The Cobordism Hypothesis in Dimension 1

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1 Introduction

Let $\mathcal{B}_1^{\mathrm{or}}$ denote the 1-dimensional oriented cobordism ∞-category, i.e. the symmetric monoidal ∞-category whose objects are oriented 0-dimensional closed manifolds and whose morphisms are oriented 1-dimensional cobordisms between them.

The category $\mathcal{B}_1^{\mathrm{or}}$ carries a fundamental inner symmetry expressing the fact that **cobordisms can be read in both directions**. This can be described formally in the language of **duality** as follows. Let $\mathcal{C}$ be an (ordinary) monoidal category and let X be an object. We say that and object $Y \in \mathcal{C}$ is **right dual** to X if there exist morphisms $\mathrm{ev} : X \otimes Y \longrightarrow 1$ and $\mathrm{coev} : 1 \longrightarrow Y \otimes X$ such that the compositions

$$X \stackrel{Id \otimes \mathrm{coev}}{\longrightarrow} X \otimes Y \otimes X \stackrel{\mathrm{ev} \otimes Id}{\longrightarrow} X$$

and

$$Y \stackrel{\mathrm{coev} \otimes Id}{\longrightarrow} Y \otimes X \otimes Y \stackrel{Id \otimes \mathrm{ev}}{\longrightarrow} Y$$

are the identity. In this case we also say that X is left dual to Y. If $\mathcal{C}$ is symmetric then this definitions become symmetric and we just say that X and Y are duals. If $\mathcal{C}$ is a symmetric monoidal ∞-category then we say that X and Y are duals if they are duals in the homotopy category $\mathrm{Ho}(\mathcal{C})$.

Example:

1. Let C be the symmetric monoidal category of finite dimensional vector spaces over $\mathbb{C}$ with monoidal product given by tensor product. Then for each $V \in C$ the object $\check{V} = \mathrm{Hom}(V, \mathbb{C})$ is dual to V. The evaluation map $V \otimes \check{V} \longrightarrow \mathbb{C}$ is clear and the coevaluation $\mathbb{C} \longrightarrow \check{V} \otimes V \cong \mathrm{End}(V)$ is given by sending $1 \in \mathbb{C}$ to the identity $I \in \mathrm{End}(V)$.

2. Let $X_+, X_- \in \mathcal{B}_1^{\mathrm{or}}$ be the points with positive and negative orientations respectively. Then X_+ and X_- are duals in $\mathcal{B}_1^{\mathrm{or}}$. The evaluation map is the "right-half-circle" cobordism

$$\mathrm{ev} : X_+ \otimes X_- \longrightarrow \emptyset$$

and the coevaluation is the "left-half-curcle" cobordism

$$\mathrm{coev} : \emptyset \longrightarrow X_+ \otimes X_-$$

Definition 1.1. We say that a symmetric monoidal ∞-category has duals if every object has a dual.

We now observe the following simple lemma

Lemma 1.2. *Let D, C be two symmetric monoidal (ordinary) categories with duals, $F, G : D \longrightarrow C$ two symmetric monoidal functors and $T : F \longrightarrow G$ a natural transformation. Then T is a natural isomorphism.*

Now let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals. The 1-dimensional cobordism hypothesis concerns the ∞-category

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$$

of symmetric monoidal functors $\varphi : \mathcal{B}_1^{\mathrm{or}} \longrightarrow \mathcal{D}$. If $X_+ \in \mathcal{B}_1^{\mathrm{or}}$ is the object corresponding to a point with positive orientation then the evaluation map $Z \mapsto Z(X_+)$ induces a functor

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D}) \longrightarrow \mathcal{D}$$

From Lemma 1.2 we see that the ∞-category $\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$ is in fact an ∞-groupoid. This means that the evaluation map $Z \mapsto Z(X_+)$ actually factors through a map

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D}) \longrightarrow \widetilde{\mathcal{D}}$$

where $\widetilde{\mathcal{D}}$ is the maximal ∞-groupoid of $\mathcal{D}$. The cobordism hypothesis then states

Theorem 1.3. *The evaluation map*

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D}) \longrightarrow \widetilde{\mathcal{D}}$$

is an equivalence of ∞-groupoids.

Remark 1.4. From the consideration above we see that we could have written the cobordism hypothesis as an equivalence

$$\widetilde{\mathrm{Fun}}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D}) \xrightarrow{\simeq} \widetilde{\mathcal{D}}$$

where $\widetilde{\mathrm{Fun}}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$ is the maximal ∞-groupoid of $\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$ (which in this case happens to coincide with $\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$). This ∞-groupoid is the fundamental groupoid of the space of maps from $\mathcal{B}_1^{\mathrm{or}}$ to $\mathcal{D}$ in the ∞-category $\mathrm{Cat}^{\otimes}$ of symmetric monoidal ∞-categories.

In his paper [Lur1] Lurie gives an elaborate sketch of proof for a higher dimensional generalization of the 1-dimensional cobordism hypothesis. For this one needs to generalize the notion of ∞-categories to (∞, n)-categories. The strategy of proof described in [Lur1] is inductive in nature. In particular in order to understand the $n = 1$ case, one should start by considering the $n = 0$ case.

