The Classification Problem of Smooth Manifolds

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1 Introduction

In this lecture we wish to survey the problem of classifying up to diffeomorphism all smooth oriented connected closed manifolds (which we shall call **standard manifolds** from now on). This problem is of course too hard to be fully solved, but there is a fairly sophisticated theory which is devoted to this subject.

One of the interesting aspects of this theory is the strong dependence on the dimension of the manifold. It appears that dimensions 2 and 3 have a theory of their own, dimension 4 has unique characteristics and dimensions 5 and up have a completely different theory. Dimensions 0 and 1 are of course trivial - the only standard manifolds in these dimensions are the point and S^1 respectively. This can be proven by a fairly elementary methods of topology.

This survey will not go into a lot of mathematical details and proofs, but rather try in convey the main ideas in the theory.

2 Basic Notions - Cobordism, Surgery and Connected Sum

Before we start with the survey we need to explain some of the basic notions and tools we are about to use. The first notion which is fundamental to the study of manifolds is that of a **cobordism**. This notion will admit several variants throughout the survey, but in order to understand them we need to first give the basic definition.

Definition 1. *Let M_1, M_2 be two closed, oriented, smooth manifolds of dimension n. An (oriented)* ***cobordism*** *between M_1 and M_2 is an $(n+1)$-dimensional smooth oriented manifold W with boundary ∂W and an orientation preserving diffeomorphism $f : M_1 \coprod \overline{M_2} \to W$ (where $\overline{M_2}$ denotes the manifold M_2 with the reverse orientation).*

If there exists a cobordism between M_1 and M_2 we say that they are **cobordant** to one another. This is easily seen to be an equivalence relation. A fundamental idea in the so called **surgery theory** is that two manifolds M_1, M_2

are cobordant if and only if M_2 can be obtained from M_1 by a finite sequence of operations called **surgery steps**:

Let M be an oriented n-dimensional standard manifold. Take some $k = 0, ..., n-1$ and let $f : S^k \times D^{n-k} \to M$ be an embedding. Let M' be the manifold obtained from M by removing the interiour of the $S^k \times D^{n-k}$ embedded in M. Thus M' is a manifold with boundary $S^k \times S^{n-k-1}$. Let

$$N = M' \coprod_{S^k \times S^{n-k-1}} \overline{D^{k+1} \times S^{n-k-1}}$$

be the manifold obtained by gluing M' to $\overline{D^{k+1} \times S^{n-k-1}}$ along their common boundary. The reverse orientation is needed to make the orientation of N consistent. We then say that N is obtained form M by a k-dimensional **surgery step**.

We claim that in that situation N is also cobordant to M. The appropriate W can be constructed as follows: Define

$$W_1 = M \times I$$

$$W_2 = N \times I$$

$$D = D^{k+1} \times D^{n-k}$$

Then

$$\partial W_1 = M \coprod \overline{M} = M \coprod \left[\overline{M'} \coprod_{S^k \times S^{n-k-1}} \overline{S^k \times D^{n-k}} \right]$$

$$\partial W_2 = N \coprod \overline{N} = \left[M' \coprod_{S^k \times S^{n-k-1}} \overline{D^{k+1} \times S^{n-k-1}} \right] \coprod \overline{N}$$

$$\partial D = \left[S^k \times D^{n-k} \right] \coprod_{S^k \times S^{n-k-1}} \left[D^{k+1} \times S^{n-k-1} \right]$$

Thus we can consistently glue these three manifolds together:

$$W = W_1 \coprod_{M'} W_2 \coprod_{\partial D} D$$

and obtain an oriented manifold with boundary $\partial W = M \coprod \overline{N}$, so M and N are cobordant. By using Morse theory, one can show that each cobordism can be partitioned into a finite composition of cobordisms of the form above, hence two manifolds are cobordant if and only if one can be obtained from the other by a finite sequence of surgery steps.

The last operation we wish to mention is the **connected sum** operation, which is a particular case of a surgery step. Let M_1, M_2 be two standard oriented manifolds of dimension n. Consider an embedding

$$f : S^0 \times D^n \to M_1 \coprod M_2$$

such that $\{0\} \times D^n$ is embedded in M_1 and $\{1\} \times D^n$ is embedded in M_2. Then the manifold N which is obtained from f by the corresponding 0-dimensional surgery step is called the **connected sum** of M_1 and M_2. Note that the resulting manifold N is standard as well, and in particular connected, hence the name connected sum. Further more, its oriented diffeomorphism type does not depend on the choice of f.

3 Low Dimensional Theory - Geometrization

3.1 2 Dimensional Manifolds

The well known classification of standard 2 dimensional manifolds was completed already in the 19th century. They are classified by a single complete invariant $g \in \mathbb{N} \cup \{0\}$ which is called the **genus** of the manifold. In genus 0 we have the sphere S^2, in genus 1 we have the torus $\mathbb{T}^2$ and similarly in genus g we have the surface obtained from S^2 by attaching g "handles" (or more precisely, g surgery steps of dimension 0)

3.2 Geometrization

A **model geometry** is a simply connected, smooth orientable manifold M together with a transitive orientation preserving action of a connected Lie group G on it with compact stabilzer. A model geometry is called **maximal** if there is no $G' \supsetneq G$ acting on M which satisfies these conditions.

It turns out that M is a model geometry if and only if it admits a complete homogenious riemannian metric, in which case G is the identity component of the isometry group $Iso(M)$ (this is why it is called a "geometry"). A maximal model geometry admits a unique riemannian metric which is G-invariant. In particular M admits a unique G invariant measure induced by that riemannian metric.

A geometric structure on a smooth oriented manifold M is a covering $X \to M$ such that X is a maximal model geometry. A manifold is called **geometric** if it has such a structure (or, equivalently, if its universal cover is a maximal model geometry).

Since M is naturally locally diffeomorphic to X it inherits its riemannian structure and the corresponding measure form. We say that a geometric manifold has **finite volume** if that measure is finite. This is a weaker condition then M being compact.

In dimension 2 there are exactly 3 model geometries which admit free finite-volume quotients:

1. Spherical Geometry: $X = S^2, G = SO(3)$.
2. Euclidean Geometry: $X = \mathbb{R}^2, G = Iso(\mathbb{R}^2)^+$.
3. Hyperbolic Geometry: $X = \mathbb{H}^2, G = SL_2(\mathbb{R}) = SO(1,2)^+$.

