Limits, colimits and adjunctions in ∞-categories

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1 Limits and colimits

Let $\mathcal{C}$ be an ordinary category, $\mathcal{I}$ a small category and $p : \mathcal{I} \longrightarrow \mathcal{C}$ a functor. The process of defining limits and colimits of p goes through the notions of over and under **slice categories**. In particular, in order to define what is a limit of p we need to define the category $\mathcal{C}_{/p}$ of **objects over** p: these are objects $C \in \mathcal{C}$ equipped with a compatible collection of structure maps $\alpha_i : C \longrightarrow p(i)$. A map in $\mathcal{C}_{/p}$ from $(C, \{\alpha_i\})$ to $(D, \{\beta_i\})$ is a map $f : C \longrightarrow D$ such that $\beta_i \circ f = \alpha_i$ for every i. One can then define a limit of p to be a terminal object of $C_{/p}$. Similarly, one can define the category $C_{p/}$ of **objects under** p and define a colimit of p to be an initial object of $C_{p/}$.

Remark 1.0.1. When $\mathcal{I} = \{*\}$ is the trivial category then functors $p : \mathcal{I} \longrightarrow \mathcal{C}$ are simply given by the objects $p(*) = C \in \mathcal{C}$. In this case we will denote $\mathcal{C}_{/p}, \mathcal{C}_{p/}$ simply by $\mathcal{C}_{/C}, \mathcal{C}_{C/}$.

In order to generalize these construction to ∞-categories we first need to understand how to construct over and under slice categories. This will be done by using an appropriate notion of **join** between simplicial sets. Unlike the case for topological spaces, there is more than more natural way to define the join of simplicial sets. In [8] Lurie describes two different (yet equivalent) constructions, each with its own technical advantages and disadvantages. In this lecture we have chosen to work with the less common construction (the one called "the alternative construction" in [8]) because it will facilitate the comparison between the notions of limits/colimits in ∞-categories and the corresponding notion of homotopy limits/colimits in simplicial categories.

Let K, L be two simplicial sets. We define their **join** $K \diamond L \in \mathrm{Set}_\Delta$ by

$$K \diamond L = K \coprod_{K \times L \times \{0\}} K \times L \times \Delta^1 \coprod_{K \times L \times \{1\}} L$$

For a fixed simplicial set K, the functors $(\bullet) \diamond K$ and $K \diamond (\bullet)$ are colimit preserving functors from Set_Δ to $(\mathrm{Set}_\Delta)_{K/}$. The right adjoints of these functors will give us the desired over and under slice constructions. More explicitly, given a map of simplicial sets $p : K \longrightarrow \mathcal{C}$ (considered as an element of $(\mathrm{Set}_\Delta)_{K/}$) we will denote by $\mathcal{C}^{/p}, \mathcal{C}^{p/}$ the simplicial sets given by

$$(\mathcal{C}^{/p})_n = \mathrm{Hom}_{(\mathrm{Set}_\Delta)_{K/}}(\Delta^n \diamond K, \mathcal{C})$$

and

$$(\mathcal{C}^{p/})_n = \mathrm{Hom}_{(\mathrm{Set}_\Delta)_{K/}}(K \diamond \Delta^n, \mathcal{C})$$

Remark 1.0.2. The slice construction commutes with mapping objects in the following sense. Let $p : K \longrightarrow \mathcal{C}$ be a map of simplicial sets and let L be a simplicial set. Let $p_K : K \times L \longrightarrow K$ be the natural projection and let $p' : K \longrightarrow \mathcal{C}^L$ the map associated to $p \circ p_K$. Then one has a natural isomorphism

$$(\mathcal{C}^{p/})^L \xrightarrow{\cong} (\mathcal{C}^L)^{p'/}$$

Furthermore, this isomorphism is compatible with the natural projections to $\mathcal{C}^L$.

Remark 1.0.3. In the notation of Remark 1.0.2 let $q : L \longrightarrow \mathcal{C}$ be an additional map. Then we see that the fiber of the natural map

$$(\mathcal{C}^{p/})^L \longrightarrow \mathcal{C}^L$$

over q is isomorphic to the fiber of the natural map

$$(\mathcal{C}^{/q})^K \longrightarrow \mathcal{C}^K$$

over p, and both are isomorphic to the mapping space

$$\mathrm{Map}_{\mathcal{C}^{K \times L}}(p \circ p_K, p \circ p_L)$$

To establish a solid ground one should first prove that $\mathcal{C}^{p/}$ and $\mathcal{C}^{/p}$ are ∞-categories. In fact, a stronger claim is true:

