GOODWILLIE APPROXIMATION TO HIGHER CATEGORIES AFTER GIJS HEUTS

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Contents

A useful idea in group theory is to study a group via its various abelian, or, more generally, nilpotent, quotients. For a given group G, these can all be obtained as quotients of the groups appearing in the central tower

$$G \to \dots \to G/G_n \to \dots \to G/G_1 \to G/G_0 = *,$$

where $G_0 = G$ and $G_{i+1} = [G, G_i]$ for $i \geq 0$. In particular, G/G_1 is the abelianization of G, and more generally $G \to G/G_n$ is the universal map from G to a nilpotent group of degree n (that is, a group whose lower central series has length n). One of the useful features of this tower is that each successive map $G/G_{n+1} \to G/G_n$ is a *central extension* with kernel G_n/G_{n+1}. In particular, its first step is an abelian group, and each further step is in some sense linear (or more precisely, affine) over the previous step. The isomorphism type of G/G_n is hence completely determined by the isomorphism type of G/G_0, together with, for each $i = 1, \dots, n-1$, the class

$$\alpha_i \in H^2(G/G_i, G_i/G_{i+1})$$

classifying the central extension $G/G_{i+1} \to G/G_i$. When the central tower converges, that is, when the map $G \to \lim_n G/G_n$ is an isomorphism, this data also determines G itself.

In homotopy theory, it is natural to ask for the analogue of this construction in the setting of E_1-*groups*, that is, group-like E_1-monoids in the ∞-category $\mathcal{S}$ of spaces. These are all obtained as the loop E_1-groups of pointed connected spaces; in fact, the formation of loops induces an equivalences between the ∞-category $\mathcal{S}_*^{\geq 1}$ of pointed connected spaces and that of E_1-groups. The modern approach to this problem is that for a pointed space X, the analogue of the universal degree n nilpotent quotient for the loop E_1-group ΩX is given by $\Omega \mathcal{P}_n(X)$, where $\mathcal{P}_n(X)$ is the n'th (pointed) space in the *Goodwillie tower*

$$X \to \dots \to \mathcal{P}_n(X) \to \dots \to \mathcal{P}_1(X) \to *$$

of X, as considered in the previous talks (here, it does not matter if we take the Goodwillie tower in $\mathcal{S}_*$ or $\mathcal{S}_*^{\geq 1}$, the two towers coincide). For example, one can show that $\pi_1\mathcal{P}_n(X)$ is the universal degree n nilpotent quotient of $\pi_1(X)$, so that on the level of fundamental groups the Goodwillie tower reproduces the central tower of π_1, see the works of Biederman and Dwyer [BD10, Bie17] on the topic. The first step in the tower $\mathcal{P}_1(X) = \Omega^\infty\Sigma^\infty(X)$ is the infinite loop space of the suspension spectrum of X, which can be considered as the universal map from X to a *linear object* (on the level of ΩX, this can be considered as the analogous of abelianization). In addition, the gap between each two successive terms in the Goodwillie tower is again linear in some sense: the map $\mathcal{P}_{n+1}(X) \to \mathcal{P}_n(X)$ is in fact a *principal fibration* with structure group an infinite loop space of the form $\Omega^\infty((E_n \otimes \Sigma^\infty X \otimes \ldots \otimes \Sigma^\infty X)_{\mathrm{h}\Sigma_n})$ for a certain spectrum with Σ_n-action E_n. The collection of spectra E_n are exactly the *Goodwillie derivatives* of the identity $\mathrm{id}_{\mathcal{S}_*}$. The equivalence type of $\mathcal{P}_n(X)$ is hence completely determines by the spectrum $\Sigma^\infty(X)$, together with, for each $i = 1, \ldots, n-1$, the class

$$\alpha_i \in H^1(\mathcal{P}_i(X), (E_n \otimes \Sigma^\infty X \otimes \ldots \otimes \Sigma^\infty X)_{\mathrm{h}\Sigma_n}) = \pi_0 \operatorname{Map}(X, \Omega^{\infty-1}((E_n \otimes \Sigma^\infty X \otimes \ldots \otimes \Sigma^\infty X)_{\mathrm{h}\Sigma_n}))$$

classifying the principal fibration $\mathcal{P}_{i+1}(X) \to \mathcal{P}_i(X)$.

The idea pursued in Gijs's thesis [Heu21] is to obtain a similar Goodwillie tower on the level of ∞-categories. Before we can describe this idea, let us first recall the basic set up of Goodwillie calculus, focusing on the case of pointed compactly ∞-categories and reduced functors between them.

1. Goodwillie calculus on pointed compactly generated ∞-categories

Recall that an ∞-category $\mathcal{C}$ is said to be *compactly generated* if it has small colimits and is generated under colimits by compact objects. Such a $\mathcal{C}$ is then of the form $\operatorname{Ind}(\mathcal{C}^c)$, where $\mathcal{C}^c \subseteq \mathcal{C}$ is the full subcategory spanned by compact objects. A *compactly generated functor* $f\colon \mathcal{C} \to \mathcal{D}$ between compactly generated ∞-categories is a functor which preserves colimits and compact objects. We will say that a compactly generated ∞-category $\mathcal{C}$ is *pointed* if its initial object is also terminal, in which case we will call this object a *zero object*, and write it as $0 \in \mathcal{C}$. A functor which preserves zero objects is called *reduced*. In particular, any compactily generated functor between pointed compactly generated ∞-categories is reduced. We will denote by $\mathcal{C}\mathrm{at}_*^\omega$ the ∞-category of pointed compactly generated ∞-categories and compactly generated functors between them.