It turns out that every standard 2-manifold is geometric. S^2 is model geometry on its own is the universal cover of itself. The universal cover of $\mathbb{T}^2$ is $\mathbb{R}^2$ and the universal cover of each standard 2-manifold with $g > 1$ can be identified with the hyperbolic plane $\mathbb{H}^2$.

The question now is, can this geometrization method be extended to higher dimensions?

3.3 3 Dimensional Manifolds

In dimension 3 we can still work with the geometrization technic. In contrast to the simple 2-dimensional case, here not every standard 3-manifold is geometric. First we need to decompose our manifold into a connected sum of **prime** manifolds (those which can't be non-trivially decomposed as connected sums). This can be done in a unique way up to order and copies of S^3. We then have Thurston geometric conjecture:

Conjecture 1. *Let M be a standard 3-dimensional manifold. Then we can embed in M a finite number of incompressible tori (i.e. tori embeddings which induce injections on the fundamental group) such that each component of the complement has a finite-volume geometric structure.*

Thurston also classified all the model geometries in 3 dimensions which have finite-volume free quotients. It turns out that there are 8 such model geometries. For every model geometry, except for the hyperbolic one, there exists a complete classification of finite-volume geometric manifolds with that geometry.

The 3 isotropic ($H = SO(3)$) geometries are:

1. Spherical: $X = S^3, G = SO(4)$.

   The geometric manifolds with this model geometry are all compact and are charaterized by the fact that they have a finite fundamental group. They admit a riemannian structure with constant positive sectional curvature.

2. Euclidean: $X = \mathbb{R}^3, G = SO(3)\widetilde{\times}\mathbb{R}^3$.

   The finite-volume manifolds of this geometry are also all compact. They admit a flat riemannian metric.

3. Hyperbolic: $X = \mathbb{H}^3, G = SO(1,3)^+$.

   This is the less understood class of geometric manifolds and is under vivid investigation. These manifolds admit a riemannian metric with constant negative sectional curvature.

There are 4 geometries with stabilizer $SO(2)$:

1. Spherical-Euclidean: $X = S^2 \times \mathbb{R}, G = SO(3) \times \mathbb{R}$.

2. Hyperbolic-Euclidean: $X = \mathbb{H}^2 \times \mathbb{R}, G = SO(1,3)^+ \times \mathbb{R}$.

3. $X = \widetilde{SL_2(\mathbb{R})}, G = \left(\widetilde{SL_2(\mathbb{R})} \times \mathbb{R}\right)/\mathbb{Z}$

4. The nil geometry: $M = U_3(\mathbb{R}), G = U_3(\mathbb{R})\widetilde{\times}SO(2)$ (where $U_3(\mathbb{R})$ is the unipotent group of 3×3 matrices over $\mathbb{R}$, also known as the Heisenberg group).

The last geometry is called the **sol** geometry and its stabilizer is trivial. In that geometry $X = G$ is the the solvable Lie group given by the extension

$$1 \to \mathbb{R}^2 \to G \to \mathbb{R} \to 1$$

where $\mathbb{R}$ acts on $\mathbb{R}^2$ by

$$t \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

In 2002 a ground breaking article of Gregory Perelman appeared which proves the geometrization conjecture. This article is better known for proving the Poincare conjecture which is a particular case of the geometrization conjecture (because the geometrization conjecture implies that each simply connected 3-manifold is a connected sum of model geometries, but the only compact model geometry is S^3).

The idea behind Perelman's proof is the following: Suppose that M is a riemannian manifold with metric tensor g. One can calculate another symmetric 2-tensor $Ric(g) \in \Gamma(Sym^2(T^*M))$ which is called the Ricci curvature. Then one can consider a differential equation on a time dependent metric g_t defined by

$$\frac{\partial g_t}{\partial t} = Ric(g)$$

It turns out that a short time solution of this equation exists and it makes the metric g_t more and more **homogeneous** (for some reason it behaves like the heat equations, for which the heat distribution becomes more and more uniform).

The problem is that a long term solution may not exists because the metric g_t will become singular in finite time. The proof then proceeds to use surgery in order to remove the singular part from the manifold and cut the manifold into several pieces such that each piece has a long term solution which converges to a locally homogeneous metric. Then one shows that the universal cover of such a piece is a model geometry.

4 The Notorious Dimension 4

In dimension 4 the geometrization technic no longer works, and we are faced with a lot of manifolds which do not admit a geometric structure. Further more, the full classification problem is virtually impossible, because it can be shown that every finitely presented group can be realized as a fundamental group of a standard 4-manifold. Thus the classification of standard 4-manifold is at least as hard as the classification of finitely presented groups, which is not only extremely complicated but also not **computationally decidable**.

Our hope now is to try to classify the **simply connected** standard 4-manifolds. It turns out that this task can be fully accomplished for (closed) **topological** manifolds, but is far from being solved in the smooth category.

Let us start with a classification of homotopy types of simply-connected 4-dimensional orientable closed topological manifolds. If M is such a manifolds then

$$H_0(M) = H_4(M) = \mathbb{Z}$$

and

$$H_1(M) = H_3(M) = 0$$

from poincare duality. Since M is simply connected $\pi_2(M)$ must be torsion free and isomorphic to $H_2(M)$. Thus $H_2(M) = \mathbb{Z}^m$ for some m. We have the intersection pairing $Q : H_2(M) \times H_2(M) \to \mathbb{Z}$ which is a unimodular symmetric bilinear form. It turns out that this Q characterizes the homotopy type of M completely:

Theorem 1. *Two closed simply connected orientable 4-manifolds are homotopic if and only if their intersection forms are isomorphic.*

Proof. Our first step will be to pass from a manifold M to a homotopy equivalent CW complex X. Let

$$f : \bigvee_m S^2 \to M$$

be a map which induces an isomorphism on π_2. We can move f a bit so it won't be onto, and then we have some open ball $\overset{o}{D^4} \subseteq M$ which is not in the image of f. Consider

$$M_0 = M \setminus \overset{o}{D^4}$$

which is a manifold with boundary $\partial M = S^3$. By Mayer-Vietoris we get that $H_4(M_0) = 0$ and $H_i(M_0) = H_i(M)$ for any $i \neq 4$. Further more by Van-Kampen's theorem we see that M_0 is simply-connected as well. Thus the map f is a map between two simply-connected CW-complexes which induces isomorphism on all the homology groups. By Whitehead's theorem it induces a homotopy equivalence

$$\bigvee_m S^2 \cong M^0$$

Let

$$\varphi : M^0 \to \bigvee_m S^2$$

be a homotopy inverse. Let X a CW-complex obtained by gluing a D^4 to $\bigvee_m S^2$ via the map $\varphi|_{\partial M}$. Then $X \simeq M$ and the data of the intersection form of M is encoded in the cup product on the second cohomology of X.

