The Cobordism Hypothesis in Dimension 1

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1 Introduction

Let \( \mathcal{B}^\text{or}_1 \) denote the 1-dimensional oriented cobordism ∞-category, i.e. the symmetric monoidal ∞-category whose objects are oriented 0-dimensional closed manifolds and whose morphisms are oriented 1-dimensional cobordisms between them.

The category \( \mathcal{B}^\text{or}_1 \) carries a fundamental inner symmetry expressing the fact that \textit{cobordisms can be read in both directions}. This can be described formally in the language of \textit{duality} as follows. Let \( \mathcal{C} \) be an (ordinary) monoidal category and let \( X \) be an object. We say that an object \( Y \in \mathcal{C} \) is \textbf{right dual} to \( X \) if there exist morphisms \( \text{ev} : X \otimes Y \to 1 \) and \( \text{coev} : 1 \to Y \otimes X \) such that the compositions

\[
X \xrightarrow{\text{Id} \otimes \text{coev}} X \otimes Y \otimes X \xrightarrow{\text{ev} \otimes \text{Id}} X
\]

and

\[
Y \xrightarrow{\text{coev} \otimes \text{Id}} Y \otimes X \otimes Y \xrightarrow{\text{Id} \otimes \text{ev}} Y
\]

are the identity. In this case we also say that \( X \) is left dual to \( Y \). If \( \mathcal{C} \) is symmetric then this definitions become symmetric and we just say that \( X \) and \( Y \) are duals. If \( \mathcal{C} \) is a symmetric monoidal ∞-category then we say that \( X \) and \( Y \) are duals if they are duals in the homotopy category \( \text{Ho}(\mathcal{C}) \).

\textbf{Example}:
1. Let $C$ be the symmetric monoidal category of finite dimensional vector spaces over $\mathbb{C}$ with monoidal product given by tensor product. Then for each $V \in C$ the object $\mathcal{V} = \text{Hom}(V, \mathbb{C})$ is dual to $V$. The evaluation map $V \otimes \mathcal{V} \to \mathbb{C}$ is clear and the coevaluation $\mathbb{C} \to \mathcal{V} \otimes V \cong \text{End}(V)$ is given by sending $1 \in \mathbb{C}$ to the identity $I \in \text{End}(V)$.

2. Let $X_+, X_- \in \mathbb{B}_1^{\text{or}}$ be the points with positive and negative orientations respectively. Then $X_+$ and $X_-$ are duals in $\mathbb{B}_1^{\text{or}}$. The evaluation map is the "right-half-circle" cobordism

$$\text{ev} : X_+ \otimes X_- \to \emptyset$$

and the coevaluation is the "left-half-circle" cobordism

$$\text{coev} : \emptyset \to X_+ \otimes X_-$$

**Definition 1.1.** We say that a symmetric monoidal $\infty$-category has duals if every object has a dual.

We now observe the following simple lemma

**Lemma 1.2.** Let $D, C$ be two symmetric monoidal (ordinary) categories with duals, $F, G : D \to C$ two symmetric monoidal functors and $T : F \to G$ a natural transformation. Then $T$ is a natural isomorphism.

Now let $D$ be a symmetric monoidal $\infty$-category with duals. The 1-dimensional cobordism hypothesis concerns the $\infty$-category

$$\text{Fun}^\otimes(\mathbb{B}_1^{\text{or}}, D)$$

of symmetric monoidal functors $\varphi : \mathbb{B}_1^{\text{or}} \to D$. If $X_+ \in \mathbb{B}_1^{\text{or}}$ is the object corresponding to a point with positive orientation then the evaluation map $Z \mapsto Z(X_+)$ induces a functor

$$\text{Fun}^\otimes(\mathbb{B}_1^{\text{or}}, D) \to D$$

From Lemma 1.2 we see that the $\infty$-category $\text{Fun}^\otimes(\mathbb{B}_1^{\text{or}}, D)$ is in fact an $\infty$-groupoid. This means that the evaluation map $Z \mapsto Z(X_+)$ actually factors through a map

$$\text{Fun}^\otimes(\mathbb{B}_1^{\text{or}}, D) \to \tilde{D}$$

where $\tilde{D}$ is the maximal $\infty$-groupoid of $D$. The cobordism hypothesis then states

**Theorem 1.3.** The evaluation map

$$\text{Fun}^\otimes(\mathbb{B}_1^{\text{or}}, D) \to \tilde{D}$$

is an equivalence of $\infty$-groupoids.
Remark 1.4. From the consideration above we see that we could have written
the cobordism hypothesis as an equivalence
\[ \widetilde{\text{Fun}}^\otimes (B^\text{or}_1, D) \xrightarrow{\sim} \tilde{D} \]
where \( \widetilde{\text{Fun}}^\otimes (B^\text{or}_1, D) \) is the maximal \( \infty \)-groupoid of \( \text{Fun}^\otimes (B^\text{or}_1, D) \) (which in this case happens to coincide with \( \text{Fun}^\otimes (B^\text{or}_1, D) \)). This \( \infty \)-groupoid is the fundamental groupoid of the space of maps from \( B^\text{or}_1 \) to \( D \) in the \( \infty \)-category \( \text{Cat}^\otimes \) of symmetric monoidal \( \infty \)-categories.

