Lecture for London

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Let us recall the end of Ilan's lecture. Let $\mathcal{C}$ be a Grothendieck site. Then there exists a canonical geometric morphism

$$\Gamma_* : \mathrm{Sh}(\mathcal{C}) \leftrightarrows \mathrm{Set} : \Gamma^*$$

where Γ_* is the global sections functor and Γ^* is the constant sheaf functor. In particular the functor Γ^* is left exact, i.e. it commutes with finite limits. According to Tomer and Ilan's theorem there exists a Quillen adjunction

$$L_\Gamma : \mathrm{Pro}\left(\mathrm{Sh}(\mathcal{C})_\Delta\right) \leftrightarrows \mathrm{Pro}\left(\mathrm{Set}_\Delta\right) : \mathrm{Pro}(\Gamma^*_\Delta)$$

with respect to the model structure constructed by Tomer and Ilan. Let $* \in \mathrm{Sh}(\mathcal{C})$ be the terminal sheaf. We will call the pro-object $L_\Gamma(*)$ the **shape** of $\mathcal{C}$. Let us consider some examples:

1. If $\mathcal{C}$ carries the trivial Grothendieck topology then $L_\Gamma(*)$ is a constant pro-space which is equal to the **nerve** of $\mathcal{C}$.

2. Let X be a topological space and let $\mathcal{C} = \mathrm{Op}(X)$ be the Grothendieck site of open subsets of X. Then the image of $L_\Gamma(*)$ in $\mathrm{Ho}(\mathrm{Set}_\Delta)$ is a known object from topology called the **shape** of $\mathbf{X}$. This object is useful when studying spaces which are not homotopic to CW complexes. In particular this construction provides a lift of this classical construction form a pro-homotopy-type to a pro-simplicial set. Note that when $\mathbf{X}$ is locally contractible then the cofibrant replacement of the terminal sheaf is a constant pro-sheaf and so the shape of $\mathbf{X}$ is a simplicial set.

3. Let $\mathcal{C}$ be the étale site of a scheme X. Then the image of $L_\Gamma(*)$ on $\mathrm{Ho}(\mathrm{Set}_\Delta)$ is the étale homotopy type of Artin and Mazur. In particular this construction provides a lift of this notion from a pro-homotopy-type to a pro-simplicial set.

One should think of $L_\Gamma(*)$ as a topological manifestation of the étale site $\mathcal{C}$.

Note that the promotion of the étale homotopy type to a pro-simplicial set is not new. It has been done before by Friedlander via the **étale topological type** construction. However, Tomer and Ilan's construction has the following advantages:

1. The passage from the non-directed category HC of hypercoverings to a directed category is via a pre-cofinal functor

$$f : J \longrightarrow \mathrm{HC}$$

   In fact, f can be chosen to be surjective on objects, so that the pro-simplicial promotion of the étale homotopy type is still "parameterized by hypercoverings". In particular the various spaces in the diagram still admit very concrete descriptions.

   In some sense one can say that this promotion from pro-homotopy type to a pro-simplicial set has virtually no price in terms of concreteness. In Friedlander's construction, however, the notion of a hypercovering is replaced by a different and more complicated objects called rigid hypercoverings. In particular, rigid hypercoverings are much "bigger" objects.

2. The new construction generalizes to the **relative setting**.

Let us now explain more carefully what we mean by a relative setting. Suppose we have a base scheme S and we want to study schemes over S. In particular, given a scheme $p : X \longrightarrow S$ over S, what would like to understand the **sections of** p, i.e. maps $\iota : S \longrightarrow X$ such that $p \circ \iota = Id$. For example, if $S = \mathrm{Spec}(K)$ for a field K then sections correspond to K-rational points. If $S = \mathrm{Spec}(O_K)$ where O_K is the ring of integers of a number field then sections correspond to K-integral points.