Let $\mathcal{B}_0^{\mathrm{un}}$ be the 0-dimensional unoriented cobordism category, i.e. the objects of $\mathcal{B}_0^{\mathrm{un}}$ are 0-dimensional closed manifolds (or equivalently, finite sets) and the morphisms are diffeomorphisms (or equivalently, isomorphisms of finite sets). Note that $\mathcal{B}_0^{\mathrm{un}}$ is a (discrete) ∞-groupoid.

Let $X \in \mathcal{B}_0^{\mathrm{un}}$ be the object corresponding to one point. Then the 0-dimensional cobordism hypothesis states that $\mathcal{B}_0^{\mathrm{un}}$ is in fact the free ∞-groupoid (or $(\infty, 0)$-category) on one object, i.e. if $\mathcal{G}$ is any other ∞-groupoid then the evaluation map $Z \mapsto Z(X)$ induces an equivalence of ∞-groupoids

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_0^{\mathrm{un}}, \mathcal{G}) \xrightarrow{\simeq} \mathcal{G}$$

Remark 1.5. At this point one can wonder what is the justification for considering non-oriented manifolds in the $n = 0$ case oriented ones in the $n = 1$ case. As is explained in [Lur1] the desired notion when working in the n-dimensional cobordism (∞, n)-category is that of **n-framed** manifolds. One then observes that 0-framed 0-manifolds are unoriented manifolds, while taking 1-framed 1-manifolds (and 1-framed 0-manifolds) is equivalent to taking the respective manifolds with orientation.

Now the 0-dimensional cobordism hypothesis is not hard to verify. In fact, it holds in a slightly more general context - we do not have to assume that $\mathcal{G}$ is an ∞-groupoid. In fact, if $\mathcal{G}$ is **any symmetric monoidal ∞-category** then the evaluation map induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_0^{\mathrm{un}}, \mathcal{G}) \xrightarrow{\simeq} \mathcal{G}$$

and hence also an equivalence of ∞-groupoids

$$\widetilde{\mathrm{Fun}}^{\otimes}(\mathcal{B}_0^{\mathrm{un}}, \mathcal{G}) \xrightarrow{\simeq} \widetilde{\mathcal{G}}$$

Now consider the under-category $\mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$ of symmetric monoidal ∞-categories $\mathcal{D}$ equipped with a functor $\mathcal{B}_0^{\mathrm{un}} \longrightarrow \mathcal{D}$. Since $\mathcal{B}_0^{\mathrm{un}}$ is free on one generator this category can be identified with the ∞-category of **pointed** symmetric monoidal ∞-categories, i.e. symmetric monoidal ∞-categories with a chosen object. We will often not distinguish between these two notions.

Now the point of positive orientation $X_+ \in \mathcal{B}_1^{\text{or}}$ determines a functor $\mathcal{B}_0^{\text{un}} \longrightarrow \mathcal{B}_1^{\text{or}}$, i.e. an object in $\text{Cat}^{\otimes}_{\mathcal{B}_0^{\text{un}}/}$, which we shall denote by $\mathcal{B}_1^+$. The 1-dimensional coborodism hypothesis is then equivalent to the following statement:

Theorem 1.6. *[Cobordism Hypothesis* 0*-to-*1*] Let $\mathcal{D} \in \text{Cat}^{\otimes}_{\mathcal{B}_0^{\text{un}}/}$ be a pointed symmetric monoidal ∞-category with duals. Then the ∞-groupoid*

$$\widetilde{\text{Fun}}^{\otimes}_{\mathcal{B}_0^{\text{un}}/}(\mathcal{B}_1^+, \mathcal{D})$$

is ***contractible****.*

Theorem 1.6 can be considered as the inductive step from the 0-dimensional cobordism hypothesis to the 1-dimensional one. Now the strategy outlines in [Lur1] proceeds to bridge the gap between $\mathcal{B}_0^{\text{un}}$ to $\mathcal{B}_1^{\text{or}}$ by considering an intermediate ∞-category

$$\mathcal{B}_0^{\text{un}} \hookrightarrow \mathcal{B}_1^{\text{ev}} \hookrightarrow \mathcal{B}_1^{\text{or}}$$

This intermediate ∞-category is defined in [Lur1] in terms of framed functions and index restriction. However in the 1-dimensional case one can describe it without going into the theory of framed functors. In particular we will use the following definition:

Definition 1.7. Let $\iota : \mathcal{B}_1^{\text{ev}} \hookrightarrow \mathcal{B}_1^{\text{or}}$ be the subcategory containing all objects and only the cobordisms M in which every connected component $M_0 \subseteq M$ is either an identity segment or an evaluation segment (i.e. a "right-half-circle" as above).