Theorem 1.0.4. *Let $\mathcal{C}$ be an ∞-category, let $\iota : K_0 \hookrightarrow K$ be an inclusion of simplicial sets and let $p : K \longrightarrow \mathcal{C}$ be a map. Then the induced map of under categories*

$$f : \mathcal{C}^{p/} \longrightarrow \mathcal{C}^{p\iota/}$$

is a left fibration. Furthermore, if ι is right anodyne then f is a trivial Kan fibration. The dual statement holds for the map of over categories

$$g : \mathcal{C}^{/p} \longrightarrow \mathcal{C}^{/p\iota}$$

Theorem 1.0.4 tells us that $\mathcal{C}^{p/} \longrightarrow \mathcal{C}$ is not only an inner fibration (implying in particular that $\mathcal{C}^{p/}$ is an ∞-category) but that it actually a left fibration, i.e. that it corresponds to some functor from $\mathcal{C}$ to spaces. This functors can be recovered by looking at the fiber of $\mathcal{C}^{p/} \longrightarrow \mathcal{C}$ over some point $\mathcal{C} \in \mathcal{C}$. In light of Remark 1.0.3 (with $L = \Delta^0$) this fiber in turn is naturally isomorphic to the mapping space

$$\mathrm{Map}_{\mathcal{C}^K}(p, \overline{C})$$

where $\overline{C} : K \longrightarrow \mathcal{C}$ is the constant map taking value C.

One now defines limits and colimits as follows.

Definition 1.0.5. Let $\mathcal{C}$ be an ∞-category and $p_0 : K \longrightarrow \mathcal{C}$ a map. A diagram $p : K \diamond \Delta^0 \longrightarrow \mathcal{C}$ extending p_0 is a **colimit diagram** if the corresponding object $p \in \mathcal{C}^{p/}$ is initial. Similarly, a diagram $p : \Delta^0 \diamond K \longrightarrow \mathcal{C}$ extending p_0 is called a **limit diagram** if the corresponding object $p \in \mathcal{C}^{/p}$ is terminal.

Remark 1.0.6. Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ has an initial object then the sub-simplicial set $\mathcal{C}_{\text{initial}} \subseteq \mathcal{C}$ spanned by initial objects is an ∞-category whose mapping spaces are contractible. It follows from the basic theory of ∞-categories that $\mathcal{C}_{\text{initial}}$ is a contractible Kan complex. Similarly, the subcategory of terminal objects is either empty or contractible. This means that the limit and colimit are unique up to a contractible choice.

Remark 1.0.7. Let $\mathcal{C}$ be an ∞-category and $C \in \mathcal{C}$ be an object. Then one can show that C is initial if and only if the map

$$f : \mathcal{C}^{C/} \longrightarrow \mathcal{C}$$

is a trivial Kan fibration. It follows that a diagram $p : K \diamond \Delta^0 \longrightarrow \mathcal{C}$ extending p_0 is a colimit diagram if and only if the natural map

$$\mathcal{C}^{p/} \longrightarrow \mathcal{C}^{p_0/}$$

is a trivial Kan fibration.

Our purpose now is to relate this definition of limit/colimit to the more classical definition of homotopy limits/colimits in the setting of simplicial categories. To fix notations let us consider the case of colimits.

We hence obtain the following result:

Theorem 1.0.8. *Let $\mathcal{C}$ be a fibrant simplicial category. Let $p_0 : K \longrightarrow \mathrm{N}(\mathcal{C})$ be a diagram and $p_0' : \mathfrak{C}(K) \longrightarrow \mathcal{C}$ the corresponding functor of simplicial categories. Let $C \in \mathcal{C}$ be an object and*

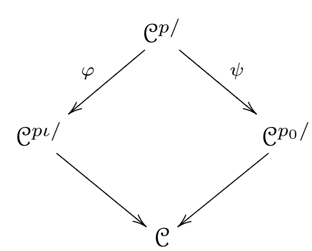
$$T : p_0' \longrightarrow \overline{C}$$

a natural transformation (where $\overline{C} : \mathfrak{C}(K) \longrightarrow \mathcal{C}$ is the constant functor at C) and let $p : K \diamond \Delta^0 \longrightarrow C$ be the map determined by T. Then p is a colimit diagram if and only if T exhibits C as a homotopy colimit of p_0', i.e., if and only if for each $D \in \mathcal{C}$ the induced transformation

$$T_* : \overline{\mathrm{Map}_{\mathcal{C}}(C, D)} \longrightarrow \mathrm{Map}_{\mathcal{C}}(p_0'(-), D)$$

exhibits $\mathrm{Map}_{\mathcal{C}}(C, D)$ *as the homotopy limit* $\mathrm{holim}_{k \in \mathfrak{C}(k)} \mathrm{Map}_{\mathcal{C}}(p_0'(k), D)$.

