{C}$ can be endowed with a canonical symmetric monoidal structure $\mathcal{C}^{\times} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ in which the monoidal operation is given by cartesian products (see §3.3). Since $\mathcal{C}^{\times}$ is canonically determined by $\mathcal{C}$ (given that it has finite products), we might hope that the notion of an algebra object in $\mathcal{C}$ (with respect to some ∞-operad $\mathcal{O}^{\otimes}$) could be phrased without making a reference to the theory of symmetric monoidal ∞-categories. To make this precise let us introduce the following notion:

Definition 4.1.20. Let $\mathcal{C}$ be an ∞-category which admits finite products and let $\mathcal{O}^{\otimes}$ be an ∞-operad. An *$\mathcal{O}$-monoid* in $\mathcal{C}$ is a functor $\varphi : \mathcal{O}^{\otimes} \longrightarrow \mathcal{C}$, such that for every

$\overline{x} \in \mathcal{O}^{\otimes}$ and every collection of inert maps $f_i : \overline{x} \longrightarrow x_i$ lying above $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$ the maps $(f_i)_* : \varphi(\overline{x}) \longrightarrow \varphi(x_i)$ exhibit $\varphi(\overline{x})$ as the product of $\varphi(x_1), ..., \varphi(x_n)$ in $\mathcal{C}$. We will denote by $\mathrm{Mon}_{\mathcal{C}}(\mathcal{E}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ the full subcategory spanned by the $\mathcal{C}$-monoid objects.

Given an ∞-category $\mathcal{C}$ which admits finite products, we may consider the associated cartesian symmetric monoidal ∞-category $\mathcal{C}^{\times}$ as *representing* the functor $\mathcal{O}^{\otimes} \mapsto \mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$. More precisely, observe that the forgetful functor $\Gamma^{\times} \longrightarrow \mathrm{Fin}_*$ admits a canonical section sending $\langle n \rangle$ to the pair $(\langle n \rangle, \langle n \rangle^{\circ})$, and so by construction we have a canonical map $\varphi : \mathcal{C}^{\times} \longrightarrow \mathcal{C}$. We then have the following universal characterization of $\mathcal{C}^{\times}$:

Proposition 4.1.21. *Let $\mathcal{C}$ be an ∞-category which admits finite products. Then the functor $\varphi : \mathcal{C}^{\times} \longrightarrow \mathcal{C}$ is a monoid object in $\mathcal{C}$. Moreover, this monoid object is universal in the following sense: for every ∞-operad $\mathcal{O}^{\otimes}$, composition with φ induces an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$$

where on the left hand side we consider $\mathcal{C}$ as endowed with the cartesian symmetric monoidal structure $\mathcal{C}^{\times}$.

We finish this section with the following proposition which gives a convenient criterion for when a map of ∞-operads is an equivalence.

Definition 4.1.22. Let $\varphi : \mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ be a map of ∞-operads. We will say that φ is a *Morita equivalence* if restriction along φ induces an equivalence of ∞-categories

$$\mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{\simeq} \mathrm{Mon}_{\mathcal{P}}(\mathcal{S}).$$

Proposition 4.1.23. *Let $\varphi : \mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ be a map of ∞-operads. Then the following are equivalent:*

(1) φ is an operadic equivalence.
(2) φ is an equivalence in the ∞-category $\mathcal{O}\mathrm{p}_{\infty}$.
(3) For every ∞-operad $\mathcal{Q}^{\otimes}$, restriction along φ induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{P}}(\mathcal{Q}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}}(\mathcal{Q})$.
(4) The map $\mathcal{O} \longrightarrow \mathcal{P}$ on underlying ∞-categories is essentially surjective, and for every symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, restriction along φ induces an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.
(5) φ is essentially surjective and a Morita equivalence (see Definition 4.1.22).

Proof (sketch). (1), (2) and (3) are equivalent essentially by construction and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) by Proposition 4.1.21. To finish the proof we will show that (5) implies (3). Let us hence assume that φ satisfies (5). Then φ is essentially surjective and since $\mathcal{O}, \mathcal{P}$ are ∞-operad it follows that φ is also essentially surjective on inert arrows. It then follows that if $\mathcal{Q}^{\otimes}$ is an ∞-operad then a functor $\mathcal{P}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes}$ over $\mathrm{N}(\mathrm{Fin}_*)$ is an ∞-operad map if and only if the composition $\mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes}$ is an ∞-operad map. We are hence reduced to showing that the restriction functor $\mathrm{Fun}_{\mathrm{N}(\mathrm{Fin}_*)}(\mathcal{P}^{\otimes}, \mathcal{Q}^{\otimes}) \longrightarrow \mathrm{Fin}_{\mathrm{N}(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{Q}^{\otimes})$ is an equivalence of ∞-categories for every $\mathcal{Q}^{\otimes}$. For this it will suffice to show that the map $\mathcal{O}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ is an equivalence of ∞-categories. Since we assumed that φ is essentially surjective it is left to check that φ is fully-faithful. Since $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are ∞-operads it will suffice to check that if $\overline{x} \in \mathcal{O}^{\otimes}$ and $y \in \mathcal{O}$ are objects then the induced map

$$\mathrm{Map}_{\mathcal{O}^{\otimes}}(\overline{x}, y) \longrightarrow \mathrm{Map}_{\mathcal{P}^{\otimes}}(\varphi(\overline{x}), \varphi(y)) \tag{4.2}$$

is an equivalence of spaces. For $i = 1, ..., n$ let $f_i : \overline{x} \longrightarrow x_i$ be an inert map lying above $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$, so that we can think of the mapping spaces appearing in (4.2) as the spaces of multimaps $\{x_1, ..., x_n\} \longrightarrow y$ and $\{\varphi(x_1), ..., \varphi(x_n)\} \longrightarrow \varphi(y)$. Summarizing the discussion so far what we need to check is that if φ satisfies (5) then it induces an equivalences on all spaces of multimaps.

We now invoke some more advanced ∞-categorical machinery which states that given $x \in \mathcal{O}$ the forgetful functor

$$\mathrm{ev}_x : \mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \longrightarrow \mathcal{S}$$

which sends an $\mathcal{O}$-monoid in spaces to the space associated to the object x, admits a *left adjoint* $\mathcal{F}_x : \mathcal{S} \longrightarrow \mathrm{Mon}_{\mathcal{O}}(\mathcal{S})$ whose value on a given space Z is usually called the *free* $\mathcal{O}$*-monoid generated from* Z *at* x. Furthermore, these free monoids admit an explicit description (see [6, §3.1.3]):

$$\mathrm{ev}_y \mathcal{F}_x(Z) = \coprod_{n \geq 0} [\mathrm{Map}_{\mathcal{O}}((x, ..., x), y) \cdot Z^n]_{h\Sigma_n}$$

where $(x, ..., x)$ denotes an object of $\mathcal{O}$ which corresponds to the tuple $(x, ..., x)$ under the (canonical) equivalence $\mathcal{O}_{\langle n \rangle} \simeq \mathcal{O}^n_{\langle 1 \rangle}$, and Σ_n is the permutation group on n elements which naturally act on $\mathrm{Map}_{\mathcal{O}}((x, ..., x), y) \cdot Z^n$. In particular, if φ induces an equivalence $\mathrm{Mon}_{\mathcal{P}}(\mathcal{S}) \longrightarrow \mathrm{Mon}_{\mathcal{O}}(\mathcal{S})$ then it also induces an equivalence $\mathcal{F}_x(Z) \simeq \varphi^* \mathcal{F}_{\varphi(x)}(Z)$ for every space Z. Setting $Z = *$ we obtain that φ induces an equivalence

$$\mathrm{Map}_{\mathcal{O}}((x, ..., x), y)_{h\Sigma_n} \longrightarrow \mathrm{Map}_{\mathcal{P}}((\varphi(x), ..., \varphi(x)), \varphi(y))_{h\Sigma_n}$$

for every n and hence an equivalence

$$\mathrm{Map}_{\mathcal{O}}((x, ..., x), y) \simeq \mathrm{Map}_{\mathcal{O}}((x, ..., x), y)_{h\Sigma_n} \times_{(*)_{h\Sigma_n}} \{*\} \longrightarrow$$
$$\mathrm{Map}_{\mathcal{O}}((\varphi(x), ..., \varphi(x)), \varphi(y))_{h\Sigma_n} \times_{(*)_{h\Sigma_n}} \{*\} \simeq \mathrm{Map}_{\mathcal{P}}((\varphi(x), ..., \varphi(x)), \varphi(y))$$

for every n. This finishes the proof if the underlying ∞-category $\mathcal{O}$ is a connected Kan complex. In general, to handle non-equivalent $x_1, ..., x_n$ one can do a similar argument using the left adjoint of the forgetful functor

$$\mathrm{ev}_{x_1,...,x_n} : \mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \longrightarrow \mathcal{S}^n$$

which sends an $\mathcal{O}$-monoid in spaces to the n-tuple of spaces associated to the objects $x_1, ..., x_n$. □

4.2. **Weak ∞-operads and approximations.** The notion of an ∞-operad defined above is somewhat rigid. It is hence useful sometimes to consider mild generalizations, which are more amenable to various constructions. In this section we will focus on one such notion, which we call a *weak* ∞*-operad*. This notion is somewhat add-hoc (we are not aware of any literature using this particular definition), and for the purposes of these notes it should be considered as merely a tool in order to prove results on ∞-operads. However, we note that it does admits quite a few interesting examples which arise naturally, and can be considered as a variant on the theory of *operator categories*.

Recall that a map $\iota : \mathcal{C}_0 \longrightarrow \mathcal{C}$ of ∞-categories is called a *subcategory* of $\mathcal{C}$ if the induced map $\mathrm{Map}_{\mathcal{C}_0}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(\iota(x), \iota(y))$ has (-1)-truncated homotopy fibers (that is, it is equivalent to an inclusion of a set of components). In this case we will often omit ι and simply write $\mathcal{C}_0 \subseteq \mathcal{C}$ (though we do not mean by this that ι is injective on the level of simplicial sets). A subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is called *wide* if the induced map $\mathcal{C}_0^{\simeq} \longrightarrow \mathcal{C}^{\simeq}$ is an equivalence of ∞-groupoids. In this case we will say that an arrow of $\mathcal{C}$ belongs to $\mathcal{C}_0$ if is equivalent in $\mathcal{C}^{\Delta^1}$ to an arrow which is in the image of $\mathcal{C}_0$. In particular, when $\mathcal{C}_0 \subseteq \mathcal{C}$ is a wide subcategory then every equivalence in $\mathcal{C}$ belongs to $\mathcal{C}_0$.

Definition 4.2.1. Let $\mathcal{C}$ be an ∞-category. By a *factorization system* on $\mathcal{C}$ we will mean two wide subcategories $\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}} \subseteq \mathcal{C}$, whose maps we call the *inert* and *active* maps respectively, such that the following two conditions hold:

(1) Every morphism $h : x \longrightarrow z$ in $\mathcal{C}$ can be factored as a composition $x \xrightarrow{f} y \xrightarrow{g} z$ such that f is inert and g is active.
(2) For every commutative square in $\mathcal{C}$ of the form

(4.3)
$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] & z \arrow[d, "g"] \\ y \arrow[r] \arrow[ur, dashed] & w \end{tikzcd}$$

such that f is inert and g is active the space of dotted lifts is contractible.

Remark 4.2.2. The notion of a factorization system given in Definition 4.2.1 is also sometimes called an *orthogonal factorization system* or a *unique factorization system*. It is stronger than the notion of a *weak factorization system* commonly found in litterature on model categories, in which the lifting solutions in (4.3) are not required to be essentially unique.

Exercise 4.2.3. Let $(\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}})$ be a unique factorization system on $\mathcal{C}$.

(1) For a map $h : x \longrightarrow z$ let us denote by $\mathcal{C}_{x//z} := (\mathcal{C}_{x/})_{/h}$ the ∞-category of factorizations
$$x \xrightarrow{f} y \xrightarrow{g} z$$
of f. Consider the full subcategory $\mathcal{C}_{x//z} \times_{\mathcal{C}_{x/}} \mathcal{C}^{\mathrm{in}}_{x/} \subseteq \mathcal{C}_{x//z}$ consisting of those factorizations $h = g \circ f$ such that f is inert and the full subcategory $\mathcal{C}_{x//z} \times_{\mathcal{C}_{x/}} \mathcal{C}^{\mathrm{act}}_{/z} \subseteq \mathcal{C}_{x//z}$ consisting of those factorizations $h = g \circ f$ such that g is active. Show that if $h = g \circ f$ is a factorization such that f is inert and g is active then (f, g) is final when considered as an object of $\mathcal{C}_{x//z} \times_{\mathcal{C}_{x/}} \mathcal{C}^{\mathrm{in}}_{x/}$ and initial when considered as an object of $\mathcal{C}_{x//z} \times_{\mathcal{C}_{/z}} \mathcal{C}^{\mathrm{act}}_{/z}$.
(2) Deduce that for every morphism $h : x \longrightarrow z$ in $\mathcal{C}$ the full subcategory $\mathcal{E}_h \subseteq \mathcal{C}_{x//y}$ consisting of those factorizations
$$x \xrightarrow{f} y \xrightarrow{g} z$$
of h such that $f \in \mathcal{C}^{\mathrm{in}}$ and $g \in \mathcal{C}^{\mathrm{act}}$, is a contractible Kan complex.
(3) Show that if $x \xrightarrow{f} y \xrightarrow{g} z$ are two composable arrows such that f is inert then g is inert if and only if $g \circ f$ is inert. Similarly, if g is active then f is active if and only if $g \circ f$ is active.

Definition 4.2.4. A *weak ∞-operad* is an ∞-category $\mathcal{C}$ equipped with a factorization system $(\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}})$ and a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ such that the following condition hold: for every $x \in \mathcal{C}$ there exists a finite set I_x and a collection of maps $f_i : x \longrightarrow x_i$ in $\mathcal{C}^{\mathrm{in}}$ indexed by $i \in I_x$, such that $x_i \in \mathcal{C}_0^{\mathrm{in}}$ and the associated functor

$$I_x \xrightarrow{\simeq} \mathcal{C}^{\mathrm{in}}_{x/} \times_{\mathcal{C}^{\mathrm{in}}} \mathcal{C}^{\mathrm{in}}_0$$

is coinitial (where $\mathcal{C}^{\mathrm{in}}_0 = \mathcal{C}_0 \cap \mathcal{C}^{\mathrm{in}}$ is the preimage of $\mathcal{C}_0$ in $\mathcal{C}^{\mathrm{in}}$).

We will refer to maps of $\mathcal{C}$ which belong to $\mathcal{C}^{\mathrm{in}}$ and as *inert* and *active* respectively, and to the objects of $\mathcal{C}^{\mathrm{in}}_0$ as the *basics*.

Definition 4.2.5. By a *functor of weak ∞-operads* we will mean a functor between the underlying ∞-categories $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ such that

- φ sends inert maps to inert maps, active maps to active maps, and basics to basics;
- for every $x \in \mathcal{C}$, the induced map $\mathcal{C}^{\mathrm{in}}_{x/} \times_{\mathcal{C}} \mathcal{C}^{\mathrm{in}}_0 \longrightarrow \mathcal{D}^{\mathrm{in}}_{x/} \times_{\mathcal{D}} \mathcal{D}^{\mathrm{in}}_0$ is coinitial.

Remark 4.2.6. Let $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}}, \mathcal{C}_0)$ be a weak ∞-operad. The condition that $\mathcal{C}^{\mathrm{in}}_{/x} \times_{\mathcal{C}^{\mathrm{in}}} \mathcal{C}^{\mathrm{in}}_0$ admits a coinitial map from a finite set is equivalent to the condition that $\mathcal{C}^{\mathrm{in}}_{/x} \times_{\mathcal{C}^{\mathrm{in}}} \mathcal{C}^{\mathrm{in}}_0$ decomposes as a finite disjoint union of ∞-categories, each of which possesses an initial object. Let

$$I_x \subseteq \mathcal{C}^{\mathrm{in}}_{/x} \times_{\mathcal{C}^{\mathrm{in}}} \mathcal{C}^{\mathrm{in}}_0 \tag{4.4}$$

be the full subcategory spanned by those objects which are initial in their component. Then I_x is categorically equivalent to a finite set, and the inclusion (4.4) is coinitial. In particular, up to categorical equivalence we may assume that the I_x in Definition (4.2.4) are given by the canonical choice of (4.4). If $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ is now a functor that sends inerts to inerts, actives to actives and basics to basics then the condition that φ is a map of weak ∞-operads is equivalent to the condition that for every $x \in \mathcal{C}$ the functor φ maps I_x to $I_{\varphi(x)}$ bijectively.

Definition 4.2.7. Let $(\mathcal{C}, \mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}}, \mathcal{C}^{\mathrm{in}}_0)$ be a weak ∞-operad and let $\mathcal{E}$ be an ∞-category with finite products. A $\mathcal{C}$*-monoid object in* $\mathcal{E}$ is a functor $\varphi : \mathcal{C} \longrightarrow \mathcal{E}$ such that $\varphi|_{\mathcal{C}^{\mathrm{in}}}$ is a right Kan extension if $\varphi|_{\mathcal{C}^{\mathrm{in}}_0}$ (see §2.7). By Theorem 2.7.5 this is equivalent to saying that for every $x \in \mathcal{C}$ the collection of maps $f_i : \varphi(x) \longrightarrow \varphi(x_i)$ for $i \in I_x$ exhibit $\varphi(x)$ as the product in $\mathcal{E}$ of the objects $\{\varphi(x_i)\}_{i \in I_x}$.

Remark 4.2.8. If $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor of weak ∞-operads in the sense of Definition 4.2.5 and $\mathcal{E}$ is any ∞-category with finite products then restriction along φ sends $\mathcal{D}$-monoids in $\mathcal{E}$ to $\mathcal{C}$-monoids in $\mathcal{E}$.

Example 4.2.9. For any factorization system $(\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}})$ on $\mathcal{C}$ we can choose $\mathcal{C}_0 = \mathcal{C}$ as our subcategory of basics and get an associated weak ∞-operad. We then have $\mathrm{Mon}_{\mathcal{C}}(\mathcal{E}) = \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ for every $\mathcal{E}$.

Example 4.2.10. In any ∞-category $\mathcal{C}$ we have two canonical factorization systems: the one where the inerts are the equivalences and the actives are all maps and the one where the inerts are all maps and the actives are the equivalences. In addition, any full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ can serve as a subcategory of basics for the former, in which case $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ is a monoid object if and only if $\psi(c)$ is terminal in $\mathcal{E}$ for every $c \notin \mathcal{C}_0$.

Example 4.2.11. If $\mathcal{O}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ is an ∞-operad then the ∞-category $\mathcal{O}^{\otimes}$ has an associated structure of a weak ∞-operad where $(\mathcal{O}^{\otimes})^{\mathrm{in}}$ consists of the inert maps,

$(\mathcal{O}^{\otimes})^{\mathrm{act}}$ consists of the active maps, and the subcategory of basics consists of the objects which lie above $\langle 1\rangle$.

Example 4.2.12. The category Δ^{op} admits a structure of a weak ∞-operad, where $\rho : [n] \longrightarrow [m]$ is active if $\rho(0) = 0$ and $\rho(n) = m$, is inert if it is of the form $\rho(i) = i + a$ for some a, and the full subcategory of basics is $\{[1]\}$. More generally, the Leinster category of any perfect operator category is a weak ∞-operad.

There is a map of weak ∞-operads $\Delta^{\mathrm{op}} \longrightarrow \mathrm{Ass}^{\otimes}$ which sends $[n]$ to the pointed set $\{*, (0,1), (1,2), ..., (n-1,n)\} \cong \langle n-1\rangle$ of consecutive edges in $[n]$ (plus an extra base point). Restriction along this map induces an equivalence

$$\mathrm{Mon}_{\mathrm{Ass}^{\otimes}}(\mathcal{E}) \xrightarrow{\simeq} \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathcal{E})$$

for every ∞-category $\mathcal{E}$ with finite products.

Example 4.2.13. If $\mathcal{C},\mathcal{D}$ are two weak ∞-operads then the cartesian product $\mathcal{C}\times\mathcal{D}$ has a naturally associated weak ∞-operad structure, where a map $(f,g) : (x,y) \longrightarrow (x',y')$ is inert (resp. active) if and only if f is inert (resp. active) in $\mathcal{C}$ and g is inert (resp. active) in $\mathcal{D}$, and the full subcategory of basics is $\mathcal{C}_0 \times \mathcal{D}_0 \subseteq \mathcal{C}\times\mathcal{D}$. If $\mathcal{E}$ is an ∞-category with finite products then $\mathrm{Mon}_{\mathcal{C}\times\mathcal{D}}(\mathcal{E}) \simeq \mathrm{Mon}_{\mathcal{C}}(\mathrm{Mon}_{\mathcal{D}}(\mathcal{E}))$.

Definition 4.2.14. Let $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of weak ∞-operads. We will say that φ is a *weak approximation* if the following conditions holds:

(1) For every $y \in \mathcal{C}$ the homotopy fibers of $\mathcal{C}^{\mathrm{act}}_{/y} \longrightarrow \mathcal{D}^{\mathrm{act}}_{/\varphi(y)}$ are weakly contractible.
(2) The map $\varphi^{-1}\mathcal{D}_0^{\mathrm{in}} \longrightarrow \mathcal{D}_0^{\mathrm{in}}$ is a localization map (Definition 2.2.13).

We will say that φ is a *strong approximation* if it is a weak approximation and the map $\varphi^{-1}\mathcal{D}_0^{\mathrm{in}} \longrightarrow \mathcal{D}_0^{\mathrm{in}}$ is an equivalence of ∞-categories.

Example 4.2.15. The map $\varphi : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Ass}^{\otimes}$ of Example 4.2.12 is a strong approximation. This follows from the fact that for every $[n] \in \Delta^{\mathrm{op}}$ the functor $(\Delta^{\mathrm{op}})^{\mathrm{act}}_{/[n]} \longrightarrow (\mathrm{Ass}^{\otimes})^{\mathrm{act}}_{/\varphi([n])}$ is an equivalence of categories and $(\mathrm{Ass}^{\otimes}_{\langle 1\rangle})^{\mathrm{in}} \simeq \varphi^{-1}(\mathrm{Ass}^{\otimes}_{\langle 1\rangle})^{\mathrm{in}} \simeq *$.

Remark 4.2.16. If $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ and $\varphi' : \mathcal{C}' \longrightarrow \mathcal{D}'$ are weak approximations of weak ∞-operads then $(\varphi,\varphi') : \mathcal{C}\times\mathcal{C}' \longrightarrow \mathcal{D}\times\mathcal{D}'$ is a weak approximation as well.

Remark 4.2.17. Consider a homotopy pullback square of ∞-categories

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] & \mathcal{C} \arrow[d, "\psi"] \\ \mathcal{D} \arrow[r, "\varphi"] & \mathcal{E} \end{tikzcd}$$

If $\mathcal{C},\mathcal{D}$ and $\mathcal{E}$ are given weak ∞-operad structures such that φ and ψ are weak ∞-operad maps then $\mathcal{P}$ inherits an associated weak ∞-operad structure where a map in $\mathcal{P}$ is inert (resp. active) if and only if its image in $\mathcal{C}$ and $\mathcal{D}$ is inert (resp. active) and $\mathcal{P}_0 := \mathcal{D}_0 \times^h_{\mathcal{E}_0} \mathcal{C}_0$. In this case, if ψ is a strong approximation then ψ' is a strong approximation.

Our main interest in weak approximations comes from the following result:

Proposition 4.2.18. *Let $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ be a weak approximation map of weak ∞-operads and let $\mathcal{E}$ be an ∞-category which admits small limits. Then the restriction*

$$\rho^* : \mathrm{Mon}_{\mathcal{D}}(\mathcal{E}) \longrightarrow \mathrm{Mon}_{\mathcal{C}}(\mathcal{E}) \tag{4.5}$$

is fully-faithful and its essential image consists of those monoid objects $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ such that $\psi|_{\varphi^{-1}\mathcal{D}_0^{\mathrm{in}}}$ factors through $\mathcal{D}_0^{\mathrm{in}}$. In particular, if φ is a strong approximation then (4.5) is an equivalence of ∞-categories.

The proof of Proposition 4.2.18 will require two lemmas:

Lemma 4.2.19. *Let $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ be a map of weak approximation of weak ∞-operads. Then for every object $y \in \mathcal{C}$ and every map $x \longrightarrow \varphi(y)$ in $\mathcal{D}$, the full subcategory $\mathcal{X} \subseteq \mathcal{C}_{/y} \times_{\mathcal{D}_{/\varphi(y)}} \mathcal{D}_{x//\varphi(y)}$ spanned by those $(g : y' \longrightarrow y, h : x \longrightarrow \varphi(y'))$ such that h is inert, is weakly contractible.*

Proof. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the full subcategory spanned by those $(g : y' \longrightarrow y, h : x \longrightarrow \varphi(y'))$ such that in addition g is active in $\mathcal{C}$. We then note that if (g, h) is an object of $\mathcal{X}$ then the comma ∞-category $(\mathcal{X}_0)_{(g,h)/}$ is equivalent to the ∞-category of factorizations of g as $g = g'' \circ g'$ such that $\varphi(g')$ is inert in $\mathcal{D}$ and g'' is active in $\mathcal{C}$. This ∞-category admits an initial object, given by any factorization $g = g'' \circ g'$ such that g' is inert in $\mathcal{C}$ and g'' is active in $\mathcal{C}$ (see Exercise 4.2.3). It then follows that the inclusion $\mathcal{X}_0 \subseteq \mathcal{X}$ is cofinal (in fact, that it admits a left adjoint) and is hence a weak homotopy equivalence. To show that $\mathcal{X}$ is weakly contractible it will hence suffice to show that $\mathcal{X}_0$ is weakly contractible. We note that $\mathcal{X}_0$ maps to the ∞-category of inert-active factorizations of f in $\mathcal{D}$. The latter category is a contractible Kan complex by Exercise 4.2.3 and so it will suffice to check that the homotopy fiber over a given factorization $x \longrightarrow x' \longrightarrow \varphi(c)$ is weakly contractible. But this follows directly from our hypothesis that φ is a weak approximation. $\square$

Lemma 4.2.20. *Let $\varphi : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of weak ∞-operads and set $\mathcal{C}^{\varphi} := \varphi^{-1}\mathcal{D}^{\mathrm{in}}$ and $\mathcal{C}_0^{\varphi} := \varphi^{-1}\mathcal{D}_0^{\mathrm{in}}$. Then for every ∞-category $\mathcal{E}$ with limits and every $\mathcal{C}$-monoid object $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ we have that $\psi|_{\mathcal{C}^{\varphi}}$ is a right Kan extension of $\varphi|_{\mathcal{C}_0^{\varphi}}$.*

Proof. Consider the full subcategory inclusion

$$\mathcal{C}_{y/}^{\mathrm{in}} \times_{\mathcal{C}} (\mathcal{C}_0^{\varphi})^{\mathrm{in}} \subseteq \mathcal{C}_{y/}^{\varphi} \times_{\mathcal{C}} \mathcal{C}_0^{\varphi} \tag{4.6}$$

where the left hand consists of those $g : y \longrightarrow y'$ on the right hand side which are furthermore inert in $\mathcal{C}$ (this subcategory is indeed full by Exercise 4.2.3(1)). We claim that the inclusion (4.6) is coinitial. To see this, observe that for $g : y \longrightarrow y'$ in $\mathcal{C}_{y/}^{\varphi} \times_{\mathcal{C}} \mathcal{C}_0^{\varphi}$ the comma ∞-category $[\mathcal{C}_{y/}^{\mathrm{in}} \times_{\mathcal{C}} (\mathcal{C}_0^{\varphi})^{\mathrm{in}}]_{/g}$ can be identified with the ∞-category of factorizations

$$\begin{tikzcd} & y'' \arrow[dr, "g''"] & \\ y \arrow[ur, "g'"] \arrow[rr, "g"] & & y' \end{tikzcd} \tag{4.7}$$

of g such that g' is inert and $\varphi(y'') \in \mathcal{D}^0$. This ∞-category embeds in the ∞-category $\mathcal{C}_{y//y'} \times_{\mathcal{C}_{y/}} \mathcal{C}_{y/}^{\mathrm{in}}$ of all factorizations $g = g'' \circ g'$ with g' inert. By Exercise 4.2.3(2) the latter has a final object given by any factorzation $g = g'' \circ g'$ as in (4.7) such that g' is inert and g'' is active. In this case, since g and g' map to inert maps in $\mathcal{D}$ we have by Exercise 4.2.3(1) that g'' must map to a map in $\mathcal{D}$ which is both inert and active, and hence an equivalence. It then follows that $\varphi(y'')\mathcal{D}_0$ and so the factorization (g', g'') belongs to $[\mathcal{C}_{y/}^{\mathrm{in}} \times_{\mathcal{C}} (\mathcal{C}_0^{\varphi})^{\mathrm{in}}]_{/g}$. We may thus conclude that the latter ∞-category has final objects and is therefore weakly contractible. The inclusion (4.6) is consequently coinitial.

It now follows from Theorem 2.7.5 that $\psi|_{\mathcal{C}^{\varphi}}$ is a right Kan extension of $\psi|_{\mathcal{C}^{\varphi}_0}$ if and only if $\psi|_{\mathcal{C}^{\mathrm{in}}}$ is a right Kan extension of $\psi|_{(\mathcal{C}^{\varphi}_0)^{\mathrm{in}}}$. Since $(\mathcal{C}^{\varphi}_0)^{\mathrm{in}}$ is a full subcategory of $\mathcal{C}^{\mathrm{in}}$ which contains $\mathcal{C}^{\mathrm{in}}_0$ the latter condition holds whenever ψ is a $\mathcal{C}$-monoid object, as desired. □

Proof of Proposition 4.2.18. $\mathcal{C}^{\varphi} \coloneqq \varphi^{-1}\mathcal{D}^{\mathrm{in}}$ and $\mathcal{C}^{\varphi}_0 \coloneqq \varphi^{-1}\mathcal{D}^{\mathrm{in}}_0$ be as in Lemma 4.2.20. Let us say that a $\mathcal{C}$-monoid $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ is *locally constant* (relative to $\mathcal{D}$) if $\psi|_{\mathcal{C}^{\varphi}_0}$ factors through $\mathcal{D}^{\mathrm{in}}_0$. We will denote by $\mathrm{Mon}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{E}) \subseteq \mathrm{Mon}_{\mathcal{C}}(\mathcal{E})$ the full subcategory spanned by the locally constant monoid objects. By definition the restriction functor (4.5) lands in locally constant $\mathcal{C}$-monoids, thus yielding a functor

$$\rho^* : \mathrm{Mon}_{\mathcal{D}}(\mathcal{E}) \longrightarrow \mathrm{Mon}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{E}) \tag{4.8}$$

The key idea of the proof is that (4.8) admits an inverse which is given by *right Kan extension* along φ. More precisely, let

$$\pi : \mathcal{M} \coloneqq \mathrm{N}^{[\varphi]}([1]) \longrightarrow \Delta^1$$

be the cartesian fibration (2.39) classified by the diagram $(\Delta^1)^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ corresponding to $[\mathcal{D} \xleftarrow{\varphi} \mathcal{C}]$. As we outlined in §2.7, the ∞-category of functors $\mathcal{M} \longrightarrow \mathcal{E}$ is equivalent to the ∞-category of triples (ψ, ψ', δ) consisting of a functor $\psi : \mathcal{C} \longrightarrow \mathcal{E}$, a functor $\psi' : \mathcal{D} \longrightarrow \mathcal{E}$, and a natural transformation $\delta : \psi' \circ \varphi \Rightarrow \psi$. Consider the following two properties a functor $\overline{\psi} : \mathcal{M} \longrightarrow \mathcal{E}$ can have:

(1) $\overline{\psi}|_{\mathcal{C}}$ is a locally constant monoid object and the natural transformation $\delta : \psi' \circ \varphi \Rightarrow \psi$ exhibits ψ' as a right Kan extension of ψ along φ.
(2) $\overline{\psi}|_{\mathcal{D}}$ is a monoid object and the natural transformation $\delta : \psi' \circ \varphi \Rightarrow \psi$ is an equivalence.

To prove that (4.8) is an equivalence it will suffice to show that for a given $\overline{\psi} : \mathcal{M}^{\mathrm{R}}_{\varphi} \longrightarrow \mathcal{E}$, Condition (1) is equivalent to Condition (2). Indeed, assume this claim for the moment and let $\mathcal{X} \subseteq \mathrm{Fun}(\mathcal{M}^{\mathrm{R}}_{\varphi}, \mathcal{E})$ be the full subcategory spanned by those $\overline{\psi}$ which satisfy either of those equivalent conditions. Consider the diagram

$$\begin{tikzcd} & \mathcal{X} \arrow[dl] \arrow[dr] & \\ \mathrm{Mon}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{E}) & & \mathrm{Mon}_{\mathcal{D}}(\mathcal{E}) \end{tikzcd} \tag{4.9}$$

Since we assumed that $\mathcal{E}$ admits limits we have by Theorem 2.7.5 that every $\varphi : \mathcal{C} \longrightarrow \mathcal{E}$ admits a right Kan extension $\overline{\psi} : \mathcal{M} \longrightarrow \mathcal{E}$. Since $\mathcal{X}$ is determines by Condition (1) the left diagonal projection in (4.9) is obtained from the cocartesian fibration $\mathrm{Fun}(\mathcal{M}^{\mathrm{R}}_{\varphi}, \mathcal{E}) \times_{\mathrm{Fun}(\mathcal{C},\mathcal{E})} \mathrm{Mon}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{E}) \longrightarrow \mathrm{Mon}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{E})$ by restricting to those objects which are final in their fiber, and is hence a trivial Kan fibration by Exercise 2.6.10. On the other hand, since $\mathcal{X}$ is determined by Condition (2) the right diagonal map in 4.9 is obtained by restricting the cartesian fibration $\mathrm{Fun}(\mathcal{M}^{\mathrm{R}}_{\varphi}, \mathcal{E}) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{E})} \mathrm{Mon}_{\mathcal{D}}(\mathcal{E}) \longrightarrow \mathrm{Mon}_{\mathcal{D}}(\mathcal{E})$ to the full subcategory spanned by those objects which are initial in their fiber, and is hence a trivial Kan fibration as well. In particular, (4.9) determines an equivalence between $\mathrm{Mon}_{\mathcal{D}}(\mathcal{E})$ and $\mathrm{Mon}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{E})$. This implies that (4.8) is an equivalence is well, since we can identify (4.8) up to homotopy with the composition of (any) section $\mathrm{Mon}_{\mathcal{D}}(\mathcal{E}) \longrightarrow \mathcal{X}$ and the projection $\mathcal{X} \longrightarrow \mathrm{Mon}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{E})$.