Let us now describe how to bridge the gap between $\mathcal{B}_0^{\text{un}}$ and $\mathcal{B}_1^{\text{ev}}$. Let $\mathcal{D}$ be an ∞-category with duals and let

$$\varphi : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{D}$$

be a symmetric monoidal functor. We will say that φ is **non-degenerate** if for each $X \in \mathcal{B}_1^{\text{ev}}$ the map

$$\varphi\left(\text{ev}_X\right) : \varphi(X) \otimes \varphi\left(\check{X}\right) \simeq \varphi\left(X \otimes \check{X}\right) \longrightarrow \varphi(1) \simeq 1$$

is **non-degenerate**, i.e. identifies $\varphi\left(\check{X}\right)$ with a dual of $\varphi(X)$. We will denote by

$$\text{Cat}^{\text{nd}}_{\mathcal{B}_1^{\text{ev}}/} \subseteq \text{Cat}^{\otimes}_{\mathcal{B}_1^{\text{ev}}/}$$

the full subcategory spanned by objects $\varphi : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{D}$ such that $\mathcal{D}$ has duals and φ is non-degenerate.

Let $X_+ \in \mathcal{B}_1^{\text{ev}}$ be the point with positive orientation. Then X_+ determines a functor

$$\mathcal{B}_0^{\text{un}} \longrightarrow \mathcal{B}_1^{\text{ev}}$$

The restriction map $\varphi \mapsto \varphi|_{\mathcal{B}_0^{\mathrm{un}}}$ then induces a functor

$$\mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/} \longrightarrow \mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$$

Now the gap between $\mathcal{B}_1^{\mathrm{ev}}$ and $\mathcal{B}_0^{\mathrm{un}}$ can be climbed using the following lemma (see [Lur1]):

Lemma 1.8. *The functor*

$$\mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/} \longrightarrow \mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$$

is fully faithful. Its essential image consists of points symmetric monoidal ∞-categories in which the pointed object admits a dual.

Now consider the natural inclusion $\iota : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{B}_1^{\mathrm{or}}$ as an object in $\mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/}$. Then by Lemma 1.8 we see that the 1-dimensional cobordism hypothesis will be established once we make the following last step:

Theorem 1.9 (Cobordism Hypothesis - Last Step)**.** *Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals and let $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ be a* ***non-degenerate*** *functor. Then the ∞-groupoid*

$$\widetilde{\mathrm{Fun}}^{\otimes}_{\mathcal{B}_1^{\mathrm{ev}}/}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$$

is contractible.

Note that since $\mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{B}_1^{\mathrm{or}}$ is essentially surjective all the functors in

$$\widetilde{\mathrm{Fun}}^{\otimes}_{\mathcal{B}_1^{\mathrm{ev}}/}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$$

will have the same essential image of φ. Hence it will be enough to prove for the claim for the case where $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ is **essentially surjective**. We will denote by

$$\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/} \subseteq \mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/}$$

the full subcategory spanned by essentially surjective functors $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$. Hence we can phrase Theorem 1.9 as follows:

Theorem 1.10 (Cobordism Hypothesis - Last Step 2)**.** *Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals and let $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ be an* ***essentially surjective non-degenerate*** *functor. Then the space of maps*

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/}}(\iota, \varphi)$$

is contractible.

2 Non-degenerate Fiber Functors

Let $\mathcal{D}$ be a symmetric monoidal ∞-categories with duals. The fact of having duals forces a strong symmetry on $\mathcal{D}$. This means, in some sense, that the information in $\mathcal{D}$ is packed with extreme **redundancy**. For example, given two objects $X, Y \in \mathcal{D}$ (with corresponding dual objects $\check{X}, \check{Y}$) all the mapping spaces

$$\mathrm{Map}_{\mathcal{D}}\left(1, \check{X} \otimes Y\right) \simeq \mathrm{Map}_{\mathcal{D}}(X, Y) \simeq \mathrm{Map}_{\mathcal{D}}\left(X \otimes \check{Y}, 1\right) \simeq \mathrm{Map}_{\mathcal{D}}\left(\check{Y}, \check{X}\right)$$

are equivalent. For example, in the case of $\mathcal{B}_1^{\mathrm{or}}$ all these spaces can be identified with the classifying space of oriented 1-manifolds M together with an identification $\partial M \simeq \check{X} \otimes Y$. This observation leads one to try to pack the information of $\mathcal{B}_1^{\mathrm{or}}$ (or a general $\mathcal{D}$ with duals) in a more efficient way. For example, one might like to remember only the mapping spaces of the form $\mathrm{Map}_{\mathcal{D}}(1, X)$, together with some additional structural data.

More precisely, suppose that we are given a non-degenerate essentially surjective functor $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$. We can define a lax symmetric functor $M_\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ (where Grp_∞ denotes the ∞-category of ∞-groupoids) by setting

$$M_\varphi(X) = \mathrm{Map}_{\mathcal{D}}(1, \varphi(X))$$

We will refer to M_φ as the **fiber functor** of φ. This functor can be considered as (at least a partial) codification of $\mathcal{D}$ which remembers the various mapping spaces of $\mathcal{D}$ "without repetitions".