In his paper \[Lur1\] Lurie gives an elaborate sketch of proof for a higher
dimensional generalization of the 1-dimensional cobordism hypothesis. For this
one needs to generalize the notion of \( \infty \)-categories to \((\infty,n)\)-categories. The
strategy of proof described in \[Lur1\] is inductive in nature. In particular in
order to understand the \( n = 1 \) case, one should start by considering the \( n = 0 \) case.

Let \( B^\text{un}_0 \) be the 0-dimensional unoriented cobordism category, i.e. the objects
of \( B^\text{un}_0 \) are 0-dimensional closed manifolds (or equivalently, finite sets) and the
morphisms are diffeomorphisms (or equivalently, isomorphisms of finite sets).
Note that \( B^\text{un}_0 \) is a (discrete) \( \infty \)-groupoid.

Let \( X \in B^\text{un}_0 \) be the object corresponding to one point. Then the 0-
dimensional cobordism hypothesis states that \( B^\text{un}_0 \) is in fact the free \( \infty \)-groupoid
(or \((\infty,0)\)-category) on one object, i.e. if \( \mathcal{G} \) is any other \( \infty \)-groupoid then the
evaluation map \( Z \mapsto Z(X) \) induces an equivalence of \( \infty \)-groupoids
\[ \text{Fun}^\otimes (B^\text{un}_0, \mathcal{G}) \xrightarrow{\sim} \mathcal{G} \]

Remark 1.5. At this point one can wonder what is the justification for con-
sidering non-oriented manifolds in the \( n = 0 \) case oriented ones in the \( n = 1 \)
case. As is explained in \[Lur1\] the desired notion when working in the \( n \)-
dimensional cobordism \((\infty,n)\)-category is that of \textit{n-framed} manifolds. One
then observes that 0-framed 0-manifolds are unoriented manifolds, while taking
1-framed 1-manifolds (and 1-framed 0-manifolds) is equivalent to taking the
respective manifolds with orientation.

Now the 0-dimensional cobordism hypothesis is not hard to verify. In fact, it
holds in a slightly more general context - we do not have to assume that \( \mathcal{G} \) is
an \( \infty \)-groupoid. In fact, if \( \mathcal{G} \) is any \textbf{symmetric monoidal} \( \infty \)-category then
the evaluation map induces an equivalence of \( \infty \)-categories
\[ \text{Fun}^\otimes (B^\text{un}_0, \mathcal{G}) \xrightarrow{\sim} \mathcal{G} \]
and hence also an equivalence of \( \infty \)-groupoids
\[ \widetilde{\text{Fun}}^\otimes (B^\text{un}_0, \mathcal{G}) \xrightarrow{\sim} \tilde{\mathcal{G}} \]

Now consider the under-category \( \text{Cat}^\otimes_{B^\text{un}_0} \) of symmetric monoidal \( \infty \)-categories
\( \mathcal{D} \) equipped with a functor \( B^\text{un}_0 \longrightarrow \mathcal{D} \). Since \( B^\text{un}_0 \) is free on one generator this
category can be identified with the \( \infty \)-category of \textbf{pointed} symmetric monoidal \( \infty \)-categories, i.e. symmetric monoidal \( \infty \)-categories with a chosen object. We will often not distinguish between these two notions.

Now the point of positive orientation \( X_+ \in \mathcal{B}_1^{\text{or}} \) determines a functor \( \mathcal{B}_0^{\text{un}} \to \mathcal{B}_1^{\text{or}} \), i.e. an object in \( \text{Cat}_{\mathcal{B}_0^{\text{un}}/}^\otimes \), which we shall denote by \( \mathcal{B}_1^+ \). The 1-dimensional cobordism hypothesis is then equivalent to the following statement:

**Theorem 1.6. [Cobordism Hypothesis 0-to-1]** Let \( D \in \text{Cat}_{\mathcal{B}_0^{\text{un}}/}^\otimes \) be a pointed symmetric monoidal \( \infty \)-category with duals. Then the \( \infty \)-groupoid

\[ \widetilde{\text{Fun}}_{\mathcal{B}_0^{\text{un}}/}(\mathcal{B}_1^+, D) \]

is \textit{contractible}.

Theorem 1.6 can be considered as the inductive step from the 0-dimensional cobordism hypothesis to the 1-dimensional one. Now the strategy outlines in \[\text{Lur1}\] proceeds to bridge the gap between \( \mathcal{B}_0^{\text{un}} \) to \( \mathcal{B}_1^{\text{or}} \) by considering an intermediate \( \infty \)-category

\[ \mathcal{B}_0^{\text{un}} \hookrightarrow \mathcal{B}_1^{\text{ev}} \hookrightarrow \mathcal{B}_1^{\text{or}} \]

This intermediate \( \infty \)-category is defined in \[\text{Lur1}\] in terms of framed functions and index restriction. However in the 1-dimensional case one can describe it without going into the theory of framed functors. In particular we will use the following definition:

**Definition 1.7.** Let \( i : \mathcal{B}_1^{\text{ev}} \hookrightarrow \mathcal{B}_1^{\text{or}} \) be the subcategory containing all objects and only the cobordisms \( M \) in which every connected component \( M_0 \subseteq M \) is either an identity segment or an evaluation segment (i.e. a ”right-half-circle” as above).