Proof. Let $\iota : \Delta^0 \longrightarrow K \diamond \Delta^0$ be the natural inclusion and consider the diagram of slice categories



We first observe that ι is **right anodyne**. This is due to the fact that ι is a pushout of the map

$$K \times \Delta^{\{1\}} \hookrightarrow K \times \Delta^1$$

which in turn is a product of K and the right anodyne map $\Delta^{\{1\}} \hookrightarrow \Delta^1$ (note the class of right anodyne maps is closed under pushout-products against inclusions). Hence we get from Theorem 1.0.4 that φ is a trivial Kan fibration. Now let $D \in \mathcal{C}$ be an object, determining a vertex $D \in \mathrm{N}(\mathcal{C})$ (which we will denote by the same name). Taking fibers over D we obtain a diagram of Kan simplicial sets

$$\mathrm{Map}_{\mathrm{N}(\mathcal{C})}(C, D) \xleftarrow{\ \varphi_D\ } \mathrm{Map}_{\mathrm{N}(\mathcal{C})^{(K \diamond \Delta^0)}}(p, \overline{D}) \xrightarrow{\ \psi_D\ } \mathrm{Map}_{\mathrm{N}(C)^K}(p_0, \overline{D})$$

where φ_D is a weak equivalence. By inverting φ_D in $\mathrm{Ho}(\mathrm{Set}_\Delta)$ we obtain a well defined morphism

$$\rho_D \stackrel{\text{def}}{=} [\psi_D] \circ [\varphi_D]^{-1} : \mathrm{Map}_{\mathrm{N}(\mathcal{C})}(C, D) \longrightarrow \mathrm{Map}_{\mathrm{N}(\mathcal{C})^K}(p_0, \overline{D})$$

in $\mathrm{Ho}(\mathrm{Set}_\Delta)$. This morphism describes the effect of pre-composing with the map $p_0 \longrightarrow \overline{C}$ determined by T. We now observe that by Remark 1.0.7 the diagram p is a colimit diagram if and only if φ is a trivial Kan fibration, i.e. if and only if the map φ_D is a weak equivalence for every D. Alternatively, we can phrase this as saying that p is a colimit diagram if and only if ρ_D is an isomorphism for every D. We now employ a final manipulation to the data above by observing that $\mathrm{Map}_{\mathrm{N}(\mathcal{C})^K}(p_0, \overline{D})$ can also be identified with the fiber of the map

$$(\mathrm{N}(\mathcal{C})^{/D})^K \longrightarrow \mathrm{N}(\mathcal{C})^K$$

over the point $p_0 \in \mathcal{C}^K$. In other words, $\mathrm{Map}_{\mathcal{C}^K}(p_0, \overline{D})$ can be identified with the space

$$\mathrm{Map}_{\mathrm{Set}_{/K}}\left(K, \mathrm{N}(\mathcal{C})^{/D} \times_{\mathrm{N}(\mathcal{C})} K\right)$$

of sections of the right fibration $f : (\mathrm{N}(\mathcal{C}))^{/D} \times_{\mathrm{N}(\mathcal{C})} K \longrightarrow K$. Now recall that $\mathrm{Set}_{/K}$ can be endowed with the contravariant model structure in which every object is cofibrant and the fibrant objects are exactly the right fibration. As a result, we can identify the mapping space above with the corresponding **derived mapping space**.

We have a Quillen equivalence between $\mathrm{Set}_{/K}$ and the projective model structure on ${\mathrm{Set}_\Delta}^{\mathfrak{C}(K)}$. Furthermore, under this Quillen equivalence the object $K \longrightarrow K$ maps to the terminal functor and the right fibration $\mathrm{N}(\mathcal{C})^{/D} \times_{\mathcal{C}} K$ corresponds (up to equivalence) to the functor $k \mapsto \mathrm{Map}_{\mathfrak{C}(\mathcal{C})}(p_0'(k), D)$ from $\mathfrak{C}(K)$ to Set_Δ. Hence we obtain a natural weak equivalence

$$\mathrm{Map}_{\mathrm{Set}_{/K}}\left(K, \mathrm{N}(\mathcal{C})^{/D} \times_{\mathcal{C}} K\right) \simeq \mathrm{Map}^h_{{\mathrm{Set}_\Delta}^{\mathfrak{C}(K)}}(*, \mathrm{Map}(p_0'(-), D)) = \mathrm{holim}_{k \in \mathfrak{C}(k)} \mathrm{Map}(p_0'(k), D)$$

The resulted morphism

$$\mathrm{Map}_{\mathcal{C}}(C,D) \simeq \mathrm{Map}_{\mathrm{N}(\mathcal{C})}(C,D) \longrightarrow \mathrm{holim}_{k\in\mathfrak{C}(k)} \mathrm{Map}(p_0'(k),D)$$

in $\mathrm{Ho}(\mathrm{Set}_\Delta)$ induced by ρ_D can be identified with the one induced by T. The desired result now follows.