Now since φ is non-degenerate the functor M_φ is not completely arbitrary. More precisely, we have the following notion:

Definition 2.1. Let $M : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ be a lax symmetric monoidal functor. An object $Z \in M(X \otimes \check{X})$ is called **non-degenerate** if for each object $Y \in \mathcal{B}_1^{\mathrm{ev}}$ the natural map

$$M(Y \otimes \check{X}) \stackrel{Id \times Z}{\longrightarrow} M(Y \otimes \check{X}) \times M(X \otimes \check{X}) \longrightarrow M(Y \otimes \check{X} \otimes X \otimes \check{X}) \stackrel{M(Id \otimes \mathrm{ev} \otimes Id)}{\longrightarrow} M(Y \otimes \check{X})$$

is an equivalence of ∞-groupoids.

Remark 2.2. If a non-degenerate element $Z \in M(X \otimes \check{X})$ exists then it is unique up to a (non-canonical) equivalence.

Example 1. *Let $M : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ be a lax symmetric monoidal functor. The lax symmetric structure of M includes a structure map $1_{\mathrm{Grp}_\infty} \longrightarrow M(1)$ which can be described by choosing an object $Z_1 \in M(1)$. The axioms of lax monoidality then ensure that Z_1 is non-degenerate.*

Definition 2.3. A lax symmetric monoidal functor $M : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ will be called **non-degenerate** if for each object $X \in \mathcal{B}_1^{\mathrm{ev}}$ there exists a non-degenerate object $Z \in M(X \otimes \check{X})$.

Definition 2.4. Let $M_1, M_2 : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ be two non-degenerate lax symmetric monoidal functors. A lax symmetric natural transformation $T : M_1 \longrightarrow M_2$ will be called **non-degenerate** if for each object $X \in \mathrm{Bord}^{\mathrm{ev}}$ and each non-degenerate object $Z \in M(X \otimes \check{X})$ the objects $T(Z) \in M_2(X \otimes \check{X})$ is non-degerate.

Remark 2.5. From remark 2.2 we see that if $T(Z) \in M_2(X \otimes \check{X})$ is non-degenerate for **at least one** non-degenerate $Z \in M_1(X \otimes \check{X})$ then it will be true for all non-degenerate $Z \in M_1(X \otimes \check{X})$.

Now we claim that if $\mathcal{D}$ has duals and $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ is non-degenerate then the fiber functor M_φ will be non-degenerate: for each object $X \in \mathcal{B}_1^{\mathrm{ev}}$ there exists a coevaluation morphism

$$\mathrm{coev}_{\varphi(X)} : 1 \longrightarrow \varphi(X) \otimes \varphi(\check{X}) \simeq \varphi(X \otimes \check{X})$$

which determines an element in $Z_X \in M_\varphi(X \otimes \check{X})$. It is not hard to see that this element is non-degenerate.

Let $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty)$ denote the ∞-category of lax symmetric monoidal functors $\mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ and by

$$\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty) \subseteq \mathrm{Fun}^{\mathrm{lax}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty)$$

the subcategory spanned by non-degenerate functors and non-degenerate natural transformations. Now the construction $\varphi \mapsto M_\varphi$ determines a functor

$$F : \mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/} \longrightarrow \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty)$$

In particular if $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{C}$ and $\psi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ are non-degenerate then any functor $T : \mathcal{C} \longrightarrow \mathcal{D}$ under $\mathcal{B}_1^{\mathrm{ev}}$ will induce a non-degenerate natural transformation

$$F(T) : M_\varphi \longrightarrow M_\psi$$

We can then consider the following modification of the cobordism hypothesis concerned with the behavior of our objects of interest after this process of compression:

Theorem 2.6 (Cobordism Hypothesis - Quasi-Unital)**.** *Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals, let $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ be a non-degenerate functor and let $\iota : \mathcal{B}_1^{\mathrm{ev}} \hookrightarrow \mathcal{B}_1^{\mathrm{or}}$ be the natural inclusion. Let $M_\iota, M_\varphi \in \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}$ be the corresponding fiber functors. Them the space of maps*

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}}(M_\iota, M_\varphi)$$

is contractible.

Now given that we have proven Theorem 2.6 we will need a way to tie the result back to the original cobordism hypothesis. For this we need to check to what extent one can reconstruct the full ∞-category with duals $\mathcal{D}$ from the compressed codification of M_φ (where $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ is any non-degenerate functor).