Let us now describe how to bridge the gap between \( \mathcal{B}_0^{\text{un}} \) and \( \mathcal{B}_1^{\text{ev}} \). Let \( D \) be an \( \infty \)-category with duals and let

\[ \varphi : \mathcal{B}_1^{\text{ev}} \to D \]

be a symmetric monoidal functor. We will say that \( \varphi \) is \textbf{non-degenerate} if for each \( X \in \mathcal{B}_1^{\text{ev}} \) the map

\[ \varphi(\text{ev}_X) : \varphi(X) \otimes \varphi(\hat{X}) \simeq \varphi(X \otimes \hat{X}) \to \varphi(1) \simeq 1 \]

is \textbf{non-degenerate}, i.e. identifies \( \varphi(\hat{X}) \) with a dual of \( \varphi(X) \). We will denote by

\[ \text{Cat}_{\mathcal{B}_1^{\text{ev}}/}^{\text{red}} \subseteq \text{Cat}_{\mathcal{B}_1^{\text{ev}}/}^\otimes \]

the full subcategory spanned by objects \( \varphi : \mathcal{B}_1^{\text{ev}} \to D \) such that \( D \) has duals and \( \varphi \) is non-degenerate.

Let \( X_+ \in \mathcal{B}_1^{\text{ev}} \) be the point with positive orientation. Then \( X_+ \) determines a functor

\[ \mathcal{B}_0^{\text{un}} \to \mathcal{B}_1^{\text{ev}} \]
The restriction map \( \varphi \mapsto \varphi|_{B^0} \) then induces a functor
\[
\text{Cat}^\text{nd}_{B^1} / \longrightarrow \text{Cat}^\otimes_{B^0} /
\]
Now the gap between \( B^1 \) and \( B^0 \) can be climbed using the following lemma (see [Lur1]):

**Lemma 1.8.** The functor
\[
\text{Cat}^\text{nd}_{B^1} / \longrightarrow \text{Cat}^\otimes_{B^0} /
\]
is fully faithful. Its essential image consists of points symmetric monoidal \( \infty \)-categories in which the pointed object admits a dual.

Now consider the natural inclusion \( \iota: B^1 \longrightarrow B \) as an object in \( \text{Cat}^\text{nd}_{B^1} / \). Then by Lemma 1.8 we see that the 1-dimensional cobordism hypothesis will be established once we make the following last step:

**Theorem 1.9 (Cobordism Hypothesis - Last Step).** Let \( D \) be a symmetric monoidal \( \infty \)-category with duals and let \( \varphi: B^1 \longrightarrow D \) be a non-degenerate functor. Then the \( \infty \)-groupoid
\[
\widetilde{\text{Fun}}^\otimes_{B^1} / (B, D)
\]
is contractible.

Note that since \( B^1 \longrightarrow B \) is essentially surjective all the functors in
\[
\widetilde{\text{Fun}}^\otimes_{B^1} / (B, D)
\]
will have the same essential image of \( \varphi \). Hence it will be enough to prove for the claim for the case where \( \varphi: B^1 \longrightarrow D \) is essentially surjective. We will denote by
\[
\text{Cat}^\text{sur}_{B^1} / \subseteq \text{Cat}^\text{nd}_{B^1} /
\]
the full subcategory spanned by essentially surjective functors \( \varphi: B^1 \longrightarrow D \). Hence we can phrase Theorem 1.9 as follows:

**Theorem 1.10 (Cobordism Hypothesis - Last Step 2).** Let \( D \) be a symmetric monoidal \( \infty \)-category with duals and let \( \varphi: B^1 \longrightarrow D \) be an essentially surjective non-degenerate functor. Then the space of maps
\[
\text{Map}_{\text{Cat}^\text{sur}_{B^1} / } (\iota, \varphi)
\]
is contractible.
2 Non-degenerate Fiber Functors

Let $\mathcal{D}$ be a symmetric monoidal $\infty$-categories with duals. The fact of having duals forces a strong symmetry on $\mathcal{D}$. This means, in some sense, that the information in $\mathcal{D}$ is packed with extreme redundancy. For example, given two objects $X, Y \in \mathcal{D}$ (with corresponding dual objects $\check{X}, \check{Y}$) all the mapping spaces

$$\text{Map}_\mathcal{D}(1, \check{X} \otimes Y) \simeq \text{Map}_\mathcal{D}(X \otimes \check{Y}, 1) \simeq \text{Map}_\mathcal{D}(\check{Y}, \check{X})$$

are equivalent. For example, in the case of $B_1$ or $1$ all these spaces can be identified with the classifying space of oriented 1-manifolds $M$ together with an identification $\partial M \simeq \check{X} \otimes Y$. This observation leads one to try to pack the information of $B_1$ or $1$ (or a general $\mathcal{D}$ with duals) in a more efficient way. For example, one might like to remember only the mapping spaces of the form $\text{Map}_\mathcal{D}(1, \check{X})$, together with some additional structural data.

More precisely, suppose that we are given a non-degenerate essentially surjective functor $\varphi : B_1^{ev} \rightarrow \mathcal{D}$. We can define a lax symmetric functor $M: B_1^{ev} \rightarrow \text{Grp}_\infty$ (where $\text{Grp}_\infty$ denotes the $\infty$-category of $\infty$-groupoids) by setting

$$M_{\varphi}(X) = \text{Map}_\mathcal{D}(1, \varphi(X))$$

We will refer to $M_{\varphi}$ as the fiber functor of $\varphi$. This functor can be considered as (at least a partial) codification of $\mathcal{D}$ which remembers the various mapping spaces of $\mathcal{D}$ "without repetitions".