A functor between compactly generated ∞-categories is said to be *finitary* if it preserves filtered colimits. We will write $\operatorname{Fun}_*^\omega(\mathcal{C}, \mathcal{D})$ for the ∞-category of reduced finitary functors from $\mathcal{C}$ to $\mathcal{D}$. In paticular, any compactly generated functor $f\colon \mathcal{C} \to \mathcal{D}$ is finitary and furthermore, any compactly generated functor f has a right adjoint $g\colon \mathcal{D} \to \mathcal{C}$ which is itself finitary. In fact, the compactly generated functors are exactly the finitary functors which admit a finitary right adjoint. We will consequently write $\operatorname{LFun}_*^\omega(\mathcal{C}, \mathcal{D})$ for the ∞-category of compactly generated functors $\mathcal{C} \to \mathcal{D}$.

We will now discuss Goodwillie calculus in the setting of reduced functors. We denote by $\mathbf{P}(n)$ the poset of subsets of $\{1, \ldots, n\}$, ordered by inclusion. In particular, when we consider this poset as a category, it has an initial object given by the empty set, and a final object given by $\{1, \ldots, n\}$. We write $\mathbf{P}_{\geq 1}(n) \subseteq \mathbf{P}(n)$ for the full subposet spanned by the subsets of side is at least 1 (that is, the non-empty subsets) of $\mathbf{P}(n)$, and write $\mathbf{P}_{\leq 1}(n) \subseteq \mathbf{P}(n)$ for the full subposet spanned by the subsets whose size as at most 1.

Let us now fix a pointed compactly generated ∞-category $\mathcal{C}$. We will refer to functors $\mathbf{P}(n) \to \mathcal{C}$ as n-cubes, and to functors $\mathbf{P}_{\geq 1}(n) \to \mathcal{C}$ as punctured n-cubes. We will say that an n-cube $\rho\colon \mathbf{P}(n) \to \mathcal{C}$ is *cartesian* if ρ is right Kan extended from $\rho|_{\mathbf{P}_{\geq 1}(n)}$, and *strongly cocartesian* if ρ is left Kan extended from $\rho|_{\mathbf{P}_{\leq 1}(n)}$. Equivalently, ρ is cartesian if and only if the induced map

$$\rho(\varnothing) \to \lim_{\varnothing \neq I} \rho(I)$$

is an equivalence, and ρ is strongly cocartesian if and only if the map

$$\rho(\{i_1\}) \coprod_{\rho(\{\})} \rho(\{i_2\}) \coprod_{\rho(\{\})} \dots \coprod_{\rho(\{\})} \rho(\{i_k\}) \to \rho(I)$$

determined by ρ is an equivalence. The latter condition is also equivalent to the condition that for every $I, J \subseteq \{n\}$ the square

(1)
$$\begin{tikzcd} \rho(I \cap J) \arrow[r] \arrow[d] & \rho(I) \arrow[d] \\ \rho(J) \arrow[r] & \rho(I \cup J) \end{tikzcd}$$

is cocartesian. Let us say that a punctured cube $\rho\colon \mathbf{P}_{\geq 1}(n) \to \mathcal{C}$ is *face-wise cocartesian* if for every $I, J \in \mathbf{P}_{\geq 1}(n)$ such that $I \cap J \in \mathbf{P}_{\geq 1}(n)$ the square (1) is cocartesian. Note that unlike the case of whole cubes, the face-wise condition for punctured cubes is not equivalent to ρ being left Kan extended from some subposet of $\mathbf{P}_{\geq 1}(n)$. Finally, for any subposet $P \subseteq \mathbf{P}(n)$, we will say that $\rho\colon P \to \mathcal{C}$ is *reduced* if $\rho(I)$ is a zero object of $\mathcal{C}$ for any $I \in P$ such that $|I| = 1$. We will use this term for ρ a cube, a punctured cube, or a functor defined on $\mathbf{P}_{\leq 1}(n)$. Let us thus write $\mathcal{N}_n(\mathcal{C}), \mathcal{N}_n^{\geq 1}(\mathcal{C})$ and $\mathcal{N}_n^{\leq 1}(\mathcal{C})$ for the full subcategories of $\mathrm{Fun}(\mathbf{P}(n), \mathcal{C}), \mathrm{Fun}(\mathbf{P}_{\geq 1}(n), \mathcal{C})$ and $\mathrm{Fun}(\mathbf{P}_{\leq 1}(n), \mathcal{C})$, respectively, spanned by the reduced functors.

We note that a functor $\rho\colon \mathbf{P}_{\leq 1}(n) \to \mathcal{C}$ is reduced if and only if it is right Kan extended from $\rho|_{\{\varnothing\}}$. In particular, evaluation at $\varnothing$ and right Kan extension yield inverse equivalences

(2)
$$\mathcal{C} \underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}} \mathcal{N}_n^{\leq 1}(\mathcal{C}).$$

We we may also consider the adjunctions

$$\mathcal{N}_n^{\leq 1}(\mathcal{C}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{N}_n(\mathcal{C}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{N}_n^{\geq 1}(\mathcal{C})$$

where the first left adjoint is given by left Kan extension along $\mathbf{P}_{\leq 1} \subseteq \mathbf{P}(n)$ and the second left adjoint by restriction along $\mathbf{P}_{\geq 1}(n) \subseteq \mathbf{P}(n)$ (their right adjoints are given respectively by restriction along $\mathbf{P}_{\leq 1}(n) \subseteq \mathbf{P}(n)$ and right Kan extension along $\mathbf{P}_{\geq 1}(n) \subseteq \mathbf{P}(n)$. Composing these adjunctions and identifying $\mathcal{N}_n^{\leq 1}(\mathcal{C})$ with $\mathcal{C}$ via (2) we obtain an adjunction

$$L_n\colon \mathcal{C} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{N}_n^{\geq 1}(\mathcal{C}) : R_n.$$