For this we can attempt to **invert** the construction $(\mathcal{D}, \varphi) \mapsto M_\varphi$. Let $M : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ be a non-degenerate lax symmetric monoidal functor. We can construct a pointed **non-unital** symmetric monoidal ∞-category $\mathcal{D}_M$ as follows:

1. The objects of $\mathcal{D}_M$ are the objects of $\mathcal{B}_1^{\mathrm{ev}}$. The marked point is the object X_+.
2. Given a pair of objects $X, Y \in \mathcal{D}_M$ we define

   $$\mathrm{Map}_{\mathcal{D}_M}(X, Y) = M(\check{X} \otimes Y)$$

   Given a triple of objects $X, Y, Z \in \mathcal{D}_M$ the composition law

   $$\mathrm{Map}_{\mathcal{D}_M}(\check{X}, Y) \times \mathrm{Map}_{\mathcal{D}_M}(\check{Y}, Z) \longrightarrow \mathrm{Map}_{\mathcal{D}_M}(\check{X}, Z)$$

   is given by the composition

   $$M(\check{X} \otimes Y) \times M(\check{Y} \otimes Z) \longrightarrow M(\check{X} \otimes Y \otimes \check{Y} \otimes Z) \longrightarrow M(\check{X} \otimes Z)$$

   where the first map is given by the lax symmetric monoidal structure on the functor M and the second is induced by the evaluation map

   $$\mathrm{ev}_Y : \check{Y} \otimes Y \longrightarrow 1$$

   in $\mathcal{B}_1^{\mathrm{ev}}$.
3. The symmetric monoidal structure is defined in a straight forward way using the lax monoidal structure of M.

Now for each non-degenerate functor $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ we have a natural pointed functor

$$N_\varphi : \mathcal{D}_{M_\varphi} \longrightarrow \mathcal{D}$$

defined as follows: N_φ maps the objects of $\mathcal{D}_{M_\varphi}$ (which are the objects of $\mathcal{B}_1^{\mathrm{ev}}$) to $\mathcal{D}$ via φ. Then for each $X, Y \in \mathcal{B}_1^{\mathrm{ev}}$ we can map the morphisms

$$\mathrm{Map}_{\mathcal{D}_{M_\varphi}}(X, Y) = \mathrm{Map}_{\mathcal{D}}(1, \check{X} \otimes Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(X, Y)$$

via the duality structure - to a morphism $f : 1 \longrightarrow \check{X} \otimes Y$ one associates the morphism $\widehat{f} : X \longrightarrow Y$ given as the composition

$$X \stackrel{Id \otimes f}{\longrightarrow} X \otimes \check{X} \otimes Y \stackrel{\varphi(\mathrm{ev}_X) \otimes Y}{\longrightarrow} Y$$

It is quite direct to verify that N_φ is a functor of (symmetric monoidal) non-unital ∞-categories, i.e. it respects composition and monoidal products in a natural way. Since $\mathcal{D}$ has duals we get that N_φ is fully faithful and since φ is essentially surjective φ we get that N_φ is essentially surjective. Hence N_φ is an equivalence and so $\mathcal{D}_{M_\varphi}$ is equivalent to the underlying non-unital ∞-category of $\mathcal{D}$.

Informally speaking one can say that we are **almost able** to reconstruct $\mathcal{D}$ out of M_φ - we are just missing the identity morphisms. However, note that if M is non-degenerate then $\mathcal{D}_M$ is not a completely arbitrary non-unital ∞-category. In fact it is very close to being unital - a **non-degenerate** object in $M(\check{X} \otimes X)$ gives a morphism which **behaves** like an identity map. Hence in some sense, we can reconstruct the units of $\mathcal{D}$ as well. To make this idea precise we will need a good theory of **quasi-unital ∞-categories**.

3 Quasi-unital ∞-Categories

Throughout this section we will assume that the reader is familiar with the formalism of **Segal spaces** and their connection with ∞-categories. Our purpose is to study the non-unital analogue of this construction, obtained by replacing simplicial spaces with **semi-simplicial spaces**.

Let X be a semi-simplicial space. Let $[n], [m] \in \Delta_s$ be two objects and consider the commutative (pushout) diagram

$$\begin{array}{ccc} [0] & \xrightarrow{0} & [m] \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle g_{n,m}} \\ [n] & \xrightarrow[f_{n,m}]{} & [n+m] \end{array}$$

where $f_{n,m}(i) = i$ and $g_{n,m}(i) = i + n$. We will say that X satisfies the **Segal condition** if for each $[n], [m]$ as above the induced commutative diagram

$$\begin{array}{ccc} X_{m+n} & \xrightarrow{g^*_{n,m}} & X_m \\ {\scriptstyle f^*_{n,m}}\downarrow & & \downarrow{\scriptstyle 0^*} \\ X_n & \xrightarrow{n^*} & X_0 \end{array}$$

is a **homotopy pullback** diagram. We will say that X is a **semiSegal space** if it is **Reedy fibrant** and satisfies the Segal condition. Note that in that case the above square will induce a homotopy equivalence

$$X_{m+n} \simeq X_m \times_{X_0} X_n$$

Example 2. *Let $\mathcal{C}$ be a non-unital small topological category. We can represent $\mathcal{C}$ as a semiSegal space as follows. For each n, let $\mathfrak{C}^{\mathrm{nu}}([n])$ denote the non-unital* Top*-enriched category whose objects are the numbers $0, ..., n$ and whose mapping spaces are*

$$\mathrm{Map}_{\mathfrak{C}^{\mathrm{nu}}([n])}(i, j) = \begin{cases} \emptyset & i \geq j \\ I^{(i,j)} & i < j \end{cases}$$

where $(i,j) = \{x \in \{0,...,n\} | i < x < j\}$. *The composition is given by the inclusion*

$$I^{(i,j)} \times I^{(j,k)} \cong I^{(i,j)} \times \{0\} \times I^{(j,k)} \subseteq I^{(i,k)}$$