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Now since $\varphi$ is non-degenerate the functor $M_{\varphi}$ is not completely arbitrary. More precisely, we have the following notion:

**Definition 2.1.** Let $M : B_1^{ev} \rightarrow \text{Grp}_\infty$ be a lax symmetric monoidal functor. An object $Z \in M(X \otimes \check{X})$ is called non-degenerate if for each object $Y \in B_1^{ev}$ the natural map

$$M(Y \otimes \check{X}) \xrightarrow{\text{Id} \times \check{Y}} M(Y \otimes \check{X}) \times M(X \otimes \check{X}) \rightarrow M(Y \otimes \check{X} \otimes X \otimes \check{X}) \xrightarrow{M(\text{Id} \otimes \check{Y} \otimes \text{Id})} M(Y \otimes \check{X})$$

is an equivalence of $\infty$-groupoids.

**Remark 2.2.** If a non-degenerate element $Z \in M(X \otimes \check{X})$ exists then it is unique up to a (non-canonical) equivalence.

**Example 1.** Let $M : B_1^{ev} \rightarrow \text{Grp}_\infty$ be a lax symmetric monoidal functor. The lax symmetric structure of $M$ includes a structure map $1_{\text{Grp}_\infty} \rightarrow M(1)$ which can be described by choosing an object $Z_1 \in M(1)$. The axioms of lax monoidality then ensure that $Z_1$ is non-degenerate.

**Definition 2.3.** A lax symmetric monoidal functor $M : B_1^{ev} \rightarrow \text{Grp}_\infty$ will be called non-degenerate if for each object $X \in B_1^{ev}$ there exists a non-degenerate object $Z \in M(X \otimes \check{X})$. 
Definition 2.4. Let $M_1, M_2 : \mathcal{B}_1^{\text{ev}} \to \text{Grp}_\infty$ be two non-degenerate lax symmetric monoidal functors. A lax symmetric natural transformation $T : M_1 \to M_2$ will be called non-degenerate if for each object $X \in \text{Bord}^{\text{ev}}$ and each non-degenerate object $Z \in M_1(X \otimes \bar{X})$ the objects $T(Z) \in M_2(X \otimes \bar{X})$ is non-degenerate.

Remark 2.5. From remark 2.2 we see that if $T(Z) \in M_2(X \otimes \bar{X})$ is non-degenerate for at least one non-degenerate $Z \in M_1(X \otimes \bar{X})$ then it will be true for all non-degenerate $Z \in M_1(X \otimes \bar{X})$.

Now we claim that if $D$ has duals and $\phi : \mathcal{B}_1^{\text{ev}} \to D$ is non-degenerate then the fiber functor $M_\phi$ will be non-degenerate: for each object $X \in \mathcal{B}_1^{\text{ev}}$ there exists a coevaluation morphism $\text{coev}_\phi(X) : 1 \to \phi(X) \otimes \phi(\bar{X}) \cong \phi(X \otimes \bar{X})$ which determines an element in $Z_X \in M_\phi(X \otimes \bar{X})$. It is not hard to see that this element is non-degenerate.

Let $\text{Fun}^{\text{lax}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty)$ denote the $\infty$-category of lax symmetric monoidal functors $\mathcal{B}_1^{\text{ev}} \to \text{Grp}_\infty$ and by

$$\text{Fun}^{\text{lax}}_{\text{nd}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty) \subseteq \text{Fun}^{\text{lax}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty)$$

the subcategory spanned by non-degenerate functors and non-degenerate natural transformations. Now the construction $\phi \mapsto M_\phi$ determines a functor

$$F : \text{Cat}_{\mathcal{B}_1^{\text{ev}}/}^{\text{sur}} \to \text{Fun}^{\text{lax}}_{\text{nd}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty)$$

In particular if $\phi : \mathcal{B}_1^{\text{ev}} \to \mathcal{C}$ and $\psi : \mathcal{B}_1^{\text{ev}} \to \mathcal{D}$ are non-degenerate then any functor $T : \mathcal{C} \to \mathcal{D}$ under $\mathcal{B}_1^{\text{ev}}$ will induce a non-degenerate natural transformation

$$F(T) : M_\phi \to M_\psi$$

We can then consider the following modification of the cobordism hypothesis concerned with the behavior of our objects of interest after this process of compression:

Theorem 2.6 (Cobordism Hypothesis - Quasi-Unital). Let $\mathcal{D}$ be a symmetric monoidal $\infty$-category with duals, let $\phi : \mathcal{B}_1^{\text{ev}} \to \mathcal{D}$ be a non-degenerate functor and let $\iota : \mathcal{B}_1^{\text{ev}} \to \mathcal{B}_1^{\text{ev}}$ be the natural inclusion. Let $M_\iota, M_\phi \in \text{Fun}^{\text{lax}}_{\text{nd}}$ be the corresponding fiber functors. Then the space of maps

$$\text{Map}_{\text{Fun}^{\text{lax}}_{\text{nd}}}(M_\iota, M_\phi)$$

is contractible.

Now given that we have proven Theorem 2.6 we will need a way to tie the result back to the original cobordism hypothesis. For this we need to check to what extent one can reconstruct the full $\infty$-category with duals $\mathcal{D}$ from the
compressed codification of $M_\varphi$ (where $\varphi : \mathcal{B}_1^{ev} \to \mathcal{D}$ is any non-degenerate functor).