We shall now prove that Condition (1) and Condition (2) are equivalent. Let $\overline{\psi} : \mathcal{M} \longrightarrow \mathcal{E}$ be such that $\overline{\psi}|_{\mathcal{C}}$ is a locally constant $\mathcal{C}$-monoid object. By Theorem 2.7.5 the condition that $\overline{\psi}$ is a right Kan extension is equivalent to the condition that for every $x \in \mathcal{D}$ the composed map $(\mathcal{M}_{x/} \times_{\mathcal{M}} \mathcal{C})^{\triangleleft} \longrightarrow \mathcal{M} \xrightarrow{\overline{\psi}} \mathcal{E}$ is a limit diagram. Let $\mathcal{I}_x \subseteq \mathcal{M}_{x/} \times_{\mathcal{M}} \mathcal{C}$ be the full subcategory spanned by those objects $[x \longrightarrow y] \in \mathcal{M}_{x/} \times_{\mathcal{M}} \mathcal{C}$ whose corresponding arrow $f : x \longrightarrow \varphi(y)$ in $\mathcal{D}$ is inert. We claim that the inclusion $\mathcal{I}_x \subseteq \mathcal{M}_{x/} \times_{\mathcal{M}} \mathcal{C}$ is coinitial. Concretely, what we need to check is that for every arrow in $\mathcal{D}$ of the form $f : x \longrightarrow \varphi(y)$, the comma ∞-category $(\mathcal{I}_x)_{/f}$ is weakly contractible. This comma ∞-category can be identified with the full subcategory $\mathcal{X} \subseteq \mathcal{C}_{/y} \times_{\mathcal{D}_{/\varphi(y)}} \mathcal{D}_{x//\varphi(y)}$ consisting of the pairs $(g : y' \longrightarrow y, h : x \longrightarrow \varphi(y'))$ such that h is inert, and is hence weakly contractible by Lemma 4.2.19 and our assumption that φ is a weak approximation. We then get that the second part of condition (1) is equivalent to the condition that for every $x \in \mathcal{D}$ the restricted diagram $\overline{\psi}_x : \mathcal{I}_x^{\triangleleft} \longrightarrow \mathcal{M} \xrightarrow{\overline{\psi}} \mathcal{E}$ is a limit diagram. Using again Theorem 2.7.5 this is equivalent to saying that $\delta|_{\mathcal{C}^{\varphi}} : \varphi^*\psi'|_{\mathcal{C}^{\varphi}} \Rightarrow \psi|_{\mathcal{C}^{\varphi}}$ exhibits $\psi'|_{\mathcal{D}^{\mathrm{in}}}$ as a right Kan extension of $\psi|_{\mathcal{C}^{\varphi}}$. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{C}_0^{\varphi} \arrow[r] \arrow[d] & \mathcal{D}_0^{\mathrm{in}} \arrow[d] \\ \mathcal{C}^{\varphi} \arrow[r] & \mathcal{D}^{\mathrm{in}} \end{tikzcd} \tag{4.10}$$

By Lemma 4.2.20 we have that $\psi|_{\mathcal{C}^{\varphi}}$ is a right Kan extension of $\psi|_{\mathcal{C}_0^{\varphi}}$. Since $\mathcal{D}_0^{\mathrm{in}} \subseteq \mathcal{D}^{\mathrm{in}}$ is a full inclusion a double application of Proposition 2.7.7 and an application of Remark 2.7.6 imply that Condition (1) is equivalent to the following condition:

(1') $\overline{\psi}|_{\mathcal{C}}$ is a locally constant monoid object, $\overline{\psi}|_{\mathcal{D}}$ is a monoid object, and the natural transformation $\delta|_{\mathcal{C}_0^{\varphi}} : \varphi^*\psi'|_{\mathcal{C}_0^{\varphi}} \Rightarrow \psi|_{\mathcal{C}_0^{\varphi}}$ exhibits $\psi'|_{\mathcal{D}_0^{\mathrm{in}}}$ as a right Kan extension of $\psi|_{\mathcal{C}_0^{\varphi}}$ along $\varphi_0 : \mathcal{C}_0^{\varphi} \longrightarrow \mathcal{D}_0^{\mathrm{in}}$.

Let us hence focus our attention on the restricted functor $\varphi_0 : \mathcal{C}_0^{\varphi} \longrightarrow \mathcal{D}_0^{\mathrm{in}}$. Let

$$\mathcal{M}_0 = \mathrm{N}^{[\varphi_0]}([1]) \longrightarrow \Delta^1$$

be the associated relative nerve, so that we have a natural inclusion of simplicial sets $\mathcal{M}_0 \subseteq \mathcal{M}$. Let $\overline{\psi}_0 := \overline{\psi}|_{\mathcal{M}_0}$. Since $\overline{\psi}|_{\mathcal{C}}$ is a $\mathcal{C}$-monoid object Condition (2) can be checked only for objects in $\mathcal{C}_0^{\varphi}$. To finish the proof it will hence suffice to show that $\overline{\psi}_0$ is a right Kan extension if and only if the natural transformation $\delta|_{\mathcal{C}_0^{\varphi}} : \varphi^*\psi'|_{\mathcal{C}_0^{\varphi}} \Rightarrow \psi|_{\mathcal{C}_0^{\varphi}}$ is an equivalence. Now since we assumed that $\overline{\psi}|_{\mathcal{C}}$ is a locally constant $\mathcal{C}$-monoid object it follows that $(\overline{\psi}_0)|_{\mathcal{C}_0^{\varphi}}$ factors through $\mathcal{D}_0^{\mathrm{in}}$. The desired result now follows from assumption that the restriction functor $\mathrm{Fun}(\mathcal{D}_0^{\mathrm{in}}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}_0^{\varphi}, \mathcal{E})$ is fully-faithful since $\mathcal{C}_0^{\varphi} \longrightarrow \mathcal{D}_0^{\mathrm{in}}$ is assumed to be a localization functor. □

We finish this section with the following result which allows one to relate two different weak ∞-operad structures on the same ∞-category $\mathcal{C}$.

Lemma 4.2.21. *Let $\mathcal{C}$ be an ∞-category and let $(\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}}, \mathcal{C}_0)$ and $(\mathcal{C}^{\mathrm{in}'}, \mathcal{C}^{\mathrm{act}'}, \mathcal{C}_1)$ two weak ∞-operad structures on $\mathcal{C}$ such that $\mathcal{C}^{\mathrm{in}'} \subseteq \mathcal{C}^{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{act}} \subseteq \mathcal{C}^{\mathrm{act}'}$. Let $\mathcal{E}$ be an ∞-category with finite products. Then the following assertions hold:*

(1) *Suppose that* $\mathcal{C}_1 = \mathcal{C}_0$ *and that for every morphism* $f : x \longrightarrow y$ *in* $\mathcal{C}^{\mathrm{in}}$ *such that* $y \in \mathcal{C}_0$ *and every factorization* $x \xrightarrow{f'} y' \xrightarrow{f''} y$ *of* f *such that* f' *is in* $\mathcal{C}^{\mathrm{in}'}$ *we have that* y' *is in* $\mathcal{C}_0$*. Then* $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ *is a monoid object with respect to* $(\mathcal{C}^{\mathrm{in}'}, \mathcal{C}^{\mathrm{act}'}, \mathcal{C}_0)$ *if and only if it is a monoid object with respect to* $(\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}}, \mathcal{C}_0)$*.*

(2) *Suppose that* $\mathcal{C}^{\mathrm{in}'} = \mathcal{C}^{\mathrm{in}}$*,*$\mathcal{C}^{\mathrm{act}'} = \mathcal{C}^{\mathrm{act}}$ *and* $\mathcal{C}_0 \subseteq \mathcal{C}_1$*. Then* $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ *is a monoid object with respect to* $(\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}}, \mathcal{C}_0)$ *if and only if it is a monoid object with respect to* $(\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}}, \mathcal{C}_1)$ *and in addition* $\psi|_{\mathcal{C}^{\mathrm{in}}_1}$ *is a right Kan extension of* $\psi|_{\mathcal{C}^{\mathrm{in}}_0}$*.*

Proof. Let us first prove (1). Let $x \in \mathcal{C}$ be an object and write ${\mathcal{C}^{\mathrm{in}'}}_0 = \mathcal{C}^{\mathrm{in}'} \cap \mathcal{C}_0$. Then our assumption implies that the inclusion ${\mathcal{C}^{\mathrm{in}'}}_{x/} \times_{\mathcal{C}} {\mathcal{C}^{\mathrm{in}'}}_0 \subseteq \mathcal{C}^{\mathrm{in}}_{x/} \times_{\mathcal{C}} \mathcal{C}^{\mathrm{in}}_0$ is coinitial. It then follows that $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ is a monoid object with respect to $(\mathcal{C}^{\mathrm{in}'}, \mathcal{C}^{\mathrm{act}'}, \mathcal{C}_0)$ if and only if it is a monoid object with respect to $(\mathcal{C}^{\mathrm{in}}, \mathcal{C}^{\mathrm{act}}, \mathcal{C}_0)$.

To prove (2), we first note that the "if" direction follows from the pasting lemma for right Kan extensions (Proposition 2.7.7). To show the "only if" direction note that since $\mathcal{C}^{\mathrm{in}}_1 \subseteq \mathcal{C}^{\mathrm{in}}$ is a full subcategory which contains $\mathcal{C}^{\mathrm{in}}_0$ we have by Remark 2.7.6 that if $\psi : \mathcal{C} \longrightarrow \mathcal{E}$ is a $\mathcal{C}$-monoid object then $\psi|_{\mathcal{C}^{\mathrm{in}}_1}$ is a right Kan extension of $\psi|_{\mathcal{C}^{\mathrm{in}}_0}$, in which case ψ must also be a right Kan extension of $\psi|_{\mathcal{C}^{\mathrm{in}}_1}$ by Proposition 2.7.7. □

4.3. **Tensor products of ∞-operads.** Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be two ∞-operads. In this section we will discuss how to associate to $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ a new ∞-operad $\mathcal{Q}^{\otimes}$, called the *tensor product* of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$, which enjoys the following mapping property: for every symmetric monoidal ∞-category $\mathcal{C}$ we have a canonical equivalence

$$\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{P}}(\mathcal{C})),$$

where $\mathrm{Alg}_{\mathcal{P}}(\mathcal{C})$ is endowed with a symmetric monoidal structure induced from that of $\mathcal{C}$. Recall first that if (I, i_0) and (J, j_0) are two finite pointed sets then their *smash product*

$$I \wedge J = I \times J / [\{i_0\} \times J \cup I \times \{j_0\}]$$

is the finite pointed set obtained from the cartesian product of I and J by collapsing $\{i_0\} \times J$ and $I \times \{j_0\}$ to a point (which also serves as the base point of $I \wedge J$). The operation of smash product determines a symmetric monoidal structure on the category of finite pointed sets, and hence also on its skeleton Fin_* spanned by the finite pointed sets $\langle n \rangle$ for $n \geq 0$. In particular, for $\langle n \rangle, \langle m \rangle \in \mathrm{Fin}_*$ we have $\langle n \rangle \wedge \langle m \rangle = \langle n \cdot m \rangle$.

Construction 4.3.1. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be ∞-operads. We define $(\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}))^{\otimes}$ to be the simplicial set whose n-simplices are pairs (σ, τ) where σ is an n-simplex of $\mathrm{N}(\mathrm{Fin}_*)$ and $\tau : \Delta^n \times \mathcal{P}^{\otimes} \longrightarrow \mathcal{P}^{\otimes}$ is a map which fits in a commutative square

(4.11)
$$\begin{tikzcd}
\Delta^n \times \mathcal{P}^{\otimes} \arrow[r, "\tau"] \arrow[d, "(\sigma,\pi)"'] & \mathcal{C}^{\otimes} \arrow[d] \\
\mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\mathrm{Fin}_*) \arrow[r, "\wedge"] & \mathrm{N}(\mathrm{Fin}_*)
\end{tikzcd}$$

and such that if $f : \overline{x} \longrightarrow \overline{y}$ is an inert map in $\mathcal{P}^{\otimes}$ and $i \in [n]$ then τ sends $\Delta^{\{i\}} \times f$ to an inert map in $\mathcal{P}$. One can then check that the projection $\mathrm{Alg}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ given by $(\sigma, \tau) \mapsto \sigma$ exhibits $(\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}))^{\otimes}$ as an ∞-operad. We will refer to $\mathrm{Alg}^{\otimes}_{\mathcal{O}}(\mathcal{P})$ as the ∞*-operad of* $\mathcal{O}$*-algebras in* $\mathcal{P}$.

Remark 4.3.2. When $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category $\mathrm{Alg}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is a symmetric monoidal ∞-category for every ∞-operad $\mathcal{O}$. This is the "pointwise" symmetric monoidal structure on algebras.

Definition 4.3.3. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}, \mathcal{Q}^{\otimes}$ be ∞-operads. An a *bifunctor* of ∞-operads is a commutative diagram of the form

$$\begin{tikzcd} \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[r, "\varphi"] \arrow[d] & \mathcal{Q}^{\otimes} \arrow[d] \\ \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\mathrm{Fin}_*) \arrow[r, "\wedge"] & \mathrm{N}(\mathrm{Fin}_*) \end{tikzcd} \tag{4.12}$$

such that for every inert morphism $f : \overline{x} \longrightarrow \overline{y}$ in $\mathcal{O}^{\otimes}$ and every inert morphism $g : \overline{z} \longrightarrow \overline{w}$ in $\mathcal{P}^{\otimes}$, the arrow $\varphi(f, g)$ is inert in $\mathcal{Q}^{\otimes}$. We will denote by $\mathrm{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{Q}^{\otimes})$ the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}, \mathcal{Q}^{\otimes})$ spanned by the bifunctors.

Remark 4.3.4. For ∞-operads $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}, \mathcal{Q}^{\otimes}$ we have a canonical isomorphism of simplicial sets

$$\mathrm{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{Q}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{P}}(\mathcal{Q})).$$

Definition 4.3.5. We will say that a bifunctor $\varphi : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes}$ *exhibits* $\mathcal{Q}^{\otimes}$ *as the tensor product of* $\mathcal{O}$ *and* $\mathcal{P}$ if for every ∞-operad $\mathcal{R}^{\otimes}$, composition with φ induces an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathcal{Q}}(\mathcal{R}) \xrightarrow{\simeq} \mathrm{BiFun}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}; \mathcal{R}^{\otimes}) \cong \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{P}}(\mathcal{R})).$$

It follows by standard arguments that if a tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ exists then it is essentially unique. To show that a tensor product always exists it is convenient to employ the model structure of Theorem 4.1.19. In particular, if $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are two ∞-operads then we may consider the cartesian product of marked simplicial sets $\mathcal{O}^{\otimes,\natural} \times \mathcal{P}^{\otimes,\natural}$ as an object of $(\mathrm{Set}_{\Delta}^{+})_{/(\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}})}$ via the composed map

$$\mathcal{O}^{\otimes,\natural} \times \mathcal{P}^{\otimes,\natural} \longrightarrow (\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}}) \times (\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}}) \xrightarrow{\wedge} (\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}}).$$

The following is the a formal consequence of the model categorical setup of Theorem 4.1.19:

Proposition 4.3.6. *A bifunctor $\varphi : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes}$ exhibits $\mathcal{Q}^{\otimes}$ as the tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ if and only if the associated map*

$$\mathcal{O}^{\otimes,\natural} \times \mathcal{P}^{\otimes,\natural} \longrightarrow \mathcal{Q}^{\otimes,\natural}$$

is a weak equivalence in $(\mathrm{Set}_{\Delta}^{+})_{/(\mathrm{N}(\mathrm{Fin}_*), E_{\mathrm{in}})}$ with respect to the operadic model structure of Theorem 4.1.19.

We note that Proposition 4.3.6 implies in particular that a tensor product of $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$ always exists: just take any fibrant replacement of $\mathcal{O}^{\otimes,\natural} \times \mathcal{P}^{\otimes,\natural}$ in the operadic model structure.

Remark 4.3.7. Let $\varphi : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes}$ be a bifunctor of ∞-operads such that the induced map $(\mathcal{O}^{\otimes})_{\langle 1 \rangle} \times (\mathcal{P}^{\otimes})_{\langle 1 \rangle} \longrightarrow \mathcal{Q}^{\otimes}_{\langle 1 \rangle}$ is essentially surjective. It then follows from Proposition 4.1.23 that φ exhibits $\mathcal{Q}^{\otimes}$ as the tensor product of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ if and if the induced map

$$\mathrm{Mon}_{\mathcal{Q}}(\mathcal{S}) \longrightarrow \mathrm{Mon}_{\mathcal{O}}(\mathrm{Mon}_{\mathcal{P}}(\mathcal{S}))$$

is an equivalence of ∞-categories.

Unfortunately, fibrant replacements in the operadic model structure are not very explicit in general and not easy to compute. One way to overcome this issue is by using the construction of *wreath products*.

Construction 4.3.8. Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be two ∞-operads. Then $\mathcal{P}^{\otimes}$ is in particular an ∞-category and we may defined the simplicial set $(\mathcal{P}^{\otimes})^{\amalg}$ as in definition 3.3.1. We note that $(\mathcal{P}^{\otimes})^{\amalg}$ is now equipped with *two different* maps to $\mathrm{N}(\mathrm{Fin}_*)$: one is the projection $\pi : (\mathcal{P}^{\otimes})^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ associated to the construction $(-)^{\amalg}$ as in Definition 3.3.1, and the other is the composed map

$$\rho : (\mathcal{P}^{\otimes})^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin})^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin}),$$

where the first map is induced by the ∞-operad structure $\mathcal{P}^{\otimes} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$ and the second is the map $\mathrm{N}(\mathrm{Fin})^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin})$ which sends a tuple $(\langle n_1\rangle, ..., \langle n_k\rangle)$ to $\langle n_1 + ... + n_k\rangle$. We then define the *wreath product* of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ to be the fiber product

$$\begin{tikzcd} \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \arrow[r] \arrow[d] & (\mathcal{P}^{\otimes})^{\amalg} \arrow[d, "\pi"] \\ \mathcal{O}^{\otimes} \arrow[r] & \mathrm{N}(\mathrm{Fin}_*) \end{tikzcd}$$

and we equip $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ with the map to $\mathrm{N}(\mathrm{Fin}_*)$ determined by the projection to $(\mathcal{P}^{\otimes})^{\amalg}$ followed by $\rho : (\mathcal{P}^{\otimes})^{\amalg} \longrightarrow \mathrm{N}(\mathrm{Fin}_*)$.

We can identify an object of $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ with a tuple $(\overline{x}, (\overline{y}_1, ..., \overline{y}_n))$ where $\overline{x}$ is an object of $\mathcal{O}^{\otimes}$ lying above $\langle n\rangle \in \mathrm{N}(\mathrm{Fin}_*)$ and $\overline{y}_1, ..., \overline{y}_n$ are objects of $\mathcal{P}^{\otimes}$. A morphism in $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ from $(\overline{x}, (\overline{y}_1, ..., \overline{y}_n))$ to $(\overline{x}', (\overline{y}'_1, ..., \overline{y}'_{n'}))$ is given by a map $f : \overline{x} \longrightarrow \overline{x}'$ in $\mathcal{O}^{\otimes}$ lying above a map $\alpha : \langle n\rangle \longrightarrow \langle n'\rangle$ together with a collection of maps $g_i : \overline{y}_i \longrightarrow \overline{y}'_{\alpha(i)}$ in $\mathcal{P}^{\otimes}$ for every $i \in \alpha^{-1}(\langle n'\rangle^{\circ})$. We will say that such a map is *inert* if f is inert in $\mathcal{O}^{\otimes}$ and each g_i is inert in $\mathcal{D}^{\otimes}$.

The projection $\Gamma^* \longrightarrow \mathrm{Fin}_*$ determines a map $\mathcal{P}^{\otimes} \times \mathrm{N}(\mathrm{Fin}_*) \longrightarrow (\mathcal{P}^{\otimes})^{\amalg}$ given informally by the formula $(\overline{y}, \langle n\rangle) \mapsto (\overline{y}, ..., \overline{y})$. We then obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \arrow[dl] \\ & \mathrm{N}(\mathrm{Fin}_*) & \end{tikzcd} \tag{4.13}$$

Theorem 4.3.9 ([6, Theorem 2.4.4.3])**.** *Let $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ be two ∞-operads and let E_{in} be the set of of inert arrows in $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$. Then (4.13) determines a weak equivalence*

$$\mathcal{O}^{\otimes,\natural} \times \mathcal{P}^{\otimes,\natural} \longrightarrow (\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}, E_{\mathrm{in}})$$

in the operadic model structure of Theorem 4.1.19.

We will not prove Theorem 4.3.9 here. We will however prove the following variant, which is sufficient for our needs, concerning the closely related notion of *monoid objects*. For this, let us consider the ∞-categories $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$ and $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ as *weak ∞-operads* (see §4.2). More precisely, we may consider each of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ as a weak ∞-operad by Example 4.2.11, and then consider the cartesian product $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$ as a weak ∞-operad by Example 4.2.13. To put a weak ∞-operad structure on the wreath product we use the following construction:

Definition 4.3.10. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and $\mathcal{C}$ a weak ∞-operad. Define their *wreath product* as $\mathcal{O}^{\otimes} \wr \mathcal{C} \coloneqq \mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathrm{Fin}_*)} \mathcal{C}^{\amalg}$. We then endow $\mathcal{O}^{\otimes} \wr \mathcal{C}$ with the structure of a weak ∞-operad by stating that a map $(f, \{g_i\}) : (\overline{x}, c_1, ..., c_n) \longrightarrow (\overline{y}, d_1, ..., d_n)$ is inert (resp. active) if $f : \overline{x} \longrightarrow \overline{y}$ is inert (resp. active) in $\mathcal{O}^{\otimes}$ and each $g_i : c_i \longrightarrow d_{\alpha(i)}$ is inert (resp. active) in $\mathcal{C}$ (where α is the image of f in $\mathrm{N}(\mathrm{Fin}_*)$). We define the subcategory of basics in $\mathcal{O}^{\otimes} \wr \mathcal{C}$ to consist of the objects of the form (x, c) where $x \in \mathcal{O}^{\otimes}$ lies above $\langle 1 \rangle \in \mathrm{N}(\mathrm{Fin}_*)$ and c belongs to $\mathcal{C}_0$.

We propose the following variant of Theorem 4.3.9:

Theorem 4.3.11. *Let $\mathcal{E}$ be an ∞-category which admits small limits. Then restriction along the map $\varphi : \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \longrightarrow \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ induces an equivalence*

$$\mathrm{Mon}_{\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}}(\mathcal{E}) \xrightarrow{\simeq} \mathrm{Mon}_{\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}}(\mathcal{E}) \simeq \mathrm{Mon}_{\mathcal{O}^{\otimes}}(\mathrm{Mon}_{\mathcal{P}^{\otimes}}(\mathcal{E})).$$

Proof. The idea of the proof is to first replace the weak ∞-operad structures on $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$ and $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ by coarser ones (i.e., ones in which there are more inert maps). With these new weak ∞-operad structures the map φ will becomes a *strong approximation* (see Definition 4.2.14), and so we will be able to deduce a comparison from Proposition 4.2.18. We will then use Lemma 4.2.21 to relate the result back to the original weak ∞-operad structures.

We define the coarse weak ∞-operad structure on $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$ to be the product structure (Example 4.2.13) associated to the operadic structure on $\mathcal{O}^{\otimes}$ (Example 4.2.11) and the trivial weak ∞-operad structure on $\mathcal{P}^{\otimes}$ in which all maps are inert and $\mathcal{P}_0 = \mathcal{P}$ (see Examples 4.2.10 and 4.2.9). Similarly, we define the coarse weak ∞-operad structure on $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ by taking the wreath weak ∞-operad structure (Definition 4.3.10) associated to the same trivial weak ∞-operad structure on $\mathcal{P}^{\otimes}$.

We now claim that φ is a strong approximation of weak ∞-operads with respect to the coarse structures. For this we first need to check φ that for every $(\overline{x}, \overline{y}) \in \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$ the induced map

$$\left(\left(\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}\right)^{\mathrm{coarse}}_{\mathrm{act}}\right)_{/(\overline{x}, \overline{y})} \longrightarrow \left(\left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}\right)^{\mathrm{coarse}}_{\mathrm{act}}\right)_{/(\overline{x}, (\overline{y}, ..., \overline{y}))} \tag{4.14}$$

has weakly contractible homotopy fibers. But this is true because the map (4.14) is in fact an equivalence of ∞-categories: both the left and right hand side are equivalent to $(\mathcal{O}^{\otimes}_{\mathrm{act}})_{/\overline{x}}$ because the active maps in the coarse structure on $\mathcal{P}^{\otimes}$ are only the equivalences. To show that φ is in fact a strong approximation we now note that the subcategory of coarse-basics and coarse-inert maps in $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ is $(\mathcal{O}^{\otimes}_{\langle 1 \rangle})^{\simeq} \times_{\{\langle 1 \rangle\}} \mathcal{P}^{\otimes} \subseteq \mathcal{O}^{\otimes} \times_{\mathrm{N}(\mathrm{Fin})} (\mathcal{P}^{\otimes})^{\amalg}$, and its inverse image in $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$ is exactly the same ∞-category, considered as a subcategory of $\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}$ in the obvious way.

We thus established that φ is a strong approximation. It then follows from Proposition 4.2.18 that restriction along φ induces an equivalence

$$\mathrm{Mon}^{\mathrm{coarse}}_{\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}}(\mathcal{E}) \xrightarrow{\simeq} \mathrm{Mon}^{\mathrm{coarse}}_{\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes}}(\mathcal{E}) \simeq \mathrm{Mon}_{\mathcal{O}^{\otimes}}(\mathrm{Fun}_{\mathcal{P}^{\otimes}}(\mathcal{E})). \tag{4.15}$$

We now wish to compare coarse $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$-monoid objects with fine $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$-monoid objects. Define the *mixed* weak ∞-operad structure $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ to be the one in which the active-inert factorization is the one of the fine structure while the subcategory of basics is the one of the coarse. In particular, the subcategory of mixed basics with mixed inert maps can be identified with $\left(\mathcal{O}^{\otimes}_{\langle 1 \rangle}\right)^{\simeq} \times (\mathcal{P}^{\otimes})^{\mathrm{in}}$. We may readily verify that this is indeed a weak ∞-operad structure. Applying Lemma 4.2.21(1) we may deduce that a functor $\psi : \mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \longrightarrow \mathcal{E}$ is a monoid object with respect to the coarse structure if and only if it is a monoid object with respect to the mixed structure. Applying Lemma 4.2.21(2) we may identify fine monoid objects with the full subcategory

$$\mathrm{Mon}^{\mathrm{fine}}_{\mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes}}(\mathcal{E}) \subseteq \mathrm{Mon}^{\mathrm{mixed}}_{\mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes}}(\mathcal{E}) \simeq \mathrm{Mon}^{\mathrm{coarse}}_{\mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes}}(\mathcal{E}) \tag{4.16}$$

spanned by those coarse monoid objects $\psi : \mathcal{O}^{\otimes}\wr\mathcal{P}^{\otimes} \longrightarrow \mathcal{E}$ such that $\psi|_{\left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^{\simeq}\times(\mathcal{P}^{\otimes})^{\mathrm{in}}}$ is a right Kan extension of $\psi|_{\left(\mathcal{O}^{\otimes}_{\langle 1\rangle}\right)^{\simeq}\times\left(\mathcal{P}^{\otimes}_{\langle 1\rangle}\right)^{\simeq}}$. Unwinding the definitions we see that the equivalence (4.15) identifies the full subcategory (4.16) with the full subcategory of $\mathrm{Mon}_{\mathcal{O}^{\otimes}}(\mathrm{Fun}(\mathcal{P}^{\otimes},\mathcal{E}))$ spanned by those $\mathcal{O}^{\otimes}$-monoid objects in $\mathrm{Fun}(\mathcal{P}^{\otimes},\mathcal{E})$ which take their values in the full subcategory $\mathrm{Mon}_{\mathcal{P}^{\otimes}}(\mathcal{E})$. The desired result now follows. □

5. The little cube ∞-operads

5.1. **Definitions and basic properties.**
In this section we will introduce and study the basic properties of the little n-cube ∞-operads $\mathbb{E}_n$.

Definition 5.1.1. Let $U, V \subseteq \mathbb{R}^n$ be two subsets of $\mathbb{R}^n$. We will say that a map $f : U \longrightarrow V$ is a *rectilinear embedding* if it is an open embedding which is given by a formula of the from

$$f(x_1,...,x_n) = (a_1x_1 + b_1, ..., a_nx_n + b_n) \tag{5.1}$$

with $a_i, b_i \in \mathbb{R}$ such that $a_i > 0$. If $\{U_i\}_{i\in I}$ is a collection open subsets of $\mathbb{R}^n$ indexed by a finite set I and $V \subseteq \mathbb{R}^n$ is another open subset then we will say that a map $f : \coprod_i U_i \longrightarrow V$ is a rectilinear embedding if it is an open embedding and the restriction of f to each U_i is rectilinear. We will denote by $\mathrm{Rect}(\coprod_{i\in I} U_i, V)$ the space of rectilinear embeddings endowed with the topology as a subspace of $(\mathbb{R}^{2n})^I$.

Definition 5.1.2. Let $\mathbb{E}^{\Delta}_n$ denote the simplicial operad with a single object $\square^n$ and such that

$$\mathrm{Mul}_{\mathbb{E}^{\Delta}_n}(\{\square^n\}_{i\in I}, \square^n) = \mathrm{Sing}(\mathrm{Rect}(\square^n \times I, \square^n)),$$

where Sing denotes the singular simplicial set functor. Composition is given by the composition of rectilinear embeddings.

We note that since the singular complex of a space is always a Kan the simplicial operad $\mathbb{E}^{\Delta}_n$ is locally Kan.

Definition 5.1.3. We define the *little n-cube* ∞-operad to be the operadic nerve

$$\mathbb{E}^{\otimes}_n := \mathrm{N}((\mathbb{E}^{\Delta}_n)^{\otimes}).$$

By Proposition 4.1.13 we have that $\mathbb{E}^{\otimes}_n$ is indeed an ∞-operad.

Variant 5.1.4. Definition 5.1.2 could naturally be made (and historically this was the way it was done) in the setting of *topological operads* rather than simplicial, that is, operads in which the sets of multimaps are replaced by topological spaces. In particular, we may define the topological variant $\mathbb{E}^{\mathrm{top}}_n$ of the little n-cube operad to be the topological operad with a single object $\square^n$ and such that

$$\mathrm{Mul}_{\mathbb{E}^{\mathrm{top}}_n}(\{\square^n\}_{i\in I}, \square^n) = \mathrm{Rect}(\square^n \times I, \square^n).$$

We then have that $\mathbb{E}^{\Delta}_n$ is obtained form $\mathbb{E}^{\mathrm{top}}_n$ by applying the functor Sing to all spaces of multimaps.

Variant 5.1.5. For $n \geq 1$ the little n-cube ∞-operads admit a natural variant where we also consider cubes with partial *boundary*. More precisely, let $\square^n_\partial = [0,-1) \times \square^{n-1}$ and let $\mathbb{E}^{\Delta}_{n,,\partial}$ denote the simplicial operad with two objects object $\square^n, \square^n_\partial$ and such that

$$\mathrm{Mul}_{\mathbb{E}^{\Delta}_n}(\{X_i\}_{i \in I}, Y) = \mathrm{Sing}(\mathrm{Rect}(\coprod_i X_i, Y)),$$

for $X_i, Y \in \{\square^n, \square^n_\partial\}$. We note that there are no open embeddings from $\square^n_\partial$ to $\square^n$ and that every open embedding from $\square^n$ to $\square^n_\partial$ is contained in $(0,1) \times \square^{n-1}$. We define $\mathbb{E}^{\otimes}_{n,\partial}$ to be the operadic nerve of $\mathbb{E}^{\Delta}_{n,\partial}$, and refer to it as the operad of *little n-cubes with boundary*.

Example 5.1.6. When $n = 0$ we have $\square^0 = (-1,1)^0 \simeq *$, the space with one point by convention. For a finite set I the space $\mathrm{Rect}(\square^0 \times I, \square^0)$ is empty if $|I| > 1$ and consist of a single point if $|I| \leq 1$. In particular, $\mathbb{E}^{\Delta}_0$ is isomorphic to the operad Poi of Example 4.1.7(5), whose algebras are *pointed objects*, and $\mathbb{E}^{\otimes}_0$ is isomorphic to the operadic nerve of Poi.