Note that $\mathfrak{C}^{\mathrm{nu}}([n])$ *depends functorially on* $[n] \in \Delta_s$. *Hence for every non-unital topological category* $\mathcal{C}$ *we can form a semi-simplicial space* $N(\mathcal{C})$ *by setting*

$$N(\mathcal{C})_n = \mathrm{Fun}(\mathfrak{C}^{\mathrm{nu}}([n]), \mathcal{C})$$

We endow $N(\mathcal{C})_n$ *with a natural topology that comes from the topology of the mapping space of* $\mathcal{C}$ *(while treating the set of objects of* $\mathcal{C}$ *as discrete). One can then check that* $N(\mathcal{C})$ *is a* ***semiSegal space****.*

We think of general semiSegal spaces X as relaxed versions of Example 2, i.e. as a **non-unital ∞-category**. The objects of the corresponding non-unital ∞-category are the points of X_0. Given two points $x, y \in X_0$ we define the **mapping space** between them by

$$\mathrm{Map}_X(x,y) = \{x\} \times_{X_0} X_1 \times_{X_0} \{y\}$$

i.e., as the fiber of the (Kan) fibration

$$X_1 \stackrel{(d_0,d_1)}{\longrightarrow} X_0 \times X_0$$

over the point (x, y).

To see how composition works consider first the case of a topological category $\mathcal{C}$ and assume that we are given three objects $x, y, z \in \mathcal{C}$ and a morphism $f : x \longrightarrow y$. One would then obtain a composition-by-f maps

$$f_* : \mathrm{Hom}_{\mathcal{C}}(z,x) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(z,y)$$

and

$$f^* : \mathrm{Hom}_{\mathcal{C}}(y,z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x,z)$$

In the semiSegal model we do not have such strict composition. Instead one can describe the composition-by-f maps as **correspondences**. If $x, y, z \in X_0$ are objects and $f : x \longrightarrow y$ is a morphism (i.e. an element in X_1 such that $d_0(f) = x$ and $d_1(f) = y$) one can consider the space $C^R_{f,z} \subseteq X_2$ given by

$$C^R_{f,z} = \{\sigma \in X_2 \mid \sigma|_{\Delta^{\{1,2\}}} = f, \sigma|_{\Delta^{\{0\}}} = z\}$$

Then the two restriction maps $\sigma \mapsto \sigma|_{\Delta^{\{0,1\}}}$ and $\sigma \mapsto \sigma|_{\Delta^{\{0,2\}}}$ give us a correspondence (recall that X is Reedy fibrant and so the restriction maps are fibrations):

$$\mathrm{Map}_X(z,x) \longleftarrow C^R_{f,z} \longrightarrow \mathrm{Map}_X(z,y)$$

This correspondence describes the operation of composing with f on the right. Similarly we have a correspondence

$$\begin{array}{ccc} & C^L_{f,z} & \\ \swarrow & & \searrow \\ \mathrm{Map}_X(y,z) & & \mathrm{Map}_X(x,z) \end{array}$$

describing composition with f on the left. The Segal condition ensures that both $C^R_{f,z}$ and $C^L_{f,z}$ are **map-like** correspondences, i.e. the left hand side maps are **weak equivalences**. In that sense composition is "almost" well-defined.

We want to define properties of f via analogous properties of the correspondences $C^R_{f,z}, C^L_{f,z}$. In particular we will want to define when a morphism is a **quasi-unit** and when it is **invertible**. For this we will need to first understand how to say this in terms of correspondences.

Recall that from each space X to itself we have the **identity correspondence** $X \xleftarrow{\mathrm{Id}} X \xrightarrow{\mathrm{Id}} X$. We will say that a correspondence $X \xleftarrow{\varphi} C \xrightarrow{\psi} X$ is **unital** if it is **equivalent** to the identity correspondence. It is not hard to check that a correspondence as above is unital if and only if both φ, ψ are weak equivalences and are **homotopic** to each other in the Kan model structure.

We will say that a correspondence $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ is **invertible** if it admits an **inverse**, i.e. if there exists a correspondence $Y \xleftarrow{\varphi} D \xrightarrow{\psi} X$ such that the compositions

$$X \longleftarrow C \times_Y D \longrightarrow X$$

and

$$Y \longleftarrow D \times_X C \longrightarrow Y$$

are unital.

Remark 3.1. Note that if a correspondence

$$X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$$

is map-like (i.e. if φ is invertible) then it is equivalent to a correspondence of the form

$$X \xleftarrow{\mathrm{Id}} D \xrightarrow{f} Y$$

such that f represents the class $[\psi] \circ [\varphi]^{-1}$ in the Kan homotopy category. In this case the invertibility of $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ is equivalent f being a weak equivalence, i.e. to ψ being a weak equivalence.