□

2 Adjunctions

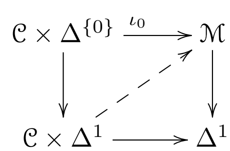
Let us start with the basic definition:

Definition 2.0.9. Let $\mathcal{C},\mathcal{D}$ be ∞-categories. An **adjunction** between them is a Cartesian-coCartesian fibration

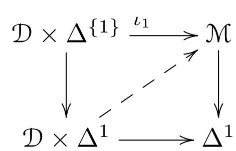
$$q : \mathcal{M} \longrightarrow \Delta^1$$

together with equivalences $\iota_0 : \mathcal{C} \longrightarrow q^{-1}(0)$ and $\iota_1 : \mathcal{C} \longrightarrow q^1(1)$.

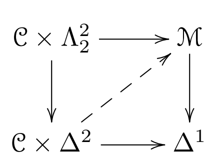
Let us begin by trying to connect the definition above with classical notion of adjunction. First observe that since q is a coCartesian fibration the diagram



admits a lift as $T : \mathcal{C} \times \Delta^1 \longrightarrow \mathcal{M}$ such that $T(\{C\} \times \Delta^1)$ is q-coCartesian for every $C \in \mathcal{C}$. Furthermore, such a lift is unique up to a contractible ambiguity. The map T can interpreted as a natural transformation between ι_0 to a map of the form $\iota_1 \circ f$ where $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a map. Similarly, since q is Cartesian the diagram



admits a Cartesian lift $S : \mathcal{D}\times\Delta^1 \longrightarrow \mathcal{M}$, leading to a functor $g : \mathcal{D} \longrightarrow \mathcal{C}$ which is again unique up to a contractible ambiguity. Now let $\mathcal{R} : C \times \Lambda^2_2 \longrightarrow \mathcal{M}$ be the map whose restriction to $\mathcal{C}\times\Delta^{\{0,2\}}$ is T and whose restriction to $\mathcal{C}\times\Delta^{\{1,2\}}$ is $S \circ (f \times \mathrm{Id}_{\Delta^1})$. Then the diagram



admits a lift as indicated since R sends $\{C\} \times \Delta^{\{1,2\}}$ to a q-Cartesian edge for each $C \in \mathcal{C}$. This results in a map $u : Id_{\mathcal{C}} \longrightarrow g \circ f$ in $\mathcal{C}^{\mathcal{C}}$, which we naturally call the unit map. Similarly, one can obtain a counit map $f \circ g \longrightarrow Id_{\mathcal{D}}$. These unit and counit maps can be shown to satisfy the unit-counit axioms up to a a choice of a map between natural transformation, which again satisfy compatibility conditions among them selves and so on. The beauty of higher category theory is that all this higher structure can be encoded by the map $\mathcal{M} \longrightarrow \Delta^1$ and can be reconstructed from it up to a contractible ambiguity.

Definition 2.0.10. In the notation above we will say that f is **left adjoint** to g and g is **right adjoint** to f. We will also that there exists an adjunction between f and g.

Let us now explain the relation between adjunctions and mapping spaces as in the classical picture. Keeping the notation from above, let $C \in \mathcal{C}$ be an object and let $\alpha : \iota_0(C) \longrightarrow \iota_1(f(C))$ be the map in $\mathcal{M}$ determined by T. Consider the diagram

$$\begin{array}{ccc} & \mathcal{M}^{\alpha/} \times_{\mathcal{M}} \mathcal{D} & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \mathcal{M}^{\iota_0(C)/} \times_{\mathcal{M}} \mathcal{D} & & \mathcal{M}^{\iota_1(f(C))/} \times_{\mathcal{M}} \mathcal{D} \end{array} \tag{2.1}$$

The map ψ is a trivial Kan fibration because the inclusion $\Delta^{\{1\}} \hookrightarrow \Delta^1$ is right anodyne. The map φ is a trivial Kan fibration because α is a q-coCartesian edge. Now let $D \in \mathcal{D}$ be an object. By taking the respective fibers over D we obtain an equivalence span, which can be transformed into an isomorphism in $\mathrm{Ho}(\mathrm{Set}_\Delta)$ of the form

$$\mathrm{Map}_{\mathcal{M}}(\iota_0(C), \iota_1(D)) \cong \mathrm{Map}_{\mathcal{D}}(f(C), D)$$