Let Z be the classifying space $(\mathbb{C}P^\infty)^m = K(\mathbb{Z}^m, 2)$. It has a CW-structure with cells only in even dimensions and its 2-skeleton is $\bigvee_m S^2$. The cohomology ring of Z is a polynomial ring on m degree 2 variables.

We identify the set of $2k$-cells as a $\mathbb{Z}$-basis for $H_{2k}(Z)$. Now consider the map

$$T : H_4(Z) \to \pi_2\left(\bigvee_m S^2\right)$$

defined by assigning to each 4-cell its gluing map. This map must be onto, because $\pi_3(Z) = 0$ and must have no kernel because $\pi_4(Z) = 0$ and Z has now cells of dimension 5.

Since $\pi_3(Z) = \pi_4(Z) = 0$ there exists a unique map up to homotopy

$$g : X \to Z$$

which extends the identity on the the 2-skeletons. Let $[X] \in H_4(X) = \mathbb{Z}$ be the generator of the unique 4-cell and let $[\varphi] \in \pi_3(\bigvee_m S^2)$ be the homotopy class of $\varphi|_{\partial M}$. Then it is easy to see that

$$[\varphi] = T(g_*[X])$$

But since g induces the identity on the second homology, and since the 4'th cohomology of Z is spanned by cup products of second cohomology classes we see that the cup product determines $g_*[X]$ completely. Thus it determines $[\varphi]$, which determines the homotopy type of X.

□

It turns out that classification up to homeomorphism is very close to the homotopy classification. This topological classification is due mainly to M. Freedman who proved in 1982 the following theorem:

Theorem 2. *For any integral symmetric unimodular form Q there exists a closed oriented simply-connected topological 4-manifold that has Q as its intersection form.*

1. *If Q is even (i.e. $Q(v, v)$ is even for every v) then there exists exactly one such manifold.*
2. *If Q is odd, then there are exactly 2 such manifolds and at least one of them does not admit any smooth structure.*

So we see that in order to understand the topological category, we need to understand unimodular symmetric forms. For **indefinite** forms, we have the following classification of Serre:

Theorem 3. *Let Q be an indefinite symmetric unimodular form.*

1. *If Q is odd then it is isomorphic to*

$$m[1] + n[-1]$$

for some $m, n \neq 0$.

2. *If Q is even then it is isomorphic to*

$$m(E_8^{\pm}) + nH$$

where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and E_8 is a form on $\mathbb{Z}^8$ which appears in the root system of the Lie group E_8.

Unfortunately, no classification exists for definite forms, and in fact there is a huge amount of irreducible definite forms, even in relatively small ranks. Interestingly, this difficulty disappears in the smooth category. We have Donaldson theorem which appeared only one year after Freedman's article:

Theorem 4. *If a (smooth) standard 4-manifold has a definite intersection form, then that form is either $m[1]$ or $m[-1]$ for some m.*

Since we can take connected sums of $\mathbb{C}P^2$ and we can tuggle the orientation we see that we can actually obtain all these forms. The proof of Donaldson theorem makes use of notions which actually come from theoretical physics, and in particular 4-dimensional QFT and Yang-Mills theory.

Now suppose that M has at least one smooth structure. Can we classify all of its smooth structures? This gap is the most difficult to understand in dimension 4. In dimensions 0-3 every topological manifold has a unique smooth structure. In dimensions higher then 4 every topological manifold has at most finitely many non-diffeomorphic smooth structures, and the surgery theory we shall learn later on works pretty well.

In dimension 4, however, many simple topological manifolds admit infinitely many and even uncountably many smooth structures. In fact, no known 4-manifolds admit only finitely many smooth structure. In particular $\mathbb{R}^4$ admits uncountably many smooth structures.

5 Dimension 5 and up - Surgery Theory

5.1 Introduction

We now come to the general theory for studying manifolds in dimensions ≥ 5. As in dimension 4 we see that classifying all the manifolds is at least as hard as classifying finitely presented groups, so we shall concentrate on simply connected manifolds. The theory we shall describe here can be generalized to some extent to the non simply connected case, but we shall not address this issue here, so for us all manifolds from now on are simply connected.

The general framework for classifying standard manifolds is to compare the smooth category to the homotopy category. First note that each smooth manifold can be triangulated, so its homotopy type is a homotopy type of a CW-complex of the same dimension. Thus we have a forgetful functor from the category of smooth manifolds to the homotopy category of spaces which have the homotopy type of CW-complexes.

Suppose for a minute that the algebraic topologists have finished all their work and have a complete classification of all the homotopy types in the CW category. The gap to classifying standard manifolds can be described in two parts:

1. Classify CW homotopy types which are obtained as the homotopy types of standard manifolds.

2. For each CW homotopy type obtained in part 1, classify all standard manifolds with that homotopy type.

The theory which tries to attack the first question is called **obstruction theory**. The idea is to define invariants of homotopy types which need to vanish in order for that homotopy type to be that of a standard manifold. Such invariants are called **obstructions**. After developing the obstruction theory we shall use a boundary-relative version of it in order attack the second question.