Now let X be a semiSegal space. Through the point of view of correspondences we have a natural way to define invertibility and unitality of morphisms:

Definition 3.2. 1. Let $x, y \in X_0$ be two objects and $f : x \longrightarrow y$ a morphism in X. We will say that f is **right-invertible** if for every $z \in X_0$ the right composition correspondence

$$\mathrm{Map}_X(z,x) \longleftarrow C^R_{f,z} \longrightarrow \mathrm{Map}_X(z,y)$$

is invertible. Similarly one says that f is **left-invertible** if for every $z \in X_0$ the left composition correspondence

$$\mathrm{Map}_X(y,z) \longleftarrow C^L_{f,z} \longrightarrow \mathrm{Map}_X(x,z)$$

is invertible. We say that f is **invertible** if it is both left invertible and right invertible.

2. Let $x \in X_0$ be an object and $f : x \longrightarrow x$ a morphism in X. We will say that f is a **quasi-unit** if for each $z \in X_0$ the correspondences

$$\mathrm{Map}_X(x,z) \longleftarrow C^R_{f,z} \longrightarrow \mathrm{Map}_X(x,z)$$

and

$$\mathrm{Map}_X(z,x) \longleftarrow C^L_{f,z} \longrightarrow \mathrm{Map}_X(z,x)$$

are **unital**.

Remark 3.3. From Remark 3.1 we see that a morphism $f : x \longrightarrow y$ in X is invertible if and only if for each $z \in X_0$ the restriction maps

$$C^R_{f,z} \longrightarrow \mathrm{Map}_X(z,y)$$

$$C^L_{f,z} \longrightarrow \mathrm{Map}_X(x,z)$$

are weak equivalences.

Invertible morphisms can be described informally as morphisms such that composition with them induces a weak equivalence on mapping spaces. Note that the notion of invertibility does not presupposed the existence of identity morphisms, i.e. it makes sense in the non-unital setting as well.

We will denote by

$$X_1^{\mathrm{inv}} \subseteq X_1$$

the maximal subspace spanned by the invertible vertices $f \in (X_1)_0$. Using Reedy fibrancy it is not hard to show that if $f, g \in X_1$ are connected by a path in X_1 then f is invertible if and only if g is invertible. Hence X_1^{inv} is just the **union of connected components of** X_1 which meet invertible edges.

We will denote by

$$\mathrm{Map}_X^{\mathrm{inv}}(x,y) = \{x\} \times_{X_0} X_1^{\mathrm{inv}} \times_{X_0} \{y\} \subseteq \mathrm{Map}_X(x,y)$$

the subspace of invertible morphisms from x to y.

Definition 3.4. Let X be a semiSegal space. We will say that X is **quasi-unital** if every $x_0 \in X_0$ admits a quasi-unit from x_0 to x_0. We say that a map $f : X \longrightarrow Y$ of quasi-unital semiSegal spaces is **unital** if it maps quasi-units to quasi-units. We will denote by

$$\mathrm{QsS}$$

the topological category of quasi-unital semi-simplicial spaces and unital maps between them.

We will be interested in studying the category QsS up to a natural notion of equivalences, given by **Dwyer-Kan equivalences**. This is a direct adaptation of the notion of DK-equivalence of ∞-categories to the quasi-unital setting.

We start with a slightly more general notion of fully-faithful maps:

Definition 3.5. Let $f : X \longrightarrow Y$ be map of semiSegal spaces. We will say that f is a **fully-faithful** for all $x, y \in X_0$ the induced map

$$\mathrm{Map}_X(x, y) \longrightarrow \mathrm{Map}_Y(f_0(x), f_0(y))$$

is a weak equivalence.

The notion of Dwyer-Kan equivalences will be obtained from the notion of fully-faithful maps by requiring the appropriate analogue of "essential surjectivity". For this let us introduce some terminology.

Definition 3.6. Let $x, y \in X_0$ be two points. We say that x and y are **equivalent** (denoted $x \simeq y$) if there exists an invertible morphism $f :\in X_1^{\mathrm{inv}}$ from x to y.

Lemma 3.7. *Let X be a quasi-unital semiSegal space. Then $\simeq$ is an equivalence relation. We will refer to the corresponding set of equivalence classes as the set of* ***equivalence-types*** *of X.*

Definition 3.8. Let $f : X \longrightarrow Y$ be a map between **quasi-unital** semiSegal spaces. We will say that f is a **Dwyer-Kan equivalence** (DK for short) if it is fully faithful and induces a surjective map on the set of equivalence-types.

Remark 3.9. A DK-equivalence $f : X \longrightarrow Y$ is automatically a unital map.

We propose to model the ∞-category of small quasi-unital ∞-categories as the localization of QsS with respect to DK-equivalence. This is analogous to modeling the ∞-category of small ∞-categories as the localization of the category of **Segal spaces** with respect to DK-equivalence. We have a natural forgetful functor between these two localizations. Our main result us that this functor is an **equivalence of categories**.