Explicitly, the left adjoint L_n sends $X \in \mathcal{C}$ to the restriction to $\mathbf{P}_{\geq 1}(n)$ of the strongly cocartesian n-cube

$$\rho_X(I) = 0 \coprod_X 0 \dots \coprod_X 0 \simeq \mathrm{cof}[\coprod_{i \in I} X \to X] \simeq \coprod_{|I|-1} \Sigma X,$$

where $0 \in \mathcal{C}$ is the zero object. The right adjoint R_n is given by $R_n(\rho) = \lim_{I \in \mathbf{P}_{\geq 1}(n)} \rho(I) \in \mathcal{C}$.

Definition 1. We will say that a reduced functor $f\colon \mathcal{C} \to \mathcal{D}$ is *n-excisive* if it sends strongly cocartesian $(n+1)$-cubes in $\mathcal{C}$ to cartesian cubes in $\mathcal{D}$.

Write

$$\mathrm{Fun}_*^{\leq n}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}_*^{\omega}(\mathcal{C}, \mathcal{D})$$

for the full subcategory spanned by the (finitary, reduced) n-excisive functors. For any finitary reduced functor $f\colon \mathcal{C} \to \mathcal{D}$ between pointed compactly generated ∞-categories we write $T_n(f)\colon \mathcal{C} \to \mathcal{D}$ for the composite

$$\mathcal{C} \xrightarrow{L_n} \mathcal{N}_n^{\geq 1}(\mathcal{C}) \xrightarrow{f_*} \mathcal{N}_n(\mathcal{D}) \xrightarrow{R_n} \mathcal{D},$$

where f_* denotes the functor induced on punctured cubes by applying f levelwise; this preserves reduced cubes since f is assumed to be reduced. The functor $T_n(f)$ is given more explicitly by the formula

$$T_n(f)(X) = \lim_{\varnothing \neq I} f(\rho_X(I)).$$

The inclusion of finitary reduced n-excisive functors inside all finitary reduced functors then admits a left adjoint

$$\mathcal{P}_n \colon \mathrm{Fun}^{\omega}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\leq n}_*(\mathcal{C}, \mathcal{D}),$$

given by the explicit formula

$$\mathcal{P}_n(f) = \mathrm{colim}[f \to T_n(f) \to T_n T_n(f) \to \ldots],$$

where the sequence is obtained by iterating the natural transformation $F \Rightarrow T_n(f)$ given by composite

$$f \Rightarrow R_n \circ L_n \circ f \Rightarrow R_n \circ f_* \circ L_n,$$

where the first map is the unit of $L_n \dashv R_n$ (pre-composed with f), and the second is induced by the canonical interchange map for colimits/left Kan extensions. We note that for $m \geq 0$ the functor $T_n^{(m)} := (T_n \circ \ldots \circ T_n)(f)$ with T_n composed m times is given by the composite

$$\mathcal{C} \xrightarrow{L_n^{(m)}} (\mathcal{N}_n^{\geq 1})^{(m)}\mathcal{C} \xrightarrow{f_*} (\mathcal{N}_n^{\geq 1})^{(m)}\mathcal{D} \xrightarrow{R_n^{(m)}} \mathcal{D}$$

where we have denoted by $(\mathcal{N}_n^{\geq 1})^{(m)}(-) := \mathcal{N}_n^{\geq 1}(\mathcal{N}_n^{\geq 1}(\ldots(\mathcal{N}_n^{\geq 1}(-))$ the m-fold iterated composite of the operation $\mathcal{N}_n^{\geq 1}(-)$, and by $L_n^{(m)}$ and $R_n^{(m)}$ the corresponding m-fold iterated composites of L_n and R_n, respectively. In these notations we may also write

$$\mathcal{P}_n(f) = \mathrm{colim}_m\, L_n^{(m)} \circ f_* \circ R_n^{(m)}.$$

Example 2. When $n = 1$ we have $T_1(f)(X) = \Omega f(\Sigma X)$, and $\mathcal{P}_1(f) = \mathrm{colim}_m\, \Omega^m f(\Sigma^m(-))$.

The resulting tower

$$f \Rightarrow \ldots \Rightarrow \mathcal{P}_n(f) \Rightarrow \ldots \Rightarrow \mathcal{P}_1(f)$$

is the Goodwillie tower of f.

Definition 3. We will say that a natural transformation $f \Rightarrow g$ of functors $\mathcal{C} \to \mathcal{D}$ is a $\mathcal{P}_n$*-equivalence* if the induced natural transformation $\mathcal{P}_n(f) \Rightarrow \mathcal{P}_n(g)$ is an equivalence.

Lemma 4. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be pointed compactly generated ∞-categories and let $f \colon \mathcal{C} \to \mathcal{D}$ and $g \colon \mathcal{D} \to \mathcal{E}$ be reduced functors which preserve filtered colimits. Then the functors*

$$(-) \circ f \colon \mathrm{Fun}^{\omega}_*(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{\omega}_*(\mathcal{C}, \mathcal{E}) \quad \textit{and} \quad g \circ (-) \colon \mathrm{Fun}^{\omega}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\omega}_*(\mathcal{C}, \mathcal{E})$$

preserve $\mathcal{P}_n$-equivalence.