5.2 Obstruction Theory

One of the characteristics of the homotopy type of a standard manifold is that its homology admits a fundamental class which induces Poincare duality. Thus if we want to take a CW-complex and ask whether it is homotopy equivalent to a standard manifold it needs to have such a fundamental class. In particular, we define:

Definition 2. *An n-dimensional **Poincare complex** is a finite n-dimensional connected CW-complex X together with a class $[X] \in H_n(X, \mathbb{Z})$, called the **fundamental class** of X, such that the cap product induces a $\mathbb{Z}[\pi_1(X)]$-module isomorphism*

$$\cap[X] : H^k(\widetilde{X}, \mathbb{Z}) \cong H_{n-k}(\widetilde{X}, \mathbb{Z})$$

where $\widetilde{X}$ is the universal cover of X.

In these notes we restrict ourselves to the simply connected case, so all Poincare complexes we shall consider will be simply connected. The question we now wish to ask is the following:

Question 1. *Given an n-dimensional Poincare complex X (for $n \geq 5$), when is it homotopy equivalent to a standard manifold?*

5.2.1 The K-Theory Obstruction

A fundamental invariant which smooth manifolds have and general topological spaces don't is the tangent bundle $TM \to M$. This bundle defines an element $[TM]$ in the stable (or reduced) K group $\widetilde{K_0}(M)$. Its inverse $[TM]^{-1}$ can be given a geometric meaning via embeddings of M in Eulidean spaces $\mathbb{R}^{n+k}$.

By a well know theorem every standard n-manifold can be embedded in a Euclidean space $i : M \to \mathbb{R}^{n+k}$ for some k. The Euclidean structure on $\mathbb{R}^{n+k}$ enables us to define the normal bundle ν on M. Since $TM \oplus \nu \cong i^*T\mathbb{R}^{n+k}$ we see that ν represents the iverse of $[TM]$ in $\widetilde{K_0}(M)$.

Suppose now that one has an n-dimensional Poincare complex X. If we had some homotopy equivalence

$$M \xrightarrow{\simeq} X$$

with M a standard n-manifold then it would induce an isomorphism

$$\widetilde{K}^0(M) \cong \widetilde{K}^0(X)$$

which would in particular identify some element in $\widetilde{K}^0(X)$ corresponding to $[TM]^{-1} \in \widetilde{K}^0(M)$. Thus our first step in order to find a homotopy equivalence $M \xrightarrow{\simeq} X$ is to identify such possible elements in $\widetilde{K}^0(X)$.

The idea is to try to mimic the construction of $[TM]^{-1}$ via embeddings. Every CW-complex X can be triangulated and thus can be embedded in some $\mathbb{R}^{n+k}$. Further more, we can find an embedding $f : X \hookrightarrow \mathbb{R}^{n+k}$ such that X is a retract of some neighborhood $f(X) \subseteq N$ which is an $n+k$-dimensional manifold with boundary ∂N.

In this scenario N is usually called a **tubular neighborhood** of X. If we restrict this retraction to $p : \partial N \to X$ we obtain a map with homotopy fiber S^k, i.e. a spherical fibration. It turns out that the class $[p]$ of this fibration in the stable spherical K-group $\widetilde{K}^0_s(X)$ is independent of the choice of embedding. It is called the **Spivak normal fibration** of X.

The group $\widetilde{K}^0_s(X)$ is defined as the group of isomorphism classes of orientable spherical fibrations with the join product operation modulu trivial fibrations. It can be defined via classifying spaces as follows: let $SG(k)$ be the group of orientation preserving self homotopy equivalences of S^k. Then we have functorial operation induced by the suspension map

$$\Sigma : [S^k, S^k] \to [\Sigma S^k, \Sigma S^k] =\!\!\longrightarrow [S^{k+1}, S^{k+1}]$$

which takes orientation preserving homotopy equivalences to orientation preserving homotopy equivalences, hence inducing a map $\Sigma : SG(k) \to SG(k+1)$. Taking the direct limit

$$SG = \lim_{k\to\infty} SG(k)$$

we obtain the structure group of spherical fibration. We then have

$$\widetilde{K}^0_s(X) = [X, BSG]$$

We have a natural operation which takes a k-dimensional vector bundle $E \to X$ and turns it into a spherical bundle $SE \to X$. Concretely we can construct it by putting a continuous riemannian metric on E and defining

$$SE = \{(x,v) | \|v\| = 1\}$$

The isomorphism class of the spherical fibration obtained is independent of the choice of metric. This induces a group homomorphism

$$S : \widetilde{K}^0(X) \to \widetilde{K}_s^0(X)$$

This map can also be obtained via the map of classifying spaces

$$BSO \to BSG$$

induced by the natural inclusion

$$SO \hookrightarrow SG$$

corresponding to the natural action of $SO(n)$ on S^n.

In the case of a standard manifold M and an embedding $i : M \hookrightarrow \mathbb{R}^{n+k}$ with tobular neighborhood N and normal bundle ν we can see that the spherical fibration $\partial N \to M$ can be identified with $S\nu$. This means that $[s] = S([TM]^{-1})$ which is a purely K-theoretical property of M hence depending only on the homotopy type of M.

We thus obtain the first obstruction for a Poincare complex to be a homotopy equivalent to a manifold: **its Spivak normal fibration must factor through some vector bundle**. This obstruction can be formulated as follows. We have a fibration of structure groups which are actually infinite loop spaces:

$$1 \to SO \to SG \to SG/SO \to 1$$

Thus SG/SO is also an infinite loop space and for each X we obtain a long exact sequence of abelian groups

$$[X, SO] \to [X, SG] \stackrel{\pi^X}{\to} [X \to SG/SO] \stackrel{\partial_0^X}{\to}$$

$$[X, BSO] \stackrel{S}{\to} [X, BSG] \stackrel{B\pi^X}{\to} [X \to B(SG/SO)] \stackrel{\partial_1^X}{\to}$$
$$[X, BBSO] \to ...$$

Note that $[X, BSO] = \widetilde{K}^0(X), [X, BSG] = \widetilde{K}_S^0(X)$ and the map between them is S. Thus we can write an exact sequence

$$0 \to coker(\pi^X) \stackrel{\partial_0^X}{\to} \widetilde{K}^0(X) \stackrel{S}{\to} \widetilde{K}_s^0(X) \stackrel{B\pi^X}{\to} ker(\partial_1^X) \to 0$$

We thus obtain a K-theoretical obstruction group $\ker(\partial_1^X)$ and an obstruction element $B\pi^X([s]) \in \ker(\partial_1^X)$ which needs to vanish in order for X to be homotopy equivalent to a manifold. This obstruction shall be called the K-obstruction. There exist examples X for which this obstruction can be calculated and proved to be non-zero, so this obstruction is not trivial.