Example 5.1.7. When $n = 1$ we have $\square^1 = (-1,1)$. In this case every rectilinear embedding $f : \square^1 \times I \longrightarrow \square^1$ determines a unique linear ordering on I by $i \leq i' \Leftrightarrow f(0,i) \leq f(0,i')$. This association determines a map of simplicial operads

$$\mathbb{E}^{\Delta}_1 \longrightarrow \mathrm{Ass}, \tag{5.2}$$

which in turn determines a map of simplicial categories

$$(\mathbb{E}^{\Delta}_1)^{\otimes} \longrightarrow \mathrm{Ass}^{\otimes}. \tag{5.3}$$

We claim that (5.3) is a Dwyer-Kan equivalence. Since it is bijective on objects it will suffice to show that (5.2) induces a weak equivalence on spaces of multimaps. Suppose that $|I| = k$. Then we can identify the map

$$\mathrm{Mul}_{\mathbb{E}^{\Delta}_1}(\{\square^1\}_{i \in I}, \square^1) \longrightarrow \mathrm{Mul}_{\mathrm{Ass}}(\{*\}_{i \in I}, *)$$

with the projection

$$Z \times \mathrm{Iso}(\{1,...,k\}, I) \longrightarrow \mathrm{Iso}(\{1,...,k\}, I),$$

where $Z \subseteq \mathrm{Rect}(\square^1 \times \{1,...,k\}, \square^1)$ is the subspace consisting of those rectilinear embeddings $f : \square^1 \times \{1,...,k\} \longrightarrow \square^1$ such that $f(0,i) \leq f(0,i')$ whenever $i \leq i'$. It will hence suffice to show that Z is contractible. Indeed, Z can be identified with the convex subset of $\mathbb{R}^{2k}$ consisting of those $(a^1, b^1, ..., a^k, b^k) \in \mathbb{R}^{2k}$ such that $a^i > 0$, $b^1 > -1 + a^1$, $b^i > b^{i-1} + a^{i-1} + a^i$ and $b^k < 1 - a^k$. We consequently obtain an equivalence of ∞-operads

$$\mathbb{E}^{\otimes}_1 \xrightarrow{\simeq} \mathrm{N}(\mathrm{Ass}^{\otimes}).$$

Example 5.1.8. Similarly to $\mathbb{E}^{\otimes}_1$, the ∞-operad $\mathbb{E}^{\otimes}_{1,\partial}$ also has discrete spaces of multimaps, and is equivalent in turn to the nerve of the ordinary operad PRAss of Example 4.1.8(4), whose algebras are given by a pair of an associative algebra object and a pointed right module over it.

Definition 5.1.9. Given an open n-manifold M (see Definition 6.1.1), we will denote by $\mathrm{Conf}(I, M)$ the space of all injective maps $I \longrightarrow M$, endowed with the topology as a subspace of M^I. We will refer to the points of $\mathrm{Conf}(I, M)$ as *configurations* to the $\mathrm{Conf}(I, M)$ itself as the *configuration space* of M.

Lemma 5.1.10. *Let I be a finite set. Then the map* $\mathrm{ev}_0 : \mathrm{Rect}(\square^n \times I, \square^n) \longrightarrow \mathrm{Conf}(I, M)$ *obtained by restricting along* $I = \{(0, ..., 0)\} \times I \hookrightarrow \square^n \times I$ *is a homotopy equivalence.*

Proof. Given a configuration $f : I \longrightarrow \square^n$ let us denote by

$$\varepsilon(f) = \min_{i,i' \in I} \min_{j=1,\dots,n} |f_j(i) - f_j(i')|$$

the minimum distance between the points in $f(I)$ (in the taxi-driver metric) and by

$$\varepsilon'(f) = \min_{i \in I} \min_{j=1,\dots,n} (1 - |f_j(i')|)$$

the minimum distance between the points in $f(I)$ and the boundary of the cube. Let $\varepsilon''(f) = \frac{\min(\varepsilon(f), \varepsilon'(f))}{2}$ and let

$$\varphi : \mathrm{Conf}(I, M) \longrightarrow \mathrm{Rect}(\square^n \times I, \square^n)$$

be the map which sends $f : I \longrightarrow \square^n$ to the rectilinear embedding

$$\varphi_f(x_1, ..., x_n, i) = f(i) + \varepsilon''(f)(x_1, ..., x_n).$$

Then φ is continuous (since min is continuous) and $\mathrm{ev}_0\, \varphi_f = f$. To finish the proof it suffices to note that the map $g \mapsto \varphi_{\mathrm{ev}_0(g)}$ from $\mathrm{Rect}(\square^n \times I, \square^n)$ to $\mathrm{Rect}(\square^n \times I, \square^n)$ is homotopic to the identity. An explicit homotopy is given by

$$H_t(g)(x_1, ..., x_n, i) = (b^i_1 + a^i_1(t)x_1, ..., b^i_n + a^i_n(t)x_n),$$

where $a^i_j(t) = ta^i_j + (1-t)\varepsilon''(\mathrm{ev}_0(g)$ and a^i_j, b^i_j are the constants determining $g|_{\square^n \times \{i\}}$. □

Corollary 5.1.11. *The space* $\mathrm{Rect}(\square^n \times I, \square^n) \simeq \mathrm{Conf}(I, \square^n)$ *is* $(n-2)$*-connected.*

Proof. We argue by induction on $|I|$. If $I = \varnothing$ then the space $\mathrm{Conf}(I, \square^n)$ is a singleton. Otherwise, let $i_0 \in I$ be an element and consider the projection

$$\mathrm{Conf}(I, M) \longrightarrow \mathrm{Conf}(I', M). \tag{5.4}$$

where $I' = I \smallsetminus \{i_0\}$. Then (5.4) is a Serre fibration whose base is $(n-2)$-connected by the induction hypothesis. By the long exact sequence in homotopy groups it will suffice to show that the fibers are $(n-2)$-connected. Now the fiber over a configuration $f : I' \longrightarrow \square^n$ is given by $\square^n \smallsetminus f(I')$. To finish the proof we hence need to verify that the complement of a finite number of points in $\square^n$ is $(n-2)$-connected. In fact, it is a classical result that the complement of k points in $\square^n$ is homotopy equivalent to a wedge of k spheres of dimension $n-1$. We note that when $n \leq 2$ this can be verified by hand. When $n \geq 3$ the proof is classical using Van-Kampen's theorem and Alexander's duality. □

For every $n \geq 0$ the identification $\square^{n+1} \cong \square^n \times (-1, 1)$ determines a map of simplicial operads $\mathbb{E}^\Delta_n \longrightarrow \mathbb{E}^\Delta_{n+1}$ which acts on spaces of operations by taking the product with $(-1, 1)$. We then obtain a map of ∞-operads $\mathbb{E}^\otimes_n \longrightarrow \mathbb{E}^\otimes_{n+1}$. Using Corollary 5.1.11 and the fact that weak equivalences in the operadic model structure of Theorem 4.1.19 are closed under filtered colimits we can now deduce the following:

Corollary 5.1.12. *The colimit of the sequence*

$$\mathbb{E}_0 \longrightarrow \mathbb{E}_1 \longrightarrow \mathbb{E}_2 \longrightarrow \dots \tag{5.5}$$

in the ∞-category $\mathcal{O}\mathrm{p}_\infty$ is equivalent to the terminal ∞-operad $\mathrm{Com}^\otimes$*.*

With Corollary 5.1.12 in mind we will sometimes denote $\mathrm{N}(\mathrm{Com}^{\otimes})$ also by $\mathbb{E}_{\infty}$. We can think of (5.5) as a sequence interpolating between the associative operad $\mathrm{Ass}^{\otimes} \simeq \mathbb{E}_1$ and the commutative operad $\mathrm{Com}^{\otimes} \simeq \mathbb{E}_{\infty}$. In particular, it encodes algebraic structures with higher and higher levels of commutativity.

5.2. **Dunn's additivity theorem.** Our goal in this section is to formulate and prove Dunn's additivity theorem, which identifies $\mathbb{E}_n$ as the n-fold tensor product of $\mathbb{E}_1$ with itself. We can phrase this idea informally as follows: providing an $\mathbb{E}_n$-algebra structure on an object x in a symmetric monoidal ∞-category $\mathcal{C}$ is equivalent to providing a collection of n associative algebra structures which commute with each other.

In the definition below we use the fact that the simplicial operad $\mathbb{E}_n^{\Delta}$ has a single object, and so we can identify the objects of the simplicial category $(\mathbb{E}_n^{\Delta})^{\otimes}$ of Construction 4.1.9 with the objects of Fin_*.

Definition 5.2.1. For integers $n, k \geq 0$ let

$$\varphi^{\Delta}_{n,k} : (\mathbb{E}^{\Delta}_n)^{\otimes} \times (\mathbb{E}^{\Delta}_k)^{\otimes} \longrightarrow (\mathbb{E}^{\Delta}_{n+k})^{\otimes}$$

be the map of simplicial categories which sends the object $(\langle m \rangle, \langle l \rangle)$ to the object $\langle m \cdot l \rangle$ and for every pair of maps $\alpha : \langle m \rangle \longrightarrow \langle m' \rangle$ and $\beta : \langle l \rangle \longrightarrow \langle l' \rangle$ sends the component

$$\prod_{i \in \langle m' \rangle^{\circ}} \mathrm{Sing}\,\mathrm{Rect}(\square^n \times \alpha^{-1}(i), \square^n) \times \prod_{j \in \langle l' \rangle^{\circ}} \mathrm{Sing}\,\mathrm{Rect}(\square^k \times \beta^{-1}(j), \square^k)$$

of $\mathrm{Map}_{(\mathbb{E}^{\Delta}_n)^{\otimes} \times (\mathbb{E}^{\Delta}_k)^{\otimes}}((\langle m \rangle, \langle l \rangle), (\langle m' \rangle, \langle l' \rangle))$ to the component

$$\mathrm{Sing}\,\mathrm{Rect}(\square^{n+k} \times \alpha^{-1}(i) \times \beta^{-1}(j), \square^{n+k}) \subseteq \mathrm{Map}_{(\mathbb{E}^{\Delta}_{n+k})^{\otimes}}(\langle m \cdot l \rangle, \langle m' \cdot l' \rangle)$$

via the rule $(f_i, g_j) \mapsto f_i \times g_j$. Passing to coherent nerves we obtain a bifunctor of ∞-operads.

$$\begin{tikzcd}
\mathbb{E}^{\otimes}_n \times \mathbb{E}^{\otimes}_k \arrow[r, "\varphi_{n,k}"] \arrow[d] & \mathbb{E}^{\otimes}_{n+k} \arrow[d] \\
\mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\mathrm{Fin}_*) \arrow[r, "\wedge"] & \mathrm{N}(\mathrm{Fin}_*)
\end{tikzcd} \tag{5.6}$$

Theorem 5.2.2 (Dunn's additivity theorem)**.** *The bifunctor* (5.6) *exhibits* $\mathbb{E}_{n+k}$ *as the tensor product of* $\mathbb{E}_n$ *and* $\mathbb{E}_k$*.*

The remainder of this section is devoted to the proof of Theorem 5.2.2. Let us consider $\mathbb{E}_{n+k}$ as a *weak ∞-operad* with its operadic weak structure of Example 4.2.11, and $\mathbb{E}_n \times \mathbb{E}_k$ as a weak ∞-operad endowed with the product structure of Example 4.2.13. Combining Proposition 4.1.23 and Proposition 4.1.21 it will suffice to show that restriction along (5.6) induces an equivalence of ∞-categories

$$\mathrm{Mon}_{\mathbb{E}^{\otimes}_{n+k}}(\mathcal{S}) \xrightarrow{\simeq} \mathrm{Mon}_{\mathbb{E}^{\otimes}_n \times \mathbb{E}^{\otimes}_k}(\mathcal{S}) \simeq \mathrm{Mon}_{\mathbb{E}^{\otimes}_n}(\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{S})), \tag{5.7}$$

where $\mathcal{S}$ denotes the ∞-category of spaces. Our first step is to reduce the higher categorical complexity of the problem by replacing $\mathbb{E}^{\otimes}_n$ by a suitable *discrete model.*

Definition 5.2.3. Let $\mathbb{D}^{\otimes}_n$ be the operadic nerve of the ordinary colored operad whose objects are the open subcubes of $\square^n$ (i.e., the images of rectilinear embeddings). Given a collection of subcubes $U_1, ..., U_n \subseteq \square^n$ and an additional subcube

$V \subseteq \square^n$ we let $\mathrm{Mul}(\{U_i\}, V)$ be a singleton if all the U_i's are contained in V and are pairwise disjoint and empty otherwise.

We note that each subcube of $\square^n$ is homeomorphic to $\square^n$ via a unique rectilinear homeomorphism (by which we mean a map given by a formula as in (5.1)). This yields a map of ∞-operads

$$\rho : \mathbb{D}_n^{\otimes} \longrightarrow \mathbb{E}_n^{\otimes}. \tag{5.8}$$

We now have the following lemma:

Proposition 5.2.4. *The map (5.8) is a weak approximation in the sense of Definition 4.2.14 (where we consider both sides as endowed with the weak operadic structure of Example 4.2.11).*

Before we prove Proposition 5.2.4 we will need to require some results concerning nerves of coverings of topological spaces. For this, let X be a topological space. We will denote by $O(X)$ the poset of open subsets of X. Let $P \subseteq O(X)$ be an open covering of X. We may then consider P as partially ordered set, with the order of inclusion. Suppose that for every $x \in X$ the subposet $\{U \in P | x \in U\}$ is *connected*. Then the canonical map

$$\operatorname*{colim}_{U \in P} U \longrightarrow X \tag{5.9}$$

is a homeomorphism of topological spaces. We can informally express this statement by saying that X is obtained by gluing the various opens $U \in P$. We note that the colimit in (5.9) is the strict colimit calculated in the ordinary category of topological spaces. It is natural to ask for a homotopical analogue of this statement, that is, to look for conditions which insure that the canonical map

$$\operatorname*{hocolim}_{U \in P} U \longrightarrow X \tag{5.10}$$

will be a weak homotopy equivalence. We then recall the following result:

Theorem 5.2.5 ([6, A.3.1])**.** *Let X be a topological spaces, let P be a poset and $P \longrightarrow O(X)$ a map of poset. Suppose that for every $x \in X$ the subposet $\{\alpha \in P | x \in U_\alpha\}$ is weakly contractible. Then the map (5.10) is a weak homotopy equivalence.*

Remark 5.2.6. In Theorem 5.2.5 one can replace P by an arbitrary ∞-category equipped with a functor $P \longrightarrow O(X)$. The reason why we chose the formulation with posets is for simplicity of exposition, and since this covers all the cases we will need in these notes.

The proof of this statement uses some machinery related to the theory of ∞-topoi, and we will not recall its proof. Let us note however that this question was also addressed in classical algebraic topology, usually in the setting where P is assumed to be *closed under intersections* (this condition automatically insures that subposet $\{U \in P | x \in U\}$ is weakly contractible for every $x \in X$. In this case the homotopy colimit appearing in (5.9) was then described using a specific model, known as the *Čech nerve* of the covering, as follows. Given P, one first forms the simplicial space $C_\bullet(P)$ such that

$$C_n(P) = \coprod_{(U_1, \dots, U_n) \in P^{n+1}} \bigcap_{i=1}^{n} U_i.$$

It can then be shown that the *geometric realization* of this simplicial space is a model for the homotopy colimit 5.10. The following can the be proven using classical methods (in particular, partitions of unity, which is why the assumption of paracompactness was needed):

Theorem 5.2.7 (see, e.g.,[3, Corollary 4G.2])**.** *Let X be a paracompact topological spaces and $P \subseteq O(X)$ an open covering. Then the natural map*

$$|C_\bullet(P)| \longrightarrow X$$

is a homotopy equivalence.

Remark 5.2.8. In Theorem 5.2.7 we did not assume that P is closed under intersection since this was not assumed in the classical setting and is not needed for the claim of Theorem 5.2.7 to hold. The condition that P is closed under finite intersections is only needed for the identification of $|C_\bullet(P)|$ with $\operatorname{hocolim}_{U\in P} U$.

Remark 5.2.9. Suppose that $P \subseteq O(X)$ is an open covering which satisfies the following property: for every $x \in X$ and every $U, V \in P$ there exists a $W \in P$ such that $x \in W \subseteq U \cap V$. Then for every $x \in X$ the subposet $\{U \in P | x \in U\}$ is *filtered* (every finite subset has a lower bound) and is hence weakly contractible. In this particular case one can construct a *hypercovering* $U_\bullet \longrightarrow X$ such that $|U_\bullet| \simeq \operatorname{hocolim}_{U\in P} U$ and prove Theorem 5.2.5 using direct arguments (not invoking ∞-topos machinery), see [2].

Corollary 5.2.10. *Let M be a manifold and $O(M)$ the poset of open subsets of M. Let I be a finite set and $P \subseteq O(M)^I$ be a subposet satisfying the following properties:*

(1) For every $(U_i)_{i\in I} \in P$, we have $U_i \cap U_j = \varnothing$ for $i \neq j$.
(2) For each configuration $f : I \longrightarrow M$ there exists an element $(U_i)_{i\in I} \in P$ such that $f(i) \in U_i$.
(3) If $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ are two elements in P and $f : I \longrightarrow M$ is a cofiguration such that $f(i) \in U_i \cap V_j$ then there exists a $(W_i)_{i\in I} \in P$ such that $f(i) \in W_i$.

Then the natural map

$$\operatorname*{hocolim}_{(U_i)_{i\in I}\in P} \prod_{i\in I} \operatorname{Conf}(\{i\}, U_i) \longrightarrow \operatorname{Conf}(I, M)$$

is a homotopy equivalence.

Proof. The association $(U_i)_{i\in I} \mapsto \prod_{i\in I} \operatorname{Conf}(\{i\}, U_i)$ determines an injective fully-faithful map from the poset P to the poset of open subsets of $\operatorname{Conf}(I, M)$. The desired result is now provided by Theorem 5.2.5 (in the situation of Remark 5.2.9). ∎

Proof of Proposition 5.2.4. We first note that $\mathbb{E}^\otimes_{\langle 1\rangle} \simeq *$ and $\varphi^{-1}\mathbb{E}^\otimes_{\langle 1\rangle} = \mathbb{D}^\otimes_{\langle 1\rangle}$ has a final object and is hence weakly contractible. This implies that the map $\mathbb{D}^\otimes_{\langle 1\rangle} \longrightarrow \mathbb{E}^\otimes_{\langle 1\rangle}$ is a localization map. We hence just need to check Condition (1) of Definition 4.2.14, namely, that for every collection of subcubes $U_1, ..., U_m$ the functor

$$(\mathbb{D}^\otimes_n)^{\text{act}}_{/(U_1,...,U_m)} \longrightarrow (\mathbb{E}^\otimes_n)^{\text{act}}_{/\langle m\rangle} \tag{5.11}$$

has weakly contractible homotopy fibers. We first observe that both the left and right hand sides are products of the corresponding ∞-categories for the individual

U_i's. In other words, we may assume that $m = 1$. Consider the commutative triangle

$$\begin{tikzcd} (\mathbb{D}_n^{\otimes})^{\mathrm{act}}_{/U} \arrow[rr] \arrow[dr] & & (\mathbb{E}_n^{\otimes})^{\mathrm{act}}_{/\langle 1\rangle} \arrow[dl] \\ & (\mathbb{E}_n^{\otimes})^{\mathrm{act}} & \end{tikzcd} \tag{5.12}$$

The right diagonal map is a right fibration whose fiber over the object $\langle l\rangle \in (\mathbb{E}_n^{\otimes})^{\mathrm{act}}$ is a Kan complex which is naturally equivalent to the singular complex of $\mathrm{Rect}(\square^n \times \langle l\rangle^{\circ}, \square^n)$. It will hence suffice to show that (5.12) induces a weak homotopy equivalence between homotopy fibers above every $\langle l\rangle \in (\mathbb{E}_n^{\otimes})^{\mathrm{act}}$. Let $\mathcal{X}_l \longrightarrow (\mathbb{D}_n^{\otimes})^{\mathrm{act}}_{/U}$ be the homotopy fiber of the left diagonal map over $\langle l\rangle \in (\mathbb{E}_n^{\otimes})^{\mathrm{act}}$. Unwinding the definitions we see that $(\mathbb{D}_n^{\otimes})^{\mathrm{act}}_{/U}$ can be identified with the subposet of $O(U)$ spanned by those open subsets which are finite disjoint unions of subcubes. Let us call such open subsets *multi-subcubes*. Under this identification the left diagonal map can be written as $V \mapsto \pi_0(V)$. We may then identify $\mathcal{X}_l$ with the ∞-category of pairs (V, ρ) where V is a multi-subcube of U and $\rho : \square^n \times \langle l\rangle^{\circ} \longrightarrow V$ is a rectilinear embedding which induces a bijection of π_0. Equivalently, we can describe the objects of $\mathcal{X}_l$ as pairs $((V_1, ..., V_l), (\rho_1, ..., \rho_l))$ where $V_1, ..., V_l$ are pairwise disjoint open subcubes of U and $\rho_i : \square^n \longrightarrow V_i$ is a rectilinear embedding. Let us denote by $\overline{X}_l \subseteq O(U)^{\langle l\rangle^{\circ}}$ the subposet spanned by the pairwise disjoint tuples of subcubes $(V_1, ..., V_l) \in O(U)^{\langle l\rangle^{\circ}}$, so that we have a forgetful functor $X_l \longrightarrow \overline{X}_l$ (which is in fact an equivalence) given by $((V_1, ..., V_l), (\rho_1, ..., \rho_l)) \mapsto (V_1, ..., V_l)$. To finish the proof we now need to check that the canonical map

$$\operatorname*{hocolim}_{(V_1,...,V_l)\in\overline{X}_l} \prod_{i\in\langle l\rangle^{\circ}} \mathrm{Rect}(\square^n \times \{i\}, V_i) \longrightarrow \mathrm{Rect}(\square^n \times \langle l\rangle^{\circ}, U)$$

is a weak homotopy equivalence. In light of Lemma 5.1.10 we may replace rectilinear embeddings by configuration spaces and prove instead that the canonical map

$$\operatorname*{hocolim}_{(V_1,...,V_l)\in\overline{X}_l} \prod_{i\in\langle l\rangle^{\circ}} \mathrm{Conf}(\{i\}, V_i) \longrightarrow \mathrm{Conf}(\langle l\rangle^{\circ}, U)$$

is a weak homotopy equivalence. But this now follows from Corollar 5.2.10 since every intersection of cubes is either empty or a cube, and every cofiguration of points is covered by at least one tuple of subcubes. $\square$

Corollary 5.2.11. *The restriction functor*

$$\mathrm{Mon}_{\mathbb{E}_n^{\otimes}}(\mathcal{S}) \longrightarrow \mathrm{Mon}_{\mathbb{D}_n^{\otimes}}(\mathcal{S})$$

is fully-faithful and its essential image is spanned by those $\mathbb{D}_n^{\otimes}$-monoid objects $\psi : \mathbb{D}_n^{\otimes} \longrightarrow \mathcal{S}$ which are locally constant *in the sense that the restriction to $(\mathbb{D}_n^{\otimes})_{\langle 1\rangle}$ sends every arrow to an equivalence of spaces.*

We now note that the bifunctor of ∞-operads of Definition 5.2.1 can also be defined on the level of $\mathbb{D}_n^{\otimes}$. Indeed, it is simply given by the functor of ordinary categories

$$\mathbb{D}_n^{\otimes} \times \mathbb{D}_k^{\otimes} \longrightarrow \mathbb{D}_{n+k}^{\otimes} \tag{5.13}$$

which sends $((U_i)_{i\in\langle m\rangle^{\circ}}, (V_j)_{j\in\langle l\rangle^{\circ}}$ to $(U_i \times V_j)_{(i,j)\in\langle m\rangle^{\circ}\times\langle l\rangle^{\circ}}$ (where we identified $\langle m\rangle^{\circ} \times \langle l\rangle^{\circ}$ with $(\langle m\rangle \wedge \langle l\rangle)^{\circ}$).

Let us now recall from §4.3 the formation of *wreath products*. We then observe that (5.13) factors as

$$\mathbb{D}^{\otimes}_n \times \mathbb{D}^{\otimes}_k \longrightarrow \mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k \longrightarrow \mathbb{D}^{\otimes}_{n+k}$$

where the second functor sends

$$((U_1,...,U_m),(\{V^1_1,...,V^1_{l_1}\},...,\{V^m_1,...,V^m_{l_m}\}) \mapsto (U_i \times V^i_j)_{i\in\langle m\rangle^\circ, j\in\langle l_i\rangle^\circ}.$$

We can consider $\mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k$ as a weak ∞-operad with respect to the wreath product structure of Definition 4.3.10.

Proposition 5.2.12. *The composed map*

$$\mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_m \longrightarrow \mathbb{D}^{\otimes}_{n+k} \longrightarrow \mathbb{E}^{\otimes}_{n+k} \tag{5.14}$$

is a weak approximation of weak ∞-operads.

Proof. Let us first establish Condition (2) of Definition 4.2.14. For this, we note that the functor $\mathbb{D}^{\otimes}_n \times \mathbb{D}^{\otimes}_k \longrightarrow \mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k$ identifies $(\mathbb{D}^{\otimes}_n)_{\langle 1\rangle} \times (\mathbb{D}^{\otimes}_k)_{\langle 1\rangle}$ with a full subcategory of $(\mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k)_{\langle 1\rangle}$, and this inclusion is *coinitial*. In addition $(\mathbb{D}^{\otimes}_n)_{\langle 1\rangle} \times (\mathbb{D}^{\otimes}_k)_{\langle 1\rangle}$ has a final object. It then follows that $\mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k$ is weakly contractible and so the map $(\mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k)_{\langle 1\rangle} \longrightarrow (\mathbb{E}^{\otimes}_{n+k})_{\langle 1\rangle} \simeq *$ is a localization map.

We shall now prove that Condition (2) of Definition 4.2.14 holds. Fix an object $X = ((U_1,...,U_m),(\{V^1_1,...,V^1_{l_1}\},...,\{V^m_1,...,V^m_{l_m}\})) \in \mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k$, and set $l = \sum_i l_i$. We need to show that the functor

$$(\mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k)^{\mathrm{act}}_{/X} \longrightarrow (\mathbb{E}^{\otimes}_{n+k})^{\mathrm{act}}_{/\langle l\rangle} \tag{5.15}$$

has weakly contractible homotopy fibers. We first observe that both the left and right hand sides of (5.15) factor as the product over $i = 1,...,m$ of the respective ∞-categories for $X_i := (U_i,\{V^i_1,...,V^1_{i_1}\})$ on the left and l_i on the right. We may hence assume that $m = 1$. Consider the commutative triangle

$$\begin{tikzcd} (\mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k)^{\mathrm{act}}_{/(U,\{V_1,...,V_l\})} \arrow[rr] \arrow[dr] & & (\mathbb{E}^{\otimes}_{n+k})^{\mathrm{act}}_{/\langle l\rangle} \arrow[dl] \\ & (\mathbb{E}^{\otimes}_{n+k})^{\mathrm{act}} & \end{tikzcd} \tag{5.16}$$

The right diagonal map is a right fibration whose fiber over the object $\langle l'\rangle \in (\mathbb{E}^{\otimes}_{n+k})^{\mathrm{act}}$ is a Kan complex which is naturally equivalent to the singular complex of $\mathrm{Rect}(\square^n \times \langle l'\rangle^\circ, \square^n \times \langle l\rangle^\circ)$. It will hence suffice to show that (5.12) induces a weak homotopy equivalence between homotopy fibers above every object $\langle l'\rangle \in (\mathbb{E}^{\otimes}_{n+k})^{\mathrm{act}}$. Let us denote by $\mathcal{X}_{l'} \longrightarrow (\mathbb{D}^{\otimes}_n \wr \mathbb{D}^{\otimes}_k)^{\mathrm{act}}_{/(U,\{V_1,...,V_l\})}$ the homotopy fiber of the left diagonal map over $\langle l'\rangle \in (\mathbb{E}^{\otimes}_n)^{\mathrm{act}}$. To identify this homotopy fiber let us define $V := \coprod_{j\in\langle l\rangle^\circ} V_j \subseteq \square^k \times \langle l\rangle^\circ$ and consider the open subset

$$X = U \times V \subseteq \square^n \times \square^k \times \langle l\rangle^\circ = \square^{n+k} \times \langle l\rangle^\circ$$

equipped with its two projections $p : X \longrightarrow U$ and $q : X \longrightarrow V$. Let $O(X,U)$ denote the poset of pairs (W,U') where W is an open subset of X and U' is open subset of U which contains the image of W via the projection $p : X \longrightarrow U$, where we consider $O(X,U)$ as ordered by inclusion. The association

$$((U_1,...,U_m),(\{V^1_1,...,V^1_{l_1}\},...,\{V^m_1,...,V^m_{l_m}\})) \mapsto (\bigcup_{i,j} U_i \times V^i_j, \bigcup_i U_i)$$

identifies $(\mathbb{D}_n^{\otimes} \wr \mathbb{D}_k^{\otimes})^{\mathrm{act}}_{/(U,\{V_1,...,V_l\})}$ with the subposet $O^{\mathrm{cube}}(X,U) \subseteq O(X,U)$ spanned by those pairs (W,U') which satisfy the following properties:

(1) U' is a disjoint union of subcubes $U_1,...,U_1 \subseteq U$.
(2) For each component $U_i \subseteq U'$ the inverse image $W \cap p^{-1}(U_i) \subseteq X$ of U_i in W factors as a product $U_i \times V'$ where $V' \subseteq V$ is a (possibly empty) union of subcubes $V_1^i,...,V_{l_i}^i$.

Under the identification $(\mathbb{D}_n^{\otimes} \wr \mathbb{D}_k^{\otimes})^{\mathrm{act}}_{/(U,\{V_1,...,V_l\})} \simeq O^{\mathrm{cube}}(X,U)$ the left diagonal map in (5.16) can be written as $(W,U') \mapsto \pi_0(W)$. We may then identify $\mathcal{X}_{l'}$ with the ∞-category of pairs (W,U',ρ) where $(W,U') \in \mathcal{O}^{\mathrm{cube}}(X,U)$ and $\rho : \square^n \times \langle l' \rangle^{\circ} \longrightarrow W$ is a rectilinear embedding which induces a bijection of π_0. Equivalently, we can describe the objects of $\mathcal{X}_l$ as tuples $((W,U'),\eta,(\rho_1,...,\rho_l))$ where $(W,U') \in O^{\mathrm{cube}}(X,U)$, $\eta : \langle l' \rangle^{\circ} \xrightarrow{\cong} \pi_0(W)$ is a bijection of sets and $\rho_i : \square^{n+k} \longrightarrow W_i^{\eta}$ is a rectilinear embedding, where by W_i^{η} we mean the component of W determined by $\eta(i) \in \pi_0(W)$.

Let $\mathcal{X}^0_{l'}$ be the full subcategory of $\mathcal{X}_{l'}$ spanned by those $((W,U'),\eta,(\rho_1,...,\rho_l))$ such that the map $W \longrightarrow U'$ is surjective on π_0. The inclusion $\mathcal{X}^0_{l'} \subseteq \mathcal{X}_{l'}$ admits a right adjoint $\mathcal{X}_{l'} \longrightarrow \mathcal{X}^0_{l'}$ which sends $((W,U'),\eta,(\rho_1,...,\rho_l))$ to $((W,p(W)),\eta,(\rho_1,...,\rho_l))$, and is hence a weak homotopy equivalence. It will hence suffice to show that the map

$$\mathcal{X}^0_{l'} \longrightarrow (\mathbb{E}^{\otimes}_{n+k})^{\mathrm{act}}_{/\langle l \rangle} \times_{(\mathbb{E}^{\otimes}_{n+k})^{\mathrm{act}}} \{\langle l' \rangle\}$$

is a weak homotopy equivalence.