Proof. We prove the claim for $(-) \circ f$, the case of $g \circ (-)$ is proven in an analogous manner (using the fact that post-composition with g preserves filtered colimits). Since g is arbitrary, by 2-out-of-3 it will suffice to show that the natural transformation $g \circ f \Rightarrow \mathcal{P}_n(g) \circ f$ is a $\mathcal{P}_n$-equivalence, that is, induces an equivalence $\mathcal{P}_n(g \circ f) \xrightarrow{\simeq} \mathcal{P}_n(\mathcal{P}_n(g) \circ f)$. Since each of the functors $R_n^{(m)}$ commutes with filtered colimits this last map can be identified with the map

$$\begin{aligned} \mathrm{colim}_{m \geq 0}\, R_n^{(m)} g_* f_* L_n^{(m)} &\to \mathrm{colim}_m\, R_n^{(m)}\, \mathrm{colim}_{k \geq 0}[R_n^{(k)} g_* L_n^{(k)}] f_* L_n^{(m)} \\ &= \mathrm{colim}_{m,k \geq 0}\, R_n^{(m+k)} g_* L_n^{(k)} f_* L_n^{(m)} \\ &=: \mathrm{colim}_{m,k}\, h_{m,k} \end{aligned}$$

induced on colimits by the poset inclusion $\mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ sending m to $(m, 0)$. This poset map is not cofinal, so in principle it is not supposed to induce an equivalence on colimits. We argue this step by constructing a poset Q equipped with a cofinal map $\mathbb{N} \times \mathbb{N} \to Q$, such that the composite $\mathbb{N} \to \mathbb{N} \times \mathbb{N} \to Q$ is cofinal as well, and then show that the functor $(m, k) \mapsto h_{m,k}$ extends to Q. We define Q to have the same elements as $\mathbb{N} \times \mathbb{N}$, namely, all pairs (m, k) with $m, k \geq 0$, but with a weaker order relation, namely, we set $(m, k) \leq_Q (m', k')$ if and only if $m \leq m'$ and $m + k \leq m' + k'$. We then have a map of poset $\mathbb{N} \times \mathbb{N} \to Q$ which is the identity on underlying sets. This map is cofinal since $\mathbb{N} \times \mathbb{N}$ is cofiltered and the relevant comma posets are non-empty, and hence also cofiltered. The same argument shows that the composite map $\mathbb{N} \to \mathbb{N} \times \mathbb{N} \to Q$ is cofinal. To finish the proof it will suffice to show that the functor $(m, k) \mapsto h_{m,k}$ extends to Q. Indeed, on objects it will be defined in the same manner and for $m \le m'$ and $k + m \le k' + m'$ the natural transformation $h_{m,k} \Rightarrow h_{m',k'}$ is given by the composite

$$\begin{aligned} R_n^{(k+m)} g_* L_n^{(k)} f_* L_n^{(m)} &\Rightarrow R_n^{(k+m)} R_n^{(k'+m'-k-m)} L_n^{(k'+m'-k-m)} g_* L_n^{(k)} f_* L_n^{(m)} \\ &\Rightarrow R_n^{(k'+m')} g_* L_n^{(k'+m'-m)} f_* L_n^{(m)} \\ &\Rightarrow R_n^{(k'+m')} g_* L_n^{(k')} f_* L_n^{(m'-m)} L_n^{(m)} \\ &= R_n^{(k'+m')} g_* L_n^{(k')} f_* L_n^{(m')}. \end{aligned}$$

where the first map is induced by the unit of $L_n^{(k'+m'-k-m)} \dashv R_n^{(k'+m'-k-m)}$, and all the other maps are given by the canonical interchange map for colimits/left Kan extensions. □

2. Categorical Goodwillie calculus

We now arrive to the work of [Heu21].

2.1. **Excisive ∞-categories and excisive equivalences.**

Definition 5. Let $\mathcal{C}$ be a pointed compactly generated ∞-category. We will say that $\mathcal{C}$ is *n-excisive* if it satisfies the following two properties:

(1) Every strongly cocartesian $(n+1)$-cube in $\mathcal{C}$ is cartesian. In other words, $\mathrm{id}_{\mathcal{C}}$ is n-excisive.
(2) Every face-wise cocartesian punctured $(n+1)$-cube $\rho\colon \mathbf{P}_{\ge 1}(n+1) \to \mathcal{C}^c$ extends to an $(n+1)$-cube $\overline{\rho}\colon \mathbf{P}(n+1) \to \mathcal{C}^c$ which is both cartesian and strongly cocartesian.

We then write $\mathcal{C}\mathrm{at}_*^{\le n} \subseteq \mathcal{C}\mathrm{at}_*^{\omega}$ for the full subcategory spanned by the n-excisive ∞-categories.

Remark 6. Any $\mathbf{P}_{\ge 1}(n+1)$-indexed diagram in $\mathcal{C}$ is a filtered colimit of diagrams valued in $\mathcal{C}^c$, and hence any strongly cocartesian $(n+1)$-cube in $\mathcal{C}^c$ is a filtered colimit of strongly cocartesian cubes taking values in $\mathcal{C}^c$. In verifying Condition 1, one may hence restrict attention to strongly cocartesian $(n+1)$-cubes which are entry-wise compact.

Example 7. For $n = 1$ the condition that a punctured 2-cube be face-wise cocartesian is vacuous. Using Remark 6 we then get that a pointed compactly generated $\mathcal{C}$ is 1-excisive if and only if $\mathcal{C}^c$ has pullbacks and a commutative square in $\mathcal{C}^c$ is a pushout square if and only if it is a pullback square. In other words, $\mathcal{C}$ is 1-excisive if and only if $\mathcal{C}^c$ is stable, which in turn is equivalent to $\mathcal{C}$ being stable.