For this we can attempt to invert the construction $(\mathcal{D}, \varphi) \mapsto M_\varphi$. Let $M : \mathcal{B}_1^{ev} \to \text{Grp}_\infty$ be a non-degenerate lax symmetric monoidal functor. We can construct a pointed non-unital symmetric monoidal $\infty$-category $\mathcal{D}_M$ as follows:

1. The objects of $\mathcal{D}_M$ are the objects of $\mathcal{B}_1^{ev}$. The marked point is the object $X_+$. 

2. Given a pair of objects $X, Y \in \mathcal{D}_M$ we define

$$\text{Map}_{\mathcal{D}_M}(X, Y) = M(\hat{X} \otimes Y)$$

Given a triple of objects $X, Y, Z \in \mathcal{D}_M$ the composition law

$$\text{Map}_{\mathcal{D}_M}(\hat{X}, Y) \times \text{Map}_{\mathcal{D}_M}(\hat{Y}, Z) \to \text{Map}_{\mathcal{D}_M}(\hat{X}, Z)$$

is given by the composition

$$M(\hat{X} \otimes Y) \times M(\hat{Y} \otimes Z) \to M(\hat{X} \otimes Y \otimes \hat{Y} \otimes Z) \to M(\hat{X} \otimes Z)$$

where the first map is given by the lax symmetric monoidal structure on the functor $M$ and the second is induced by the evaluation map $\text{ev}_Y : \hat{Y} \otimes Y \to 1$ in $\mathcal{B}_1^{ev}$.

3. The symmetric monoidal structure is defined in a straightforward way using the lax monoidal structure of $M$.

Now for each non-degenerate functor $\varphi : \mathcal{B}_1^{ev} \to \mathcal{D}$ we have a natural pointed functor

$$N_\varphi : \mathcal{D}_{M_\varphi} \to \mathcal{D}$$

defined as follows: $N_\varphi$ maps the objects of $\mathcal{D}_{M_\varphi}$ (which are the objects of $\mathcal{B}_1^{ev}$) to $\mathcal{D}$ via $\varphi$. Then for each $X, Y \in \mathcal{B}_1^{ev}$ we can map the morphisms

$$\text{Map}_{\mathcal{D}_{M_\varphi}}(X, Y) = \text{Map}_{\mathcal{D}}(1, \hat{X} \otimes Y) \to \text{Map}_{\mathcal{D}}(X, Y)$$

via the duality structure - to a morphism $f : 1 \to \hat{X} \otimes Y$ one associates the morphism $\hat{f} : X \to Y$ given as the composition

$$X \xrightarrow{1d \otimes f} X \otimes \hat{X} \otimes Y \xrightarrow{\varphi(\text{ev}_X) \otimes Y} Y$$

It is quite direct to verify that $N_\varphi$ is a functor of (symmetric monoidal) non-unital $\infty$-categories, i.e. it respects composition and monoidal products in a natural way. Since $\mathcal{D}$ has duals we get that $N_\varphi$ is fully faithful and since $\varphi$ is essentially surjective $\varphi$ we get that $N_\varphi$ is essentially surjective. Hence $N_\varphi$ is an
equivalence and so $D_M$ is equivalent to the underlying non-unital $\infty$-category of $D$.

Informally speaking one can say that we are almost able to reconstruct $D$ out of $M_M$ - we are just missing the identity morphisms. However, note that if $M$ is non-degenerate then $D_M$ is not a completely arbitrary non-unital $\infty$-category. In fact it is very close to being unital - a non-degenerate object in $M(\tilde{X} \otimes X)$ gives a morphism which behaves like an identity map. Hence in some sense, we can reconstruct the units of $D$ as well. To make this idea precise we will need a good theory of quasi-unital $\infty$-categories.

3 Quasi-unital $\infty$-Categories

Throughout this section we will assume that the reader is familiar with the formalism of Segal spaces and their connection with $\infty$-categories. Our purpose is to study the non-unital analogue of this construction, obtained by replacing simplicial spaces with semi-simplicial spaces.

Let $X$ be a semi-simplicial space. Let $[n], [m] \in \Delta_s$ be two objects and consider the commutative (pushout) diagram

$$
\begin{array}{ccc}
[n] & \xrightarrow{f_{n,m}} & [n+m] \\
\downarrow^{g_{n,m}} & & \downarrow_{n+m} \\
[m] & \xrightarrow{g_{n,m}} & [n+m]
\end{array}
$$

where $f_{n,m}(i) = i$ and $g_{n,m}(i) = i + n$. We will say that $X$ satisfies the Segal condition if for each $[n], [m]$ as above the induced commutative diagram

$$
\begin{array}{ccc}
X_{m+n} & \xrightarrow{g_{n,m}} & X_m \\
\downarrow^{f_{n,m}} & & \downarrow_{0^*} \\
X_n & \xrightarrow{n^*} & X_0
\end{array}
$$

is a homotopy pullback diagram. We will say that $X$ is a semiSegal space if it is Reedy fibrant and satisfies the Segal condition. Note that in that case the above square will induce a homotopy equivalence

$$X_{m+n} \simeq X_m \times_{X_0} X_n$$

Example 2. Let $\mathcal{C}$ be a non-unital small topological category. We can represent $\mathcal{C}$ as a semiSegal space as follows. For each $n$, let $\mathcal{C}^m([n])$ denote the non-unital Top-enriched category whose objects are the numbers $0, \ldots, n$ and whose mapping spaces are