Let us denote by $\overline{\mathcal{X}}^0_{l'}$ the poset of pairs $((W,U'),\eta)$ where $(W,U') \in O^{\mathrm{cube}}(X,U)$, $W \longrightarrow U'$ is surjective on π_0 and $\eta : \langle l' \rangle^{\circ} \xrightarrow{\cong} \pi_0(W)$ is a bijection. In particular we have a natural forgetful functor $\overline{X}^0_{l'} \longrightarrow \overline{X}_{l'}$ (which is an equivalence). Given $((W,U'),\eta) \in \overline{\mathcal{X}}^0_{l'}$ the multi-subcube $U' \subseteq U$ is completely determined by W (as its image). We then see that the association $((W,U'),\eta) \mapsto (W_i^{\eta})_{i \in \langle l' \rangle^{\circ}}$ identifies $\overline{\mathcal{X}}^0_{l'}$ with a certain subposet of $O(X)^{\langle l' \rangle^{\circ}}$. We shall henceforth adopt this identification implicitly. To finish the proof it is left to check that the canonical map

$$\operatorname*{hocolim}_{(W_1,...,W_{l'}) \in \overline{\mathcal{X}}^0_{l'}} \prod_{i \in \langle l' \rangle^{\circ}} \mathrm{Rect}(\square^n \times \{i\}, W_i) \longrightarrow \mathrm{Rect}(\square^n \times \langle l' \rangle^{\circ}, X)$$

is a weak homotopy equivalence. In light of Lemma 5.1.10 we may replace rectilinear embeddings by configuration spaces and prove instead that the canonical map

$$\operatorname*{hocolim}_{(W_1,...,W_{l'}) \in \overline{\mathcal{X}}^0_{l'}} \prod_{i \in \langle l' \rangle^{\circ}} \mathrm{Conf}(\{i\}, W_i) \longrightarrow \mathrm{Conf}(\langle l' \rangle^{\circ}, X)$$

is a weak homotopy equivalence. We would like now to invoke Corollary 5.2.10 as in the proof of Proposition 5.2.4. The only additional detail we need to verify (which was obvious in the case of Proposition 5.2.4) is that for every configuration $f : \langle l' \rangle^{\circ} \longrightarrow X$ there exists a $(W_1,...,W_{l'}) \in \overline{\mathcal{X}}^0_{l'}$ such that $f(i) \in W_i$. To see this, let $I \subseteq U$ be the image of the composed map $\langle l' \rangle^{\circ} \xrightarrow{f} X \xrightarrow{p} U$. We may then identify I with $\langle m \rangle^{\circ}$ for some m (embedded in U) and we let $\alpha : \langle l' \rangle \longrightarrow \langle m \rangle$ be the corresponding active surjective map. Then we may choose small n-cubes neighborhoods $\{U_i\}_{i \in \langle m \rangle^{\circ}}$ for each $i \in \langle m \rangle^{\circ}$ such that $U_i \cap U_{i'} = \varnothing$ when $i \neq i'$. For each $i \in \langle m \rangle^{\circ}$ let $J_i \subseteq V$ be the image of the subset $\alpha^{-1}(i) \subseteq \langle l' \rangle^{\circ}$ under the composed map $\langle l' \rangle^{\circ} \longrightarrow X \xrightarrow{q} V$. We may then choose small k-cubes neighborhoods $\{V_j\}_{j \in J_i}$ for each $j \in J_i$ such that $V_j \cap V_{j'} = \varnothing$ when $j \neq j'$. The object $(U_{\alpha(j)} \times V_j)_{j\in\langle l'\rangle^\circ} \in O^{\mathrm{cube}}(X)^{\langle l'\rangle^\circ}$ then belongs to $\overline{\mathcal{X}}^0_{l'}$ and the corresponding open subset of $\mathrm{Conf}(\langle l'\rangle^\circ, X)$ contains the configuration f, as desired. □

Proof of Theorem 5.2.2. We would like to show that (5.7) is an equivalence of ∞-categories. Consider the commutative diagram

$$(5.17)\qquad \begin{tikzcd} \mathrm{Mon}_{\mathbb{E}^\otimes_{n+k}}(\mathcal{S}) \arrow[r] \arrow[dr] & \mathrm{Mon}_{\mathbb{E}^\otimes_n \wr \mathbb{E}^\otimes_k}(\mathcal{S}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Mon}_{\mathbb{E}^\otimes_n}(\mathrm{Mon}_{\mathbb{E}^\otimes_k}(\mathcal{S})) \arrow[d] \\ & \mathrm{Mon}_{\mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k}(\mathcal{S}) \arrow[r, "\simeq"] & \mathrm{Mon}_{\mathbb{D}^\otimes_n}(\mathrm{Mon}_{\mathbb{D}^\otimes_k}(\mathcal{S})) \end{tikzcd}$$

where the right horizontal maps are equivalences by Theorem 4.3.11. By Proposition 4.2.18 and Proposition 5.2.12 the left diagonal map in (5.17) is fully-faithful and its essential image is spanned by those monoid objects $\psi : \mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k \longrightarrow \mathcal{S}$ which send every arrow in $(\mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k)_{\langle 1\rangle}$ to an equivalence. On the other hand, by Proposition 4.2.18, Proposition 5.2.4 and Remark 4.2.16 the right most vertical arrow is fully-faithful and its essential image is spanned by those monoid objects $\psi : \mathbb{D}^\otimes_n \times \mathbb{D}^\otimes_k \longrightarrow \mathcal{S}$ which send every arrow in $(\mathbb{D}^\otimes_n)_{\langle 1\rangle} \times (\mathbb{D}^\otimes_k)_{\langle 1\rangle}$ to an equivalence. To finish the proof it will hence suffice to show that if a monoid object $\psi : \mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k \longrightarrow \mathcal{S}$ sends every arrow in $(\mathbb{D}^\otimes_n)_{\langle 1\rangle} \times (\mathbb{D}^\otimes_k)_{\langle 1\rangle}$ to an equivalence then it sends every arrow in $(\mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k)_{\langle 1\rangle}$ to an equivalence. Indeed, let $\psi : \mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k \longrightarrow \mathcal{S}$ be such a monoid object. Then for every object $((U_1, ..., U_m), (\{\}, ..., \{V\}, ..., \{\}))$ in $(\mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k)_{\langle 1\rangle}$ the map

$$\psi((U_1, ..., U_m), (\{\}, ..., \{V\}, ..., \{\})) \xrightarrow{\simeq} \psi(U_{i_0}, V) \prod_{i_0 \neq i \in \langle m\rangle^\circ} \psi(U_i, \{\})$$

induced by the inert map $(U_1, ..., U_m), (\{\}, ..., \{V\}, ..., \{\}) \longrightarrow (U_{i_0}, V)$ and the inert maps $(U_1, ..., U_m), (\{\}, ..., \{V\}, ..., \{\}) \longrightarrow (U_i, \{\})$ for $i \neq i_0$, is an equivalence. In addition, the monoid condition also implies that $\psi(U_i, \{\}) \simeq *$. We may then conclude that ψ sends every inert map of the form

$$(5.18)\qquad ((U_1, ..., U_m), (\{\}, ..., \{V\}, ..., \{\})) \longrightarrow (U_{i_0}, V)$$

to an equivalence. By the 2-out-of-3 rule we then get that ψ sends every map

$$(5.19)\qquad (U', V') \longrightarrow ((U_1, ..., U_m), (\{\}, ..., \{V\}, ..., \{\}))$$

whose domain is in $(\mathbb{D}^\otimes_n)_{\langle 1\rangle} \times (\mathbb{D}^\otimes_k)_{\langle 1\rangle}$ to an equivalence. We now observe that *any* object in $(\mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k)_{\langle 1\rangle}$ receives at least one map from an object in $(\mathbb{D}^\otimes_n)_{\langle 1\rangle} \times (\mathbb{D}^\otimes_k)_{\langle 1\rangle}$. Applying again the 2-out-of-3 argument we may conclude that ψ sends every map in $(\mathbb{D}^\otimes_n \wr \mathbb{D}^\otimes_k)_{\langle 1\rangle}$ to an equivalence, as desired. □

Remark 5.2.13. Letting either k or n to be 0 in Theorem 5.2.2 we recover the claim that $\mathbb{E}_0 \otimes \mathbb{E}_n \simeq \mathbb{E}_n \otimes \mathbb{E}_0 \simeq \mathbb{E}_n$, and so the forgetful functors $\mathrm{Mon}_{\mathbb{E}_n}(\mathrm{Mon}_{\mathbb{E}_0}(\mathcal{E})) \longrightarrow \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{E})$ and $\mathrm{Mon}_{\mathbb{E}_0}(\mathrm{Mon}_{\mathbb{E}_n}(\mathcal{E})) \longrightarrow \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{E})$ are equivalences for every ∞-category with finite products $\mathcal{E}$. More generally, if $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category then the forgetful functors

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \xleftarrow{\simeq} \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}))$$

are equivalences.

Remark 5.2.14. A variant of the argument in the proof of Theorem 5.2.2 shows that $\mathbb{E}^{\otimes}_{n+k,\partial} \simeq \mathbb{E}^{\otimes}_{n,\partial} \otimes \mathbb{E}^{\otimes}_{k}$. In particular, by example 5.1.8 we may identify $\mathbb{E}_{n,\partial}$-algebras in $\mathcal{C}$ with $\mathbb{E}_n$-algebras in $\mathrm{Alg}_{\mathbb{E}_{1,\partial}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathrm{PRAss}}(\mathcal{C})$.

5.3. **May's recognition principle.** Let (X, x_0) be a pointed connected topological space. Then the loop space

$$\Omega X := \{p : [-1,1] \longrightarrow X | p(-1) = p(1) = x_0\},$$

equipped with the compact open topology, acquires a natural action of the topological little 1-cube operad $\mathbb{E}^{\mathrm{top}}_1$ (see Variant 5.1.4), which can be described as follows: given a rectilinear embedding $f : \square^1 \times I \longrightarrow \square^1$, the corresponding map

$$f_* : \prod_{i \in I} \Omega X \longrightarrow \Omega X$$

sends $\{p_i\}_{i \in I}$ to the path $q : [-1,1] \longrightarrow X$ which is given by $q(x) = p_I(f^{-1}(t))$ if t is in the image of f and $q(t) = x_0$ otherwise. Here we consider $\square^1 = (-1,1)$ as a subspace of $[-1,1]$ and $p_I : \square^1 \times I \longrightarrow X$ is the map $p_I(t,i) = p_i(t)$. More generally, the n-fold loop space

$$\Omega^n X := \Omega(\Omega(\cdots(\Omega X))) \cong \{p : [-1,1]^n \longrightarrow X | p(\partial[-1,1]^n) = x_0\}$$

acquires a similarly defined action of $\mathbb{E}^{\mathrm{top}}_n$. A famous theorem of May (related to a large body of work of Boardman and Vogt on the topic) identifies the $\mathbb{E}^{\mathrm{top}}_n$-monoid objects in Top which can be obtained in this way. To formulate May's result we will need the following definition:

Definition 5.3.1. Let X be a topological space equipped with an action of $\mathbb{E}^{\mathrm{top}}_n$ for $n \geq 1$. We will say that X is *group-like* if for every rectilinear embedding $f : \square^n \times \{1,2\} \longrightarrow \square^n$ the map

$$X \times X \xrightarrow{(\mathrm{pr}_1, f_*)} X \times X$$

is a homotopy equivalence, where $\mathrm{pr}_1 : X \times X \longrightarrow X$ is the projection to the first coordinate and $f_* : X \times X \longrightarrow X$ is the map associated to f by the action of $\mathbb{E}^{\mathrm{top}}_n$.

Theorem 5.3.2 (May [7])**.** *Let X be a space equipped with an action of $\mathbb{E}^{\mathrm{top}}_n$. Then the following conditions are equivalent:*

(1) X is of the form $\Omega^n Y$ for some pointed $(n-1)$-connected topological space Y.
(2) X is group-like.

Theorem 5.3.2 gives an important relation between the theory of $\mathbb{E}^{\mathrm{top}}_n$-monoids in spaces and the theory of iterated loop spaces, and was historically one of the motivations behind the introduction of the little n-cube operads. It is often referred to as *May's recognition principle*, since it can be considered as giving an internal characterization of n-fold loop spaces in terms of the structure they acquire, which then allows one to recognize them without knowing any explicit delooping. We would like to formulate and prove May's theorem in the setting of ∞-operads. For this, let us first define the notion of being group-like in a more general context:

Definition 5.3.3. Let $\psi : \mathbb{E}^{\otimes}_n \longrightarrow \mathcal{E}$ be an $\mathbb{E}_n$-monoid object in $\mathcal{E}$ for $n \geq 1$. We will say that ψ is *group-like* if for every active $f : \langle 2 \rangle \longrightarrow \langle 1 \rangle$ in $\mathbb{E}^{\otimes}_n$ the map

$$\psi(\langle 2 \rangle) \xrightarrow{(\rho^1_*, f_*)} \psi(\langle 1 \rangle) \times \psi(\langle 1 \rangle) \tag{5.20}$$

is an equivalence.

Example 5.3.4. In the situation of Definition 5.3.3, suppose that $\mathcal{E} = \mathcal{S}$ is the ∞-category of spaces. Given a monoid object $\psi : \mathbb{E}_n^{\otimes} \longrightarrow \mathcal{S}$ let $X := \psi(\langle 1 \rangle) \in \mathcal{S}$ be its underlying space. The map (5.20) can then be identified up to equivalence with a map of the form

$$X \times X \xrightarrow{(\mathrm{pr}_1, f_*)} X \times X \tag{5.21}$$

where $\mathrm{pr}_1 : X \times X \longrightarrow X$ is the projection on the first component and $f_* : X \times X \longrightarrow X$ is the multiplication induced by f. Now the map (5.21) respects the projection on the first component on both sides. By the long exact sequence in homotopy groups we have that (5.21) is an equivalence if and only if it induces an equivalence on homotopy fibers of pr_1 over every $x \in X$. Given a particular $x \in X$, the induced map on homotopy fibers over x can be identified with the map

$$\{x\} \times X \xrightarrow{f_x} \{x\} \times X \tag{5.22}$$

given by f-multiplication by x. We may thus conclude that X is group-like if and only if for every active $f : \langle 2 \rangle \longrightarrow \langle 1 \rangle$ in $\mathbb{E}_n^{\otimes}$, the operation of f-multiplication by any point $x \in X$ is an equivalence. Replacing f with the composition $\langle 2 \rangle \xrightarrow{\tau} \langle 2 \rangle \xrightarrow{f} \langle 1 \rangle$, where τ lies over the map in $\mathrm{N}(\mathrm{Fin}_*)$ which switches 1 and 2, we see that this is equivalent to saying that f-multiplication by x from the left or from the right is an equivalence for every f and x. Finally, this condition does not depend on f, since $\mathrm{Map}_{(\mathbb{E}_n^{\otimes})^{\mathrm{act}}}(\langle 2 \rangle, \langle 1 \rangle)$ is either connected (when $n > 1$) or has exactly two components (when $n = 1$) which are then switched by pre-composition with τ. We note that any choice of f induces a structure of a *monoid* on $\pi_0(X)$, and the condition that every point acts invertibly from the left and from the right is equivalent to the condition that $\pi_0(X)$ is in fact a *group*.

In order to prove May's recognition principle it will be convenient to replace $\mathbb{E}_n^{\otimes}$ by a suitable strong approximation of it. Recall first that $\mathbb{E}_1^{\otimes} \simeq \mathrm{N}(\mathrm{Ass}^{\otimes})$ (see Example 5.1.7). In addition, the category Δ^{op} has a natural structure of a *weak ∞-operad* (see Example 4.2.12) and we have a strong approximation map $\Delta^{\mathrm{op}} \longrightarrow \mathrm{Ass}^{\otimes}$ (see Example 4.2.15). By Proposition 4.2.18 the restriction functor

$$\mathrm{Mon}_{\mathrm{Ass}}(\mathcal{E}) \xrightarrow{\simeq} \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathcal{E})$$

is an equivalence of ∞-categories. We note that we may also model $\mathbb{E}_n$-monoids in this way using the additivity theorem. More precisely, let us consider the n-fold product $(\Delta^{\mathrm{op}})^n = \Delta^{\mathrm{op}} \times ... \times \Delta^{\mathrm{op}}$ equipped with the product weak ∞-operad structure of Example 4.2.13. By Remark 4.2.16 the map $(\Delta^{\mathrm{op}})^n \longrightarrow (\mathrm{Ass}^{\otimes})^n$ is again a strong approximation and so

$$\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{E}) \simeq \mathrm{Mon}_{(\mathrm{Ass}^{\otimes})^n}(\mathcal{E}) \simeq \mathrm{Mon}_{(\mathbb{E}_1^{\otimes})^n}(\mathcal{E}) \simeq \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{E}) \tag{5.23}$$

by Theorem 5.2.2.

We can also construct a similar type of strong approximation for the ∞-operad $\mathbb{E}_0$. Consider the one-arrow category $[1] = \bullet \longrightarrow \bullet$ equipped with the weak ∞-operad structure in which all maps are active, only the isomorphisms are inert, and the basics are $\{1\} \subseteq [1]$. We have a natural map of weak ∞-operads $[1] \longrightarrow \mathrm{Poi}^{\otimes}$ which sends 0 to $\langle 0 \rangle$ and 1 to $\langle 1 \rangle$. This map a strong approximation since it induces an equivalence on active over-categories (and the second condition of

Definition 4.2.14 is clear). We may then conclude that

$$\mathrm{Mon}_{[1]}(\mathcal{E}) \simeq \mathrm{Mon}_{\mathrm{Poi}}(\mathcal{E}) \simeq \mathrm{Mon}_{\mathbb{E}_0}(\mathcal{E}).$$

Unwinding the definitions we see that $\mathrm{Mon}_{[1]}(\mathcal{E}) \subseteq \mathrm{Fun}([1], \mathcal{E})$ is the full subcategory spanned by those arrows $x \longrightarrow y$ in $\mathcal{E}$ such that x is final. Fixing a final object $*$ we then have a natural equivalence $\mathrm{Mon}_{[1]}(\mathcal{E}) \simeq \mathcal{E}_{*/}$.

Definition 5.3.5. Let $\mathcal{E}$ be an ∞-category with finite products. We will say that a $(\Delta^{\mathrm{op}})^n$-monoid object $\mathrm{N}(\Delta^{\mathrm{op}})^n \longrightarrow \mathcal{E}$ is *group-like* if it corresponds to a group-like $\mathbb{E}_n$-monoid object under the equivalence (5.23). We will denote by $\mathrm{Mon}^{\mathrm{gr}}_{(\Delta^{\mathrm{op}})^n}(\mathcal{E}) \subseteq \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{E})$ the full subcategory spanned by the group-like Δ^{op}-monoid objects.

Remark 5.3.6. Unwinding the definitions we see that a Δ^{op}-monoid object $\psi : \mathrm{N}(\Delta^{\mathrm{op}}) \longrightarrow \mathcal{E}$ is group-like if and only if the map

$$\psi([2]) \xrightarrow{\simeq} \psi([1]) \times \psi([1]) \tag{5.24}$$

induced by the faces $\{0,1\} \subseteq [2]$ and $\{0,2\} \subseteq [2]$ is an equivalence.

Remark 5.3.7. In the situation of Remark 5.3.6, if $\psi : \mathrm{N}(\Delta^{\mathrm{op}}) \longrightarrow \mathcal{E}$ is group-like then an a-priori stronger condition automatically holds: for every n the collection of maps $\rho_i : \psi([n]) \longrightarrow \psi(\{0,i\})$ for $i \neq 0$ exhibit $\psi([n])$ as the product $\prod_{i\neq 0} \psi(\{0,i\})$. To see this let us argue by induction on n (the case $n = 2$ of (5.24) being the base). Indeed, suppose the claim is true for some $n \geq 2$. The monoid condition implies that $\psi([m]) \simeq \prod_{i=0}^{m-1} \psi(\{i,i+1\})$ for every m and hence that $\psi([m]) \simeq \psi(\{0,...,i\}) \times \psi(\{i,...m\})$ for every $0 \leq i \leq m$. Combining this with the induction hypothesis we then conclude that

$$\psi([n+1]) \simeq \psi(\{0,...,n\}) \times \psi(\{n,n+1\}) \simeq \prod_{i=1,\dots,n} \psi(\{0,i\}) \times \psi(\{n,n+1\}) \simeq$$

$$\prod_{i=1,\dots,n-1} \psi(\{0,i\} \times \psi(\{0,n,n+1\}) \simeq \prod_{i=1,\dots,n+1} \psi(\{0,i\}),$$

as desired.

In order to formulate May's recognition principle ∞-categorically we would like to view the formation of iterated loop spaces not as a topological construction but rather as a higher categorical one, namely, as a form of a homotopy limit.

Definition 5.3.8. Let $\mathcal{E}$ be an ∞-category. We will say that a square of the form

$$\begin{tikzcd} y \arrow[r] \arrow[d] & * \arrow[d, "p"] \\ * \arrow[r, "p"'] & x \end{tikzcd}$$

exhibits y as the loop object of x at p if $*$ is a final object and the square is cartesian in $\mathcal{E}$. In this case we will also informally write $y \simeq \Omega_p x$, or even $y \simeq \Omega x$ if the base point is implied.

Example 5.3.9. Consider the category Top of topological spaces equipped with the model structure in which the weak equivalences are the weak homotopy equivalences and the fibrations are Serre fibrations. Given a pointed space (X, x_0), the inclusion $\{x_0\} \longrightarrow X$ can be replaced with a fibration $\mathrm{ev}_1 : P_{x_0} \longrightarrow X$ where P_{x_0} is the space of paths in X starting from x_0, and ev_1 is given by evaluating at the end point.

One can then compute the homotopy pullback $\{x_0\} \times^h_X \{x_0\}$ as the fiber product $P_{x_0} \times_X \{x_0\} \cong \Omega X$. In particular, Definition 5.3.8 specializes to the classical notion of a loop space in the case $\mathcal{E} = \mathcal{S}$.

Let now $\mathcal{E}$ be an ∞-category which admits final objects and pullbacks. We would like to express the idea that loop objects $\Omega_p x$ in $\mathcal{E}$ carry a canonical structure of a group-like $\mathbb{E}_1$-monoid object (in particular, this structure has nothing specific to do with topological spaces). For this it will be convenient to work in the following setting: let Δ_+ denote the enlargement of Δ obtained by including also the *empty* linearly ordered set $[-1] := \varnothing \in \Delta_+^{\mathrm{op}}$. For $-1 \leq k \leq m$ let us denote by $\Delta_{[m,k]} \subseteq \Delta_+$ the full subcategory spanned by the objects $[k], [k+1], ..., [m]$, so that $\Delta_{[k,m]} \subseteq \Delta$ when $k \geq 0$. We note that $\Delta_{[-1,0]}$ has two objects and a unique non-zero arrow $[-1] \longrightarrow [0]$. In particular, $\mathrm{N}(\Delta_{[-1,0]}) \cong \Delta^1$. We then have the following:

Proposition 5.3.10. *Let $\mathcal{E}$ be an ∞-category with a final object $* \in \mathcal{E}$ and let $\psi : \mathrm{N}(\Delta_+^{\mathrm{op}}) \longrightarrow \mathcal{E}$ be a diagram. Then the following are equivalent:*

(1) $\psi([0]) \simeq *$ and ψ is a right Kan extension of $\psi|_{\Delta^{\mathrm{op}}_{[-1,0]}}$.

(2) $\psi|_{\Delta^{\mathrm{op}}}$ is a group-like Δ^{op}-monoid object and $\psi|_{(\Delta^{\mathrm{op}}_{[-1,1]})^{\mathrm{in}}}$ is a right Kan extension of $\psi|_{(\Delta^{\mathrm{op}}_{[0,1]})^{\mathrm{in}}}$.

Proof. Let us say that a map in Δ_+^{op} is inert if it is either an inert map of Δ^{op} or a map whose codomain is $[-1]$. We note that any map whose codomain is $[0]$ is inert and hence any map in Δ_+^{op} whose codomain is in $\Delta^{\mathrm{op}}_{[-1,0]}$ is inert. In particular, for $n \geq 1$ the functor

$$(\Delta_+^{\mathrm{op}})^{\mathrm{in}}_{[n]/} \times_{\Delta_+^{\mathrm{op}}} \Delta^{\mathrm{op}}_{[-1,0]} \longrightarrow (\Delta_+^{\mathrm{op}})_{[n]/} \times_{\Delta_+^{\mathrm{op}}} \Delta^{\mathrm{op}}_{[-1,0]}$$

is an isomorphism, and hence the second part of Condition (1) is equivalent to the condition that $\psi'|_{(\Delta_+^{\mathrm{op}})^{\mathrm{in}}}$ is a right Kan extension of $\psi'|_{\Delta^{\mathrm{op}}_{[-1,0]}}$. By the pasting lemma for Kan extensions (Proposition 2.7.7) and Remark 2.7.6 we see that Condition (1) is equivalent to the conjunction of the following three conditions

(1a) $\psi([0]) \simeq *$;

(1b) $\psi|_{(\Delta_+^{\mathrm{op}})^{\mathrm{in}}}$ is a right Kan extension of $\psi|_{(\Delta^{\mathrm{op}}_{[-1,1]})^{\mathrm{in}}}$; and

(1c) $\psi|_{(\Delta^{\mathrm{op}}_{[-1,1]})^{\mathrm{in}}}$ is a right Kan extension of $\psi|_{\Delta^{\mathrm{op}}_{[0,1]}}$.

Similarly, Condition (2) above is equivalent to the conjunction of Condition (1c) and the following three conditions:

(2a) $\psi|_{(\Delta^{\mathrm{op}})^{\mathrm{in}}}$ is a right Kan extension of $\psi|_{(\Delta^{\mathrm{op}}_{[0,1]})^{\mathrm{in}}}$;

(2b) $\psi|_{(\Delta^{\mathrm{op}}_{[0,1]})^{\mathrm{in}}}$ is a right Kan extension of $\psi|_{\{1\}}$; and

(2c) the map $\psi([2]) \longrightarrow \psi(\{0,1\}) \times \psi(\{0,2\})$ of (5.24) is an equivalence.

Now for $n \geq 2$ the full subcategory inclusions

$$(\Delta^{\mathrm{op}})^{\mathrm{in}}_{[n]/} \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{[0,1]} \subseteq (\Delta_+^{\mathrm{op}})^{\mathrm{in}}_{[n]/} \times_{\Delta_+^{\mathrm{op}}} \Delta^{\mathrm{op}}_{[-1,1]} \tag{5.25}$$

is coinitial: this follows from the fact that the only object in the right hand side which is not in the left hand side is final and the left hand side is weakly contractible. It then follows from Theorem 2.7.5 that Condition (1b) is equivalent to Condition (2a). Furthermore, since there are no inert maps from $[0]$ to $[1]$ in Δ^{op}, the same theorem implies that Condition (1a) is equivalent to Condition (2b). To finish the proof it will suffice to show that Condition (1) implies Condition (2c).

Let $\mathcal{I} \subseteq (\Delta_+^{\mathrm{op}})^{\mathrm{in}}_{[2]/}$ be the full subcategory containing $(\Delta_+^{\mathrm{op}})_{[2]/} \times_{\Delta_+^{\mathrm{op}}} \Delta^{\mathrm{op}}_{[-1,0]}$ and in addition the objects corresponding to the inclusions $\{0,1\}, \{0,2\} \subseteq \{0,1,2\}$. Let

$$\psi' : \mathcal{I} \longrightarrow \Delta_+^{\mathrm{op}} \xrightarrow{\psi} \mathcal{E}$$

be the composed diagram. Condition (1) implies that ψ' is a right Kan extension of its restriction to $(\Delta_+^{\mathrm{op}})_{[2]/} \times_{\Delta_+^{\mathrm{op}}} \Delta^{\mathrm{op}}_{[-1,0]}$ and so by Theorem 2.7.5 and Proposition 2.7.7 the composed map

$$\mathcal{I}^{\triangleleft} \longrightarrow \Delta_+^{\mathrm{op}} \xrightarrow{\psi} \mathcal{E}$$

is a limit diagram. Let $\mathcal{J} \subseteq \mathcal{I}$ be the full subcategory spanned by the objects corresponding to the inclusions $\{0\}, \{0,1\}, \{0,2\} \subseteq \{0,1,2\}$. Then it is easy to check that $\mathcal{J}$ is coinitial in $\mathcal{I}$. We may hence conclude that the composed map

$$\mathcal{J}^{\triangleleft} \longrightarrow \Delta_+^{\mathrm{op}} \xrightarrow{\psi} \mathcal{E}$$

is a limit diagram. Since $\psi([0]) \simeq *$ is final this exactly means that the map $\psi([2]) \longrightarrow \psi(\{0,1\}) \times \psi(\{0,2\}) \simeq \psi(\{0,1\}) \times_{\psi(\{0\})} \psi(\{0,2\}$ is an equivalence, as desired. □

Definition 5.3.11. Given an ∞-category $\mathcal{E}$ with a final object we will denote by

$$\mathcal{E}_* := \mathrm{Mon}_{[1]}(\mathcal{E})$$

the ∞-category of $[1]$-monoids in $\mathcal{E}$. We will refer to these monoids as *pointed objects* in $\mathcal{E}$.

Construction 5.3.12. Let $\mathcal{E}$ be an ∞-category with a terminal object $* \in \mathcal{E}$. Define $\mathcal{X}_{\mathcal{E}} \subseteq \mathrm{Fun}(\mathrm{N}(\Delta_+^{\mathrm{op}}), \mathcal{E})$ to be the full subcategory spanned by those functors $\psi : \mathrm{N}(\Delta_+^{\mathrm{op}}) \longrightarrow \mathcal{E}$ such that $\psi([0]) \simeq *$ and ψ is a right Kan extension of $\psi|_{\Delta^{\mathrm{op}}_{[-1,0]}}$. We may then identify the ∞-category $\mathcal{E}_*$ of Definition 5.3.11 with the full subcategory $\mathcal{E}_* \subseteq \mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}_{[-1,0]}, \mathcal{E})$ spanned by those $\psi : \mathrm{N}(\Delta^{\mathrm{op}}_{[-1,0]}) \longrightarrow \mathcal{E}$ such that $\psi([0]) \simeq *$. Consider the diagram

$$\begin{tikzcd} & \mathcal{X}_{\mathcal{E}} \arrow[dl] \arrow[dr, "\simeq"] & \\ \mathrm{Mon}_{\mathrm{N}(\Delta^{\mathrm{op}})}(\mathcal{E}) & & \mathcal{E}_* \end{tikzcd}$$

where the left diagonal map is given by restriction along $\Delta^{\mathrm{op}} \subseteq \Delta_+^{\mathrm{op}}$ (and is well-defined in light of Proposition 5.3.10), and the right diagonal map is given by restriction along $\Delta^{\mathrm{op}}_{[-1,0]} \subseteq \Delta_+^{\mathrm{op}}$. By Remark 2.7.2 the right diagonal map is a trivial Kan fibration. Choosing a section for this trivial Kan fibration we hence obtain a functor

$$\widetilde{\Omega}_{\mathcal{E}} : \mathcal{E}_* \longrightarrow \mathrm{Mon}_{\mathrm{N}(\Delta^{\mathrm{op}})}(\mathcal{E}),$$

well-defined up to a contractible space of choices. Let us denote by

$$\Omega_{\mathcal{E}} : \mathcal{E}_* \xrightarrow{\widetilde{\Omega}_{\mathcal{E}}} \mathrm{Mon}_{\mathrm{N}(\Delta^{\mathrm{op}})}(\mathcal{E}) \longrightarrow \mathcal{E}$$

the composed functor, where the second functor is given by evaluation at $[1] \in \Delta^{\mathrm{op}}$.

Lemma 5.3.13. *Let* $\mathrm{Forget} : \mathcal{E}_* \longrightarrow \mathcal{E}$ *be the functor which sends* $[* \longrightarrow x] \in \mathcal{E}_*$ *to* x*. Then there exists a cartesian square of the form*

$$\begin{tikzcd} \Omega_{\mathcal{E}} \arrow[r] \arrow[d] & \overline{*} \arrow[d, "\tau"] \\ \overline{*} \arrow[r, "\tau"'] & \mathrm{Forget} \end{tikzcd} \tag{5.26}$$

in the ∞*-category* $\mathrm{Fun}(\mathcal{E}_*, \mathcal{E})$ *such that for every* $[* \longrightarrow x] \in \mathcal{E}_*$ *the square*

$$\begin{tikzcd} \Omega_{\mathcal{E}}(x) \arrow[r] \arrow[d] & \overline{*}(x) \arrow[d, "\tau_x"] \\ \overline{*}(x) \arrow[r, "\tau_x"'] & x \end{tikzcd} \tag{5.27}$$

exhibits $\Omega_{\mathcal{E}}(x)$ *as the loop object of* x *at* τ_x*.*

Proof. Recall that $\widetilde{\Omega}_{\mathcal{E}}$ is given by first choosing a section $s : \mathcal{E}_* \longrightarrow \mathcal{X}_{\mathcal{E}}$ for the trivial Kan fibration $\mathcal{X}_{\mathcal{E}} \longrightarrow \mathcal{E}_*$ and then applying the restriction functor $\mathcal{X}_{\mathcal{E}} \longrightarrow \mathrm{Mon}_{\mathrm{N}(\Delta^{\mathrm{op}})}(\mathcal{E})$. In particular, for every $[* \longrightarrow x] \in \mathcal{E}_*$ the functor $s_x : \mathrm{N}(\Delta^{\mathrm{op}}_+) \longrightarrow \mathcal{E}$ is a right Kan extension of $(s_x)|_{\mathrm{N}(\Delta^{\mathrm{op}}_{[-1,0]})}$ and $s_x([0]) \simeq *$. Let $\mathcal{I} := (\Delta^{\mathrm{op}}_+)_{[1]/} \times_{\Delta^{\mathrm{op}}_+} \Delta^{\mathrm{op}}_{[-1,0]}$, so that we can identify $\mathcal{I}$ with the category

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

The adjunction between the cone and slice constructions yields a natural map $\sigma : \mathcal{I}^{\triangleleft} \longrightarrow \Delta^{\mathrm{op}}_+$ which sends the cone point to $[1]$. We may then consider the map

$$\rho : \mathcal{E}_* \times \mathcal{I}^{\triangleleft} \longrightarrow \mathcal{E}$$

given by the formula $\rho(x,i) = s_x(\sigma(i))$. The adjoint map $\mathcal{I}^{\triangleleft} \longrightarrow \mathrm{Fun}(\mathcal{E}_*, \mathcal{E})$ can then be depicted as a square of the form (5.26) which has the desired properties by Theorem 2.7.5, since s_x is a right Kan extension of $(s_x)|_{\mathrm{N}(\Delta^{\mathrm{op}}_{[-1,0]})}$ and $s_x([0])$ is final in $\mathcal{E}$. □

In light of Lemma 5.3.13 we can consider the functor $\widetilde{\Omega}_{\mathcal{E}} : \mathcal{E}_* \longrightarrow \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathcal{E})$ as a refinement of the "loop functor" $\mathcal{E}_* \longrightarrow \mathcal{E}$ (which is well-defined by up to a contractible space of choices) to a functor which takes values in Δ^{op}-monoids. In other words, $\widetilde{\Omega}_{\mathcal{E}}$ encodes a natural Δ^{op}-monoid structure (equivalently, associative monoid structure, or an $\mathbb{E}_1$-structure) on the loop object of every $x \in \mathcal{E}$.

We shall now apply the machinery above to the case $\mathcal{E} := \mathcal{S}$ of spaces. Let us denote by $\mathcal{S}^0_* \subseteq \mathcal{S}_*$ the full subcategory spanned by the pointed connected spaces.

Proposition 5.3.14. *The functor*

$$\widetilde{\Omega}_0 := \widetilde{\Omega}_{\mathcal{S}} : \mathcal{S}_* \longrightarrow \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathcal{S}) \tag{5.28}$$

of Construction 5.3.12 restricts to an equivalence between the full subcategory $\mathcal{S}^0_* \subseteq \mathcal{S}_*$ *on the left hand side and the full subcategory* $\mathrm{Mon}^{\mathrm{gr}}_{\Delta^{\mathrm{op}}}(\mathcal{S}) \subseteq \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathcal{S})$ *on the right hand side.*

Before we prove Proposition 5.3.14 let us recall a certain important property of the ∞-category of spaces.

Definition 5.3.15. Let $\mathcal{C}, \mathcal{D}$ be two ∞-categories and $\varphi, \psi : \mathcal{C} \longrightarrow \mathcal{D}$ two functors. We will say that a natural transformation $\tau : \phi \Rightarrow \psi$ is *cartesian* if for every arrow $f : x \longrightarrow y$ in $\mathcal{C}$ the square

$$\begin{tikzcd} \phi(x) \arrow[r] \arrow[d] & \psi(x) \arrow[d] \\ \phi(y) \arrow[r] & \psi(y) \end{tikzcd} \tag{5.29}$$

is cartesian in $\mathcal{D}$.