A natural question now rises: is this the only obstruction to X being homotopy equivalent to a standard manifold?

5.2.2 Normal Maps

Suppose that we have an n-dimensional poincare complex X for which the K-obstruction vanishes. Let $p : E \to X$ be an (orientable) vector bundle such that $[SE]$ is the spivak normal fibration. We can define the disk bundle DE as either the mapping cylinder of $p|SE$ or by putting a continuous riemannian metric and defining

$$DE = \{(x, v) | \|v\| \leq 1\}$$

We then define the **Thom space** to be DE/SE. The point $[SE] \in Th(E)$ is denoted by ∞.

Now suppose we choose an orientation on E. We then have a prefered orientation on each fiber E_x which gives us a prefered generator to

$$H^k(DE_x, SE_x) \cong \mathbb{Z}$$

Define the **Thom class** of E to be the unique class $\omega \in H^k(DE, SE)$ which restrict to our prefered generator on each $H^k(DE_x, SE_x)$.

The fundamental property of Thom spaces is the **Thom isomrphism theorem**:

Theorem 5. *Let $p : E \to X$ be an oriented k-dimensional vector bundle over a finite connected CW complex, with Thom class $\omega \in H^k(Th(E))$. Then the cap product induces isomorphisms*

$$\omega\cap : H_{m+k}(DE, SE) \cong H_m(DE)$$

for each $m \geq 0$.

This theorem is true for more general base spaces X, but we will not need this here. Note that we have an inclusion $X \hookrightarrow DE$ as the zero section, and this inclusion is a homotopy equivalence. Furhter more we can identify $H_{m+k}(DE, SE)$ with the reduced homology $\widetilde{H}_{m+k}(Th(E))$. Thus we can write this isomorphism as

$$H_m(X) \cong \widetilde{H}_{m+k}(Th(E)) \cong H_{m+k}(Th(E))$$

where the last equality is because we assume $k > 0$. Note in particular that a choice of a fundamental class $[X] \in H_n(X)$ and a choice of an orientation on E forces a choice of a fundamental class $[Th(E)] \in H_{n+k}(Th(E))$.

Now consider an element $[\varphi] \in \pi_{n+k}(Th(E), \infty)$ represented by a map $\varphi : S^{n+k} \to Th(E)$ which is transversal at X, i.e. we can locally trivialize the bundle so that after projecting φ to the fiber $D\mathbb{R}^k$ we get a submersion.

This transversality condition implies that the preimage of the zero section $\varphi^{-1}(X) \subseteq S^{n+k}$ is an n-dimensional manifold M, so we get a map

$$\varphi|_M = f : M \to X$$

The pullback f^*E of E can be identified with the normal bundle ν of the inclusion $M \hookrightarrow S^{n+k}$ under the standard riemannian metric on S^{n+k}. Thus the normal bundle is orientable, which means that M inherits an orientation $[M] \in H_n(M)$ from the standard orientation of S^{n+k}. Thus M is a standard manifold, and we have a bundle map

$$\begin{array}{ccc} \nu & \xrightarrow{\overline{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

which is an isomorphism on fibers. Such objects are called **Normal maps** with respect to E. We can also define cobordisms of such objects $(M_1, \nu_1, f, \overline{f})$, $(M_2, \nu_1, f, \overline{f})$ by taking some normal bundle ν of rank k on a cobordism $W : M_1 \sim M_2$ which extends ν_1, ν_2.

$$\begin{array}{ccc} \nu & \xrightarrow{\overline{f}} & E \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & X \end{array}$$

It turns out that the image $f_*[M]$ is exactly the image of $\varphi_*[S^{n+k}]$ under the Thom isomorphism. Since we want f to be as close as possible to a homotopy equivalence, it is natural to want that $f_*[M] = [X]$. This can be achieved in following way:

Recall that the Spivak fibration was obtained via an embedding $i : X \hookrightarrow \mathbb{R}^{n+k}$ and a tubular neighborhood $i(X) \subseteq N$ such that $s : \partial N \to X$ was homotopy equivalent to our fibration. It is easy to see that the retraction $N \to X$ is the associated disk bundle of s and thus $N/\partial N$ is homotopy equivalent to $Th(s)$.

Construct a collapse map $c : \mathbb{R}^{n+k} \to Th(s)$ by extending the quotient map $N \to N/\partial N$ to $\mathbb{R}^{n+k}$ such that all the points outside n are mapped to $[\partial N]$. Extend c so $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\}$ by sending ∞ to $[\partial N]$ as well.

It can be shown that the resulting element $[c] \in \pi_{n+k}(Th(s))$ satisfies the requirement that the image of $c^*[S^{n+k}]$ under the Thom isomorphism is exactly $[X]$. Thus we can use $[c]$ in order to construct a normal map $f : M \to X$ satisfying $f_*[M] = [X]$. Such maps are called normal maps of degree 1.

In fact, it turns out that the existence of such $[c] \in \pi_{n+k}(Th(s))$ characterizes the Spivak fibration. Thus every degree 1 map from a manifold to X can be obtained in this way.

5.2.3 The L-Theory Obstruction

Now suppose that we have a normal map of degree 1

$$\begin{array}{ccc} \nu & \xrightarrow{\overline{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

We wish to know whether or not its cobordism class contains a homotopy equivalence. Just like regular cobordism, here we can also produce each normal cobordism by a finite sequence of **normal surgery steps**. A k-dimensional normal surgery step on a normal map $(M, \nu, f, \overline{f})$ is constructed from a commutative diagram

$$\begin{array}{ccc} S^k \times D^{n-k} & \to & D^{k+1} \times D^{n-k} \\ q \downarrow & & \downarrow Q \\ M & \xrightarrow{f} & X \end{array}$$

such that the left vertical arrow is an embedding. Let

$$M' = M \backslash Int(q(S^k \times D^{n-k}))$$

and

$$N = M' \coprod_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k}$$

the manifold obtained from M via regular surgery on q. The map Q enables us to extend the map f from M' to whole of N, and it can be showed that this map is also of degree 1.

