Second descent and rational points on Kummer surfaces

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IHÉS

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- Probably not in general.
- Conditional counter-examples constructed by Sarnak & Wang, Poonen and Smeets.
- No unconditional counter-example is known.

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Conjecture (Skorobogatov)

The Brauer-Manin obstruction is the only obstruction for the Hasse principle on smooth and proper K3 surfaces.

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Diophantine problem of interest

Assuming finiteness of Ш for all associated abelian surfaces, find sufficient conditions for the Hasse principle to hold on X_α.

Previous results

Skorobogatov & Swinnerton-Dyer (2005) - sufficient conditions when $A = E_1 \times E_2$ and $E_1[2], E_2[2]$ have trivial Galois action. X is then a smooth and proper model for $y^2 = f(x)g(z)$ where $\deg(f) = \deg(g) = 4$ whose cubic resolvants define E_1 and E_2.

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Applies in principle to surfaces which are fibered into curves of genus 1, typically requires the assumption finiteness of Ш and Schinzel's hypothesis (not needed for the Kummer surface variant)

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$H^1(k, A[2]) \cong \{(b_0, ..., b_5) \in \mathcal{G}^6 | b_1 \cdot ... \cdot b_5 = 1\}/\mathcal{G} \quad (\mathcal{G} := k^*/(k^*)^2)$

For a class $\overline{b} \in H^1(k, A[2])$ associated Kummer surface is

$$X_{\overline{b}}: \quad \sum_i \frac{b_i x_i^2}{f'(a_i)} = \sum_i \frac{b_i a_i x_i^2}{f'(a_i)} = \sum_i \frac{b_i a_i^2 x_i^2}{f'(a_i)} = 0$$

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Theorem (H. 2016)

Let $A, \{a_i\}, \overline{b}$ be as above. Assume that $\frac{b_1}{b_0}, ..., \frac{b_4}{b_0}$ are linearly independent in $\mathcal{G}$ and that for every $i = 1, ..., 5$ there exists a place w_i such that $\mathrm{val}_{w_i}(a_i - a_0) = \mathrm{val}_{w_i} d = 1$ and $\mathrm{val}_{w_i}(b_j/b_0) = 0$. Assume that the 2-primary torsion subgroup of Ш is finite for every quadratic twist of A. Then the BM obstruction is the only one for the Hasse principle on the Kummer surfaces $X_{\overline{b}}$.

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- Step 1: find a quadratic extension $F = k(\sqrt{a})$ such that Y_α^F has points everywhere locally. Replacing Y_α with Y_α^F we may assume without loss of generality that Y_α itself has points everywhere locally. In particular, $\alpha \in \mathrm{Sel}_2(A)$.

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- Step 2: find a quadratic extension such that $\mathrm{Sel}_2(A^F)$ is generated by α and the image of the 2-torsion.
- $\Rightarrow$ conclude that $Ш(A^F)[2]$ is generated by the image of α. By assumption $Ш(A^F)$ is finite and the Cassels-Tate pairing is alternating (Poonen-Stoll) $\Rightarrow$ the dimension of $Ш(A^F)[2]$ is even $\Rightarrow Ш(A^F)[2] = 0 \Rightarrow Y_\alpha^F(k) \neq \emptyset \Rightarrow X_\alpha(k) \neq \emptyset$.

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Problem

Sometimes step 2 is not possible (without further assumptions)

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- Step 3: find a quadratic extension such that the subgroup $\mathrm{Sel}_2^{\circ}(A^F) \subseteq \mathrm{Sel}_2(A^F)$ generated by those elements which are orthogonal to all of $\mathrm{Sel}_2(A)$ with respect to Cassels-Tate pairing is generated by α and the image of the 2-torsion. Assume WLOG that this holds already for A itself.

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- $\Rightarrow$ conclude that the subgroup $Ш^{\circ} \subseteq Ш[2]$ generated by those elements which are orthogonal to $Ш[2]$ is generated by the image of α. By assumption $Ш(A)$ is finite and the Cassels-Tate pairing is alternating $\Rightarrow$ the dimension of $Ш^{\circ} \cong Ш(A)[4]/Ш(A)[2]$ is even $\Rightarrow Ш^{\circ} = 0 \Rightarrow Y_{\alpha}^{F}(k) \neq \emptyset \Rightarrow X_{\alpha}(k) \neq \emptyset$.

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- Note $\dim_2 W_v = \dim_2 W_v^F = 4 = \frac{1}{2}\dim_2 H^1(k_v, M)$.