For $n = 1$ the inclusion $\mathcal{C}\mathrm{at}_*^{\le 1} \subseteq \mathcal{C}\mathrm{at}_*^{\omega}$ admits a left adjoint

$$\mathrm{Sp}(-)\colon \mathcal{C}\mathrm{at}_*^{\omega} \to \mathcal{C}\mathrm{at}_*^{\le 1}$$

given by the stabilization $\mathrm{Sp}(\mathcal{C}) = \lim_n[\mathcal{C} \xleftarrow{\Omega} \mathcal{C} \xleftarrow{\Omega} \mathcal{C} \xleftarrow{\Omega} \dots]$. This always results in a compactly generated ∞-category with the ∞-category of compact objects consisting of the Spanier-whitehead stabilization of $\mathcal{C}^c$, namely

$$\mathrm{Sp}(\mathcal{C}) = \mathrm{Ind}(\mathrm{Sp}(\mathcal{C})^c) = \mathrm{Ind}(\mathrm{Sp}^{\mathrm{SW}}(\mathcal{C}^c)) = \mathrm{Ind}\,\mathrm{colim}_n[\mathcal{C}^c \xrightarrow{\Sigma} \mathcal{C}^c \xrightarrow{\Sigma} \dots].$$

There is a canonical adjunction

$$\Sigma^\infty_{\mathcal{C}}\colon \mathcal{C} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathrm{Sp}(\mathcal{C}) : \Omega^\infty_{\mathcal{C}}.$$

where $\Sigma^\infty_{\mathcal{C}}$ is a compactly generated functor induced by the canonical map $\mathcal{C}^c \to \mathrm{Sp}^{\mathrm{SW}}(\mathcal{C}^c)$, and $\Omega^\infty_{\mathcal{C}}$ is a finitary right adjoint induced by projecting to $\mathcal{C}$ from its Ω-tower. For every stable compactly generated ∞-category $\mathcal{D}$ we then have that restriction along $\Sigma^\infty_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ induces an equivalence

$$\mathrm{LFun}_*^{\omega}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{LFun}_*^{\omega}(\mathcal{C}, \mathcal{D}).$$

The main goal of the work [Heu21] is to obtain a similar picture for higher n.

Definition 8. Let $f: \mathcal{C} \to \mathcal{D}$ be a compactly generated functor between pointed compactly generated ∞-categories and let $g: \mathcal{D} \to \mathcal{C}$ be its finitary right adjoint. We will say that f is an *n-excisive eqiuvalence* if the unit $\mathrm{id}_{\mathcal{C}} \Rightarrow gf$ and counit $fg \Rightarrow \mathrm{id}_{\mathcal{D}}$ are $\mathcal{P}_n$-equivalences.

Proposition 9. *Let $f: \mathcal{C} \to \mathcal{D}$ be a compactly generated functor between pointed compactly generated ∞-categories. Then the following conditions are equivalent:*

(1) f is an n-excisive equivalence.
(2) f induces an equivalence on stabilizations and the natural transformation

$$\Sigma^{\infty}_{\mathcal{D}} f g \Omega^{\infty}_{\mathcal{D}} \Rightarrow \Sigma^{\infty}_{\mathcal{D}} \Omega^{\infty}_{\mathcal{D}}$$

of functors $\mathrm{Sp}(\mathcal{D}) \to \mathrm{Sp}(\mathcal{D})$ induced by the counit of $f \dashv g$ is a $\mathcal{P}_n$-equivalence.
(3) f induces an equivalence on stabilizations and the unit $\mathrm{id}_{\mathcal{C}} \Rightarrow gf$ is a $\mathcal{P}_n$-equivalence.

Proof. We begin with (1) $\Rightarrow$ (2). Let $f: \mathcal{C} \to \mathcal{D}$ be a n-excisive equivalence and $g: \mathcal{D} \to \mathcal{C}$ its finitary right adjoint. We first prove that f induces an equivalence on stabilizations. Consider the 1-excisive approximations $\mathcal{P}_1(f)$ and $\mathcal{P}_1(g)$. They factor canonically as composites

$$\mathcal{C} \xrightarrow{\Sigma^{\infty}_{\mathcal{C}}} \mathrm{Sp}(\mathcal{C}) \xrightarrow{\partial_1 f} \mathrm{Sp}(\mathcal{D}) \xrightarrow{\Omega^{\infty}_{\mathcal{D}}} \mathcal{D}$$

and

$$\mathcal{D} \xrightarrow{\Sigma^{\infty}_{\mathcal{D}}} \mathrm{Sp}(\mathcal{D}) \xrightarrow{\partial_1 g} \mathrm{Sp}(\mathcal{C}) \xrightarrow{\Omega^{\infty}_{\mathcal{C}}} \mathcal{C},$$

where $\partial_1 f$ and $\partial_1 g$ are the first derivatives of f and g. In the case of f, the functor $\partial_1 f$ identifies with the functor $f_*: \mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$ induced by f on stabilizations; indeed, since f preserves colimits the construction $\mathrm{colim}_n \Omega^n f \Sigma^n$ which produces $\mathcal{P}_1(f)$ can also be viewed as $f \mapsto \Omega^{\infty}_{\mathcal{D}} f_* \Sigma^{\infty}_{\mathcal{C}}$. It will hence suffice to show that $\partial_1 f$ is an equivalence. Now since f is an n-excisive equivalence the unit and counit natural transformations are both $\mathcal{P}_n$-equivalence, and in particular $\mathcal{P}_1$-equivalences. Using the Klein-Rognes chain rule [Lur14, Corollary 6.2.1.24] we obtain:

$$\partial_1 f \circ \partial_1 g \simeq \partial_1 (f \circ g) \simeq \partial_1 \mathrm{Id}_{\mathcal{D}} \simeq \mathrm{Id}_{\mathrm{Sp}(\mathcal{D})}$$

and

$$\partial_1 g \circ \partial_1 f \simeq \partial_1 (g \circ f) \simeq \partial_1 \mathrm{id}_{\mathcal{C}} \simeq \mathrm{id}_{\mathrm{Sp}(\mathcal{C})}.$$