We now wish to extend the normal bundle ν from M' to N. Recall that ν was given as the normal bundle of some embedding $i : M \hookrightarrow \mathbb{R}^{n+m}$, so we wish to extend this embedding to N by maybe extending the range from $\mathbb{R}^{n+m}$ to $\mathbb{R}^{m+n+1}$ via the standard inclusion.

Note that Q gives a trivialization of $q^*\nu$ which is unique up to homotopy (i.e. determines an element in $\pi_k(SO(m))$. Let ω be the normal bundle to $q(S^k)$ in M. The the extension of $q|_{S^k}$ to $S^k \times D^{n-k}$ gives a trivialization of ω, again unique up to homotopy (i.e determines an element in $\pi_k(SO(n-k))$). These two trivializations determine a trivialization of $q^*\nu \oplus \omega$, which can be identified with the normal bundle to $i(S^k)$ in $\mathbb{R}^{n+m}$. Let us call this trivialization $x \in \pi_n(SO(m+n-k))$.

By using some standard embedding technics we can show that since this bundle is trivial we can extend the embedding $i : M' \hookrightarrow \mathbb{R}^{n+m}$ to N. But this extension also gives a trivialization of $q^*\nu \oplus \omega$ - the standard trivialization $1 \in \pi_k(SO(n+m-k))$.

thus we see that in order to be able to extend the bundle map $\overline{f}$ to ν over N we need that the element x from before will be trivial. This is a sufficient condition as well.

How can we use these normal surgery steps in order to make a normal map $f : M \to X$ into a homotopy equivalence? Since both these spaces have the homotopy types of CW-complexes, whitehead's theorem tells us it is enough to make into a weak equivalence.

Define $\pi_k(f)$ to be the group of homotopy classes of commutative diagrams of pointed maps

$$\begin{array}{ccc} S^k & \xrightarrow{i} & D^{k+1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

Then we have a long exact sequence of homotopy groups:

$$... \to \pi_k(f) \to \pi_k(M) \to \pi_k(X) \to \pi_{k-1}(f) \to ...$$

Suppose that f is $(k-1)$-connected, i.e. $\pi_m(f) = 0$ for $m = 0, ..., k-1$. What we need to do in order to make f k-connected is to kill the elements in $\pi_k(f)$ via normal surgery. This is done in the following way. Let $x \in \pi_k(f)$ be represented by the diagram:

$$\begin{array}{ccc} S^k & \to & D^{k+1} \\ q \downarrow & & \downarrow Q \\ M & \xrightarrow{f} & X \end{array}$$

The first step is homotope q into an **embedding**

$$q : S^k \to M$$

For $k = 0, 1$ this can be done "by hand". For $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ we have the **Whitney embedding theorem**:

Theorem 6. *Let $f : M^m \to N^n$ be a map of smooth manifolds satisfying the following assumptions: $2m \leq n$, $n - m > 2$, M connected and N simply-connected. Then f is homotopic to a smooth embedding.*

The second step is to extend q to an embedding:

$$\widetilde{q} : S^k \times D^{n-k} \to M$$

Here we come across an obstruction. In order for this to be done we need the normal bundle of $q(S^k)$ in M to be trivial. This obstruction lies in $\pi_{k-1}(SO(n-k))$ which classifies $(n-k)$-bundles on S^k. Note that we can choose a trivialization and each trivialization will give a different extension $\widetilde{q}$.

The third step is to check that all the trivializations are consistent, as mentioned before. It turns out that we can pack these second and third obstructions into a single obstruction group $\pi_k(SO(m+n-k)/SO(m))$ which lies in the long exact sequence:

$$... \to \pi_l(SO(m)) \to \pi_l(SO(m+n-k)) \to \pi_l(SO(m+n-k)/SO(m)) \xrightarrow{\partial}$$

$$\pi_{l-1}(SO(n-k)) \to ...$$

and the image of our obstruction under

$$\partial : \pi_k(SO(m+n-k)/SO(m)) \to \pi_{k-1}(SO(n-k))$$

is the classifying map of the normal bundle to $q(S^k)$.

It turns out that

Theorem 7. *1. If $k < n - k$ then $\pi_k(SO(m+n-k)/SO(m)) = 0$.*

2. *If $k = n-k$ and k is even then $\pi_k(SO(m+n-k)/SO(m)) = \mathbb{Z}$*

3. *If $k = n-k$ and k is odd then $\pi_k(SO(m+n-k)/SO(m)) = \mathbb{Z}_2$*

Thus as long as $k < \lfloor \frac{n}{2} \rfloor$ (and when n is odd, even when $k = \lfloor \frac{n}{2} \rfloor$) we can perform the surgery step on elements in $\pi_k(f)$. Note that doing surgery to kill elements in $\pi_k(f)$ may create non-trivial elements in $\pi_m(f)$ for $m \geq n-k-1$, but can't effect $\pi_m(f)$ for $m < n-k-1$. Thus we have the following:

Theorem 8. *Let $(M, \mu, f, \overline{f})$ be a normal map from M to X. Then it is normally bordand to a map $(N, \nu, g, \overline{g})$, such that $\pi_k(g) = 0$ for $k = 0, ..., \lfloor \frac{n}{2} \rfloor - 1$.*

We now arrive at the middle dimensions. If we succeed in doing surgery here, then from poincare duality and Hurewicz's theorem we would get that f is a weak equivalence (and thus a homotopy equivalence since we are dealing with homotopy types of CW complexes).

The behaviour at the middle dimensions depends on the value of n mod 4:

1. If n is odd, then we can extend the surgery to the middle dimension, i.e. we don't have any further obstructions.

2. If $n = 2$ mod 4 then we have an obstruction which lies in $\mathbb{Z}/2$. It is called the **Arf invariant** of the normal map.

3. If $n = 0$ mod 4 then we have an obstruction which lies in $\mathbb{Z}$. It is called the **signature** of the normal map (because it actually comes from a signature of quadratic form).

In this note we shall explain and give a sketch of proof for the case of $n = 0$ mod 4. The other cases can be read in any standard textbook on the subject.