Proposition 5.3.16. *Let $\mathcal{I}$ be an ordinary category and $p, q : \mathrm{N}(\mathcal{I}) \longrightarrow \mathcal{S}$ two $\mathcal{I}$-indexed diagram of spaces. Let $\overline{p}, \overline{q} : \mathrm{N}(\mathcal{I})^{\triangleright} \longrightarrow \mathcal{S}$ be two colimits cones extending p and q respectively and let $\overline{\tau} : \overline{p} \Rightarrow \overline{q}$ be a natural transformation. If $\tau := \overline{\tau}|_{\mathcal{I}} : p \Rightarrow q$ is cartesian then $\overline{\tau}$ is cartesian.*

Proof. This is a classical fact. One way to prove this is by representing $\overline{p}$ and $\overline{q}$ by actual colimit diagrams of simplicial sets and $\overline{\tau}$ by a natural transformation which is levelwise a *minimal Kan fibration*. In this case all the squares (5.29) will necessarily be cartesian squares in simplicial sets, at which point it will be enough to verify the 1-categorical analogue of the result for the category of simplicial sets. There we can reduce to checking it for sets, which is clear. $\square$

Proof of Proposition 5.3.14. By Remark 2.7.2 it will suffice to show that the following conditions are equivalent for a given functor $\psi : \mathrm{N}(\Delta_+^{\mathrm{op}}) \longrightarrow \mathcal{S}$:

(i) $\psi([-1])$ is connected and ψ belong to $\mathcal{X}$.
(ii) $\psi|_{\mathrm{N}(\Delta^{\mathrm{op}})}$ is a group-like monoid object and ψ is a *left* Kan extension of $\psi|_{\mathrm{N}(\Delta^{\mathrm{op}})}$.

Suppose first that $\psi|_{\Delta^{\mathrm{op}}}$ is a group-like monoid object and that ψ is a left Kan extension of $\psi|_{\Delta^{\mathrm{op}}}$. Then $\psi([-1])$ is the geometric realization of the simplicial object $\psi|_{\Delta^{\mathrm{op}}}$ and since $\psi([0]) \simeq *$ is in particular connected it follows that $\psi([-1])$ is connected. To show that Condition (i) holds it will suffice by Proposition 5.3.10 to show that $\psi|_{(\Delta^{\mathrm{op}}_{[-1,0,1]})^{\epsilon}}$ is a right Kan extension of $\psi|_{\Delta^{\mathrm{op}}_{[-1,0]}}$, which means, more concretely, that we need to check that the square

$$\begin{tikzcd} \psi([1]) \arrow[r] \arrow[d] & \psi([0]) \arrow[d] \\ \psi([0]) \arrow[r] & \psi([-1]) \end{tikzcd} \tag{5.30}$$

is cartesian. Let $\rho : \Delta_+^{\mathrm{op}} \longrightarrow \Delta_+^{\mathrm{op}}$ be the functor $[n] \mapsto [n] * [0] \simeq [n+1]$ obtained by concatenation with $[0]$, and let $\rho^*\psi : \Delta_+^{\mathrm{op}} \longrightarrow \mathcal{S}_*$ be obtained by precomposing with ρ. The natural transformation $\rho \Rightarrow \mathrm{Id}$ (whose value at $[n]$ identifies $[n]$ as the prefix of length n in $\rho(n)$) induces a natural transformation $\rho^*\psi \longrightarrow \psi$. We now claim that the restricted natural transformation $\rho^*\varphi|_{\Delta^{\mathrm{op}}} \Rightarrow \varphi_{\Delta^{\mathrm{op}}}$ is cartesian in the sense of Definition 5.3.15. Unwinding the definition of ρ what we need to check is that for every $\sigma : [n] \longrightarrow [m]$ the square

$$\begin{tikzcd} \psi([m] * [0]) \arrow[r] \arrow[d] & \psi([n] * [0]) \arrow[d] \\ \psi([m]) \arrow[r] & \psi([n]) \end{tikzcd} \tag{5.31}$$

is cartesian. We now note that the collection of $\sigma : [n] \longrightarrow [m]$ in Δ for which (5.31) is cartesian is closed under composition, and if it contains σ and $\sigma' \circ \sigma$ then it contains σ'. It will hence suffice to show that (5.31) is cartesian for maps of the form $\sigma : \{i\} \hookrightarrow [n]$. In this case, the square (5.31) becomes

$$\begin{tikzcd} \psi(\{i, n+1\}) \arrow[r] \arrow[d] & \psi(\{0, ..., n+1\}) \arrow[d] \\ \psi(\{i\}) \arrow[r] & \psi(\{0, ..., n\}) \end{tikzcd} \tag{5.32}$$

We note that the monoid condition implies that $\psi([m]) \simeq \prod_{j=0}^{m-1} \psi(\{j, j+1\})$ and hence that $\psi([n+1]) \simeq \psi(\{0, ..., i\}) \times \psi(\{i, ..., n+1\})$ and $\psi([n]) \simeq \psi(\{0, ..., i\}) \times \psi(\{i, ..., n\})$. To show that (5.32) is cartesian it will hence suffice to show that

$$\begin{tikzcd} \psi(\{i, n+1\}) \arrow[r] \arrow[d] & \psi(\{i, ..., n+1\}) \arrow[d] \\ \psi(\{i\}) \arrow[r] & \psi(\{i, ..., n\}) \end{tikzcd} \tag{5.33}$$

is cartesian. This now follows from Remark 5.3.6, according to which, when ψ is group-like, $\psi([m])$ decomposes as $\prod_j \psi(\{0, j\})$ for every $[m]$. We now note that $\rho^*\psi$ is a colimit diagram since it is a split simplicial object. By Proposition 5.3.16 we may thus conclude that $\rho^*\varphi \Rightarrow \varphi$ is a cartesian natural transformation. Taking the edge $[0] \longrightarrow [-1]$ in Δ_+^{op} we recover that (5.30) is cartesian, as desired.

Let us now assume that (i) holds. Then $\psi|_{\Delta^{\mathrm{op}}}$ is a group-like monoid object by Proposition 5.3.10. Since $\mathcal{S}$ admits colimits it follows from Theorem 2.7.5 that there exists a functor $\psi' : \Delta_+^{\mathrm{op}} \longrightarrow \mathcal{S}$ and a natural transformation $\delta : \psi|_{\Delta^{\mathrm{op}}} \Rightarrow \psi'|_{\Delta^{\mathrm{op}}}$ exhibits ψ' as a left Kan extension of $\psi|_{\Delta^{\mathrm{op}}}$ along $\Delta^{\mathrm{op}} \subseteq \Delta_+^{\mathrm{op}}$. This means, in particular, that (ψ', δ) is initial in the ∞-category of left extensions of $\psi|_{\Delta^{\mathrm{op}}}$, and so there exists a natural transformation $\delta' : \psi' \Rightarrow \psi$ such that $\delta'|_{\Delta^{\mathrm{op}}} \circ \delta = \mathrm{Id}$. To finish the proof it will suffice to show that δ' is an equivalence. Now since the inclusion $\Delta^{\mathrm{op}} \subseteq \Delta_+^{\mathrm{op}}$ is fully-faithful we know that $\delta|_{\Delta^{\mathrm{op}}}$ is an equivalence. It will then be enough to verify that $\delta'([-1]) : \psi'([-1]) \longrightarrow \psi([-1])$ is an equivalence. Now ψ' satisfies (ii) by construction and so by the argument above it also satisfies (i). By Proposition 5.3.10 we then have that $\psi'|_{(\Delta^{\mathrm{op}}_{[-1,1]})^{\mathrm{in}}}$ is a right Kan extension of $\psi'|_{\Delta^{\mathrm{op}}_{[-1,0]}}$. Since the same is true for ψ this means that the map $\delta'([1]) : \psi'([1]) \longrightarrow \psi([1])$ is equivalent to the looping of the map $\delta'([-1]) : \psi'([-1]) \longrightarrow \psi([-1])$. Since the latter is a map between connected spaces and $\delta'([1])$ is an equivalence it follows that $\delta'([-1])$ is also an equivalence, as desired. $\square$

For $n \geq 1$ let us denote by

$$\widetilde{\Omega}_n : \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})_* \longrightarrow \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})) \tag{5.34}$$

the functor of Construction 5.3.12 applied to $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})$. Let us denote by $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})^0_* \subseteq \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})$ the full subcategory spanned by those $(\Delta^{\mathrm{op}})^n$-monoid objects in $\mathcal{S}$ whose underlying object is connected. The statement of Proposition 5.3.14 can now be bootstrapped from $\mathcal{S}$ to $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})$:

Corollary 5.3.17. *For every $n \geq 1$ the functor (5.34) restricts to an equivalence between the full subcategory $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})^0_* \subseteq \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})$ on the left hand side and the full subcategory $\mathrm{Mon}^{\mathrm{gr}}_{\Delta^{\mathrm{op}}}(\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})) \subseteq \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}))$ on the right.*

Proof. Let us denote by

$$\widetilde{\Omega}_0^{(\Delta^{\mathrm{op}})^n} : \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}_*) \longrightarrow \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathcal{S})) \tag{5.35}$$

the functor induced by (5.28) upon taking $(\Delta^{\mathrm{op}})^n$-monoid objects on both sides. Since $\mathcal{S}^0_* \subseteq \mathcal{S}_*$ is a full subcategory closed under finite products Proposition 5.3.14 implies that the functor $\widetilde{\Omega}_0^{(\Delta^{\mathrm{op}})^n}$ restricts to an equivalence between $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}^0_*)$ and $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathrm{Mon}^{\mathrm{gr}}_{\Delta^{\mathrm{op}}}(\mathcal{S}_*))$. It will hence suffice to show that under the identifications

$$\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}_*) \simeq \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})_*$$

and

$$\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathcal{S})) \simeq \mathrm{Mon}_{(\Delta^{\mathrm{op}})^{n+1}}(\mathcal{S}) \simeq \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})),$$

the functor $\widetilde{\Omega}_0^{(\Delta^{\mathrm{op}})^n}$ is homotopic to $\widetilde{\Omega}_n$, and the full subcategory $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathrm{Mon}^{\mathrm{gr}}_{\Delta^{\mathrm{op}}}(\mathcal{S}))$ corresponds to the full subcategory $\mathrm{Mon}^{\mathrm{gr}}_{\Delta^{\mathrm{op}}}(\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}))$. Now the second claim follows from the fact that for every ∞-category $\mathcal{E}$ with finite products the forgetful functor $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{E}) \longrightarrow \mathcal{E}$ preserves products and detects equivalences, and so in both cases the group-like condition is detected by the forgetful functor to $\mathrm{Mon}^{\mathrm{gr}}_{\Delta^{\mathrm{op}}}(\mathcal{S})$. To prove the first claim let

$$\mathcal{X}_n := \mathcal{X}_{\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})} \subseteq \mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}_+), \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}))$$

be as in Construction 5.3.12, so that the functor $\widetilde{\Omega}_n$ is defined by choosing a section to the trivial Kan fibration $\mathcal{X}_n \longrightarrow \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})_*$ and then restricting to Δ^{op}. By the uniqueness of sections to trivial Kan fibration it will now suffice to show that under the identification

$$\mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}_+), \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})) \simeq \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}_+), \mathcal{S})),$$

the full subcategory $\mathcal{X}_n$ on the left hand side corresponds to the full subcategory $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{X}_0)$ on the right hand side. Unwinding the definitions, this is a direct consequence of Theorem 2.7.5 and the fact that the forgetful functor $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}) \longrightarrow \mathcal{S}$ detects equivalences and preserves limits (because the inclusion $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}) \subseteq \mathrm{Fun}((\Delta^{\mathrm{op}})^n, \mathcal{S})$ preserves limits and the evaluation $\mathrm{ev}_{[1]} : \mathrm{Fun}((\Delta^{\mathrm{op}})^n, \mathcal{S}) \longrightarrow \mathcal{S}$ preserves limits). □

We will say that an ∞-category $\mathcal{E}$ is *pointed* if it contains an object $* \in \mathcal{E}$ which is both initial and final. In this case we will refer to such an object as a *zero object*. Now suppose that $\mathcal{E}$ is an ∞-category with a final object. Then the forgetful functor

$$\mathcal{E}_* \longrightarrow \mathcal{E} \tag{5.36}$$

is a left fibration whose fiber over $x \in \mathcal{E}$ is equivalent to the full subgroupoid of $\mathcal{E}_{/x}$ spanned by those $y \longrightarrow x$ such that y is final. This ∞-groupoid is contractible if and only if $\mathcal{E}$ is pointed. In this case, (5.36) is even a trivial Kan fibration by (the dual of) Proposition 2.6.8.

Construction 5.3.18. Let $\mathcal{E}$ be a pointed ∞-category. Then (5.36) is a trivial Kan fibration and so we may fix a section

$$\rho : \mathcal{E} \longrightarrow \mathcal{E}_*.$$

The choice of ρ is unique up to a contractible space of choices.

Example 5.3.19. By Remark 5.2.13 it follows that $\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{E}) \simeq \mathrm{Mon}_{\mathbb{E}_n}(\mathcal{E})$ is a pointed ∞-category for every ∞-category with limits $\mathcal{E}$. A zero object is given by the the terminal object of $\mathcal{E}$ equipped with its unique $\mathbb{E}_n$-monoid structure.

Now for every $n \geq 1$ let us denote by

(5.37)

$$\widetilde{\Omega}^*_n : \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}) \xrightarrow{\rho} \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})_* \xrightarrow{\widetilde{\Omega}_n} \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})) \cong \mathrm{Mon}_{(\Delta^{\mathrm{op}})^{n+1}}(\mathcal{S})$$

the composition of $\widetilde{\Omega}_n$ with the section ρ of Construction 5.3.18. We may then define the *iterated loop functor* $\widetilde{\Omega}^n : \mathcal{S}_* \longrightarrow \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})$ to be the composed functor

$$\widetilde{\Omega}^n : \mathcal{S}_* \xrightarrow{\widetilde{\Omega}_0} \mathrm{Mon}_{\Delta^{\mathrm{op}}}(\mathcal{S}) \xrightarrow{\widetilde{\Omega}^*_1} \cdots \xrightarrow{\widetilde{\Omega}^*_{n-1}} \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}) \ . \tag{5.38}$$

Let $\mathcal{S}^n_* \subseteq \mathcal{S}_*$ be the full subcategory spanned by those pointed spaces which are n-connected (i.e., have no homotopy groups in degree $\leq n$). We may now deduce the following version of May's recognition principle:

Corollary 5.3.20. *For every $n \geq 1$ the iterated loop functor (5.38) restricts to an equivalence between the full subcategory $\mathcal{S}^n_* \subseteq \mathcal{S}_*$ of pointed n-connected spaces and the full subcategory $\mathrm{Mon}^{\mathrm{gr}}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S}) \subseteq \mathrm{Mon}_{(\Delta^{\mathrm{op}})^n}(\mathcal{S})$ of group-like $(\Delta^{\mathrm{op}})^n$-monoid objects.*

Proof. For $k = 1, ..., n$ let us denote by $\mathrm{Mon}^{n-k}_{(\Delta^{\mathrm{op}})^k}(\mathcal{S}) \subseteq \mathrm{Mon}_{(\Delta^{\mathrm{op}})^k}(\mathcal{S})$ the full subcategory spanned by those $(\Delta^{\mathrm{op}})^k$-monoids whose underlying space is $(n-k)$-connected. Combining Proposition 5.3.14 and Lemma 5.3.13 we may conclude that the functor (5.28) restricts to an equivalence

$$\widetilde{\Omega}_0 : \mathcal{S}^n_* \xrightarrow{\simeq} \mathrm{Mon}^{n-1}_{\Delta^{\mathrm{op}}}(\mathcal{S}),$$

where we use the fact that any $(n-1)$-connected (and so in particular connected) Δ^{op}-monoid in spaces is automatically group-like by Example 5.3.4. Similarly, by Proposition 5.3.14 for every $k = 1, ..., n-1$ the functor (5.37) restricts to an equivalence

$$\widetilde{\Omega}^*_k : \mathrm{Mon}^{n-k}_{(\Delta^{\mathrm{op}})^k} \xrightarrow{\simeq} \mathrm{Mon}^{n-k-1}_{(\Delta^{\mathrm{op}})^{k+1}}(\mathcal{S}).$$

The desired result now follows by composing these equivalences. □

6. Factorization homology

6.1. Manifolds and framings.

Definition 6.1.1. Let $n \geq 1$ be an integer. By an *n-manifold* we will mean a paracompact Hausdorff space M such that each $x \in M$ either has a neighborhood homeomorphic to $\square^n$ or has a neighborhood homeomorphic to $\square^n_\partial$. We will say that M is *open* if every point has a neighborhood homeomorphic to $\square^n$.

If M is an n-manifold then we will denote by $\partial M \subseteq M$ the subspace consisting of those points which have neighborhoods homeomorphic to $\square^n_\partial$, and refer to ∂M as the *boundary* of M. We note that ∂M is an open $(n-1)$-manifold and $M^\circ := M \smallsetminus \partial M$ is an open n-manifold. We emphasize that we do not require M nor ∂M to be compact. Given two n-manifolds with boundary we will denote by $\mathrm{Emb}(M,N)$ the space of open embeddings of M in N, endowed with the compact-open topology.

Definition 6.1.2. We will denote by $\mathcal{M}\mathrm{fld}^\Delta_n$ the simplicial category whose objects are the n-manifolds and such that

$$\mathrm{Map}_{\mathcal{M}\mathrm{fld}^\Delta_n}(M,N) = \mathrm{Sing}\,\mathrm{Emb}(M,N).$$

We will denote by $\mathcal{M}\mathrm{fld}_n = \mathrm{N}(\mathcal{M}\mathrm{fld}^\Delta)$ the coherent nerve of $\mathcal{M}\mathrm{fld}^\Delta$, and refer to it as the ∞*-category of n-manifolds*. We will denote by $\mathcal{M}\mathrm{fld}^\circ_n \subseteq \mathcal{M}\mathrm{fld}_n$ the full subcategory spanned by the open n-manifolds.

Let us denote by $\mathrm{Top}(n) \subseteq \mathrm{Emb}(\square^n,\square^n)$ the subspace consisting of all *homeomorphisms* from $\square^n$ to $\square^n$. Then $\mathrm{Top}(n)$ is a topological group and the inclusion $\mathrm{Top}(n) \subseteq \mathrm{Emb}(\square^n,\square^n)$ is a continuous homomorphism of topological monoids. We recall the following fundamental result in geometric topology:

Theorem 6.1.3 (Kister-Mazur)**.** *The inclusion* $\mathrm{Top}(n) \subseteq \mathrm{Emb}(\square^n,\square^n)$ *is a homotopy equivalence.*

A direct consequence of Theorem 6.1.3 is that the monoid $\pi_0\,\mathrm{Emb}(\square^n,\square^n)$ is a group, and hence every embedding $\square^n \longrightarrow \square^n$ is invertible up to isotopy. Let $\mathrm{BTop}^\Delta(n) \subseteq \mathcal{M}\mathrm{fld}^\Delta_n$ be the full subcategory spanned by $\square^n$. It then follows that the coherent nerve

$$\mathrm{BTop}(n) := \mathrm{N}(\mathrm{BTop}^\Delta(n))$$

is a *Kan complex*, which by Theorem 6.1.3 we can identify with the classifying space of the topological group $\mathrm{Top}(n)$ (hence the notation).

Construction 6.1.4. Given an n-manifold $M \in \mathcal{M}\mathrm{fld}_n$ we will denote by

$$\overline{M} := (\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} \mathrm{BTop}(n)$$

equipped with its projection

$$p : \overline{M} \longrightarrow \mathrm{BTop}(n).$$

We will refer to $\overline{M}$ as the *underlying space* of M, and say that p is the classifying map for the *tangent bundle* of M.

Remark 6.1.5. The map $p : \overline{M} \longrightarrow \mathrm{BTop}(n)$ is a right fibration by construction and since $\mathrm{BTop}(n)$ is a Kan complex it follows that $\overline{M}$ is a Kan complex and p is a Kan fibration.

Proposition 6.1.6. *The Kan complex $\overline{M}$ is naturally homotopy equivalent to* $\mathrm{Sing}(M)$.

The proof of Proposition 6.1.6 will make use of the notion of a *germ* of an embedding. For $0 < \varepsilon < 1$ let us denote by $\square^n(\varepsilon) := (-\varepsilon,\varepsilon)^n \subseteq \square^n$.

Definition 6.1.7. Let M be an n-manifold and I a finite set. We define

$$\mathrm{Germ}(I,M) := \operatorname*{colim}_k \mathrm{Sing}\,\mathrm{Emb}(\square^n(1/2^k) \times I, M),$$

and refer to it as the simplicial set of *I-germs in M*.

Lemma 6.1.8. *Let M be an n-manifold and I be a finite set. Then the natural map*

$$\mathrm{Sing}\,\mathrm{Emb}(\square^n \times I, M) \xrightarrow{\simeq} \mathrm{Germ}(I, M).$$

is a homotopy equivalence of Kan complexes.

Proof. It will suffice to show that for every $k \geq 0$ the map $\mathrm{Sing}\,\mathrm{Emb}(\square^n(1/2^k) \times I, M) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(\square^n(1/2^{k+1}) \times I, M)$ is a homotopy equivalence. Indeed, this follows from the fact that for every $\varepsilon < \varepsilon'$ the inclusion $i : \square^n(\varepsilon) \subseteq \square^n(\varepsilon')$ is an *isotopy equivalence*: it admits an inverse $p : \square^n(\varepsilon') \longrightarrow \square^n(\varepsilon)$ such that $p \circ i$ is in the path component of the identity in $\mathrm{Emb}(\square^n(\varepsilon), \square^n(\varepsilon))$ and $i \circ p$ is in the path component of the identity in $\mathrm{Emb}(\square^n(\varepsilon'), \square^n(\varepsilon'))$. □

Remark 6.1.9. Using a suitable variant of the *isotopy extension theorem* it can be shown that for every finite set I the evaluation at 0 map

$$\mathrm{Emb}(\square^n \times I, M) \longrightarrow \mathrm{Conf}(I, M)$$

is a Serre fibration, and so the map

$$\mathrm{Sing}\,\mathrm{Emb}(\square^n \times I, M) \longrightarrow \mathrm{Sing}\,\mathrm{Conf}(I, M)$$

is a Kan fibration. Since every n-simplex and every horn have finitely many non-degenerate simplices it follows that the map

$$\mathrm{Germ}(I, M) \longrightarrow \mathrm{Sing}\,\mathrm{Conf}(I, M)$$

is a Kan fibration as well.

Proof of Proposition 6.1.6. Consider the simplicial functor $\mathrm{Sing} : \mathcal{M}\mathrm{fld}_n^{\Delta} \longrightarrow \mathcal{K}\mathrm{an}$ which sends M to $\mathrm{Sing}(M)$ and acts on mapping spaces via the natural map $\mathrm{Sing}\,\mathrm{Emb}(M, N) \longrightarrow \mathrm{Map}(\mathrm{Sing}\,M, \mathrm{Sing}\,N)$. Let $\mathrm{Sing}_\infty : \mathcal{M}\mathrm{fld}_n \longrightarrow \mathcal{S}$ be the induced functor on coherent nerves. Then Sing_∞ induces a map of Kan complexes

$$\overline{M} := (\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} \mathrm{BTop}(n) \longrightarrow \mathcal{S}_{/\mathrm{Sing}\,M} \times_{\mathcal{S}} \mathcal{B}$$

where $\mathcal{B} \subseteq \mathcal{S}$ is the full subcategory spanned by the contractible Kan complex $\mathrm{Sing}(\square^n)$.

Let $j : \square^n \longrightarrow M$ be a map in $\mathcal{M}\mathrm{fld}_n$, which we can identify with an object of $(\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} \mathrm{BTop}(n)$. Then the induced map $j_* : \mathrm{Map}_{\mathcal{M}\mathrm{fld}_n}(\square^n, \square^n) \longrightarrow \mathrm{Map}_{\mathcal{M}\mathrm{fld}_n}(\square^n, M)$ can be identified with the inclusion of the homotopy fiber of the map

$$\mathrm{Map}_{\mathcal{M}\mathrm{fld}_n}(\square^n, M) = (\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} \{\square^n\} \longrightarrow (\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} \mathrm{BTop}(n)$$

over j. The same statement holds for $\mathcal{S}$ with respect the base point determined by $\mathrm{Sing}(j) : \mathrm{Sing}\,\square^n \longrightarrow \mathrm{Sing}\,M$. We then obtain a commutative diagram

(6.1)

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{M}\mathrm{fld}_n}(\square^n, \square^n) \arrow[r, "j_*"] \arrow[d] & \mathrm{Map}_{\mathcal{M}\mathrm{fld}_n}(\square^n, M) \arrow[r] \arrow[d] & (\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} \mathrm{BTop}(n) \arrow[d] \\
\mathrm{Map}_{\mathcal{S}}(\mathrm{Sing}\,\square^n, \mathrm{Sing}\,\square^n) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathcal{S}}(\mathrm{Sing}\,\square^n, \mathrm{Sing}\,M) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathcal{S}_{/\mathrm{Sing}\,M} \times_{\mathcal{S}} \mathcal{B} \\
\mathrm{Sing}\,\square^n \arrow[r] & \mathrm{Sing}\,M &
\end{tikzcd}$$

in which the first two rows are fiber sequences of spaces and the bottom vertical maps are given by evaluating at $0 \in \mathrm{Sing}\,\square^n$, and are weak equivalences since $\mathrm{Sing}\,\square^n \simeq *$ in $\mathcal{S}$. The middle right horizontal map is then an equivalence since its fibers $\mathrm{Map}_{\mathcal{S}}(\mathrm{Sing}\,\square^n, \mathrm{Sing}\,\square^n)$ are contractible. I will now suffice to show that the right most vertical map is an equivalence. For this, it will suffice to show that the left square is homotopy cartesian for every $j : \square^n \hookrightarrow M$, or alternatively, that the left external rectangle is homotopy cartesian. By Lemma 6.1.8 we may instead prove that the equivalent square

$$\begin{tikzcd} \mathrm{Germ}(*, \square^n) \arrow[r] \arrow[d] & \mathrm{Germ}(*, M) \arrow[d] \\ \mathrm{Sing}\,\square^n \arrow[r] & \mathrm{Sing}\,M \end{tikzcd} \tag{6.2}$$

is homotopy cartesian, where the horizontal maps are induced by the fixed embedding $j : \square^n \longrightarrow M$. But this is because the right vertical map in (6.2) is a Kan fibration (see Remark 6.1.9) and the square itself is strictly cartesian. □

Definition 6.1.10. By a *tangent structure* we will mean a Kan complex B equipped with a Kan fibration $\pi : B \longrightarrow \mathrm{BTop}(n)$.

Definition 6.1.11. Let $B \longrightarrow \mathrm{BTop}(n)$ be a tangent structure and let M be an open n-manifold. A *B-framing* of M is a lift of the form

$$\begin{tikzcd} & B \arrow[d] \\ \overline{M} \arrow[ur, dashed] \arrow[r, "\pi"] & \mathrm{BTop}(n) \end{tikzcd}$$

Example 6.1.12. When $B \longrightarrow \mathrm{BTop}(n)$ is an equivalence the notion of a B-framing is vacuous. On the other extreme, when $B \simeq *$ the notion of a B-framing coincides with a trivialization of the tangent bundle.

Example 6.1.13. When $B \longrightarrow \mathrm{BTop}(n)$ is the universal covering of $\mathrm{BTop}(n)$, a B-framing is the same as an orientation. Similarly, when B is the 2-connected covering of $\mathrm{BTop}(n)$ a B-framing is a topological spin structure.

Example 6.1.14. By smoothing theory, when $n \neq 5$ the data of a smooth structure on M is equivalent to the data of a framing with respect to the map $\mathrm{BGL}(n) \longrightarrow \mathrm{BTop}(n)$, and the data of a piecewise linear structure is equivalent to the data of a framing with respect to the map $\mathrm{BPL}(n) \longrightarrow \mathrm{BTop}(n)$.

Example 6.1.15. If N is an open n-manifold then we can take $B = \overline{N}$ with the map $\pi : \overline{N} \longrightarrow \mathrm{BTop}(n)$ classifying the tangent bundle, in which case we will simplify notation and write N-framing instead of $\overline{N}$-framing. Then any *open immersion* $M \longrightarrow N$ (i.e., a continuous map which is locally a homeomorphism) gives an N-framing on M, although not every N-framing is obtained this way (e.g., the $\mathbb{R}^1$-framing of S^1 cannot be obtained by an immersion of S^1 in $\mathbb{R}^1$). Note however that by the Yoneda lemma every N-framing on $\square^n$ is equivalent to one which comes from an immersion, and even an embedding, of $\square^n$ in N.

It what follows it will be useful to have a notion of a framing for n-manifolds with possibly non-empty boundary. Consider the full subcategory $\mathrm{BTop}^{\partial}(n) \subseteq \mathcal{M}\mathrm{fld}_n$

spanned by the objects $\square^n, \square^n_\partial$. Given an n-manifold M we define the right fibration

$$\pi : \overline{M}_\partial := (\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} \mathrm{BTop}^\partial(n) \longrightarrow \mathrm{BTop}^\partial(n).$$

To understand this fibration it will be useful to extend the construction of germs to cubes with boundary. For $0 < \varepsilon < 1$ let us denote by $\square^n_\partial(\varepsilon) := [0,\varepsilon) \times (-\varepsilon,\varepsilon)^{n-1} \subseteq \square^n_\partial$.

Definition 6.1.16. Let M be an n-manifold, I a finite set and $X : I \longrightarrow \{\square^n, \square^n_\partial\}$ an I-tuple of objects of $\mathrm{BTop}_\partial(n)$. We define

$$\mathrm{Germ}(X, M) := \operatorname*{colim}_k \mathrm{Sing}\,\mathrm{Emb}\big(\coprod_{i\in I} X_i(1/2^k), M\big),$$

and refer to it as the simplicial set of *boundary X-germs in M*. We will also denote by $\mathrm{Germ}_\partial(I, M) := \mathrm{Germ}(\overline{\square}^n_\partial, M)$ where $\overline{\square}^n_\partial : I \longrightarrow \{\square_n, \square^n_\partial\}$ denotes the constant tuple with value $\square^n_\partial$.

Lemma 6.1.17. *Let M be an n-manifold, I a finite set and $\overline{X} : I \longrightarrow \{\square^n, \square^n_\partial\}$ and I-tuple of objects of $\mathrm{Top}_\partial(n)$. Then the natural map*

$$\mathrm{Sing}\,\mathrm{Emb}\big(\coprod_{i\in I} X_i, M\big) \xrightarrow{\simeq} \mathrm{Germ}(\overline{X}, M).$$

is a homotopy equivalence of Kan complexes.

Proof. Same proof as Lemma 6.1.8. □

Remark 6.1.18. Using a suitable variant of the isotopy extension theorem it can be shown that for every finite set I the evaluation at $\overline{0} \in \square^n_\partial$ map

$$\mathrm{Emb}(\square^n_\partial \times I, M) \longrightarrow \mathrm{Conf}(I, \partial M)$$

is a Serre fibration, and so the map

$$\mathrm{Sing}\,\mathrm{Emb}(\square^n_\partial \times I, M) \longrightarrow \mathrm{Sing}\,\mathrm{Conf}(I, \partial M)$$

is a Kan fibration. Since every n-simplex and every horn have finitely many non-degenerate simplices it follows that the map

$$\mathrm{Germ}_\partial(I, M) \longrightarrow \mathrm{Sing}\,\mathrm{Conf}(I, \partial M)$$

is a Kan fibration as well.

Lemma 6.1.19. *Let M be an n-manifold with boundary ∂M, I a finite set and $X : I \longrightarrow \{\square^n, \square^n_\partial\}$ and I-tuple of objects of $\mathrm{Top}_\partial(n)$. Let $I_0 \subseteq I$ be the set of those indices which map to $\square^n$ and $I_1 \subseteq I$ the indices that map to $\square^n_\partial$. Then the restriction map*

$$\mathrm{Emb}\big(\coprod_{i\in I} X_i, M\big) \longrightarrow \mathrm{Emb}\big(\square^n \times I_0, M^\circ\big) \times \mathrm{Emb}\big(\square^{n-1} \times I_1, \partial M\big) \tag{6.3}$$

is a weak homotopy equivalence.