Goal

Compare the groups

$$\mathrm{Sel}_2(A) = \{\beta \in H^1(k, M) | \, \mathrm{loc}_v(\beta) \in W_v\}$$

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- $\dim_2 \mathrm{Sel}_2(A^F) - \dim_2 \mathrm{Sel}_2(A) = \dim_2 V_T^F - \dim_2 V_T$.

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Lemma (Mazur-Rubin)

$$\dim_2 V_T + \dim_2 V_T^F \leq \sum_{v \in T} r_v$$

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Proof.

The Weil pairing induces the non-degenerate local Tate pairing

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with W_v and W_v^F maximal isotropic.

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$\Rightarrow$ "natural coordinates" on $H^1(k, M)$.

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Let $F = k(\sqrt{a})$ be a quadratic extension and let $P, Q \in M$ be two 2-torsion points. Then

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$\Rightarrow$ we are in the situation of

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$\dim_2 V_T \geq 2g + 1 \Rightarrow \dim_2 \mathrm{Sel}_2(A^F) \leq \dim_2 \mathrm{Sel}_2(A) - 1.$

Unavoidable elements

Conclusion

Given $\alpha \in \mathrm{Sel}_2(A)$ we may find a quadratic extension F/k such that $\mathrm{Sel}_2(A^F)$ contains α and is generated (modulu the image of the 2-torsion) by elements which split in L_α. Let us call these the "unavoidable elements".

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Remark

Elements which split in L_α but do not split in $k(\alpha)$ can actually be avoided. The truely unavoidable elements are those which split in $k(\alpha)$. However, for the purpose of the third step it is not important what exactly is our group of unavoidable elements as long as it is a-priori bounded by a fixed finite subgroup of $H^1(k, M)$.

Step 3 - second descent

A - an abelian surface with a principal polarization coming from a symmetric line bundle. Assume $\text{Ш}(A)$ finite. Cassels-Tate pairing

$$\langle , \rangle_{\mathrm{CT}}^{A} : \text{Ш}(A) \times \text{Ш}(A) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

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There exists a class $[C_\beta] \in \mathrm{Br}(U_\alpha)/\mathrm{Br}(k)$ *(independent of* F*) whose image in* $\mathrm{Br}(W_\alpha^F)/\mathrm{Br}(k)$ *coincides with the restriction of* $[B_\beta] \in \mathrm{Br}(Y_\alpha^F)/\mathrm{Br}(k)$*. In particular, if* $(x_v) \in X_\alpha(\mathbb{A}_k)$ *is a point which lifts to* Y_α^F *then*

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Proposition (Changing Cassels-Tate pairing by a quadratic twist)

Let $\sigma, \tau \in \Gamma_k$ be two elements. Then there exists a quadratic extension F/k, unramified over S, such that $\mathrm{Sel}_2^R(A^F) = \mathrm{Sel}_2^R(A)$ and such that for every $\alpha, \beta \in \mathrm{Sel}_2^R(A)$ one has

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Corollary

Let $\alpha, \beta \in \mathrm{Sel}_2^\circ(A)$ be elements. Suppose $\exists w_0 \in R$ such that $\langle \alpha, P_{w_0} \rangle, \langle \beta, P_{w_0} \rangle \in H^1(k, \mu_2)$ linearly independent. Then $\exists F/k$ such that $\alpha \in \mathrm{Sel}_2^\circ(A^F) \subseteq \mathrm{Sel}_2^\circ(A)$ and $\beta \notin \mathrm{Sel}_2^\circ(A^F)$.

Step 3 - conclusion

A - an abelian surface with 2-structure $R \subseteq S$.

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A multiplicative place $w \in S$ is minimal if the induced map

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Let $\alpha \in \mathrm{Sel}_2^R(A)$ be a non-degenerate element which is Cassels-Tate orthogonal to all of $\mathrm{Sel}_2(A)$. Suppose there exists a minimal place $w_+ \in S \setminus R$ such that $P_{w_+} = \sum_{w \in R} P_w$. Then we may always find a quadratic extension F/k such that α is the **only** *non-zero element of $\mathrm{Sel}_2^R(A)$ which is orthogonal to all of $\mathrm{Sel}_2(A)$.*

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$\Rightarrow X_\alpha = \mathrm{Kum}(Y_\alpha)$ has a rational point.

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The method as presented above can be generalized to various other types of abelian surfaces, assuming the existence of suitable places with prescribed bad reduction (and finiteness of Ш)).

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More cases: $A = E_1 \times E_2$ with Galois action on each $E_i[2]$ either trivial, quadratic, or S_3, under suitable assumptions.

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It seems likely that a step of second descent can be added also to the original form of Swinnerton-Dyer's method. In the good of all possible worlds one can hope that this would allow one to remove "Condition (D)" appearing in one form or another in most applications of the method, at least in some geometric situations.