$$\text{Map}_{\mathcal{C}^m([n])}(i, j) = \begin{cases} \\ \emptyset & i \geq j \\ I(i,j) & i < j \end{cases}$$
where \((i,j) = \{x \in \{0,\ldots,n\}|i < x < j\}\). The composition is given by the inclusion
\[
I^{(i,j)} \times I^{(j,k)} \cong I^{(i,j)} \times \{0\} \times I^{(j,k)} \subseteq I^{(i,k)}
\]
Note that \(C^{\mathrm{nu}}([n])\) depends functorially on \([n] \in \Delta_s\). Hence for every non-unital topological category \(\mathcal{C}\) we can form a semi-simplicial space \(N(\mathcal{C})\) by setting
\[
N(\mathcal{C})_n = \text{Fun}(C^{\mathrm{nu}}([n]), \mathcal{C})
\]
We endow \(N(\mathcal{C})_n\) with a natural topology that comes from the topology of the mapping space of \(\mathcal{C}\) (while treating the set of objects of \(\mathcal{C}\) as discrete). One can then check that \(N(\mathcal{C})\) is a semiSegal space.

We think of general semiSegal spaces \(X\) as relaxed versions of Example \([\ast]\), i.e. as a non-unital \(\infty\)-category. The objects of the corresponding non-unital \(\infty\)-category are the points of \(X_0\). Given two points \(x,y \in X_0\) we define the mapping space between them by
\[
\text{Map}_X(x,y) = \{x\} \times_{X_0} X_1 \times_{X_0} \{y\}
\]
i.e., as the fiber of the (Kan) fibration
\[
X_1 \overset{(d_0,d_1)}{\longrightarrow} X_0 \times X_0
\]
over the point \((x,y)\).

To see how composition works consider first the case of a topological category \(\mathcal{C}\) and assume that we are given three objects \(x,y,z \in \mathcal{C}\) and a morphism \(f : x \to y\). One would then obtain a composition-by-\(f\) maps
\[
f_* : \text{Hom}_\mathcal{C}(z,x) \to \text{Hom}_\mathcal{C}(z,y)
\]
and
\[
f^* : \text{Hom}_\mathcal{C}(y,z) \to \text{Hom}_\mathcal{C}(x,z)
\]
In the semiSegal model we do not have such strict composition. Instead one can describe the composition-by-\(f\) maps as correspondences. If \(x,y,z \in X_0\) are objects and \(f : x \to y\) is a morphism (i.e. an element in \(X_1\) such that \(d_0(f) = x\) and \(d_1(f) = y\)) one can consider the space \(C^R_{f,z} \subseteq X_2\) given by
\[
C^R_{f,z} = \{\sigma \in X_2 | \sigma|_{\Delta(1,2)} = f, \sigma|_{\Delta(0)} = z\}
\]
Then the two restriction maps \(\sigma \mapsto \sigma|_{\Delta(0,1)}\) and \(\sigma \mapsto \sigma|_{\Delta(0,2)}\) give us a correspondence (recall that \(X\) is Reedy fibrant and so the restriction maps are fibrations):
This correspondence describes the operation of composing with $f$ on the right. Similarly we have a correspondence

\[
\begin{array}{ccc}
C^L_{f,z} & \rightarrow & \text{Map}_X(x,z) \\
\downarrow & & \downarrow \\
\text{Map}_X(y,z) & \rightarrow & \text{Map}_X(x,z)
\end{array}
\]

desccribing composition with $f$ on the left. The Segal condition ensures that both $C^R_{f,z}$ and $C^L_{f,z}$ are map-like correspondences, i.e. the left hand side maps are weak equivalences. In that sense composition is "almost" well-defined.

We want to define properties of $f$ via analogous properties of the correspondences $C^R_{f,z}, C^L_{f,z}$. In particular we will want to define when a morphism is a quasi-unit and when it is invertible. For this we will need to first understand how to say this in terms of correspondences.

Recall that from each space $X$ to itself we have the identity correspondence $X \xleftarrow{\text{id}} X \xrightarrow{\text{id}} X$. We will say that a correspondence $X \xleftarrow{\varphi} C \xrightarrow{\psi} X$ is unital if it is equivalent to the identity correspondence. It is not hard to check that a correspondence as above is unital if and only if both $\varphi, \psi$ are weak equivalences and are homotopic to each other in the Kan model structure.

We will say that a correspondence $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ is invertible if it admits an inverse, i.e. if there exists a correspondence $Y \xleftarrow{\varphi'} D \xrightarrow{\psi'} X$ such that the compositions

\[
X \xleftarrow{\varphi} C \times_Y D \rightarrow X
\]

and

\[
Y \xleftarrow{\varphi'} D \times_X C \rightarrow Y
\]

are unital.

**Remark 3.1.** Note that if a correspondence

\[
X \xleftarrow{\varphi} C \xrightarrow{\psi} Y
\]

is map-like (i.e. if $\varphi$ is invertible) then it is equivalent to a correspondence of the form

\[
X \xleftarrow{\text{id}} D \xrightarrow{f} Y
\]

such that $f$ represents the class $[\psi] \circ [\varphi]^{-1}$ in the Kan homotopy category. In this case the invertibility of $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ is equivalent $f$ being a weak equivalence, i.e. to $\psi$ being a weak equivalence.