So suppose now that $n = 4m$. Let $f : M \to X$ be a $2m-1$-connected normal degree 1 map, with normal bundle ν of rank r. Then it can be shown that

$$\pi_{2m}(f) \cong K_{2m}(f) \overset{def}{=} \ker(f_* : H_{2m}(M) \to H_{2m}(X))$$

is a free $\mathbb{Z}$-module. Since we have poincare dualities

$$\alpha_M : [M] \cap : H^{2m}(M) \cong H_{2m}(M)$$

$$\alpha_X : [X] \cap : H^{2m}(X) \cong H_{2m}(X)$$

which are preserved by f (because $f_*[M] = [X]$) we see that

$$f_* \circ \alpha_M \circ f^* = Id$$

which means that the maps f_* and f^* split, i.e. $H_{2m}(X)$ and $H^{2m}(X)$ are direct summands in $H_{2m}(M)$ and $H^{2m}(M)$ respectively. In particular, we have natural identifications

$$H_{2m}(M) \cong H_{2m}(X) \oplus \pi_{2m}(f)$$

$$H^{2m}(M) \cong H^{2m}(X) \oplus \pi^*_{2m}(f)$$

and the last decomposition respects the cup product, i.e. it gives a decomposition of Q into a direct sum of two forms: $Q(M) = Q(X) \oplus Q(f)$ where $Q(f)$ is the form induced by the cup product on $\pi^*_{2m}(f)$. Then $Q(f)$ must be unimodular as well, hence inducing an isomorphic form on $\pi_{2m}(f)$. Let $\sigma(f)$ denote the signature of that form.

We wish to show that for $m > 1$, a $2m-1$-connected normal map $(M, \nu, f, \overline{f})$ of degree 1 is normally bordant to a $2m$-connected normal map (and hence to a homotopy equivalence) if and only if $\sigma(f) = 0$.

This is not very surprising to people who are familiar with cobordism theory, because it is known that for a standard manifold of dimension $4m$, the obstruction to being null-cobordant is the signature of its intersection form Q. The proof of these two statements is analogous and relies on the following three facts:

1. If Q is an indefinite form (over $\mathbb{Z}$), then there exists an element x with $(x, x) = 0$.

2. If $x \in \pi_{2m}(f)$ satisfies $(x, x) = 0$ then one can do surgery in dimension $2m$ that kills (at least) the element x and preserves the lower homotopy groups of f.

3. Two normally cobordant normal maps of degree 1 have the same signature.

From these fact the theorem follows easily: a non-trivial form of signature 0 has to be indefinite, so it has an element $x \in \pi_{2m}(f)$ such that $(x, x) = 0$. From fact 2 we can do surgery to kill this element without ruining anything else. From fact 3 the resulting manifold still has signature 0 so we can proceed by induction. Further more from fact 3 we see that if the signature is not 0, then our map can't be normally cobordant to a homotopy equivalence.

We shall only sketch the proof of fact 2:

Let $x \in \pi_{2m}(f)$ be an element represented by a diagram:

$$\begin{array}{ccc} S^{2m} & \rightarrow & D^{2m+1} \\ q \downarrow & & \downarrow Q \\ M & \xrightarrow{f} & X \end{array}$$

where q is an embedding (this is possible from Whitney's theorem, and the assumption that $m > 1$). Let $o \in \pi_{2m}(SO(2m+r)/SO(2m)) \cong \mathbb{Z}$ be the obstruction to surgery on q, as discussed above. We wish to show that o is trivial iff $(x, x) = 0$. Consider the boundary map

$$\partial : \pi_{2m}(SO(2m+r)/SO(2m)) \rightarrow \pi_{2m-1}(SO(2m))$$

It can be shown by analizing the long exact sequence that in these circumstances ∂ is an isomorphism, and thus $o = 0$ iff $\partial o = 0$. Recall from before that ∂o can be thought of as the classifying map of the normal bundle to $q(S^{2m})$ in

M. Thus in dimensions $n = 4m$ all the obstruction is the non-triviality of the normal bundle.

Oriented $2m$-bundles $E \to S^{2m}$ are classified by their Euler class

$$\chi(E) \in H^{2m}(S^{2m}) \cong \mathbb{Z}$$

The Euler class of an oriented rank k bundle $E \to Y$ is defined as follows: consider the Thom space $Th(E)$. The orientation of E gives us a Thom class $u \in H^k(Th(E))$. The Euler class is defined as

$$i^*u \in H^k(Y)$$

where $i : Y \to Th(E)$ is the natural inclusion.

Now suppose that $q : Y \hookrightarrow M$ is an embedding of an oriented manifold Y of dimension k in a manifold M of dimension $2k$. Let ν be the normal bundle and N a tubular neighborhood. Then ν and N inherit natural orientations and we have as before $N/\partial N \cong Th(\nu)$ orientedly with natural fundamental class $[Th(E)] \in H_{2k}(Th(E))$.

We have the Thom class $u \in H^k(Th(E))$ which satisfies

$$[Th(E)] \cap u = i_*[Y]$$

In other words, u is the Poincare dual of $i_*[Y]$. Thus we have

$$[Y] \cdot \chi(\nu) = [Y] \cdot i^*u = i_*[Y] \cdot u = [Th(E)] \cdot (u \cup u)$$

We have a collapse map $c : M \to Th(\nu)$ defined by extending the quotient map $N \to Th(E)$ by a constant map which sends all the points in $M \backslash N$ to $\infty = [\partial N]$. This collapse map satisfies $c_*[M] = [Th(E)]$ and thus preserves poincare duality. Further more we can see from the definitions that

$$i_*[Y] = c_*(q_*[Y])$$

Thus since u is the Poincare dual of $c_*(q_*[Y])$, we see that c^*u is the Poincare dual of $q_*[Y]$. To finish the argument we note that

$$(q_*[Y], q_*[Y]) = [M] \cdot (c^*u \cup c^*u) = [Th(E)] \cdot (u \cup u) = [Y] \cdot \chi(\nu)$$

which shows that the triviality of the self intersection of $q_*[Y]$ is equivalent to the vanishing of the Euler class of the normal bundle. This finishes the sketch of the proof.

5.3 The Surgery Exact Sequence

We now wish to address the second classification issue. Given a standard manifold X of dimension n, how can we classify up to oriented diffeomorphism the standard n-manifolds which are homotopy equivalent to X?