We then conclude that $\partial_1 f$ and $\partial_1 g$ are inverse equivalences of ∞-categories, and so f induces an equivalence on stabilization. In addition, since the counit $fg \Rightarrow \mathrm{Id}_{\mathcal{D}}$ is a $\mathcal{P}_n$-equivalence it follows form Lemma 4 that the induced map $\Sigma^{\infty}_{\mathcal{D}} f g \Omega^{\infty} \Rightarrow \Sigma^{\infty}_{\mathcal{D}} \Omega^{\infty}_{\mathcal{D}}$ is a $\mathcal{P}_n$-equivalence. This shows (1) $\Rightarrow$ (2).

Now assume that (2) holds, so that f induces an equivalence on stabilizations and the induced map $\Sigma^{\infty}_{\mathcal{D}} f g \Omega^{\infty} \Rightarrow \Sigma^{\infty}_{\mathcal{D}} \Omega^{\infty}_{\mathcal{D}}$ is a $\mathcal{P}_n$-equivalence. We note that saying that f induces an equivalence on stabilizations is the same as saying that $\Sigma^{\infty} f: \mathcal{C} \to \mathrm{Sp}(\mathcal{D})$ exhibits $\mathrm{Sp}(\mathcal{D})$ as the stabilization of $\mathcal{C}$, in which case we may simply identify $\Sigma^{\infty}_{\mathcal{C}} = \Sigma^{\infty}_{\mathcal{D}} f$ and $\Omega^{\infty}_{\mathcal{C}} = g \Omega^{\infty}$. The assumption of (2) then says that the associated map of endo-functors $\Sigma^{\infty}_{\mathcal{C}} \Omega^{\infty}_{\mathcal{C}} \to \Sigma^{\infty}_{\mathcal{D}} \Omega^{\infty}_{\mathcal{D}}$ is an equivalence. By a theorem of Arone and Ching, generalized to the setting of compactly generated ∞-categories in [Heu21], the map $\mathcal{P}_n(\mathrm{id}_{\mathcal{C}}) \to \mathcal{P}_n(gf)$ can be identified with the map induces on totalizations by the map of cosimplicial objects. More precisely, we may form a map of coaugemented cosimplicial objects

$$\begin{tikzcd}
\mathcal{P}_n(\mathrm{id}_{\mathcal{C}}) \arrow[r] \arrow[d] & \mathcal{P}_n(\Omega^{\infty}_{\mathcal{C}} \Sigma^{\infty}_{\mathcal{C}}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{P}_n(\Omega^{\infty}_{\mathcal{C}} \Sigma^{\infty}_{\mathcal{C}} \Omega^{\infty}_{\mathcal{C}} \Sigma^{\infty}_{\mathcal{C}}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \mathcal{P}_n(\Omega^{\infty}_{\mathcal{C}} \Sigma^{\infty}_{\mathcal{C}} \Omega^{\infty}_{\mathcal{C}} \Sigma^{\infty}_{\mathcal{C}} \Omega^{\infty}_{\mathcal{C}} \Sigma^{\infty}_{\mathcal{C}}) \cdots \arrow[d] \\
\mathcal{P}_n(gf) \arrow[r] & \mathcal{P}_n(g \Omega^{\infty}_{\mathcal{D}} \Sigma^{\infty}_{\mathcal{D}} f) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{P}_n(g \Omega^{\infty}_{\mathcal{D}} \Sigma^{\infty}_{\mathcal{D}} \Omega^{\infty}_{\mathcal{D}} \Sigma^{\infty}_{\mathcal{D}} f) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{P}_n(g \Omega^{\infty}_{\mathcal{C}} \Sigma^{\infty}_{\mathcal{D}} \Omega^{\infty}_{\mathcal{D}} \Sigma^{\infty}_{\mathcal{D}} \Omega^{\infty}_{\mathcal{D}} \Sigma^{\infty}_{\mathcal{D}} f) \cdots ,
\end{tikzcd}$$

such that the two exhibit their coaugmentation as their totalization. Identifying $g \Omega^{\infty}_{\mathcal{D}} = \Omega^{\infty}_{\mathcal{C}}$ and $\Sigma^{\infty}_{\mathcal{D}} f \simeq g \Omega^{\infty}_{\mathcal{D}} = \Omega^{\infty}_{\mathcal{C}}$, the map of totalizations is an equivalence by the assumption that the map $\Sigma^{\infty}_{\mathcal{C}} \Omega^{\infty}_{\mathcal{C}} \to \Sigma^{\infty}_{\mathcal{D}} \Omega^{\infty}_{\mathcal{D}}$ is a $\mathcal{P}_n$-equivalence. This shows that (2) $\Rightarrow$ (3).