Now let $X$ be a semiSegal space. Through the point of view of correspondences we have a natural way to define invertibility and unitality of morphisms:

**Definition 3.2.** 1. Let $x, y \in X_0$ be two objects and $f : x \rightarrow y$ a morphism in $X$. We will say that $f$ is right-invertible if for every $z \in X_0$ the right composition correspondence

\[
\text{Map}_X(z, x) \xleftarrow{\text{Map}_X(z, f)} \text{Map}_X(z, y)
\]
is invertible. Similarly one says that \( f \) is **left-invertible** if for every \( z \in X_0 \) the left composition correspondence

\[
\text{Map}_X(y, z) \leftarrow C^{L}_{f,z} \rightarrow \text{Map}_X(x, z)
\]

is invertible. We say that \( f \) is **invertible** if it is both left invertible and right invertible.

2. Let \( x \in X_0 \) be an object and \( f : x \to x \) a morphism in \( X \). We will say that \( f \) is a **quasi-unit** if for each \( z \in X_0 \) the correspondences

\[
\text{Map}_X(x, z) \leftarrow C^{R}_{f,z} \rightarrow \text{Map}_X(x, z)
\]

and

\[
\text{Map}_X(z, x) \leftarrow C^{L}_{f,z} \rightarrow \text{Map}_X(z, x)
\]

are **unital**.

**Remark 3.3.** From Remark 5.1 we see that a morphism \( f : x \to y \) in \( X \) is invertible if and only if for each \( z \in X_0 \) the restriction maps

\[
C^{R}_{f,z} \rightarrow \text{Map}_X(z, y)
\]

\[
C^{L}_{f,z} \rightarrow \text{Map}_X(x, z)
\]

are weak equivalences.

Invertible morphisms can be described informally as morphisms such that composition with them induces a weak equivalence on mapping spaces. Note that the notion of invertibility does not presupposed the existence of identity morphisms, i.e. it makes sense in the non-unital setting as well.

We will denote by

\[
X_{1}^{\text{inv}} \subseteq X_1
\]

the maximal subspace spanned by the invertible vertices \( f \in (X_1)_0 \). Using Reedy fribancy it is not hard to show that if \( f, g \in X_1 \) are connected by a path in \( X_1 \) then \( f \) is invertible if and only if \( g \) is invertible. Hence \( X_{1}^{\text{inv}} \) is just the **union of connected components of** \( X_1 \) which meet invertible edges.

We will denote by

\[
\text{Map}_{X}^{\text{inv}}(x, y) = \{x\} \times_{X_0} X_{1}^{\text{inv}} \times_{X_0} \{y\} \subseteq \text{Map}_{X}(x, y)
\]

the subspace of invertible morphisms from \( x \) to \( y \).

**Definition 3.4.** Let \( X \) be a semiSegal space. We will say that \( X \) is **quasi-unital** if every \( x_0 \in X_0 \) admits a quasi-unit from \( x_0 \) to \( x_0 \). We say that a map \( f : X \to Y \) of quasi-unital semiSegal spaces is **unital** if it maps quasi-units to quasi-units. We will denote by

\[
\text{QsS}
\]

the topological category of quasi-unital semi-simplicial spaces and unital maps between them.
We will be interested in studying the category $\mathcal{Q}_{\text{S}}$ up to a natural notion of equivalences, given by **Dwyer-Kan equivalences**. This is a direct adaptation of the notion of DK-equivalence of $\infty$-categories to the quasi-unital setting.

We start with a slightly more general notion of fully-faithful maps:

**Definition 3.5.** Let $f : X \to Y$ be map of semiSegal spaces. We will say that $f$ is a **fully-faithful** for all $x, y \in X_0$ the induced map

$$\text{Map}_X(x, y) \to \text{Map}_Y(f_0(x), f_0(y))$$

is a weak equivalence.

The notion of Dwyer-Kan equivalences will be obtained from the notion of fully-faithful maps by requiring the appropriate analogue of "essential surjectivity". For this let us introduce some terminology.

**Definition 3.6.** Let $x, y \in X_0$ be two points. We say that $x$ and $y$ are **equivalent** (denoted $x \simeq y$) if there exists an invertible morphism $f : \in X_\text{inv}$ from $x$ to $y$.

**Lemma 3.7.** Let $X$ be a quasi-unital semiSegal space. Then $\simeq$ is an equivalence relation. We will refer to the corresponding set of equivalence classes as the set of **equivalence-types** of $X$.

**Definition 3.8.** Let $f : X \to Y$ be a map between quasi-unital semiSegal spaces. We will say that $f$ is a **Dwyer-Kan equivalence** (DK for short) if it is fully faithful and induces a surjective map on the set of equivalence-types.

**Remark 3.9.** A DK-equivalence $f : X \to Y$ is automatically a unital map.

We propose to model the $\infty$-category of small quasi-unital $\infty$-categories as the localization of $\mathcal{Q}_{\text{S}}$ with respect to DK-equivalence. This is analogous to modeling the $\infty$-category of small $\infty$-categories as the localization of the category of Segal spaces with respect to DK-equivalence. We have a natural forgetful functor between these two localizations. Our main result us that this functor is an **equivalence of categories**.

In his fundamental paper [Rez] Rezk constructs (using the framework of model categories) an explicit model for this localization in terms of complete Segal spaces. We propose an analogous model for the quasi-unital case as follows:

**Definition 3.10.** Let $X$ be a semiSegal space. We will say that $X$ is **complete** if the restricted maps $d_0 : X_\text{inv} \to X_0$ and $d_1 : X_\text{inv} \to X_0$ are both homotopy equivalences.