In his fundamental paper [Rez] Rezk constructs (using the framework of model categories) an explicit model for this localization in terms of **complete Segal spaces**. We propose an analogous model for the quasi-unital case as follows:

Definition 3.10. Let X be a semiSegal space. We will say that X is **complete** if the restricted maps $d_0 : X_1^{\mathrm{inv}} \longrightarrow X_0$ and $d_1 : X_1^{\mathrm{inv}} \longrightarrow X_0$ are both homotopy equivalences.

An important observation is that any complete semiSegal space is quasi-unital: since the map $X_1^{\mathrm{inv}} \longrightarrow X_0$ is a trivial fibration every object $x \in X_0$ admits an invertible morphism of the form $f : x \longrightarrow y$ for some y. This implies that x admits a quasi-unit.

Let CsS $\subseteq$ QsS denote the full topological subcategory spanned by complete semiSegal spaces. We claim that the topological category CsS can serve as a model for the localization of QsS by DK-equivalences. Formally speaking (see Definition 5.2.7.2 and Proposition 5.2.7.12 of [Lur3]) this means that there exists a functor

$$\widehat{\bullet} : \mathrm{QsS} \longrightarrow \mathrm{CsS}$$

such that:

1. $\widehat{\bullet}$ is homotopy left adjoint to the inclusion CsS $\subseteq$ QsS.
2. A map in QsS is a DK-equivalence if and only if its image under $\widehat{\bullet}$ is a homotopy equivalence.

Let CS be the topological category of complete Segal spaces. Our main result of this section can now be stated as follows:

Theorem 3.11. *The forgetful functor*

$$\mathrm{CS} \longrightarrow \mathrm{CsS}$$

is an equivalence.

4 Completion of the Proof

Let us now go back to the construction $M \mapsto \mathcal{D}_M$ described above. When M is non-degenerate we get that $\mathcal{D}_M$ is quasi-unital. Furthermore, any non-degenerate natural transformation $M \longrightarrow N$ will induce a unital functor $\mathcal{D}_M \longrightarrow \mathcal{D}_N$. Hence the construction $M \mapsto \mathcal{D}_M$ determines a functor

$$G : \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}^{\mathrm{ev}}_1, \mathrm{Grp}_\infty) \longrightarrow \mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}^{\mathrm{un}}_0/}$$

where $\mathrm{Cat}^{\mathrm{qu},\otimes}$ is the ∞-category of symmetric monoidal quasi-unital categories (i.e. commutative algebra objects in the ∞-category $\mathrm{Cat}^{\mathrm{qu}}$ of quasi-unital ∞-categories). Since the forgetful functor

$$S : \mathrm{Cat} \longrightarrow \mathrm{Cat}^{\mathrm{qu}}$$

From ∞-categories to quasi-unital ∞-categories is an **equivalence** we get that the induced forgetful functor

$$S_* : \mathrm{Cat}^{\otimes}_{\mathcal{B}^{\mathrm{un}}_0/} \longrightarrow \mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}^{\mathrm{un}}_0/}$$

is an equivalence as well.

Composing G with the $\varphi \mapsto M_\varphi$ functor

$$F : \mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/} \longrightarrow \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}^{\mathrm{ev}}_1, \mathrm{Grp}_\infty)$$

described above we get a functor

$$G \circ F : \mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/} \longrightarrow \mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}^{\mathrm{un}}_0/}$$

and we have a homotopy commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/} & & \\ & \swarrow^{F} & & \searrow^{T} & \\ \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}^{\mathrm{ev}}_1, \mathrm{Grp}_\infty) & & & & \mathrm{Cat}^{\otimes}_{\mathcal{B}^{\mathrm{un}}_0/} \\ & \searrow_{G} & & \swarrow_{S_*} & \\ & & \mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}^{\mathrm{un}}_0/} & & \end{array}$$

where T is given by restriction along $X_+ : \mathcal{B}^{\mathrm{un}} \longrightarrow \mathcal{B}^{\mathrm{ev}}_1$. Now from Lemma 1.8 we see that T is fully faithful. Since S_* is an equivalence of ∞-categories we get

Corollary 4.1. *The functor $G \circ F$ is fully faithful.*

We are now ready to complete the proof of 1.10. Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals and let $\varphi : \mathcal{B} \longrightarrow \mathcal{D}$ be a non-degenerate functor. We wish to show that the space of maps

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/}}(\iota, \varphi)$$

is contractible. Consider the sequence

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/}}(\iota, \varphi) \longrightarrow \mathrm{Map}_{\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}^{\mathrm{ev}}_1, \mathrm{Grp}_\infty)}(M_\iota, M_\varphi) \longrightarrow \mathrm{Map}_{\mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}^{\mathrm{un}}_0/}}(\mathcal{B}^{\mathrm{or}}_1, \mathcal{D})$$

By Theorem 2.6 the middle space is contractible and by lemma 4.1 the composition

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/}}(\iota, \varphi) \longrightarrow \mathrm{Map}_{\mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}^{\mathrm{un}}_0/}}(\mathcal{B}^{\mathrm{or}}_1, \mathcal{D})$$

is a weak equivalence. Hence we get that

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/}}(\iota, \varphi)$$

is contractible. This completes the proof of Theorem 1.10.

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