Applying a similar argument for $g : \mathcal{D} \longrightarrow \mathcal{C}$ we obtain another isomorphism in $\mathrm{Ho}(\mathrm{Set}_\Delta)$

$$\mathrm{Map}_{\mathcal{M}}(\iota_0(C), \iota_1(D)) \cong \mathrm{Map}_{\mathcal{D}}(C, g(D))$$

Hence we obtain the classical isomorphism

$$\mathrm{Map}_{\mathcal{D}}(f(C), D) \longrightarrow \mathrm{Map}_{\mathcal{D}}(C, g(D))$$

Note that this isomorphism in $\mathrm{Ho}(\mathrm{Set}_\Delta)$ is uniquely determined. However, if we want to construct actual maps of simplicial sets then we need to make some choices (like choosing a section for the trivial Kan fibrations φ, ψ). This choices will replacement the uniqueness of the classical picture with uniqueness up to a contractible ambiguity in the ∞-picture.

Note that diagram 2.1 can be interpreted as saying that left fibration $\mathcal{M}^{\iota_0(C)/} \times_{\mathcal{M}} \mathcal{D} \longrightarrow \mathcal{D}$ is equivalent to a corepresentable left fibration. In fact, all we needed for that is that α is q-coCartesian. In fact, this property is practically the definition of q-coCartesian edges. These considerations can be put together to form a proof of the following:

Proposition 2.0.11. *Let $q : \mathcal{M} \longrightarrow \Delta^1$ be an inner fibration and $\iota_0 : \mathcal{C} \longrightarrow q^{-1}(0), \iota_1 : \mathcal{D} \longrightarrow q^{-1}(1)$ be equivalences. Then*

1. *q is coCartesian if and only if for every $C \in \mathcal{C}$ the left fibration*

$$\mathcal{M}^{\iota_0(C)/} \times_{\mathcal{M}} \mathcal{D} \longrightarrow \mathcal{D}$$

is corepresentable (up to equivalence).

2. *q is Cartesian if and only if for every $D \in \mathcal{D}$ the right fibration*

$$\mathcal{M}^{/\iota_1(D)} \times_{\mathcal{M}} \mathcal{C} \longrightarrow \mathcal{C}$$

is representable (up to equivalence).

Corollary 2.0.12. *Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a map. Then f admits a right adjoint if and only if for every $C \in \mathcal{C}$ the right fibration*

$$\mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{f(C)/}$$

is representable. A similar criterion exists for left adjoints.

We now wish to prove that if $f : \mathcal{C} \longrightarrow \mathcal{D}$ is left adjoint to some $g : \mathcal{D} \longrightarrow \mathcal{C}$ then f respects colimits. The key point is that Cartesian and coCartesian fibrations are stable under exponentiation. Hence if $q : \mathcal{M} \longrightarrow \Delta^1$ is an adjunction between f and g and K is a simplicial set then $\mathcal{M}^K \longrightarrow (\Delta^1)^K$ is again Cartesian and coCartesian. Pulling back along the diagonal embedding $\Delta^1 \longrightarrow (\Delta^1)^K$ we obtain a Cartesian/coCartesian fibration

$$q_K : \mathcal{M}^K \times_{(\Delta^1)^K} \Delta^1 \longrightarrow \Delta^1$$

The maps ι_0, ι_1 then induce equivalences $\mathcal{C}^K \xrightarrow{\simeq} q_k^{-1}(0)$ and $\iota_1 : \mathcal{D}^K \longrightarrow q_K^{-1}(1)$ and so we get an adjunction between $f^K : \mathcal{C}^K \longrightarrow \mathcal{D}^K$ and $g^K : \mathcal{D}^K \longrightarrow \mathcal{C}^K$. The same can be done with $K \diamond \Delta^0$ instead of K. Rapping these arguments together with our insights from the previous lecture one can obtain the following result:

Theorem 2.0.13. *Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor which is left adjoint to $g : \mathcal{D} \longrightarrow \mathcal{C}$. Then f preserves all colimits which exists in $\mathcal{C}$ and g preserves all limits which exist in $\mathcal{D}$.*

References

[7] Lurie, J. *Higher Algebra*, `http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf`.

[8] Lurie, J., *Higher Topos Theory*, Annals of Mathematics Studies, 170, Princeton University Press, 2009, `http://www.math.harvard.edu/~lurie/papers/highertopoi.pdf`.