Proof. By Lemma 6.1.8 and Lemma 6.1.17 we may instead prove that the map of simplicial sets

$$\mathrm{Germ}(X, M) \longrightarrow \mathrm{Germ}(I_0, M^\circ) \times \mathrm{Germ}(I_1, \partial M) \tag{6.4}$$

is a weak homotopy equivalence. We first observe that on the level of simplicial sets $\mathrm{Germ}(X, M)$ breaks as a product of $\mathrm{Germ}_\partial(I_1, M)$ and $\mathrm{Germ}(I_0, M)$. It will hence suffice to show that the restriction map

$$\mathrm{Germ}_\partial(I_1, M) \longrightarrow \mathrm{Germ}(I_1, \partial M) \tag{6.5}$$

is a weak equivalence of Kan complexes. Let $\iota : \square^n_\partial \times I_1 \longrightarrow M$ be an open embedding and consider the commutative square

(6.6)

$$\begin{tikzcd} \prod_{i\in I_1} \mathrm{Germ}_\partial(\{i\}, \square^n_\partial \times \{i\}) \arrow[r] \arrow[d] & \prod_{i\in I_1} \mathrm{Germ}(\{i\}, \square^{n-1} \times \{i\}) \arrow[r] \arrow[d] & \prod_{i\in I_1} \mathrm{Conf}(\{i\}, \square^{n-1}_\partial \times \{i\}) \arrow[d] \\ \mathrm{Germ}_\partial(I_1, M) \arrow[r] & \mathrm{Germ}(I_1, \partial M) \arrow[r] & \mathrm{Conf}(I_1, \partial M) \end{tikzcd}$$

where the vertical maps are induced by ι. The right square is cartesian on the level of simplicial sets and is also homotopy cartesian since the right bottom horizontal map is a Kan fibration (see Remark 6.1.9). For the same reason the external rectrangle is homotopy cartesian (see Remark 6.1.18). We may hence conclude that the left square is homotopy cartesian. In particular, the homotopy fibers of the top left horizontal map are the same as those of the bottom left horizontal map. Since every vertex of $\mathrm{Germ}(I_1, \partial M)$ is in the image of the vertical map for at least one $\iota : \square^n_\partial \times I_1 \longrightarrow M$ it will now suffice to show that the top left horizontal map is a homotopy equivalence of Kan complexes. Since this map breaks as a product of maps which just need to show that

$$\mathrm{Germ}_\partial(*, \square^n_\partial) \longrightarrow \mathrm{Germ}(*, \square^{n-1}).$$

is a homotopy equivalence of Kan complexes. For this, it will suffice to show that for every $\varepsilon > 0$ the map of topological spaces

$$\mathrm{Emb}(\square^n_\partial(\varepsilon), \square^n_\partial) \longrightarrow \mathrm{Emb}(\square^{n-1}(\varepsilon), \square^{n-1}). \tag{6.7}$$

We claim that in this case the map

$$[0,\varepsilon) \times (-) : \mathrm{Emb}(\square^{n-1}, \square^{n-1}) \longrightarrow \mathrm{Emb}([0,\varepsilon) \times \square^{n-1}, [0,\varepsilon) \times \square^{n-1}) \tag{6.8}$$

given by taking the product with $[0,\varepsilon)$ is a homotopy inverse to (6.3). Indeed, the composition of (6.3) after (6.8) is the identity, and the composition of (6.8) after (6.3) is homotopic to the identity via the homotopy $f \mapsto f_t$ where

$$f_t(s, \overline{x})_i := \begin{cases} (s, f_0(\overline{x})) & s \le t \\ g_t^{-1} f(g_t(s, \overline{x})) & s \ge t \end{cases},$$

Here we use the coordinates $s \in [0,\varepsilon)$ and $\overline{x} \in \square^{n-1}$, $f_0 : \square^{n-1} \longrightarrow \square^{n-1}$ the restriction of f to $s = 0$ and $g_t : [t,\varepsilon) \times \square^{n-1} \longrightarrow [0,\varepsilon) \times \square^{n-1}$ is the homeomorphism given by $g_t(r, \overline{x}) = (\frac{r-t}{\varepsilon - t}, \overline{x})$. □

Let

$$q : \mathrm{BTop}^\partial(n) \longrightarrow \Delta^1 \tag{6.9}$$

be the unique map which sends $\square^n$ to 0 and $\square^n_\partial$ to 1.

Lemma 6.1.20. *The map* (6.9) *is a right fibration classified by the functor* $[1] \longrightarrow \mathrm{Cat}_\infty$ *depicted by the arrow* $\mathrm{BTop}(n-1) \longrightarrow \mathrm{BTop}(n)$ *induced by taking the product with* $(-1,1)$.

Proof. Let $\mathrm{BTop}^\Delta_+(n) \subseteq \mathrm{BTop}^\Delta_\partial(n)$ be the full subcategory spanned by $\square^n_\partial$ and let $\mathrm{BTop}_+(n) := \mathrm{N}(\mathrm{BTop}^\Delta_+(n))$ be its coherent nerve. Consider the functor

$$\iota : \mathrm{BTop}^\Delta_+(n) \longrightarrow \mathrm{BTop}^\Delta(n) \tag{6.10}$$

which sends $\square^n_\partial$ to $\square^n$ and any open embedding $f : \square^n_\partial \longrightarrow \square^n_\partial$ to the open embedding $\square^n \xrightarrow{\cong} (0,1) \times \square^{n-1} \longrightarrow (0,1) \times \square^{n-1}$ where the first arrow is the unique rectilinear homeomorphism of this form and the second is obtained by restricting f. We then observe that $\mathrm{BTop}^\partial(n)$ is naturally equivalent to the Grothendieck construction $\int_{[1]}[\iota]$ where $[\iota] : [1] \longrightarrow \mathrm{Cat}_\Delta$ is the functor corresponding to the arrow of simplicial categories $\iota : \mathrm{BTop}^\Delta_+(n) \longrightarrow \mathrm{BTop}^\Delta(n)$. It then follows by Proposition 2.5.9 that $\mathrm{BTop}_\partial(n) \times_{\Delta^1} \Delta^{\{0\}} \cong \mathrm{BTop}(n)$,$\mathrm{BTop}_\partial(n) \times_{\Delta^1} \Delta^{\{1\}} \cong \mathrm{BTop}_+(n)$ and $q : \mathrm{BTop}^\partial(n) \longrightarrow \Delta^1$ is a right fibration classified by the arrow

$$\mathrm{N}(\iota) : \mathrm{BTop}_+(n) \longrightarrow \mathrm{BTop}(n). \tag{6.11}$$

It will hence suffice to show that ι is weakly equivalent in the arrow category of Cat_Δ to the functor $\mathrm{BTop}^\Delta(n-1) \longrightarrow \mathrm{BTop}^\Delta(n)$ induced by taking the product with $(-1,1)$. To see this, observe that the latter functor factors as

$$\mathrm{BTop}^\Delta(n-1) \xrightarrow{\eta} \mathrm{BTop}^\Delta_+(n) \xrightarrow{\iota} \mathrm{BTop}^\Delta(n)$$

where η is induced by taking the product with $[0,1)$. It will hence suffice to show that η is a Dwyer-Kan equivalence, that is that the map

$$[0,1) \times (-) : \mathrm{Emb}(\square^{n-1}, \square^{n-1}) \longrightarrow \mathrm{Emb}([0,1) \times \square^{n-1}, [0,1) \times \square^{n-1}) \tag{6.12}$$

is a weak homotopy equivalence. But this now follows from Lemma 6.1.19 by the 2-out-of-3 property. □

By Lemma 6.1.20 the composed map $\overline{M}_\partial \longrightarrow \mathrm{BTop}^\partial(n) \longrightarrow \Delta^1$ is also a right fibration. We note that the fiber $\overline{M}_\partial \times_{\Delta^1} \Delta^{\{0\}}$ is naturally isomorphic to the simplicial set $\overline{M}$. Let us denote by

$$\partial\overline{M} := \overline{M}_\partial \times_{\Delta^1} \Delta^{\{1\}}$$

the fiber over $\Delta^{\{1\}} \subseteq \Delta^1$. The map (6.11) then naturally extends to a commutative square of Kan complexes:

$$\begin{tikzcd} \partial\overline{M} \arrow[r] \arrow[d] & \overline{M} \arrow[d] \\ \mathrm{BTop}_+(n) \arrow[r] & \mathrm{BTop}(n) \end{tikzcd} \tag{6.13}$$

We will refer to $\partial\overline{M}$ as the *underlying boundary* of M.

Remark 6.1.21. By Proposition 6.1.6 we have that $\overline{M} \simeq \mathrm{Sing}(M^\circ) \simeq \mathrm{Sing}\, M$ and by Lemma 6.1.19 the map

$$\mathrm{Emb}(\square^n_\partial, M) \longrightarrow \mathrm{Emb}(\{0\} \times \square^{n-1}, \partial M)$$

is a homotopy equivalence, and hence $\partial\overline{M}$ is naturally equivalent to the Kan complex $\overline{\partial M}$ obtained by performing Construction 6.1.4 to the $(n-1)$-manifold ∂M. In particular, $\partial\overline{M} \simeq \mathrm{Sing}\,\partial M$ by Proposition 6.1.6. Furthermore, one can show that under these identification the top horizontal map in (6.13) is homotopic to the map $\mathrm{Sing}\,\partial M \longrightarrow \mathrm{Sing}\, M$ induced by the inclusion $\partial M \longrightarrow M$.

Definition 6.1.22. By a *boundary tangent structure* we will mean a right fibration of the form

$$B_\partial \longrightarrow \mathrm{BTop}_\partial(n).$$

We will generally like to think of boundary tangent structure as a commutative diagram of ∞-groupoids

$$\begin{tikzcd} \partial B \arrow[r] \arrow[d] & B \arrow[d] \\ \mathrm{BTop}_+(n) \arrow[r] & \mathrm{BTop}(n) \end{tikzcd} \tag{6.14}$$

where $B \coloneqq B_\partial \times_{\Delta^1} \Delta^{\{0\}}$ and $\partial B \coloneqq B_\partial \times_{\Delta^1} \Delta^{\{1\}}$.

Definition 6.1.23. Given a boundary tangent structure $B_\partial \longrightarrow \mathrm{BTop}_\partial(n)$ and an n-manifold M, a B_∂-*framing* on M is a lift of the form

$$\begin{tikzcd} & B_\partial \arrow[d] \\ \overline{M}_\partial \arrow[r, "\pi"] \arrow[ur, dashed] & \mathrm{BTop}_\partial(n) \end{tikzcd}$$

Remark 6.1.24. If $\partial B = \varnothing$ then any B_∂-framed n-manifold is open, and the notion of a B_∂-framing coincides with a B-framing. In particular, we may consider the condition of being an open n-manifold as a special case of framing.

Example 6.1.25. If N is an n-manifold then we can take $B_\partial = \overline{N}_\partial$ equipped with its natural map to $\mathrm{BTop}_\partial(n)$. Similarly to Example 6.1.15, if U is either $\square^n$ or $\square^n_\partial$, then the data of an $\overline{N}_\partial$-framing on U is essentially equivalent to the data of an open embedding $U \longrightarrow N$.

We close this section with the following lemma, which we will invoke several times in the subsequent sections. Upon first contemplating the following lemma, the reader is encouraged to take $M = *$.

Lemma 6.1.26. *Let $p : N \longrightarrow M$ be a continuous map from an n-manifold N to an m-manifold M. Let I be a finite set, $\overline{X} : I \longrightarrow \{\square^n, \square^n_\partial\}$ an I-tuple of objects of $\mathrm{Top}_\partial(n)$. Let $\mathrm{Disj}(M) \subseteq O(M)$ be the full subposet spanned by those open subsets $V \subseteq M$ which are homeomorphic to disjoint unions of copies of $\square^n$ and $\square^n_\partial$ and let $P \subseteq O(N)^I \times \mathrm{Disj}(M)$ be the full subposet spanned by those pairs $(\{U_i\}_{i\in I}, V)$ such that $U_i \cong X_i$, the U_i's are pairwise disjoint, and $p(U_i) \subseteq V$. Then the canonical maps*

$$\operatorname*{hocolim}_{(\{U_i\},V)\in P} \prod_{i\in I} \mathrm{Emb}(X_i, U_i) \longrightarrow \operatorname*{hocolim}_{V\in \mathrm{Disj}(M)} \mathrm{Emb}(\coprod_{i\in I} X_i, p^{-1}(V)) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}\big(\coprod_i X_i, N\big) \tag{6.15}$$

are both weak homotopy equivalences.

Proof. It will suffice to show that the composed map in (6.15) is a weak homotopy equivalence: the left map can then be recovered by replacing N with $p^{-1}(V)$ and M with a point and the right map will follow by the 2-out-of-3 rule. Let $I_0 \subseteq I$ be the subset of those i's such that $X_i = \square^n$ and let $I_1 = I_0 \smallsetminus I_0$. By Lemma 6.1.17, to show that the composed map in (6.15) is a weak homotopy equivalence we may instead show that the canonical map

$$\operatorname*{hocolim}_{(\{U_i\},V)\in P} \prod_{i\in I_0} \mathrm{Germ}(\{i\}, U_i) \times \prod_{i\in I_1} \mathrm{Germ}_\partial(\{i\}, U_i) \longrightarrow \mathrm{Germ}(I_0, N) \times \mathrm{Germ}_\partial(I_1, N) \tag{6.16}$$

is a weak homotopy equivalence. Consider the commutative square

(6.17)
$$\begin{tikzcd}
\operatorname*{hocolim}_{(\{U_i\},V)\in P} \prod_{i\in I_0} \mathrm{Germ}(\{i\},U_i) \prod_{i\in I_1} \times \mathrm{Germ}_{\partial}(\{i\},U_i) \arrow[r] \arrow[d] & \mathrm{Germ}(I_0,N)\times \mathrm{Germ}_{\partial}(I_1,N) \arrow[d] \\
\operatorname*{hocolim}_{(\{U_i\},V)\in P} \prod_{i\in I_0} \mathrm{Sing}\,\mathrm{Conf}(\{i\},U_i)\times \prod_{i\in I_1} \mathrm{Sing}\,\mathrm{Conf}(\{i\},\partial U_i) \arrow[r] & \mathrm{Sing}\,\mathrm{Conf}(I_0,N)\times \mathrm{Sing}\,\mathrm{Conf}(I_1,\partial N)
\end{tikzcd}$$

We now observe that the natural transformation

$$\prod_{i\in I_0} \mathrm{Germ}(\{i\},U_i)\times \prod_{i\in I_1} \mathrm{Germ}_{\partial}(\{i\},U_i) \longrightarrow \prod_{i\in I_0} \mathrm{Sing}\,\mathrm{Conf}(\{i\},U_i) \prod_{i\in I_1} \mathrm{Sing}\,\mathrm{Conf}(\{i\},\partial U_i)$$

of functors from P to spaces is cartesian in the sense of Definition 5.3.15 since it is cartesian on the level of simplicial sets and is levelwise a Kan fibration by Remarks 6.1.9 and 6.1.18. By Proposition 5.3.16 we may conclude that (6.17) is homotopy cartesian. It will hence suffice to show that the bottom map in (6.17) is a weak homotopy equivalence. By Theorem 5.2.5 it will suffice to show that the for every pair of configurations $f_0 : I_0 \longrightarrow N$ and $f_1 : I_1 \longrightarrow \partial N$, the subposet

$$\{(\{U_i\},V)\in P | f_0(i)\subseteq U_i, f_1(i')\in U_{i'}\}$$

is weakly contractible. Indeed, this poset is filtered since N is an n-manifold and M is an m-manifold.

□

6.2. **Little cube algebras with tangent structures.** In this section we will discuss some variants of the little n-cube ∞-operad where we incorporate a *tangent structure*. Algebras over these variants are closely related to algebras of the little n-cube ∞-operad: they can be described as certain "twisted" families of $\mathbb{E}_n$-algebras.

Definition 6.2.1. Let $\mathbb{E}^{\Delta}_{\mathrm{Top}(n)}$ be the simplicial operad with a single object $\square^n$ and such that

$$\mathrm{Mul}_{\mathbb{E}^{\Delta}_{\mathrm{Top}(n)}}(\{\square^n\}_{i\in I},\square^n) = \mathrm{Sing}\,\mathrm{Emb}(\square^n\times I,\square^n).$$

Remark 6.2.2. We point out that the difference between Definition 6.2.1 and Definition 5.1.2 is that in the former the embeddings are not assumed to be rectilinear.

Since the singular complex of a space is always a Kan complex the simplicial operad $\mathbb{E}^{\Delta}_{\mathrm{Top}(n)}$ is locally Kan.

Definition 6.2.3. We define the *unframed little n-cube operad*

$$\mathbb{E}^{\otimes}_{\mathrm{Top}(n)} := \mathrm{N}\left(\mathbb{E}^{\Delta}_{\mathrm{Top}(n)}\right)$$

to be the operadic nerve of $\mathbb{E}^{\Delta}_{\mathrm{Top}(n)}$. Identifying rectilinear embeddings as a subspace of all embeddings we obtain a natural map of ∞-operads

$$\mathbb{E}^{\otimes}_{n} \longrightarrow \mathbb{E}^{\otimes}_{\mathrm{Top}(n)}.$$

Example 6.2.4. When $n = 0$ every embedding $\square^0\times I \longrightarrow \square^0$ is rectilinear and we have $\mathbb{E}^{\otimes}_{\mathrm{Top}(0)} = \mathbb{E}^{\otimes}_{0}$. When $n = 1$ the spaces $\mathrm{Emb}(\square^1\times I,\square^1)$ are homotopy equivalent to discrete spaces and $\mathbb{E}^{\otimes}_{\mathrm{Top}(1)}$ is equivalent to the operadic nerve of the ordinary operad AssInv of Example 4.1.7(5) which controls the theory of *algebras with involution*.

We may identify the underlying simplicial category of $\mathbb{E}^{\Delta}_{\mathrm{Top}(n)}$ with the simplicial category $\mathrm{BTop}^{\Delta}(n)$ of §6.1. We note that the natural map $\mathrm{Emb}(\square^n \times I, \square^n) \longrightarrow \prod_{i\in I} \mathrm{Emb}(\square^n, \square^n)$ induces a map of simplicial categories

$$\left(\mathbb{E}^{\Delta}_{\mathrm{Top}(n)}\right)^{\otimes} \longrightarrow \mathrm{BTop}^{\Delta}(n)^{\amalg},$$

where $\mathrm{BTop}(n)^{\amalg}$ is defined as in Exercise 3.3.2. Taking coherent nerves and using Exercise 3.3.2 we obtain a map of ∞-operads

$$\mathbb{E}^{\otimes}_{\mathrm{Top}(n)} \longrightarrow \mathrm{BTop}(n)^{\amalg}. \tag{6.18}$$

To relate this construction to the little n-cube ∞-operad $\mathbb{E}^{\otimes}_n$, let us denote by $\mathrm{Rect}^{\Delta}(n)$ the underlying simplicial category of $\mathbb{E}^{\Delta}_n$, that is, the simplicial category with a single object $\square^n$ and such that $\mathrm{Map}_{\mathrm{Rect}^{\Delta}(n)}(\square^n, \square^n) = \mathrm{Sing}\,\mathrm{Rect}(\square^n, \square^n)$. Since the mapping spaces in Rect^{Δ}_n are contractible we have that $\mathrm{Rect}(n) \coloneqq \mathrm{N}(\mathrm{Rect}^{\Delta}(n))$ is a contractible Kan complex. We then obtain a commutative square of simplicial categories

$$\begin{tikzcd} (\mathbb{E}^{\Delta}_n)^{\otimes} \arrow[r] \arrow[d] & \mathrm{Rect}^{\Delta}(n)^{\amalg} \arrow[d] \\ (\mathbb{E}^{\Delta}_{\mathrm{Top}(n)})^{\otimes} \arrow[r] & \mathrm{BTop}^{\Delta}(n)^{\amalg} \end{tikzcd} \tag{6.19}$$

which, after passing to coherent nerves, yields a square

$$\begin{tikzcd} \mathbb{E}^{\otimes}_n \arrow[r] \arrow[d] & \mathrm{Rect}(n)^{\amalg} \arrow[d] \\ \mathbb{E}^{\otimes}_{\mathrm{Top}(n)} \arrow[r] & \mathrm{BTop}(n)^{\amalg} \end{tikzcd} \tag{6.20}$$

of ∞-operads (see Example 4.1.12), in which the top right corner is equivalent to the terminal ∞-operad $\mathrm{N}(\mathrm{Com}^{\otimes})$ (since $\mathrm{Rect}(n)$ is a contractible Kan complex).

Proposition 6.2.5. *The square* (6.20) *is a homotopy pullback square of ∞-operads. In particular, we may identify $\mathbb{E}^{\otimes}_n$ with the* homotopy fiber *of the map $\mathbb{E}^{\otimes}_{\mathrm{Top}(n)} \longrightarrow \mathrm{BTop}(n)^{\amalg}$.*

Proof. We first note that all four ∞-operads appearing in (6.20) have underlying ∞-categories which are Kan complexes, and hence have the property that a map in any of them is inert if and only if its image in $\mathrm{N}(\mathrm{Fin}_*)$ is inert. It follows that if $\mathcal{O}^{\otimes}$ is an ∞-operad then any map of ∞-categories from $\mathcal{O}^{\otimes}$ to any of the ∞-operads in (6.20) will automatically preserve inert maps, and hence will automatically be a map of ∞-operads. It will hence suffice to show that (6.20) is a homotopy pullback square of ∞-categories. Since the coherent nerve is a right Quillen functor we may instead show that (6.19) is a homotopy pullback square of simplicial categories. We note that all four simplcial categories in (6.19) have the same set of objects as Fin_*. Comparing Construction 4.1.9 and the construction in Exercise 3.3.2 it will now suffice to show that for every $\langle m\rangle \in \mathrm{Fin}_*$ the square of Kan complexes

$$\begin{tikzcd} \mathrm{Sing}\,\mathrm{Rect}(\square^n \times \langle m\rangle^\circ, \square^n) \arrow[r] \arrow[d] & \prod_{i\in\langle m\rangle^\circ} \mathrm{Sing}\,\mathrm{Rect}(\square^n \times \{i\}, \square^n) \arrow[d] \\ \mathrm{Sing}\,\mathrm{Emb}(\square^n \times \langle m\rangle^\circ, \square^n) \arrow[r] & \prod_{i\in\langle m\rangle^\circ} \mathrm{Sing}\,\mathrm{Emb}(\square^n \times \{i\}, \square^n) \end{tikzcd} \tag{6.21}$$

is a homotopy cartesian. To show this, let us extend this square to a diagram of the form

$$\begin{tikzcd} \mathrm{Sing}\,\mathrm{Rect}(\square^n \times \langle m\rangle^\circ, \square^n) \arrow[r] \arrow[d] & \prod_{i\in\langle m\rangle^\circ} \mathrm{Sing}\,\mathrm{Rect}(\square^n \times \{i\}, \square^n) \arrow[d] \\ \mathrm{Sing}\,\mathrm{Emb}(\square^n \times \langle m\rangle^\circ, \square^n) \arrow[r] \arrow[d, "\simeq"] & \prod_{i\in\langle m\rangle^\circ} \mathrm{Sing}\,\mathrm{Emb}(\square^n \times \{i\}, \square^n) \arrow[d, "\simeq"] \\ \mathrm{Germ}(\langle m\rangle^\circ, \square^n) \arrow[r] \arrow[d] & \prod_i \mathrm{Germ}(\{i\}, \square^n) \arrow[d] \\ \mathrm{Sing}\,\mathrm{Conf}(\langle m\rangle^\circ, \square^n) \arrow[r] & \prod_i \mathrm{Sing}\,\mathrm{Conf}(\{i\}, \square^n) \end{tikzcd} \tag{6.22}$$

where the vertical maps in the middle square are equivalences by Lemma 6.1.8. In particular, the middle square is homotopy cartesian. Similarly, the vertical maps in the external rectangle are equivalences by Lemma 5.1.10, and so the external rectangle is homotopy cartesian. By the pasting lemma for homotopy cartesian square, to show that the top square in (6.22) is homotopy cartesian it will suffice to show that the bottom square in (6.22) is homotopy cartesian. But this is indeed true since the right vertical map in this right most square is a Kan fibration (see Remark 6.1.9) and the right most square itself is cartesian on the level of simplicial sets. □

Definition 6.2.6. Let $B \longrightarrow \mathrm{BTop}(n)$ be a tangent structure (Definition 6.1.10). We define the ∞-operad $\mathbb{E}_B^\otimes$ by the pullback square

$$\begin{tikzcd} \mathbb{E}_B^\otimes \arrow[r] \arrow[d] & B^\amalg \arrow[d] \\ \mathbb{E}_{\mathrm{Top}(n)}^\otimes \arrow[r] & \mathrm{BTop}(n)^\amalg \end{tikzcd}$$

We will refer to $\mathbb{E}_B^\otimes$ as the B-*framed* little cube ∞-operad.

Remark 6.2.7. When $B \longrightarrow \mathrm{BTop}(n)$ is an equivalence we have that $\mathbb{E}_B^\otimes \simeq \mathbb{E}_{\mathrm{Top}(n)}^\otimes$. On the other extreme, when $B \simeq *$ we have by Proposition 6.2.5 that $\mathbb{E}_B^\otimes \simeq \mathbb{E}_n^\otimes$. For this reason the ∞-operad $\mathbb{E}_n^\otimes$ is also called sometimes the *framed* little cube

∞-operad. We may hence think of the ∞-operads $\mathbb{E}_B^{\otimes}$ as interpolating between $\mathbb{E}_n^{\otimes}$ and $\mathbb{E}_{\mathrm{Top}(n)}^{\otimes}$ by allowing various intermediate tangent structures.

We would like to understand the relation between the notions of $\mathbb{E}_n$-algebras and $\mathbb{E}_{\mathrm{Top}(n)}$-algebras in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, or, more generally, $\mathbb{E}_B$-algebras for some tangent structure $B \longrightarrow \mathrm{BTop}(n)$. For this, let us restrict attention for the moment to the case where $\mathcal{C}^{\otimes} := \mathcal{E}^{\times}$ is the cartesian symmetric monoidal ∞-category associated to an ∞-category with finite products $\mathcal{E}$. In other words, let us try to understand the notion of an $\mathbb{E}_{\mathrm{Top}}^{\otimes}$-monoid in $\mathcal{E}$ in terms of the notion of an $\mathbb{E}_n^{\otimes}$-monoid in $\mathcal{E}$. For this, we will once again make use of the theory of *weak ∞-operads* introduced in §4.2.

Let us consider the Kan complex B as a weak ∞-operad in which every map is both inert and active and $B_0 = B$. We may then endow the cartesian product $\mathrm{N}(\mathrm{Fin}_*) \times B$ with the product weak ∞-operad structure of Example 4.2.13. The projection $\Gamma^* \longrightarrow \mathrm{Fin}_*$ determines a map

$$\mathrm{N}(\mathrm{Fin}_*) \times B \longrightarrow B^{\amalg} \tag{6.23}$$

given informally by the formula $(\langle n \rangle, b) \mapsto (b, ..., b)$. It is straightforward to verify that this is map of weak ∞-operads in the sense of Definition 4.2.5 (we note that in both cases the active and inert maps are determined in $\mathrm{N}(\mathrm{Fin}_*)$).

Definition 6.2.8. Let $B \longrightarrow \mathrm{BTop}(n)$ be a tangent structure. We define $\int_B \mathbb{E}_n^{\otimes}$ to be any ∞-category sitting in a *homotopy* pullback square of the form

$$\begin{tikzcd} \int_B \mathbb{E}_n^{\otimes} \arrow[r] \arrow[d] & \mathrm{N}(\mathrm{Fin}_*) \times B \arrow[d] \\ \mathbb{E}_B^{\otimes} \arrow[r] & B^{\amalg} \end{tikzcd} \tag{6.24}$$

We will consider $\int_B \mathbb{E}_n^{\otimes}$ as a *weak ∞-operad* via the pullback structure of Remark 4.2.17.

Warning 6.2.9. The maps $\mathbb{E}_B^{\otimes} \longrightarrow B^{\amalg}$ and $\mathrm{N}(\mathrm{Fin}_*) \times B \longrightarrow B^{\amalg}$ are generally not categorical fibrations. To obtain an explicit model for $\int_B \mathbb{E}_n^{\otimes}$ one needs to first replace these maps by categorical fibrations and then take the actual product of simplicial sets.

Remark 6.2.10. When $B \simeq *$ we have $\int_B \mathbb{E}_n^{\otimes} \simeq \mathbb{E}_B^{\otimes} \simeq \mathbb{E}_n^{\otimes}$ by Proposition 6.2.5.

We then have the following key observation:

Proposition 6.2.11. *The map $\int_B \mathbb{E}_n^{\otimes} \longrightarrow \mathbb{E}_B^{\otimes}$ is a* strong approximation *of weak ∞-operads (see Definition 4.2.14).*

Proof. By Remark 4.2.17 it will suffice to show that the map (6.23) is a strong approximation. We first show that (6.23) is a weak approximation, that is, we need to show that for every $(\langle m \rangle, b) \in \mathrm{N}(\mathrm{Fin}_*) \times B$ the induced map

$$(\mathrm{N}(\mathrm{Fin}_*) \times B)^{\mathrm{act}}_{/(\langle m \rangle, b)} \longrightarrow (B^{\amalg})^{\mathrm{act}}_{/(b,\dots,b)} \tag{6.25}$$

has weakly contractible homotopy fibers. But this is true because the map (6.25) is in fact an equivalence of ∞-categories: indeed, we can consider (6.25) as a map over $\mathrm{N}(\mathrm{Fin}_*)^{\mathrm{act}}_{/\langle m \rangle}$, and for a given active $\alpha : \langle m' \rangle \longrightarrow \langle m \rangle$, the map on homotopy fibers over α induced by (6.25) is given by the diagonal map $* \simeq B_{/b} \longrightarrow \prod_{i \in m'} B_{/b} \simeq *$.

To show that (6.23) is in fact a strong approximation we simply note that both for $\mathrm{N}(\mathrm{Fin}_*) \times B$ and $B^{\amalg}$ the basics are those objects which lie above $\langle 1 \rangle \in \mathrm{N}(\mathrm{Fin}_*)$, a map is inert if and only if its image in $\mathrm{N}(\mathrm{Fin}_*)$ is inert, and the subcategories of basics and inert maps identifies on both sides with the Kan complex B. □

Applying Proposition 4.2.18 we now get the following:

Corollary 6.2.12. *Let $\mathcal{E}$ be an ∞-category which admits limits. Then the restriction functor*

$$\mathrm{Mon}_{\mathbb{E}^{\otimes}_B}(\mathcal{E}) \xrightarrow{\simeq} \mathrm{Mon}_{\int_B \mathbb{E}^{\otimes}_n}(\mathcal{E})$$

is an equivalence of ∞-categories.

We now unwind the definitions to see what the notion of an $\int_B \mathbb{E}^{\otimes}_n$-monoid in $\mathcal{E}$ actually is. Consider the composed map

$$\pi : \int_B \mathbb{E}^{\otimes}_n \longrightarrow \mathrm{N}(\mathrm{Fin}_*) \times B \longrightarrow B. \tag{6.26}$$

Without loss of generality we may assume that we have chosen $\int_B \mathbb{E}^{\otimes}_n$ such that π is a categorical fibration. Since B is a Kan complex the map (6.26) is then automatically a cocartesian fibration. Let us denote by $\mathbb{E}^{\otimes}_b$ the fiber of π over $b \in B$. We then have a commutative diagram

$$\begin{tikzcd}
\mathbb{E}^{\otimes}_b \arrow[r] \arrow[d] & \{b\} \times \mathrm{N}(\mathrm{Fin}_*) \arrow[d] \\
\int_B \mathbb{E}^{\otimes}_n \arrow[r] \arrow[d] & B \times \mathrm{N}(\mathrm{Fin}_*) \arrow[d] \\
\mathbb{E}^{\otimes}_B \arrow[r] \arrow[d] & B^{\amalg} \arrow[d] \\
\mathbb{E}^{\otimes}_{\mathrm{Top}(n)} \arrow[r] & \mathrm{BTop}(n)^{\amalg}
\end{tikzcd}$$

in which all squares are homotopy cartesian, and so the external rectangle is homotopy cartesian. This yields an identification $\mathbb{E}^{\otimes}_b \simeq \mathbb{E}^{\otimes}_n$ by Proposition 6.2.5. The cocartesian fibration π then corresponds to a functor $\chi : B \longrightarrow \mathrm{Cat}_\infty$, all of whose fibers are equivalent to $\mathbb{E}^{\otimes}_n$. Furthermore, since π factors through the projection $\mathrm{N}(\mathrm{Fin}_*) \times B \longrightarrow B$ we see that the transition functors $\alpha_! : \mathbb{E}^{\otimes}_b \longrightarrow \mathbb{E}^{\otimes}_{b'}$ respect the projection to $\mathrm{N}(\mathrm{Fin}_*)$. Since they are all equivalences they must also preserve inert maps. In particular, all the transition functors are maps of ∞-operads $\mathbb{E}^{\otimes}_b \longrightarrow \mathbb{E}^{\otimes}_{b'}$. We may then consider χ as a B-indexed diagram of ∞-operads, all of which are equivalent to $\mathbb{E}^{\otimes}_n$.

To translate this into an a description of the notion of an $\int_B \mathbb{E}^{\otimes}_n$-monoid object in $\mathcal{E}$ we note first that since (6.26) is a cocartesian fibration the data of a functor $\psi : \int_B \mathbb{E}^{\otimes}_n \longrightarrow \mathcal{E}$ can be identified with the data of a compatible B-indexed family of functors $\mathbb{E}^{\otimes}_b \longrightarrow \mathcal{E}$. Unwinding the definitions we see that such a ψ is a monoid object if and only if each $\psi|_{\mathbb{E}^{\otimes}_b}$ is an $\mathbb{E}^{\otimes}_b$-monoid object. We may hence conclude that $\int_B \mathbb{E}^{\otimes}_n$-monoid objects in $\mathcal{E}$ correspond to *compatible families* $\psi_b \in \mathrm{Mon}_{\mathbb{E}^{\otimes}_b}(\mathcal{E})$

of $\mathbb{E}_b^{\otimes}$-monoid objects. By Proposition (6.2.11) this description is valid for $\mathbb{E}_B^{\otimes}$ as well.

Remark 6.2.13. Using a suitable generalization of Proposition 4.2.18 one can show that the same description holds also for $\mathbb{E}_B^{\otimes}$-algebra objects in symmetric monoidal ∞-categories which are not necessarily cartesian. More generally, for every ∞-operad $\mathcal{O}^{\otimes}$ the collection of maps $\mathbb{E}_b^{\otimes} \longrightarrow \mathbb{E}_B^{\otimes}$ induces an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_B}(\mathcal{O}) \simeq \lim_{b\in B} \mathrm{Alg}_{\mathbb{E}_b}(\mathcal{O}).$$

This, in turn, implies that the collection of maps $\mathbb{E}_b^{\otimes} \longrightarrow \mathbb{E}_B^{\otimes}$ exhibit $\mathbb{E}_B^{\otimes}$ as the *colimit* of the family $\{\mathbb{E}_b^{\otimes}\}_{b\in B}$ in $\mathcal{O}\mathrm{p}_\infty$.