An important observation is that any complete semiSegal space is quasi-unital: since the map $X_\text{inv} \to X_0$ is a trivial fibration every object $x \in X_0$ admits an invertible morphism of the form $f : x \to y$ for some $y$. This implies that $x$ admits a quasi-unit.
Let $CsS \subseteq QsS$ denote the full topological subcategory spanned by complete semiSegal spaces. We claim that the topological category $CsS$ can serve as a model for the localization of $QsS$ by DK-equivalences. Formally speaking (see Definition 5.2.7.2 and Proposition 5.2.7.12 of [Lur3]) this means that there exists a functor

\[ \hat{\bullet} : QsS \to CsS \]

such that:

1. $\hat{\bullet}$ is homotopy left adjoint to the inclusion $CsS \subseteq QsS$.

2. A map in $QsS$ is a DK-equivalence if and only if its image under $\hat{\bullet}$ is a homotopy equivalence.

Let $CS$ be the topological category of complete Segal spaces. Our main result of this section can now be stated as follows:

**Theorem 3.11.** The forgetful functor

\[ CS \to CsS \]

is an equivalence.

## 4 Completion of the Proof

Let us now go back to the construction $M \mapsto D_M$ described above. When $M$ is non-degenerate we get that $D_M$ is quasi-unital. Furthermore, any non-degenerate natural transformation $M \to N$ will induce a unital functor $D_M \to D_N$. Hence the construction $M \mapsto D_M$ determines a functor

\[ G : \text{Fun}_{\text{nd}}(B_1^{\text{ev}}, \text{Grp}_\infty) \to \text{Cat}_{\text{Bun}_0}^{\text{q}, \odot} \]

where $\text{Cat}_{\text{Bun}_0}^{\text{q}, \odot}$ is the $\infty$-category of symmetric monoidal quasi-unital categories (i.e. commutative algebra objects in the $\infty$-category $\text{Cat}_{\text{q}}^\infty$ of quasi-unital $\infty$-categories). Since the forgetful functor

\[ S : \text{Cat} \to \text{Cat}_{\text{q}}^\infty \]

From $\infty$-categories to quasi-unital $\infty$-categories is an equivalence we get that the induced forgetful functor

\[ S_* : \text{Cat}_{\text{Bun}_0}^{\odot} \to \text{Cat}_{\text{Bun}_0}^{\text{q}, \odot} \]

is an equivalence as well.

Composing $G$ with the $\varphi \mapsto M_\varphi$ functor

\[ F : \text{Cat}_{\text{Bun}_0}^{\text{sur}} \to \text{Fun}_{\text{nd}}(B_1^{\text{ev}}, \text{Grp}_\infty) \]

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described above we get a functor

\[ G \circ F : \text{Cat}^{\text{sur}}_{B^{\text{ev}}} / \longrightarrow \text{Cat}^{\text{qu.}}_{B^{\text{un}}} / \]

and we have a homotopy commutative diagram:

\[
\begin{array}{ccc}
\text{Cat}^{\text{sur}}_{B^{\text{ev}}} / & \xrightarrow{T} & \text{Cat}^{\text{qu.}}_{B^{\text{un}}} / \\
\text{Fun}_{\text{ind}}^{\text{lax}} (B^{\text{ev}}_1, \text{Grp}_\infty) & \xrightarrow{G} & \text{Cat}^{\text{qu.}}_{B^{\text{un}}} /
\end{array}
\]

where \( T \) is given by restriction along \( X_+ : B^{\text{un}} \longrightarrow B^{\text{ev}}_1 \). Now from Lemma 1.8 we see that \( T \) is fully faithful. Since \( S_* \) is an equivalence of \( \infty \)-categories we get

**Corollary 4.1.** The functor \( G \circ F \) is fully faithful.

We are now ready to complete the proof of 1.10. Let \( \mathcal{D} \) be a symmetric monoidal \( \infty \)-category with duals and let \( \varphi : B \longrightarrow \mathcal{D} \) be a non-degenerate functor. We wish to show that the space of maps

\[ \text{Map}_{\text{Cat}^{\text{sur}}_{B^{\text{ev}}} /} (t, \varphi) \]

is contractible. Consider the sequence

\[ \text{Map}_{\text{Cat}^{\text{sur}}_{B^{\text{ev}}} /} (t, \varphi) \longrightarrow \text{Map}_{\text{Fun}_{\text{ind}}^{\text{lax}} (B^{\text{ev}}_1, \text{Grp}_\infty)} (M_t, M_\varphi) \longrightarrow \text{Map}_{\text{Cat}^{\text{qu.}}_{B^{\text{un}}} /} (B^{\text{or}}_1, \mathcal{D}) \]

By Theorem 2.6 the middle space is contractible and by lemma 4.1 the composition

\[ \text{Map}_{\text{Cat}^{\text{sur}}_{B^{\text{ev}}} /} (t, \varphi) \longrightarrow \text{Map}_{\text{Cat}^{\text{qu.}}_{B^{\text{un}}} /} (B^{\text{or}}_1, \mathcal{D}) \]

is a weak equivalence. Hence we get that

\[ \text{Map}_{\text{Cat}^{\text{sur}}_{B^{\text{ev}}} /} (t, \varphi) \]

is contractible. This completes the proof of Theorem 1.10.

**References**

Harpaz, Y. Quasi-unital $\infty$-categories, PhD Thesis.