Finally, let us show that (3) ⇒ (1). We hence assume that $f\colon \mathcal{C} \to \mathcal{D}$ induces an equivalence on stabilizations and that the unit map $\mathrm{id}_{\mathcal{C}} \Rightarrow gf$ is a $\mathcal{P}_n$-equivalence. We need to show that the counit map $fg \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a $\mathcal{P}_n$-equivalence. In other words, we need to show that the induced map $\mathcal{P}_n(fg) \Rightarrow \mathcal{P}_n(\mathrm{id}_{\mathcal{D}})$ is an equivalence. To begin, note that by the triangle equalities this map is an equivalence on objects of the form $f(X)$ for $X \in \mathcal{C}$. More precisely, since $\mathrm{id}_{\mathcal{C}} \Rightarrow gf$ is a $\mathcal{P}_n$-equivalence we have by Lemma 4 that $f \Rightarrow fgf$ is a $\mathcal{P}_n$-equivalence, and since the composite $f \Rightarrow fgf \Rightarrow f$ is homotopic to the identity we conclude that $fgf \Rightarrow f$ is a $\mathcal{P}_n$-equivalence. Using that f preserves colimits this means that

$$\mathcal{P}_n(fg)f = \mathcal{P}_n(fgf) \Rightarrow \mathcal{P}_n(f) = \mathcal{P}_n(\mathrm{id}_{\mathcal{D}})f$$

is an equivalence. In other words, the natural transformation $\mathcal{P}_n(gf) \Rightarrow \mathcal{P}_n(\mathrm{id})$ is an equivalence on objects of the form $f(X)$.

For every $Y \in \mathcal{D}$, the object $\Sigma^{\infty}_{\mathcal{D}}(Y) \in \mathrm{Sp}(\mathcal{D}) \simeq \mathrm{Sp}(\mathcal{C})$ is compact, and hence of the form $\Sigma^{\infty-n}_{\mathcal{C}}(X) = \Sigma^{\infty-n}_{\mathcal{D}}(f(X))$ for some $X \in \mathcal{C}$. Since

$$\mathrm{Map}(\Sigma^{\infty-n}_{\mathcal{D}} f(X), \Sigma^{\infty}_{\mathcal{D}} Y) = \mathrm{colim}_{m \geq n} \mathrm{Map}_{\mathcal{D}}(\Sigma^{m-n} f(X), \Sigma^m Y)$$

and

$$\mathrm{Map}(\Sigma^{\infty}_{\mathcal{D}} Y, \Sigma^{\infty-n}_{\mathcal{D}} f(X)) = \mathrm{colim}_{m \geq n} \mathrm{Map}_{\mathcal{D}}(\Sigma^m Y, \Sigma^{m-n} f(X))$$

we deduce that a pair of inverse equivalences $\Sigma^{\infty-n} f(X) \underset{\longleftarrow}{\xrightarrow{\simeq}} \Sigma^{\infty}(Y)$ lifts to a pair of inverse equivalences $\Sigma^{m-n} f(X) \underset{\longleftarrow}{\xrightarrow{\simeq}} \Sigma^m Y$ for large enough m. In particular, some large enough suspension of Y is in the essential image of $f\colon \mathcal{C} \to \mathcal{D}$. Since f preserves colimits, every finite coproduct of copies of $\Sigma^m Y$ is in the image of f as well. By the above we consequently have that the map $\mathcal{P}_n(fg)Z \to \mathcal{P}_n(\mathrm{id}_{\mathcal{D}})Z$ is an equivalence whenever Z is equivalent to a finite coproduct of copies of $\Sigma^m Y$. To finish the proof we will now show that for every $0 \leq k \leq m$, the map $\mathcal{P}_n(fg)Z \to \mathcal{P}_n(\mathrm{id}_{\mathcal{D}})Z$ is an equivalence for every Z which is equivalent to a finite coproduct of copies of Σ^k. We argue by descending induction on k, the case of $k = m$ having been just established. Now assume the claim is true for a given $1 \leq k \leq m$, and let Z be an object which is equivalent to a finite coproduct of copies of $\Sigma^{k-1} Y$. Consider the strongly cocartesian $(n+1)$-cube $\rho_Z\colon \mathbf{P}(n+1) \to \mathcal{D}$, given by $\rho_Z(I) = \mathrm{cof}[\coprod_I Z \to Z]$. Since $\mathcal{P}_n(\mathrm{id}_{\mathcal{D}})$ and $\mathcal{P}_n(fg)$ are n-excisive the induced map

$$\mathcal{P}_n(gf)\rho_Z(-) \to \mathcal{P}_n(\mathrm{id}_{\mathcal{D}})\rho_Z(-)$$

is a map of cartesian $(n+1)$-cubes. On the other hand, for each $I \neq \varnothing$ we have that $\rho_Z(I)$ is equivalent to a coproduct of $|I| - 1$ copies of ΣZ, and hence to a finite coproduct of copies of $\Sigma^k Y$. By the induction hypothesis the above map of cartesian cubes is an equivalence for every $I \neq \varnothing$, and hence it is also an equivalence for $I = \varnothing$, where we have $\rho_Z(\varnothing) = Z$. This concludes the proof of the proposition. □

Corollary 10 (2-out-of-3)**.** *Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{f'} \mathcal{E}$ be a composable pair of compactly generated functors between pointed compactly generated ∞-categories. If either two of $f, f', f' \circ f$ are n-excisive equivalences then so is the third.*

Proof. This is clear from the second characterization of Proposition 9. □

Lemma 11. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an n-excisive equivalence. If $\mathcal{C}$ and $\mathcal{D}$ are both n-excisive then f is an equivalence.*

Proof. Let $g\colon \mathcal{D} \to \mathcal{C}$ be the finitary right adjoint of f. Since f is an n-excisive equivalence the unit $\mathrm{id}_{\mathcal{C}} \Rightarrow gf$ is a $\mathcal{P}_n$-equivalence. But both $\mathrm{id}_{\mathcal{C}}$ and gf are n-excisive since both $\mathcal{C}$ and $\mathcal{D}$ are n-excisive, and so the unit is an equivalence. We conclude that f is fully-faithful.