Example 6.2.14. When $B = \mathrm{BTop}(n)$ the fibration $\int_{\mathrm{BTop}(n)} \mathbb{E}_n^{\otimes} \longrightarrow \mathrm{BTop}(n)$ encodes an action of the simplicial group $\mathrm{Sing}\,\mathrm{Top}(n) \subseteq \mathrm{Sing}\,\mathrm{Emb}(\square^n, \square^n)$ on the little n-cube ∞-operad $\mathbb{E}_n^{\otimes}$. On the level of spaces of operations, under the homotopy equivalence

$$\mathrm{Map}_{\mathbb{E}_n^{\otimes}}(\langle m\rangle, \langle 1\rangle) \simeq \mathrm{Sing}\,\mathrm{Emb}(\square^n \times \langle m\rangle, \square^n) \simeq \mathrm{Sing}\,\mathrm{Conf}(\langle m\rangle^{\circ}, \square^n)$$

of lemma 5.1.10, the action of $\mathrm{Sing}\,\mathrm{Top}(n)$ is given by the natural action of the topological group $\mathrm{Top}(n)$ on the configuration space $\mathrm{Conf}(\langle m\rangle^{\circ}, \square^n)$. Given an $\mathbb{E}_n^{\otimes}$-monoid in an ∞-category with finite products $\mathcal{E}$, this action induces an action of $\mathrm{Sing}\,\mathrm{Top}(n)$ on the ∞-category $\mathrm{Mon}_{\mathbb{E}_n^{\otimes}}(\mathcal{E})$, which for $\sigma \in \mathrm{Sing}\,\mathrm{Top}(n)$ we may denote as $\psi \mapsto \psi^{\sigma}$. The data of an $\int_B \mathbb{E}_n^{\otimes}$-monoid object in $\mathcal{E}$ can then be informally described as the data of an $\mathbb{E}_n^{\otimes}$-monoid ψ together with a compatible family of equivalences $T_\sigma : \psi \simeq \psi^{\sigma}$ for $\sigma \in \mathrm{Sing}\,\mathrm{Top}(n)$.

Remark 6.2.15. In general, while each fiber $\mathbb{E}_b^{\otimes}$ is equivalent to $\mathbb{E}_n^{\otimes}$, this equivalence cannot be chosen compatibly over B. In other words, the B-family of ∞-operads $\int_B \mathbb{E}_n^{\otimes} \longrightarrow B$ is *not* equivalent to the constant family $\mathbb{E}_n^{\otimes} \times B$. In particular, when $B = \mathrm{BTop}(n)$ the corresponding action of $\mathrm{Sing}\,\mathrm{Top}(n)$ on $\mathbb{E}_n^{\otimes}$ is *not* equivalent to the constant action. For example, when $n = 1$ the group $\mathrm{Top}(2) \cong \mathrm{Homeo}(\mathbb{D}^2, \mathbb{D}^2)$ contains S^1 as the subgroup of rotations. The restricted action of S^1 on $\mathrm{Conf}(\{0,1\}, \square^n) \cong \mathrm{Conf}(\{0,1\}, \mathbb{D}^2)$ is very far from being trivial: choosing any base point in $\mathrm{Conf}(\{0,1\}, \mathbb{D}^2)$, the action of S^1 yields a homotopy equivalence $S^1 \longrightarrow \mathrm{Conf}(\{0,1\}, \mathbb{D}^2)$. We hence cannot say that a $\int_B \mathbb{E}_n^{\otimes}$ is given by a family of $\mathbb{E}_n^{\otimes}$-monoids (that is by a functor $B \longrightarrow \mathrm{Mon}_{\mathbb{E}_n^{\otimes}}(\mathcal{E})$), but rather by a *twisted family*, where the twisting is determined by the map $B \longrightarrow \mathrm{BTop}(n)$.

Remark 6.2.16. The data of a Kan fibration $\pi : B \longrightarrow \mathrm{BTop}(n)$ is the homotopy coherent way of describing a space $\widetilde{B}$ equipped with an action of $\mathrm{Sing}\,\mathrm{Top}(n)$ (the space $\widetilde{B}$ is then given by the homotopy fiber of π over $\square^n \in \mathrm{BTop}(n)$). We may then combine the descriptions of Example 6.2.14 and Remark 6.2.15 as follows: an $\int_B \mathbb{E}_n^{\otimes}$-monoid object in $\mathcal{E}$ can be informally describe as a family $\{\psi_b\}_{b\in\widetilde{B}}$ of $\mathbb{E}_n^{\otimes}$-monoids in $\mathcal{E}$, parameterized by $\widetilde{B}$, which is *equivariant* with respect to the actions of $\mathrm{Sing}\,\mathrm{Top}(n)$ on $\widetilde{B}$ and $\mathrm{Mon}_{\mathbb{E}_n^{\otimes}}(\mathcal{E})$. In particular, we have a compatible family of equivalences $T_{b,\sigma} : \psi_b \xrightarrow{\simeq} \psi^{\sigma}_{\sigma(b)}$ for $b \in \widetilde{B}$ and $\sigma \in \mathrm{Sing}\,\mathrm{Top}(n)$.

We finish this section with a useful variant of the above construction for the ∞-operads $\mathbb{E}_{n,\partial}^{\otimes}$ of little cubes with boundary (see Variant 5.1.5).

Definition 6.2.17. Let $\mathbb{E}^{\Delta}_{\mathrm{Top}_\partial(n)}$ be the simplicial operad with two objects $\square^n, \square^n_\partial$ and such that

$$\mathrm{Mul}_{\mathbb{E}^{\Delta}_{\mathrm{Top}_\partial(n)}}(\{X_i\}_{i\in I}, Y) \coloneqq \mathrm{Sing}\,\mathrm{Emb}(\coprod_i X_i, Y)$$

for $X_i, Y \in \{\square^n, \square^n_\partial\}$. We will denote by

$$\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)} \coloneqq \mathrm{N}(\mathbb{E}^{\Delta}_{\mathrm{Top}_\partial(n)})$$

the operadic nerve of $\mathbb{E}^{\Delta}_{\mathrm{Top}_\partial(n)}$.

We note that the underlying ∞-category $(\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})_{\langle 1\rangle}$ of $\mathbb{E}^{\otimes}_{\mathrm{Top}_{\partial(n)}}$ can be identified with the ∞-category $\mathrm{BTop}_\partial(n)$ described in §6.1, and that we have a natural map of ∞-operads

$$\mathbb{E}^{\otimes}_{\mathrm{BTop}_\partial(n)} \longrightarrow \mathrm{BTop}_\partial(n)^{\amalg}$$

constructed exactly like the map (6.18) above.

Construction 6.2.18. Given a boundary tangent structure $B_\partial \longrightarrow \mathrm{Top}_\partial(n)$ (see Definition 6.1.22) we define the ∞-operad $\mathbb{E}^{\otimes}_{B_\partial}$ by the pullback square

$$\begin{tikzcd}
\mathbb{E}^{\otimes}_{B_\partial} \arrow[r] \arrow[d] & B_\partial^{\amalg} \arrow[d] \\
\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)} \arrow[r] & \mathrm{BTop}_\partial(n)^{\amalg}
\end{tikzcd}$$

Example 6.2.19. If $B_\partial \times_{\Delta^1} \Delta^{\{1\}} = \varnothing$ then $\mathbb{E}^{\otimes}_{B_\partial} \simeq \mathbb{E}^{\otimes}_{B}$ where $B \coloneqq B_\partial \times_{\Delta^1} \Delta^{\{0\}}$.

Example 6.2.20. If $B_\partial \longrightarrow \mathrm{BTop}_\partial(n)$ is an equivalence then $\mathbb{E}^{\otimes}_{B_\partial} \simeq \mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)}$. On the other hand, if the composed map $B_\partial \longrightarrow \mathrm{BTop}_\partial(n) \longrightarrow \Delta^1$ is an equivalence then $\mathbb{E}^{\otimes}_{B_\partial} \simeq \mathbb{E}^{\otimes}_{n,\partial}$. This can be proven using a similar argument to the one used in the proof of Proposition 6.2.5.

Remark 6.2.21. Arguing using suitable weak ∞-operadic models as above one can extract a description of $\mathbb{E}^{\otimes}_{B_\partial}$-monoids in an ∞-category $\mathcal{E}$ in the spirit of Remark 6.2.15. More precisely, given a boundary tangent structure $B_\partial \longrightarrow \mathrm{Top}_\partial(n)$ the map

$$\chi_n : B \coloneqq B_\partial \times_{\Delta^1} \Delta^{\{0\}} \longrightarrow \mathrm{BTop}(n)$$

determines a family $\{\mathbb{E}^{\otimes}_b\}_{b\in B}$ of ∞-operads (all equivalent to $\mathbb{E}^{\otimes}_n$), and the map

$$\chi_{n-1} : \partial B \coloneqq B_\partial \times_{\Delta^1} \Delta^{\{1\}} \longrightarrow \mathrm{BTop}_{+}(n) \simeq \mathrm{BTop}(n-1)$$

determines a family $\{\mathbb{E}^{\otimes}_c\}_{c\in\partial B}$ of ∞-operads (all equivalent to $\mathbb{E}^{\otimes}_{n-1}$). Furthermore, the commutativity of the square (6.14), determines, in a compatible manner, for each $c \in \partial B$ with image $b \in B$, a bifunctor of ∞-operads $\mathbb{E}^{\otimes}_1 \times \mathbb{E}^{\otimes}_c \longrightarrow \mathbb{E}^{\otimes}_b$, which is equivalent to the standard bifunctor (5.6), and hence exhibits $\mathbb{E}^{\otimes}_b$ as the tensor product $\mathbb{E}_1 \otimes \mathbb{E}_c$ (by Theorem 5.2.2). The data of an $\mathbb{E}^{\otimes}_{B_\partial}$-monoid in $\mathcal{E}$ then consists of two families $\{X_b\}_{b\in B}$ and $\{Y_c\}_{c\in\partial B}$, where each X_b is an $\mathbb{E}_b$-monoid, each Y_c is an $\mathbb{E}_c$-monoid, and such that for every $c \in \partial B$ with image $b \in B$, we have a right action of X_b on Y_c, where we identify X_b with an $\mathbb{E}_1$-monoid in $\mathbb{E}_c$-monoids via the given identification $\mathbb{E}_b \simeq \mathbb{E}_1 \otimes \mathbb{E}_c$ at c.

Remark 6.2.22. Given a boundary tangent structure $\pi : B \longrightarrow \mathrm{BTop}_\partial(n)$, the description in Remark 6.2.21 can be used in order to express $\mathbb{E}^\otimes_{B_\partial}$ as the colimit in $\mathcal{O}\mathrm{p}_\infty$ of a certain diagram

$$B_\partial \xrightarrow{\pi} \mathrm{BTop}_\partial(n) \longrightarrow \mathcal{O}\mathrm{p}_\infty$$

where the second arrow can be informally described by the formula $\square^n \mapsto \mathbb{E}^\otimes_n$ and $\square^n_\partial \mapsto \mathbb{E}^\otimes_{n,\partial}$.

6.3. **Little cube algebras over manifolds.**

Definition 6.3.1. Let M be an n-manifold. We define

$$\mathbb{E}^\otimes_M \coloneqq \mathbb{E}^\otimes_{\mathrm{Top}_\partial(n)} \times_{\mathrm{BTop}_\partial(n)^\amalg} \overline{M}^\amalg_\partial$$

to be the ∞-operad associated to the boundary tangent structure $\overline{M}_\partial \longrightarrow \mathrm{BTop}_\partial(n)$ as in Construction 6.2.18. We will refer to $\mathbb{E}^\otimes_M$ as the *M-framed* little cube ∞-operad.

Example 6.3.2. When $M = \square^n$ we have $\mathbb{E}^\otimes_M \simeq \mathbb{E}^\otimes_n$ and when $M = \square^n_\partial$ we have $\mathbb{E}^\otimes_M \simeq \mathbb{E}^\otimes_{n,\partial}$.

Example 6.3.3. If $n = 1$ and $M = I = [0,1]$ is the unit interval then the notion of an $\mathbb{E}^\otimes_I$-algebra in a symmetric monoidal ∞-category $\mathcal{C}$ is equivalent to the data of a triple (A, M_0, M_1) where A is an associative algebra object, M_0 is a pointed right A-module and M_1 is a pointed left A-module (see Example 4.1.8(4)).

As discussed in §6.2, the notion of an $\mathbb{E}_M$-algebra in a symmetric monoidal ∞-category $\mathcal{C}^\otimes$ can be described via certain *twisted families* of $\mathbb{E}_n$-algebras and $\mathbb{E}_{n-1}$-algebras, parameterized by $\overline{M}$ and $\partial\overline{M}$ respectively, where the nature of the twisting is determined by the map $\pi : \overline{M}_\partial \longrightarrow \mathrm{BTop}_\partial(n)$ which classifies the topological tangent bundle of $(M, \partial M)$. In light of 6.1.6, we may expect that $\mathbb{E}_M$-algebras will admit a more geometric description in terms of the manifold M itself (as opposed to its homotopical avatar $\overline{M}_\partial$). To see this, we will define a certain discrete approximation to the ∞-operad $\mathbb{E}^\otimes_M$.

Definition 6.3.4. Let $\mathbb{D}^{\mathrm{ord}}_M$ be the *ordinary* operad whose objects are pairs (U, ρ) where $U \in \{\square^n, \square^n_\partial\}$ and $\rho : U \longrightarrow M$ is an open embedding. Given objects $(X_1, \rho_1), ..., (X_m, \rho_m), (Y, \eta) \in \mathbb{D}^{\mathrm{ord}}_M$, a multimap $\{(X_i, \rho_i)\} \longrightarrow (Y, \eta)$ is given by a commutative diagram

$$\begin{tikzcd} \coprod_i X_i \arrow[rr] \arrow[dr, "(\rho_1,\dots,\rho_m)"'] & & Y \arrow[dl, "\eta"] \\ & M & \end{tikzcd} \tag{6.27}$$

where the horizontal arrow is an open embedding. We note that for any given $(X_1, \rho_1), ..., (X_m, \rho_m), (Y, \eta)$, there is at most one multimap $\{(X_i, \rho_i\} \longrightarrow (Y, \eta)$ (which is the case exactly when the images of $\rho_1, ..., \rho_m$ are disjoint and contained in the image of η). We will denote by $\mathbb{D}^\otimes_M \coloneqq \mathrm{N}((\mathbb{D}^{\mathrm{ord}}_M)^\otimes)$ the *operadic nerve* of $\mathbb{D}^{\mathrm{ord}}_M$.

Construction 6.3.5. We define $\mathcal{M}\mathrm{fld}^{\mathrm{ord}}_n$ to be the ordinary category with the same objects as $\mathcal{M}\mathrm{fld}_n$ and such that $\mathrm{Hom}_{\mathcal{M}\mathrm{fld}^{\mathrm{ord}}_n}(M, N)$ is the *set* $\mathrm{Emb}^{\mathrm{ord}}(M, N)$ of open embeddings $M \longrightarrow N$ (considered without any topology). In other words, $\mathcal{M}\mathrm{fld}^{\mathrm{ord}}_M$ is obtained from $\mathcal{M}\mathrm{fld}^\Delta_M$ by replacing each mapping simplicial set by its set of vertices. We will denote by $\mathrm{BTop}^{\mathrm{ord}}_{\partial}(n) \subseteq \mathcal{M}\mathrm{fld}^{\mathrm{ord}}_n$ the full subcategory spanned by $\{\square^n, \square^n_{\partial}\}$, and by $\mathrm{BTop}^{\mathrm{ord}}(n), \mathrm{BTop}^{\mathrm{ord}}_{+}(n) \subseteq \mathrm{BTop}^{\mathrm{ord}}_{\partial}(n)$ the full subcategories spanned by $\square^n$ and $\square^n_{\partial}$ respectively. Given an n-manifold M we will then denote by

$$\overline{M}^{\mathrm{ord}}_{\partial} \coloneqq (\mathcal{M}\mathrm{fld}^{\mathrm{ord}})_{/M} \times_{\mathcal{M}\mathrm{fld}^{\mathrm{ord}}_n} \mathrm{BTop}^{\mathrm{ord}}_{\partial}(n).$$

By construction we have a natural map $\mathcal{M}\mathrm{fld}^{\mathrm{ord}}_n \longrightarrow \mathcal{M}\mathrm{fld}^{\Delta}_n$ which induces a map

$$\mathrm{N}(\overline{M}^{\mathrm{ord}}_{\partial}) \longrightarrow \overline{M}_{\partial}.$$

Remark 6.3.6. The map $\mathrm{BTop}^{\mathrm{ord}}_{\partial}(n) \longrightarrow \Delta^1$ which sends $\square^n$ to 0 and $\square^n_{\partial}$ to 1 is a cartesian fibration corresponding to the arrow $\iota^{\mathrm{ord}} : \mathrm{BTop}^{\mathrm{ord}}_{+}(n) \longrightarrow \mathrm{BTop}^{\mathrm{ord}}(n)$ defined analogously to (6.10). It then follows that the composed map

$$\overline{M}^{\mathrm{ord}}_{\partial} \longrightarrow \mathrm{BTop}^{\mathrm{ord}}_{\partial}(n) \longrightarrow \Delta^1 \tag{6.28}$$

is a cartesian fibration. We may identify the fiber of (6.28) over $\Delta^{\{0\}}$ with poset of opens in M which are homeomorphic to $\square^n$ and the fiber over $\Delta^{\{1\}}$ with the poset of opens in M which are homeomorphic to $\square^n_{\partial}$.

For $X_1, ..., X_n, Y \in \{\square^n, \square^n_{\partial}\}$ the map $\mathrm{Emb}^{\mathrm{ord}}(\coprod_i X_i, Y) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(\coprod_i X_i, Y)$ determines a map of simplicial operads $\mathbb{D}^{\mathrm{ord}}_M \longrightarrow \mathbb{E}^{\Delta}_{\mathrm{Top}_{\partial}(n)}$ (which, in particular, forgets the embedding in M) and hence a map of ∞-operads

$$\mathbb{D}^{\otimes}_M \longrightarrow \mathbb{E}^{\otimes}_{\mathrm{Top}_{\partial}(n)}.$$

In addition, the underlying ∞-category $(\mathbb{D}^{\otimes}_M)_{\langle 1\rangle}$ is naturally isomorphic to the nerve of $\overline{M}^{\mathrm{ord}}_{\partial}$, and we have a natural map $\mathbb{D}^{\otimes}_M \longrightarrow (\overline{M}^{\mathrm{ord}}_{\partial})^{\amalg}$ which sends a multimap as in (6.27) to the collection of commutative triangles

$$\begin{tikzcd} X_i \arrow[rr] \arrow[dr, "\rho_i"'] & & Y \arrow[dl, "\eta"] \\ & M & \end{tikzcd} \tag{6.29}$$

Now consider the commutative square of ∞-operads

$$\begin{tikzcd} \mathbb{D}^{\otimes}_M \arrow[r] \arrow[d] & \mathbb{E}^{\otimes}_{\mathrm{Top}_{\partial}(n)} \arrow[d] \\ \overline{M}^{\amalg}_{\partial} \arrow[r] & \mathrm{BTop}_{\partial}(n)^{\amalg} \end{tikzcd} \tag{6.30}$$

where the left vertical map is given by the composition $\mathbb{D}^{\otimes}_M \longrightarrow (\overline{M}^{\mathrm{ord}}_{\partial})^{\amalg} \longrightarrow \overline{M}^{\amalg}_{\partial}$. The square (6.30) then induces a map of ∞-operads

$$\mathbb{D}^{\otimes}_M \longrightarrow \overline{M}^{\amalg}_{\partial} \times_{\mathrm{BTop}(n)^{\amalg}} \mathbb{E}^{\otimes}_{\mathrm{Top}(n)} = \mathbb{E}^{\otimes}_M \tag{6.31}$$

Proposition 6.3.7. *The functor*

$$\iota : \overline{M}^{\mathrm{ord}}_{\partial} \longrightarrow \overline{M}_{\partial} \tag{6.32}$$

has weakly contractible homotopy fibers.

Proof. Consider the commutative triangle

$$\begin{tikzcd} \overline{M}_\partial^{\mathrm{ord}} \arrow[rr, "\iota"] \arrow[dr] & & \overline{M}_\partial \arrow[dl] \\ & \mathrm{BTop}_\partial(n) & \end{tikzcd} \tag{6.33}$$

The right vertical map is right fibration and hence its homotopy fibers, which are also its actual fibers, are ∞-groupoids. It will hence suffice to show that (6.33) induces an equivalence on homotopy fibers over every object of $\mathrm{BTop}_\partial(n)$. Let us hence fix such an object $V \in \{\square^n, \square^n_\partial\}$. Then the (homotopy) fiber of the right diagonal map in (6.33) can be identified with $\mathrm{Sing}\,\mathrm{Emb}(V, M)$, while the homotopy fiber $\mathcal{X}_V \longrightarrow \overline{M}_\partial^{\mathrm{ord}}$ of the left diagonal map over V can be modeled by the full subcategory of the comma ∞-category $(\overline{M}_\partial^{\mathrm{ord}})_{V/} \coloneqq \mathrm{BTop}_\partial(n)_{V/} \times_{\mathrm{BTop}_\partial(n)} \overline{M}_\partial^{\mathrm{ord}}$ spanned by those $(\rho : U \longrightarrow M, \eta : V \longrightarrow U)$ such that η is an equivalence in $\mathrm{BTop}_\partial(n)$. Let $\overline{\mathcal{X}}_V \subseteq \overline{M}_\partial^{\mathrm{ord}}$ be the full subcategory spanned by those $\rho : U \longrightarrow M$ such that $U = V \in \{\square^n, \square^n_\partial\}$. We may identify $\overline{X}_V$ with the full subposet of $O(M)$ spanned by those open subsets which are homeomorphic to either $\square^n$ or $\square^n_\partial$. Then the forgetful map $\mathcal{X}_V \longrightarrow \overline{\mathcal{X}}_V$ is a left fibration which corresponds to the functor $\chi([\rho : U \longrightarrow M]) = \mathrm{Sing}\,\mathrm{Emb}(V, U)$ (where we note that any embedding $V \longrightarrow U$ is an isotopy equivalence when $U = V$). In particular, $\mathcal{X}_V$ is a model for the homotopy colimit of $\chi : \overline{\mathcal{X}}_V \longrightarrow \mathcal{S}$. We may then identify the map on homotopy fibers over V induced by (6.33) with the natural map

$$\operatorname*{hocolim}_{[\rho:U\longrightarrow M]\in\overline{\mathcal{X}}_V} \mathrm{Sing}\,\mathrm{Emb}(V, U) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(V, M) \tag{6.34}$$

This map is a weak equivalence by Lemma 6.1.26 applied to the map $p : M \longrightarrow *$ (with $|I| = 1$), and so the desired result follows. $\square$

Proposition 6.3.8. *The map* (6.31) *is a weak approximation in the sense of Definition 4.2.14.*

Proof. Condition (2) of Definition 4.2.14 follows from (6.3.7). We shall now prove that Condition (1) holds. Arguing as in the proof of Proposition 5.2.4 it will suffice to show that for every $\rho : V \longrightarrow M$ in $(\mathbb{D}^{\otimes}_M)_{\langle 1\rangle} \subseteq \mathbb{D}^{\otimes}_M$, the map

$$(\mathbb{D}^{\otimes}_M)^{\mathrm{act}}_{/\rho} \longrightarrow (\mathbb{E}^{\otimes}_M)^{\mathrm{act}}_{/\rho}$$

has weakly contractible homotopy fibers. We first note that the map $(\mathbb{E}^{\otimes}_M)^{\mathrm{act}} \longrightarrow (\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})^{\mathrm{act}}$ is a right fibration (since it is the pullback of a right fibration), and hence the map

$$(\mathbb{E}^{\otimes}_M)^{\mathrm{act}}_{/\rho} \longrightarrow (\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})^{\mathrm{act}}_{/V}$$

is a trivial Kan fibration. On the other hand, a direct inspection shows that the map $(\mathbb{D}^{\otimes}_V)^{\mathrm{act}}_{/\mathrm{Id}} \longrightarrow (\mathbb{D}^{\otimes}_M)^{\mathrm{act}}_{/\rho}$ induced by ρ is an isomorphism of simplicial sets. It will hence suffice to show that the map

$$(\mathbb{D}^{\otimes}_V)^{\mathrm{act}}_{/\mathrm{Id}} \longrightarrow (\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})^{\mathrm{act}}_{/V}$$

has weakly contractible homotopy fibers. Consider the commutative diagram

$$\begin{tikzcd} (\mathbb{D}^{\otimes}_V)^{\mathrm{act}}_{/\mathrm{Id}} \arrow[rr] \arrow[dr] & & (\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})^{\mathrm{act}}_{/V} \arrow[dl] \\ & (\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})^{\mathrm{act}} & \end{tikzcd} \tag{6.35}$$

Since the right diagonal map is a right fibration it will suffice to show that the horizontal arrow in (6.35) induces a weak equivalence on homotopy fibers over any object $\overline{U} \in (\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})^{\mathrm{act}}$, which we can identify with a tuple $(U_1, ..., U_l)$ of objects of $\mathrm{Top}_\partial(n)$ by choosing cocartesian lifts to $\rho^i : \langle n\rangle \longrightarrow \langle 1\rangle$. We now note that the (homotopy) fiber of the right diagonal map over $\overline{U} = (U_1, ..., U_l)$ is naturally equivalent to $\mathrm{Sing}\,\mathrm{Emb}(\coprod_i U_i, V)$. To describe the homotopy fiber of the left diagonal map, let $P_V \subseteq O(V)$ be the subposet consisting of those open subsets $W \subseteq V$ which are (abstractly) homeomorphic to either $\square^n$ or $\square^n_\partial$. Let $\overline{\mathcal{X}}_{\overline{U}} \subseteq P_V^{\langle l\rangle^\circ}$ be the subposet spanned by those tuples $(W_1, ..., W_l) \in (P_V)^{\langle l\rangle^\circ}$ such that $W_i = U_i \in \{\square^n, \square^n_\partial\}$ and the W_i's are pairwise disjoint. Unwinding the definitions as in the proof of Proposition 5.2.4 we see that the map induced by (6.35) on homotopy fibers over $(U_1, ..., U_l)$ can be identified with the canonical map

$$\operatorname*{hocolim}_{(W_1,...,W_l)\in\overline{\mathcal{X}}_{\overline{U}}} \prod_{i\in\langle l\rangle^\circ} \mathrm{Sing}\,\mathrm{Emb}(U_i, W_i) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(\coprod_i U_i, V) \tag{6.36}$$

This map is a weak equivalence by Lemma 6.1.26 applied to the map $p : V \longrightarrow *$ (with $I = \langle l\rangle^\circ$), and so the desired result follows.

□

Let us say that a map

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & M & \end{tikzcd}$$

in $(\mathbb{D}^{\otimes}_M)_{\langle 1\rangle}$ is an *isotopy equivalence* if the open embedding $U \longrightarrow V$ is an equivalence when considered as an arrow in $\mathcal{M}\mathrm{fld}_n$. Combining Proposition 6.3.8 and Proposition 4.2.18 we may now conclude the following:

Corollary 6.3.9. *Let $\mathcal{E}$ be an ∞-category which admits limits. Then the restriction map*

$$\mathrm{Mon}_{\mathbb{E}^{\otimes}_M}(\mathcal{E}) \longrightarrow \mathrm{Mon}_{\mathbb{D}^{\otimes}_M}(\mathcal{E})$$

is fully-faithful and its essential image consists of those $\mathbb{D}^{\otimes}_M$-monoids $\psi : \mathbb{D}^{\otimes}_M \longrightarrow \mathcal{E}$ whose restriction to $(\mathbb{D}^{\otimes}_M)_{\langle 1\rangle}$ sends isotopy equivalences to equivalences.

Remark 6.3.10. The content of Corollary 6.3.9 can be informally summarized as follows: the data of an $\mathbb{E}_M$-monoid in $\mathcal{E}$ is given associating to every open subset $U \subseteq M$ homeomorphic to either $\square^n$ or $\square^n_\partial$ an object $A_U \in \mathcal{E}$ and for every inclusion $V_1 \cup ... \cup V_m \subseteq U$ such that $V_i \cap V_j = \varnothing$ a map $A_{V_1} \times ... \times A_{V_m} \longrightarrow A_U$ which is an equivalence when the inclusion is an isotopy equivalence.

Remark 6.3.11. The statement of Corollary 6.3.9 also holds for $\mathbb{E}_M$-algebra objects in a general symmetric monoidal ∞-category. This can be proven using a suitable generalization of Proposition 4.2.18.

6.4. **Factorization homology.** Throughout this section, let us fix a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$. We wish to impose certain conditions regarding colimits in $\mathcal{C}$. For this we will need the following notion:

Definition 6.4.1. Let K be a simplicial set. We will say that K is *sifted* if the diagonal map $K \longrightarrow K \times K$ is cofinal (see Definition 2.6.11).

Example 6.4.2. The category Δ^{op} is sifted.

We now impose the following hypothesis on $\mathcal{C}$:

Hypothesis 6.4.3. $\mathcal{C}$ admits sifted colimits (that is, colimits indexed by sifted simplicial sets). In addition, the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ preserves sifted colimits in each variable separately.

We are now almost ready to define factorization homology. We first observe that the active part $(\mathbb{E}^{\otimes}_{\mathrm{Top}(n)})^{\mathrm{act}}$ of $\mathbb{E}^{\otimes}_{\mathrm{Top}(n)}$ can be identified with the full subcategory of $\mathcal{M}\mathrm{fld}_n$ spanned by those n-manifolds which are homeomorphic to a finite disjoint union of n-cubes. We will denote by

$$\mathbb{E}(M) := (\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} (\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})^{\mathrm{act}}.$$

It will be useful to also have a discrete version of $\mathbb{E}(M)$. Let $\mathbb{D}^{\otimes}_{\mathrm{Top}_\partial(n)}$ be the ∞-operad corresponding to the ordinary operad whose objects are $\square^n, \square^n_\partial$ and whose multimaps are given by the sets

$$\mathrm{Mul}_{\mathbb{D}^{\otimes}_{\mathrm{Top}_\partial(n)}}(\{X_i\}_{i\in I}, Y) = \mathrm{Emb}^{\mathrm{ord}}(\coprod_i X_i, Y).$$

Define $\mathrm{Disj}(M) := (\mathcal{M}\mathrm{fld}_n)_{/M} \times_{\mathcal{M}\mathrm{fld}_n} (\mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)})^{\mathrm{act}}$, so that the natural map

$$\mathbb{D}^{\otimes}_{\mathrm{Top}_\partial(n)} \longrightarrow \mathbb{E}^{\otimes}_{\mathrm{Top}_\partial(n)}$$

induces a natural map

$$\mathrm{Disj}(M) \longrightarrow \mathbb{E}(M). \tag{6.37}$$

Lemma 6.4.4. *Let M be an n-manifold. Then the following holds:*

(1) The map (6.37) is cofinal.
(2) The ∞-category $\mathbb{E}(M)$ is sifted.

Proof. Consider the pair of functors

$$\mathrm{Disj}(M) \longrightarrow \mathbb{E}(M) \longrightarrow \mathbb{E}(M) \times \mathbb{E}(M). \tag{6.38}$$

We wish to show that both are cofinal. For this, it will suffice to show that first map $\mathrm{Disj}(M) \longrightarrow \mathbb{E}(M)$ and the composed map $\mathrm{Disj}(M) \longrightarrow \mathbb{E}(M) \times \mathbb{E}(M)$ are both cofinal. To make the proof more efficient we can organize these two claims under a common generalization. Suppose that G is a finite group and $p : \widetilde{M} \longrightarrow M$ is a G-covering map (so that $\widetilde{M}$ is also an n-manifold). Then for every open embedding $\rho : \coprod_i U_i \longrightarrow M$ with I a finite set and $U_i \in \{\square^n, \square^n_\partial\}$ the pullback $U_i \times_M \widetilde{M} \longrightarrow U_i$ of p then canonically splits as the projection $U_i \times J_i \longrightarrow U_i$ where $J_i = \pi_0(U_i \times_M \widetilde{M})$ is naturally endowed with the structure of a G-torsor. The association $(\{U_i\}_{i\in I}, \rho) \mapsto (\{U_i \times \{j\}\}_{i\in I, j\in J_i}, p^*\rho)$ then determines a functor from $\mathrm{Disj}(M)$ to $\mathrm{Disj}(N)$, where $p^*\rho : X \times_M \widetilde{M} \longrightarrow N$ denotes the projection on the right component. It will then suffice to show that the composed map

$$\mathrm{Disj}(M) \longrightarrow \mathrm{Disj}(\widetilde{M}) \longrightarrow \mathbb{E}(\widetilde{M}) \tag{6.39}$$

is cofinal. Indeed, taking G to be the trivial group and p to be the identity map $M \longrightarrow M$ we get that the first map in (6.38) is cofinal, and taking $G = \mathbb{Z}/2$ and p to be the split 2-covering $M \coprod M \longrightarrow M$ we get that the composed map in (6.38) is cofinal (where we note that $\mathbb{E}(M \coprod M)$ can be naturally identified with $\mathbb{E}(M) \times \mathbb{E}(M)$). Let us now prove that the composed map in (6.39) is indeed cofinal.