As we saw above, the set of normal maps of rank r is a principle homogenous space of the group

$$[X, SG(r)/SO(r)]$$

If we mod out normal maps with trivial bundles we get a principle homogenous space of
$$[X, SG/SO]$$

Let L_n denote the L-obstruction groups described in the last section, i.e.:

1. $L_n = 0$, For n odd.
2. $L_n = \mathbb{Z}_2$, For $n = 2 \bmod 4$.
3. $L_n = \mathbb{Z}$, For $n = 0 \bmod 4$.

Now let $S_n(X)$ denote the set of oriented homotopy equivalence $f : M \to X$ (where M is a standard n-manifold) up to conjugation by orientated diffeomorphisms. Note that $S_n(X)$ has a prefered element, which is the identity $Id : X \to X$.

For each normal bundle ν on X, i.e.
$$[\nu] = [TX]^{-1} \in \widetilde{K}^0(X)$$
we can pull it back to M and obtain a normal map $(M, f, \overline{f}, f^*\nu)$. This normal map is well defined up to cobordism and stabilization. This gives us a preferred element equivalence class of normal maps which is generated by the identity $X \to X$. Thus we can identify the set of equivalence classes of normal maps with $[X, SG/SO]$, this construction gives us a map
$$T_n : S_n(X) \to [X, SG/SO]$$
which sends $Id \in S_n(X)$ to the neutral element in $[X, SG/SO]$. We saw before that we can associate with each normal map an element in the obstruction group L_n. This gives us an obstruction map
$$\sigma_n : [X, SG/SO] \to L_n$$
This is a map of pointed sets, but not necessarily a homomorphism of groups. The obstruction theory of the previous section asserts that the sequence
$$S_n(X) \overset{T_n}{\to} [X, SG/SO] \overset{\sigma_n}{\to} L_n$$
of pointed sets is **exact**.

The main theorem of this section is the **surgery exact sequence**:

Theorem 9. *The exact sequence above can be extended to*
$$[\Sigma X, SG/SO] \overset{\sigma_{n+1}}{\to} L_{n+1} \overset{\partial}{\to} S_n(X) \overset{T_n}{\to} [X, SG/SO] \overset{\sigma_n}{\to} L_n$$

The space $[\Sigma X, SG/SO]$ can be identified with stable cobordism classes of normal maps
$$F : (M, \partial M) \to (X \times I, X \times \{0, 1\})$$

Defined in a boundary relative way, and σ_{n-1} is the L-obstruction of this normal map. The reason that we have a well defined map

$$L_n \xrightarrow{\partial} S_n(X)$$

Is the celebrated h-cobordism theorem:

Theorem 10. *Let M, N be two simply connected standard manifolds of dimension $n \geq 5$. Let W be a cobordism between them such that the boundary inclusions $M \hookrightarrow W, N \hookrightarrow W$ are deformation retracts. Then M is orientedly diffeomorphic to N and W oriently diffeomorphic to $M \times I$.*

5.4 Smooth Structures on Spheres

Spheres have the property that we have a canonical abelian group structure on $\theta_n = S_n(S^n)$ given by connected sum. Further more we have $S^{n+1} = \Sigma S^n$. This enables us to turn the surgery exact sequence into a **long exact sequence of abelian groups**:

$$... \to \Theta_n \xrightarrow{T_n} \pi_n(SG/SO) \xrightarrow{\sigma_n} L_n \xrightarrow{\partial}$$

$$\Theta_{n-1} \xrightarrow{T_{n-1}} \pi_{n-1}(SG/SO) \xrightarrow{\sigma_{n-1}} L_{n-1} \to ... \to L_5$$

Thus in order to calculate θ_n we need to understand the homotopy groups of SG/SO and the maps T_n and σ_n.

This strategy was used by milnor in order to calculate θ_n in the sixties. In order to understand the homoropy groups of SG/SO we need to look at the long exact sequence

$$... \to \pi_n(SO) \to \pi_n(SG) \to \pi_n(SG/SO) \to \pi_{n-1}(SO) \to ...$$

Theorem 11. *The homotopy group $\pi_n(SG)$ is isomorphic to the stable homotopy group*

$$\pi_n^s = \lim_{k \to \infty} \pi_{n+k}(S^k)$$

Proof. This isomorphism is given by the following sequence of maps:

$$[S^n, SG(k)] \to [S^n, Map(S^k, S^k)] = [S^n \times S^k, S^k] \xrightarrow{h}$$

$$[S^n * S^k, \Sigma S^k] = [S^{n+k+1}, S^{k+1}]$$

The map h is given by first extending a map from $S^n \times S^k$ to S^k to a map from $S^n \times S^k \times I$ to $S^k \times I$ and then collapsing it. □

The first stable homotopy groups are:

0	1	2	3	4	5	6	7	8
$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

The homotopy groups of SO are 8-periodic and completely known. They are:

0	1	2	3	4	5	6	7
$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

The isomorphism of theorem 10 gives us a map

$$J_n : \pi_n(SO) \to \pi_n^s$$

called the J-homomorphism. In order to understand $\pi_n(SG/SO)$ we need to understand its kernel and cokernel. We have a deep theorem by Adams which states:

Theorem 12. *1. If $n \neq 3 \bmod 4$ then J_n is injective.*

*2. If $n = 3 \bmod 4$ then the order of the **image** of J_n is Denominator($B_k/4k$) where $n = 4k-1$ and B_k is the k'th Bernoulli number.*

Now suppose we wish to calculate Θ_7. The Bernoulli number $B_2 = \frac{1}{30}$. From Adams's theorem and the tables above we see that J_7 is surjective and J_6 is injective, thus $\pi_7(SG/SO) = 0$. Since $L_8 = \mathbb{Z}$ we see that Θ_7 is cyclic, and all is left is to calculate its order.

The general formula for the order of $\partial_{4k}(L_{4k})$ is:

$$\frac{2^{2k}(2^{2k-1}-1)B_k \cdot (3-(-1)^k)}{32k} \cdot Denominator(B_k/4k) =$$

$$2^{2k-3}(2^{2k-1}-1) \cdot (3-(-1)^k) \cdot Nominator(B_k/4k)$$