The fundamental idea is to understand the sections $\iota : S \longrightarrow X$ by translating the situation into the realem of homotopy theory. Recall that in homotopy theory when one is presented with a map of topological spaces $p : Y \longrightarrow Z$ one can use obstruction theory in order to study the sections $Z \longrightarrow Y$. In particular one has a sequence of obstructions which live in

$$H^{n+1}(Z, \pi_n(F))$$

where F is the homotopy fiber of p. If all these obstructions vanish then in good cases this will imply the existence of a section $\iota : Z \longrightarrow Y$. Further more, one can compute the homotopy groups of the **space of sections** $\mathrm{Sec}(p)$ via a spectral sequence of the forms

$$H^s(Z, \pi_t(F)) \Rightarrow \pi_{t-s}(\mathrm{Sec}(p))$$

Now the first attempt at translating an algebraic-geometric setting $p : X \longrightarrow S$ to the homotopical setting can be by replacing both X and S by their homotopical realization, i.e. by the promotion of their étale homotopy type. However, this approach will loose a lot of geometric information. In order to understand why this happens consider the following example. Let p be the map

$$p : \mathrm{Spec}(\mathbb{C}) \longrightarrow \mathbb{A}^1_{\mathbb{C}}$$

obtained by inclusion in 0. Since $\mathbb{A}^1_{\mathbb{C}}$ has a contractible étale homotopy type the homotopical picture is trivial. However, p does not admit any section. More generally, one can explain this phenomenon by thinking of a map $p : X \longrightarrow S$ as a **sheaf over** S whose stacks are the geometric fibers of p. In homotopy theory, one is working with homotopy fibers instead of actual fibers. This is equivalent to saying that homotopy theory makes the sheaf into a "homotopically locally constant sheaf". However in the setting of algebraic geometry a map $p : X \longrightarrow S$ will rarely encode a locally constant sheaf, because it can have very different fibers. For example there can be bad fibers, or empty fibers.

In order to remedy this situation we will encode the setting $p : X \longrightarrow S$ by applying the construction of Tomer and Ilan to the geometric morphism

$$p_* : \mathrm{Sh}(X) \leftrightarrows \mathrm{Sh}(S) : p^*$$

One then have a Quillen adjunction

$$L_p : \mathrm{Pro}\,(\mathrm{Sh}(X)_\Delta) \leftrightarrows \mathrm{Pro}\,(\mathrm{Sh}(S)_\Delta) : \mathrm{Pro}\,(p^*_\Delta)$$

We will call the pro-object $L_p(*) \in \mathrm{Pro}\,(\mathrm{Sh}(S)_\Delta)$ the **relative étale homotoyp type** of X with respect to S. This is a pro-simplicial sheaf over S. We can study then study sections $\iota : S \longrightarrow X$ by comparing them to global sections of the pro-simplicial sheaf $L_p(*)$ (i.e. to derived maps in $\mathrm{Pro}\,(\mathrm{Sh}(X)_\Delta)$ from the terminal pro-sheaf to $L_p(*)$).

By work of Tomer and Ilan this space of derived global sections will still be relatively tractable. In particular one will have an obstruction theory with obstructions lying in the continuous sheaf cohomology groups

$$H^{n+1}_c(S, \pi_n(L_p(*)))$$

Further more if a derived global section exists then the homotopy groups of the space $\mathrm{Map}(*, L_p(*))$ of derived global sections can be computed by a spectral sequence of the form

$$H^s_c(S, \pi_t(L_p(*))) \Rightarrow \pi_{t-s}\,(\mathrm{Map}(*, L_p(*)))$$

Note that these sheaf of homotopy groups can in good cases be computed as well. By a unpublished results of Tomer and the author one can identify the stacks of $\pi_t(L_p(*))$ with the (absolute) étale homotopy type of the geometric fibers of p.

Here are some applications

Theorem 0.1. *Let K be number field and X, Y two smooth geometrically connected K-varieties. Then*

$$(X \times Y)(\mathbb{A})^{fin,\mathrm{Br}} = X(\mathbb{A})^{fin,\mathrm{Br}} \times Y(\mathbb{A})^{fin,\mathrm{Br}}$$

Theorem 0.2. *Let K be a number field and X a smooth geometrically connected variety over K. Assume further that*

$$\pi_2^{\acute{e}t}(\overline{X}) = 0$$

Then

$$X(\mathbb{A})^{fin} = X(\mathbb{A})^{fin,\mathrm{Br}}$$

Theorem 0.3. *Let X and K be as above and assume that $\pi_1^{\acute{e}t}(\overline{X})$ is abelian and that $\pi_2^{\acute{e}t}(\overline{X}) = 0$ (e.g. X is an abelian variety or an algebraic torus). Then*

$$\begin{array}{ccc} X(\mathbb{A})^{fin-ab} & = & X(\mathbb{A})^{fin} \\ \| & & \| \\ X(\mathbb{A})^{\mathrm{Br}} & = & X(\mathbb{A})^{fin,\mathrm{Br}} \end{array}$$