We fix a finite set I and an object $\eta : V := \coprod_{i \in I} V_i \longrightarrow \widetilde{M}$ of $\mathbb{E}(\widetilde{M})$. Consider the comma ∞-category $\mathrm{Disj}(M)_{\eta/} := \mathbb{E}(\widetilde{M})_{\eta/} \times_{\mathbb{E}(\widetilde{M})} \mathrm{Disj}(M)$. We need to show that $\mathrm{Disj}(M)_{\eta/}$ is weakly contractible. We remark that the category $\mathrm{Disj}(M)$ is in fact a poset which can be identified with the poset of open subsets of M which are homeomorphic to a finite disjoint union of $\square^n$ and $\square^n_\partial$. To simplify notation, let us hence denote a general object $(J, \{U_j\}_{j \in J}, \rho : \coprod_j U_j \longrightarrow M)$ of $\mathrm{Disj}(M)$ simply as $U \subseteq M$, where $U := \coprod_j U_j$ is understood as an open subset of M of the above form. The projection $\mathrm{Disj}(M)_{\eta/} \longrightarrow \mathrm{Disj}(M)$ is then a left fibration corresponding to the functor

$$U \mapsto \mathrm{Map}_{(\mathcal{M}\mathrm{fld}_n)_{/N}}(V, U \times_M \widetilde{M}) \simeq \mathrm{hofib}_\eta[\mathrm{Sing}\,\mathrm{Emb}(V, U \times_M \widetilde{M}) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(V, \widetilde{M})].$$

Since the base change functor $\mathrm{hofib}_\eta : \mathcal{S}_{/\mathrm{Sing}\,\mathrm{Emb}(V,\widetilde{M})} \longrightarrow \mathcal{S}$ preserves homotopy colimits (this is a general property of base change functors in spaces) and using Theorem 2.6.6 it will suffice to show that the canonical map

$$\operatorname*{hocolim}_{U \in \mathrm{Disj}(M)} \mathrm{Sing}\,\mathrm{Emb}(V, U \times_M \widetilde{M}) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(V, \widetilde{M})$$

is a weak homotopy equivalence. This map is a weak equivalence by Lemma 6.1.26 applied to the map $p : \widetilde{M} \longrightarrow M$, and so the desired result follows. □

Given two n-manifolds M, N, let us denote by $\mathrm{Emb}^{\mathrm{loc}}(M, N)$ the space of those maps $M \longrightarrow N$ which restrict to an open embedding on each connected component of M (endowed with the compact open topology). We define $\mathcal{M}\mathrm{fld}_n^{\mathrm{loc}}$ be the coherent nerve of the simplicial category whose objects are n-manifolds and whose mapping spaces are given by $\mathrm{Sing}\,\mathrm{Emb}^{\mathrm{loc}}(M, N)$. We then see that the active part $(\mathrm{BTop}(n)^{\amalg})^{\mathrm{act}}$ of $\mathrm{BTop}(n)^{\amalg}$ can be identified with the full subcategory of $\mathcal{M}\mathrm{fld}_n^{\mathrm{loc}}$ spanned by the finite disjoint unions of n-cubes. In addition, given an n-manifold M we have a natural isomorphism of simplicial sets

$$(\overline{M}^{\amalg})^{\mathrm{act}} \cong (\mathcal{M}\mathrm{fld}_n^{\mathrm{loc}})_{/M} \times_{\mathcal{M}\mathrm{fld}_n^{\mathrm{loc}}} (\mathrm{BTop}(n)^{\amalg})^{\mathrm{act}}.$$

The functor $\mathcal{M}\mathrm{fld}_n \longrightarrow \mathcal{M}\mathrm{fld}_n^{\mathrm{loc}}$ then induces a commutative square

$$\begin{tikzcd} \mathbb{E}(M) \arrow[r] \arrow[d] & (\mathbb{E}^{\otimes}_{\mathrm{Top}(n)})^{\mathrm{act}} \arrow[d] \\ (\overline{M}^{\amalg})^{\mathrm{act}} \arrow[r] & (\mathrm{BTop}(n)^{\amalg})^{\mathrm{act}} \end{tikzcd}$$

which determines a map of ∞-categories

$$r : \mathbb{E}(M) \longrightarrow (\mathbb{E}^{\otimes}_M)^{\mathrm{act}}.$$

Suppose now that $A : \mathbb{E}^{\otimes}_M \longrightarrow \mathcal{C}^{\otimes}$ is an $\mathbb{E}_M$-algebra in $\mathcal{C}$. We would like to consider A as a functor that takes values in $\mathcal{C}$, as apposed to $\mathcal{C}^{\otimes}$. For this, we observe that $\langle 1 \rangle$ is final when considered as an object of $\mathrm{Fin}^{\mathrm{act}}_*$, and so there is a unique natural transformation between $(\mathbb{E}_M^{\otimes})^{\mathrm{act}} \longrightarrow \mathrm{N}(\mathrm{Fin}_*^{\mathrm{act}})$ and the constant functor $(\mathbb{E}_M^{\otimes})^{\mathrm{act}} \longrightarrow \{\langle 1\rangle\} \subseteq \mathrm{N}(\mathrm{Fin}_*^{\mathrm{act}})$. We hence obtain a diagram of the form

$$\begin{tikzcd} (\mathbb{E}_M^{\otimes})^{\mathrm{act}} \times \Delta^{\{0\}} \arrow[r, "A"] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d] \\ (\mathbb{E}_M^{\otimes})^{\mathrm{act}} \times \Delta^{1} \arrow[r] \arrow[ur, dashed] & \mathrm{N}(\mathrm{Fin}_*^{\mathrm{act}}) \end{tikzcd} \tag{6.40}$$

Since $\pi : (\mathcal{C}^{\otimes})^{\mathrm{act}} \longrightarrow \mathrm{N}(\mathrm{Fin}_*^{\mathrm{act}})$ is a cocartesian fibration we can lift this natural transformation in an essentially unique way to a π-pointwise natural transformation in $\mathcal{C}^{\otimes}$ from A to other functor $(\mathbb{E}_M^{\otimes})^{\mathrm{act}} \longrightarrow \mathcal{C}^{\otimes}$ which factors through the fiber $\mathcal{C} = (\mathcal{C}^{\otimes})_{\langle 1\rangle} \subseteq \mathcal{C}^{\otimes}$. We will denote the resulting functor by

$$\overline{A} : (\mathbb{E}_M^{\otimes})^{\mathrm{act}} \longrightarrow \mathcal{C}.$$

We note that $\overline{A}$ can be informally described by the formula $\overline{x} \mapsto A(x_1)\otimes \ldots \otimes A(x_m)$, where $\overline{x} \in (\mathbb{E}_M^{\otimes})^{\mathrm{act}}$ denotes an object lying over $\langle m\rangle$ which corresponds to the tuple $(x_1, \ldots, x_m)$ under the equivalence $(\mathbb{E}_M^{\otimes})_{\langle m\rangle} \simeq \prod_{i\in\langle m\rangle^{\circ}} (\mathbb{E}_M^{\otimes})_{\langle 1\rangle}$.

Definition 6.4.5. We define the *factorization homology* $\int_M A$ of M with coefficients in A as the colimit

$$\int_M A := \operatorname*{colim}_{\mathbb{E}(M)} r^*\overline{A} \in \mathcal{C}.$$

Remark 6.4.6. In light of Lemma 6.4.4 restriction along $\mathrm{Disj}(M) \longrightarrow \mathbb{E}(M)$ induces an equivalence

$$\int_M A \simeq \operatorname*{colim}_{U\in\mathrm{Disj}(M)} \overline{A}(U).$$

Remark 6.4.7. If $B_\partial \longrightarrow \mathrm{BTop}^{\partial}(n)$ be a boundary tangent structure and M carried a B_∂-framing then we have a map of ∞-operads $\varphi : \mathbb{E}_{M_\partial}^{\otimes} \longrightarrow \mathbb{E}_{B_\partial}^{\otimes}$, and so any $\mathbb{E}_{B_\partial}^{\otimes}$-algebra can be pulled back to an $\mathbb{E}_M^{\otimes}$-algebra to which we can take factorization homology. In this case we will denote $\int_M \varphi^* A$ simply by $\int_M A$. for example, if A is an $\mathbb{E}_{\mathrm{Top}(n)}^{\otimes}$-algebra then A admits a factorization homology along any open n-manifold, and if M is an open $*$-framed manifold then any $\mathbb{E}_n^{\otimes}$-algebra admits a factorization homology along M.

Remark 6.4.8. The formation of factorization homology can be made functorial in M in the following sense. Suppose that $A : \mathbb{E}_{\mathrm{Top}(n)}^{\otimes} \longrightarrow \mathcal{C}$ is an $\mathbb{E}_{\mathrm{Top}(n)}$-algebra object. By Theorem 2.7.5, Lemma 6.4.4 and our Hypothesis 6.4.3 the functor $\overline{A} : (\mathbb{E}_{\mathrm{Top}(n)}^{\otimes})^{\mathrm{act}} \longrightarrow \mathcal{C}$ associated to A as above admits a left Kan extension

$$\int_{(-)} A : \mathcal{M}\mathrm{fld}_n \longrightarrow \mathcal{C} \tag{6.41}$$

along $(\mathbb{E}_{\mathrm{Top}(n)}^{\otimes})^{\mathrm{act}} \longrightarrow \mathcal{M}\mathrm{fld}_n$ which is given on objects by the factorization homology $M \mapsto \int_M A$ (see Remark 6.4.7).

Remark 6.4.9. In the situation of Remark 2.7.5, it follows from Lurie's theory of *operadic left Kan extensions* (using our base hypothesis 6.4.3 and Lemma 6.4.4) that the left Kan extension (6.41) refines to a map of ∞-operads

$$\int_{(-)} A : (\mathcal{M}\mathrm{fld}_n)^{\otimes} \longrightarrow \mathcal{C}^{\otimes} \tag{6.42}$$

In this particular case one can even show that (6.42) is a *symmetric monoidal functor*, that is, it preserves all cocartesian edges over $\mathrm{N}(\mathrm{Fin}_*)$. This eventually follows from the fact that if $M = M_0 \coprod M_1$ then the rule that sends $(X_0 \longrightarrow M_0, X_1 \longrightarrow M_1)$ to $X_0 \coprod X_1 \longrightarrow M$ determines an equivalence

$$(\mathbb{E}^{\otimes}_{\mathrm{Top}(n)})^{\mathrm{act}}_{/M_0} \times (\mathbb{E}^{\otimes}_{\mathrm{Top}(n)})^{\mathrm{act}}_{/M_1} \xrightarrow{\simeq} (\mathbb{E}^{\otimes}_{\mathrm{Top}(n)})^{\mathrm{act}}_{/M}.$$

Remark 6.4.10. In the situation of Remark 6.4.9, if we replace $\mathcal{M}\mathrm{fld}_n$ by the ∞-category $\mathcal{M}\mathrm{fld}_n^{B_\partial}$ of B_∂-framed manifolds for some boundary tangent structure B_∂ and $A : \mathbb{E}^{\otimes}_{B_\partial} \longrightarrow \mathcal{C}$ is an $\mathbb{E}_{B_\partial}$-algebra object then the association $M \mapsto \int_M A$ can also be made into a symmetric monoidal functor

$$\int_{(-)} A : (\mathcal{M}\mathrm{fld}_n^{B_\partial})^{\otimes} \longrightarrow \mathcal{C}^{\otimes}.$$

Example 6.4.11 ([1, Corllary 3.12]). Let I be the unit interval. Recall (see Example 6.3.3) that an $\mathbb{E}^{\otimes}_I$-disk algebra in $\mathcal{C}$ is equivalent to the data of a triple (A, M_0, M_1) where A is an associative algebra object in $\mathcal{C}$, M_0 is a right A-module in $\mathcal{C}_{1_{\mathcal{C}}/}$ and M_1 is a left A-module in $\mathcal{C}_{1_{\mathcal{C}}/}$. In this case we have a natural equivalence

$$\int_M (A, M_0, M_1) \simeq M_0 \otimes_A M_1$$

Our next goal is to discuss the *Fubini property* of factorization homology. For this, we will need to introduce the notion of a *bundle map* of manifolds with boundary.

Suppose first that N is an open k-manifold and E is an n-manifold, possibly with boundary. Then we have the notion of a *manifold bundle map* from E to N, which is, by definition a map $p : E \longrightarrow N$ such that for every open embedding $\rho : \square^k \longrightarrow N$ the pullback $\rho^* E := E \times_N \square^k \longrightarrow \square^k$ admits a trivialization of the form $\tau : \rho^* E \cong P \times \square^k$ with P an $(n-k)$-manifold (here by trivialization we simply mean that τ commutes with the respective projections to $\square^k$). In this case the association $[\rho : \square^k \longrightarrow N] \mapsto E \times_N \square^k$ determines a $\mathbb{D}_N$-algebra object in $\mathcal{M}\mathrm{fld}_n^{\partial}$, which we shall call $[p^{-1}]$. The local triviality of E now implies that all the 1-ary operations act on $[p^{-1}]$ by equivalences, and by Remark 6.3.11 we have that $[p^{-1}]$ descends to an essentially unique N-disk algebra object in $\mathcal{M}\mathrm{fld}_n^{\partial}$.

We would like to have a similar story when N is a k-manifold which is not necessarily open (i.e., can have a boundary).

Definition 6.4.12. Let N be an k-manifold. By a *manifold ∂-bundle* over N we shall mean a map $E \longrightarrow N$ with E an n-manifold and such that the following conditions hold:

(1) For every open embedding $\rho : \square^k \longrightarrow N$ the pullback $\rho^* E \longrightarrow \square^k$ admits a trivialization of the form $\rho^* E \cong P \times \square^k$ with P a $(n-k)$-manifold.
(2) For every open embedding $\rho : \square^k_\partial \longrightarrow N$ the pullback $\rho^* E \longrightarrow \square^k_\partial$ admits an identification of the form

$$\begin{tikzcd} \rho^* E \arrow[rr, "\cong"] \arrow[dr] & & P \times \square^{k-1} \arrow[dl, "{(f,\mathrm{Id})}"] \\ & \square^k_\partial & \end{tikzcd}$$

where P is an $(n-k+1)$-manifold equipped with a continuous map $f : P \longrightarrow [0,1)$.

In this case we will also say that $p : E \longrightarrow N$ is a *∂-bundle map*.

Remark 6.4.13. If $E \longrightarrow N$ is a manifold ∂-bundle then $E \times_N (N \smallsetminus \partial N) \longrightarrow (N \smallsetminus \partial N)$ is a manifold bundle in the usual sense. Furthermore, if $\rho : \partial N \times [0,1) \longrightarrow N$ is a tubular neighborhood of the boundary of N then the composed map $\rho^* E \longrightarrow \partial N \times [0,1) \longrightarrow \partial N$ is a manifold bundle as well.

Example 6.4.14. If N is 1-dimensional then the boundary of N is 0-dimensional. In this case condition (2) of Definition 6.4.12 is vacuous, and so $p : E \longrightarrow N$ is a ∂-bundle map if and only if it restricts to a bundle map over the interior of N.

Remark 6.4.15. If we consider fiber bundles over N as analogous to locally constant sheaves, then the notion of a ∂-bundle can be considered as analogous to sheaves on N which are *constructible* with respect to the stratification $\partial N \subseteq N$.

Example 6.4.16. It is worthwhile to spell out what do ∂-manifold bundles over the unit interval I look like. Let M be a n-manifold and P an $(n-1)$-manifold. We will say that an open embedding $\rho : (0,1) \times P \longrightarrow M$ is a *right P-collar* if $\rho([\varepsilon, 1) \times P)$ is closed in M for every $\varepsilon \in (0,1)$. Similarly, we will say that ρ is a *left P-collar* if $\rho((0,\varepsilon] \times P)$ is closed in M for every $\varepsilon \in (0,1)$.

If M_0, M_1 are two n-manifolds, $\rho_0 : (0,1) \times P \hookrightarrow M_0$ a right P-collar and $\rho_1 : (0,1) \times P \hookrightarrow M_1$ a left P-collar then the topological space $M := M_0 \coprod_{(0,1)\times P} M_1$ is again an n-manifold which contains M_0 and M_1 as submanifolds. Following [1] we will refer to M as the *collar gluing* of M_0 and M_1 along $(0,1) \times P$. In this case, M admits a natural ∂-bundle map $M \longrightarrow [0,1]$ which extends the projection $(0,1) \times P \longrightarrow (0,1)$ and maps $M_0 \smallsetminus \mathrm{Im}(\rho_0)$ and $M_1 \smallsetminus \mathrm{Im}(\rho_1)$ to 0 and 1, respectively.

On the other hand, if $p : M \longrightarrow [0,1]$ is any ∂-bundle then by definition $p|_{(0,1)} : (0,1) \times_I M \longrightarrow (0,1)$ splits as a product $(0,1) \times_I M \cong (0,1) \times P$. If we now set $M_0 = p^{-1}[0,1)$ and $M_1 = p^{-1}(0,1]$ then the embedding $(0,1) \times P \hookrightarrow M_0$ is a right collar, the embedding $(0,1) \times P \hookrightarrow M_1$ is a left collar and $M \cong M_0 \coprod_{(0,1)\times P} M_1$ is a collar gluing of M_0 and M_1 along $(0,1) \times P$.

We shall now explain how the notion of a ∂-bundle can be used to construct $\mathbb{E}_N$-algebras. Suppose that $p : E \longrightarrow N$ is a manifold ∂-bundle and that E is equipped with a B_∂-framing for some boundary tangent structure $B_\partial \longrightarrow \mathrm{BTop}_\partial(n)$. Then the association $[\rho : U \hookrightarrow N] \mapsto U \times_N E$ for $U \cong \square^n, \square^n_\partial$ determines a $\mathbb{D}_N$-algebra object $[p^{-1}]$ in $\mathcal{M}\mathrm{fld}_n^{B_\partial}$. The local models of Definition 6.4.12 imply that the 1-ary operations coming from inclusions of $\square^n$ in $\square^n$ or $\square^n_\partial$ in $\square^n_\partial$ act on $[p^{-1}]$ by equivalences, and so by Remark 6.3.11 we have that $[p^{-1}]$ descends to an essentially unique $\mathbb{E}_N$-algebra object in $\mathcal{M}\mathrm{fld}_n^{B_\partial}$.

If $\mathcal{F} : (\mathcal{M}\mathrm{fld}_n^{\partial B})^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ is a symmetric monoidal functor then the composed functor $p_*\mathcal{F} := \mathcal{F} \circ [p^{-1}] : \mathbb{E}_N^{\otimes} \longrightarrow \mathcal{C}$ gives an $[\partial N \to N]$-algebra in $\mathcal{C}$. This construction can be used to produce a variety of interesting examples of cube algebras over manifolds. We will also make use of it in order to formulate the Fubini property of factorization homology and to define the property of being a homology theory for manifolds in §6.5.

Let $B_\partial \longrightarrow \mathrm{BTop}_\partial(n)$ be a boundary tangent structure and let A be an $\mathbb{E}_{B_\partial}$-algebra. Let M be a B_∂-framed n-manifold and N a k-manifold. Given a ∂-bundle map $p : M \longrightarrow N$, let us denote by $p_* A : \mathbb{E}_N^{\otimes} \longrightarrow \mathcal{C}$ the composed functor

$$\mathbb{E}_N^{\otimes} \xrightarrow{[p^{-1}]} (\mathcal{M}\mathrm{fld}_n^{B_\partial})^{\otimes} \xrightarrow{\int_{(-)} A} \mathcal{C}^{\otimes}$$

where $[p^{-1}]$ is the $\mathbb{E}_N$-algebra object in $\mathcal{M}\mathrm{fld}_n^{B_\partial}$ associated to $p : M \longrightarrow N$ as above.

Proposition 6.4.17 (The Fubini property, [1, Proposition 3.23])**.** *In the above setting the natural map*

$$\int_N p_* A \longrightarrow \int_M A \tag{6.43}$$

is an equivalence.

Proof. By Remark 6.4.6 the left hand side of (6.43) is given by an iterated colimit

$$\int_N p_* A \simeq \operatorname*{colim}_{U \in \mathrm{Disj}(N)} p_* A(U) \simeq \operatorname*{colim}_{U \in \mathrm{Disj}(M)} \operatorname*{colim}_{V \in \mathrm{Disj}(p^{-1}(U))} A(V)$$

which can be assembled to a single colimit indexed by the full subposet $\mathrm{Disj}(p) \subseteq \mathrm{Disj}(M) \times \mathrm{Disj}(N)$ spanned by those pairs (V, U) such that $p(V) \subseteq U$, since the forgetful functor $\mathrm{Disj}(p) \longrightarrow \mathrm{Disj}(N)$ is the cocartesian fibration classifying the functor $U \mapsto \mathrm{Disj}(p^{-1}(U))$ (this follows from a general fubini property of colimits along cocartesian fibrations). To show that (6.43) is an equivalence it will hence suffice to show that the composed functor

$$\mathrm{Disj}(p) \longrightarrow \mathrm{Disj}(M) \longrightarrow \mathbb{E}(M),$$

is cofinal. Let us hence fix a finite set I and an object $\iota : W := \coprod_i W_i \longrightarrow M$ of $\mathbb{E}(M)$. Then the comma ∞-category $\mathrm{Disj}(p)_{\iota/}$ sits in a left fibration $\mathrm{Disj}(p)_{\rho/} \longrightarrow \mathrm{Disj}(p)$ which classifies the functor

$$(U, V) \mapsto \mathrm{hofib}_\iota \left[\mathrm{Sing}\,\mathrm{Emb}(W, V) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(W, M)\right].$$

Since the base change functor $\mathrm{hofib}_\iota : \mathcal{S}_{/\mathrm{Sing}\,\mathrm{Emb}(W,M)} \longrightarrow \mathcal{S}$ preserves homotopy colimits we may instead show that the natural map

$$\operatorname*{hocolim}_{(U,V) \in \mathrm{Disj}(p)} \mathrm{Sing}\,\mathrm{Emb}(W, V) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(W, M) \tag{6.44}$$

is a weak homotopy equivalence. The homotopy colimit in (6.44) can then be rebroken into an iterated colimit:

$$\operatorname*{hocolim}_{U \in \mathrm{Disj}(N)} \operatorname*{hocolim}_{V \in \mathrm{Disj}(p^{-1}(U)} \mathrm{Sing}\,\mathrm{Emb}(W, V) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(W, M). \tag{6.45}$$

We may then factor 6.45 into a composition of two maps

(6.46)

$$\operatorname*{hocolim}_{U \in \mathrm{Disj}(N)} \operatorname*{hocolim}_{V \in \mathrm{Disj}(p^{-1}(U))} \mathrm{Sing}\,\mathrm{Emb}(W, V) \longrightarrow \operatorname*{hocolim}_{U \in \mathrm{Disj}(N)} \mathrm{Emb}(W, p^{-1}(U)) \longrightarrow \mathrm{Sing}\,\mathrm{Emb}(W, M).$$

We now observe that the first map in (6.46) is a weak homotopy equivalence by Lemma 6.1.26 applied to the identity map $p^{-1}(V) \longrightarrow p^{-1}(V)$ and the second map is a weak homotopy equivalence by Lemma 6.1.26 applied to the identity map $N \longrightarrow N$. □

Example 6.4.18. The Fubini property can help us to decipher what is the factorization homology along the circle. We first note that by Remark 6.2.13 the notion of an $\mathbb{E}_{S^1}$-algebra object in $\mathcal{C}$ is equivalent to that of a pair (A, τ) where A is an associative algebra A and $\tau : A \longrightarrow A$ is an automorphism (associated to the monodromy along the circle). The projection $p : S^1 \longrightarrow [-1, 1]$ on the x-axis is a ∂-bundle map and the $\mathbb{E}_{[-1,1]}$-algebra $p_*(A, \tau)$ can be identified with the triple $(A^{\mathrm{op}} \otimes A, A_{-1}, A_1)$ where A_{-1} is a copy of A considered as a pointed right $A^{\mathrm{op}} \otimes A$-module and A_1 is a copy of A considered as a pointed left $A^{\mathrm{op}} \otimes A$-module via the equivalence

$(\mathrm{Id}\otimes\tau) : A^{\mathrm{op}}\otimes A \longrightarrow A^{\mathrm{op}}\otimes A$. By the Fubini property and Example 6.4.11 we then have that

$$\int_{S^1}(A,\tau) \simeq \int_{[-1,1]} p_*(A,\tau) \simeq A\otimes_{A^{\mathrm{op}}\otimes A} A$$

is the τ-twisted Hochschild homology of A.

6.5. **Axiomatic characterization of factorization homology.** In this section we will focus attention on *open* n-manifolds, i.e., those which do not have boundary. Following [1], our goal is to consider *homology theories* on suitably framed open n-manifolds. For this we will need to isolate a particular full subcategory of $\mathcal{M}\mathrm{fld}_n$ spanned by manifolds which can be built in finitely many steps by gluing disks of various dimensions. This gluing is defined via the notion of a *collar gluing* spelled out in Example 6.4.16.

Let $\partial\square^k$ denote the boundary of the n-cube, which is homeomorphic to the $(k-1)$-sphere (where $\partial\square^0=\varnothing$ by convention). We have a canonical right $\partial\square^k$-collar $(0,1)\times\partial\square^k \hookrightarrow \square^k$ embedded as the complement of $\{0\}\subset\square^k$. Let M_0 be an open n-manifold. If $\rho : (0,1)\times\partial\,\square^k\times\square^{n-k}\hookrightarrow M_0$ is a left $[\partial\,\square^k\times\square^{n-k}]$-collar then we will say that

$$M = \square^k\times\square^{n-k}\coprod_{(0,1)\times\partial\square^k\times\square^{n-k}} M_0$$

is obtained from M_0 by adding an *open handle* of index k.

Definition 6.5.1. Let M be an open n-manifold. We will say that M is of *finite type* if it can be obtained from $\varnothing$ by adding finitely many open handles. We will denote by $\mathcal{M}\mathrm{fld}_n^{\mathrm{fin}}\subseteq\mathcal{M}\mathrm{fld}_n$ the full subcategory spanned by open n-manifolds of finite type. Similarly, if $B\longrightarrow \mathrm{BTop}(n)$ is a tangent structure then we will denote by $\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} := \mathcal{M}\mathrm{fld}_n^B\times_{\mathcal{M}\mathrm{fld}_n}\mathcal{M}\mathrm{fld}_n^{\mathrm{fin}}$. We note that the inclusion $\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}}\subseteq\mathcal{M}\mathrm{fld}_n^B$ is fully faithful and its essential image is spanned by those B-framed open n-manifolds which are of finite type.

Example 6.5.2. Adding to M_0 an open handle of index 0 is simply taking the coproduct $M = M_0\coprod\square^n$. In particular, the n-disk $\square^n$ is an n-manifold of finite type.

Example 6.5.3. $\partial\,\square^{k+1}\times\square^{n-k}$ is obtained from $\square^k\times\square^{n-k}\cong\square^n$ by adding a single open handle of index k. In particular, $\partial\,\square^{k+1}\times\square^{n-k}$ is an n-manifold of finite type.

Warning 6.5.4. The notion of an open handle is closely related, but not identical, to the notion of a handle studied in classical geometric topology, which is usually applied only to compact manifolds. However, if M is a compact manifold with a finite handle decomposition in the classical sense, then the interior of M is of finite type in the sense of Definition 6.5.1. In particular, it is known that any closed manifold of dimension $\neq 4$ has a finite handle decomposition, and is hence of finite type. On the other hand, in dimension 4 a closed manifold admits a finite handle decomposition if and only if it is smoothable.

Definition 6.5.5. We will say that a manifold ∂-bundle $p : M\longrightarrow N$ is *open* if M is open, and we will say that p has *finite type* if for every $U\subseteq N$ which is homeomorphic to either $\square^k$ or $\square^k_\partial$ the fiber product $M\times_N U$ has finite type.

Now let $\mathcal{C}$ be as in Hypothesis 6.4.3 and let $B\longrightarrow\mathrm{BTop}(n)$ be a tangent structure. In this section we will describe a certain class of symmetric monoidal functors

$\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ which are called *homology theories* in [1]. The defining property of these functors is that they satisfy $\otimes$-*excision*, a term we shall now define.

Definition 6.5.6. Let $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ be a symmetric monoidal functor. We will say that $\mathcal{F}$ satisfies $\otimes$-*excision* if for every open finite type ∂-bundle $p : M \longrightarrow I$ the induced map

$$\int_I p_*\mathcal{F} \longrightarrow \mathcal{F}(M)$$

is an equivalence, where $p_*\mathcal{F}$ is the composed functor $\mathbb{E}_B^{\otimes} \xrightarrow{[p^{-1}]} (\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}})^{\otimes} \xrightarrow{\mathcal{F}} \mathcal{C}^{\otimes}$ as above.

Remark 6.5.7. In light of Example 6.4.11 and Example 6.4.16 we may also (somewhat informally) phrase the $\otimes$-excision property as saying that for every collar gluing $M = M_0 \coprod_{(0,1)\times P} M_1$ of finite type open n-manifolds the induced map

$$\mathcal{F}(M_0) \otimes_{\mathcal{F}((0,1)\times P)} \mathcal{F}(M_1) \longrightarrow \mathcal{F}(M)$$

is an equivalence.

Definition 6.5.8. Let $B \longrightarrow \mathrm{BTop}(n)$ be a tangent structure and $\mathcal{C}$ a presentably symmetric monoidal ∞-category. A B-framed *homology theory* is a symmetric monoidal functor $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ which satisfies $\otimes$-excision. We will denote by $\mathbf{H}(\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}}, \mathcal{C}) \subseteq \mathrm{Fun}^{\otimes}(\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}}, \mathcal{C})$ the full subcategory spanned by the B-framed homology theories.

Example 6.5.9. Let $B \longrightarrow \mathrm{BTop}(n)$ be a boundary tangent structure and let A be a B-disk algebra. Then the symmetric monoidal functor $\int_{(-)} : (\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}})^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$ described in Remark 6.4.9 is a B-framed homology theory. This follows immediately from Proposition 6.4.17.

Remark 6.5.10. If $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ is a symmetric monoidal functor then $\mathcal{F}|_{\mathbb{E}_B^{\otimes}}$ is by definition an $\mathbb{E}_B^{\otimes}$-algebra object. Remark 6.4.8 then furnishes a symmetric monoidal natural transformation

$$\int_{(-)} \mathcal{F}|_{\mathbb{E}_B^{\otimes}} \Rightarrow \mathcal{F}(-)$$

of symmetric monoidal functors $\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$.

We now come to the main result of this talk.

Theorem 6.5.11 ([1])**.** *Let $\mathcal{F} : \mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}} \longrightarrow \mathcal{C}$ be a homology theory for manifolds and let $A = \mathcal{F}|_{\mathbb{E}_B^{\otimes}}$ be the associated B-disk algebra in $\mathcal{C}$. Then the natural map*

$$\int_M A \longrightarrow \mathcal{F}(M) \tag{6.47}$$

of Remark 6.5.10 is an equivalence for every B-framed n-manifold M of finite type.

Proof. We prove by double induction on the open handle decomposition of M. For integers $0 \leq k, m$ let us say that an open manifold M has *type* (k, m) if it can be obtained from $\varnothing$ by adding finitely many handles of index $\leq k$ out of which at most m handles are of index exactly k. We first note that an open n-manifold is of type $(0, 1)$ if and only if it is the n-disk, and the map (6.47) is an equivalence in this case by definition.

Now suppose we have proven that (6.47) is an equivalence for every B-framed manifold of type (k,m) where either $k>0$ or $k=0$ and $m\geq 1$, and let M be a B-framed manifold of type $(k,m+1)$. Then by definition there exists an n-manifold M_0 of type (k,m) and a left $[\partial\,\square^k\times\square^{n-k}]$-collar $\rho:(0,1)\times\partial\,\square^k\times\square^{n-k}\hookrightarrow M_0$ such that

$$M := \square^k\times\square^{n-k}\coprod_{(0,1)\times\partial\square^k\times\square^{n-k}} M_0.$$

In this case the B-framing on M restricts to B-framings on M_0, $(0,1)\times\partial\,\square^k\times\square^{n-k}$ and $\square^k\times\square^{n-k}$, so that we can consider all of them as B-framed sub-manifolds of M. Let $p: M\longrightarrow[0,1]$ be the manifold ∂-bundle of Example 6.4.16, which is open and of finite type by construction (see Example 6.5.3 and Example 6.5.2), and consider the diagram

$$\begin{tikzcd} \int_I p_*A \arrow[r] \arrow[d, "\simeq"] & \int_I p_*\mathcal{F} \arrow[d, "\simeq"] \\ \int_M A \arrow[r] & \mathcal{F}(M) \end{tikzcd} \tag{6.48}$$

in which the vertical maps are equivalences since $\mathcal{F}$ and $\int_{(-)}A$ are homology theories. To show that the bottom horizontal map is an equivalence it will hence suffice to show that the top vertical map is an equivalence. We now observe that if $U\subseteq I$ is an open subset homeomorphic to either $\square^1$ or $\square^1_\partial$ then $p^{-1}(U)$ is an open manifold which is homeomorphic to either M_0, $\square^k\times\square^{n-k}\cong\square^n$ or $(0,1)\times\partial\,\square^k\times\square^{n-k}$, which are manifolds of types (k,m), $(k-1,2)$ and $(0,1)$, respectively By the induction hypothesis the map

$$\int_U p_*A\longrightarrow\mathcal{F}(U)$$

is an equivalence for every such $U\subseteq I$, and so the top vertical map of (6.48) is an equivalence, as desired. We may hence conclude that (6.47) is an equivalence for every manifold of type $(k,m+1)$. By induction on m we now get that (6.47) is an equivalence for every manifold of type (k,m') for $m'\geq 0$, and hence for every manifold of type $(k+1,0)$. By induction on k we now get that (6.47) is an equivalence for any open n-manifold of finite type, as desired. $\square$

Corollary 6.5.12 ([1, Theorem 3.24]). *Restriction along $\mathbb{E}_B^{\otimes}\hookrightarrow(\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}})^{\otimes}$ determines an equivalence*

$$\mathbf{H}(\mathcal{M}\mathrm{fld}_n^{B,\mathrm{fin}},\mathcal{C})\xrightarrow{\simeq}\mathrm{Alg}_{\mathbb{E}_B}(\mathcal{C})$$

between B-framed homology theories with values in $\mathcal{C}$ and $\mathbb{E}_B$-algebra objects in $\mathcal{C}$.

Remark 6.5.13. We focused in this section on open n-manifolds, but there is no obstacle in extending the notion of homology theories, as well as Theorem 6.5.11 and Corollary 6.5.12, to the case of n-manifolds with boundary. To do so, one just needs to spell out what it means for an n-manifold with boundary to be of *finite type*, a notion which can be formulated using suitable handles with boundary.

7. Poincaré-Koszul duality

7.1. **Factorization homology with support.**

7.2. **$\mathbb{E}_n$-suspension and $\mathbb{E}_n$-loops.**

7.3. Nonabelian Poincaré duality (after Lurie).

7.4. Poincaré-Koszul duality for dg-algebras (after Francis and Ayala).

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