We now show that f is essentially surjective. Since f is an n-equivalence it induces an equivalence on stabilizations by Proposition 9. Since in the proof of that proposition this implies that for every compact object Y in $\mathcal{D}$, a large enough suspension $\Sigma^m Y$ is in the essential image of f, and similarly, any finite coproduct of copies of $\Sigma^m Y$ is in the image of $\mathcal{C}^c$. We now prove by descending induction on $1 \le k \le m$ that any object Z which is equivalent to finite coproduct of copies of $\Sigma^k Y$ is in the essential image of $\mathcal{C}^c$. Indeed, suppose that this is has been established for some $2 \le k \le m$ and let Z be an object which is equivalent to a finite coproduct of copies of $\Sigma^{k-1}Y$. Consider the strongly cocartesian cube $\rho_Z(I) = \mathrm{cof}[\coprod_{i\in I} Z \to Z]$. Since $\mathcal{D}$ is n-excisive ρ is also cartesian. In addition, for every $I \neq \varnothing$ we have that $\rho_Z(I)$ is equivalent to a finite coproduct of copies of ΣZ, and hence to a finite coproduct of copies of ΣY. We conclude that $\rho(I)$ is in the essential image of $\mathcal{C}^c$ for every $I \neq \varnothing$, and hence $\rho|_{\mathbf{P}^{\ge 1}(n)}$ is equivalent to the image under f of a punctured cube $\varphi\colon \mathbf{P}^{\ge 1}(n+1) \to \mathcal{C}^c$. Since f is fully-faithful and colimit preserving it also detects colimits, and hence φ is face-wise cocartesian. Since $\mathcal{C}$ is n-excisive φ extends to a cartesian $(n+1)$-cube $\overline{\varphi}\colon \mathbf{P}(n+1) \to \mathcal{C}^c$ which is also strongly cocartesian. Then $f(\overline{\varphi})$ is a strongly cocartesian n-cube and is hence cartesian, since $\mathcal{D}$ is n-excisive. We conclude that $f(\varphi)$ and ρ are two cartesian cubes whose associated punctured cubes are equivalent, and hence $f(\overline{\varphi})$ and ρ themselves must be equivalent. We then have that $\rho(\varnothing) = Z$ is in the essential image of $\mathcal{C}^c$. □

2.2. Weakly excisive ∞-categories.

Definition 12. Let $\mathcal{C}$ be a pointed compactly generated ∞-category. We will say that $\mathcal{C}$ is *weakly n-excisive* if every strongly cocartesian $(n+1)$-cube in $\mathcal{C}$ is cartesian. In other words, $\mathcal{C}$ is weakly n-excisive if the identity functor $\mathrm{id}_{\mathcal{C}}$ is n-excisive.

Lemma 13. *Let $\mathcal{C}$ be a pointed compactly generated ∞-category. The following conditions are equivalent:*
(1) $\mathcal{C}$ is weakly 1-excisive.
(2) Every pushout square in $\mathcal{C}$ is a pullback square.
(3) The suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is fully-faithful.
(4) The functor $\Sigma^\infty_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$ is fully-faithful.
(5) $\mathcal{C}$ is a compactly generated full subcategory of a stable compactly generated ∞-category.

Proposition 14. *A compactly generated functor $f\colon \mathcal{C} \to \mathcal{D}$ between weak n-excisive ∞-categories is an n-excisive equivalence if and only if it is fully-faithful and induces an equivalence on stabilizations.*

Proof. Let $g\colon \mathcal{D} \to \mathcal{C}$ be the finitary right adjoint of f. Since f preserves colimits, g preserevs limits, and $\mathcal{D}$ is weakly n-excisive we get that gf is n-excisive. In addition, $\mathrm{id}_{\mathcal{C}}$ is also n-excisive, and so $\mathrm{id}_{\mathcal{C}} \to gf$ is a $\mathcal{P}_n$-equivalence if and only if it is an equivalence, that is, if and only if f is fully-faithful. The desired result now follows from the third characterization in Proposition 9. □

For a pointed compactly generated ∞-category $\mathcal{C}$, let us write
$$\mathcal{Q}_n(\mathcal{C}) = \mathrm{Ind}[\mathrm{colim}[\mathcal{C}^c \to \mathcal{N}_n^{\ge 1}(\mathcal{C})^c \to \mathcal{N}_n^{\ge 1}\mathcal{N}_n^{\ge 1}(\mathcal{C})^c \to \ldots].$$
The map from the first term $\mathcal{C}^c$ of the above colimit then induces a compactly generated functor $L_n^{(\infty)}\colon \mathcal{C} \to \mathcal{Q}_n(\mathcal{C})$ with finitary right adjoint $R_n^{(\infty)}\colon \mathcal{Q}_n(\mathcal{C}) \to \mathcal{C}$ and we may identify the unit map $\mathrm{id}_{\mathcal{C}} \Rightarrow R_n^{(\infty)} L_n^{(\infty)}$ with the map $\mathrm{id}_{\mathcal{C}} \to \mathrm{colim}_n R_n^{(m)} L_n^{(m)} = \mathcal{P}_n(\mathrm{id})$. Consider the square
$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{Q}_n(\mathcal{C}) \arrow[d] \\ \mathrm{Sp}(\mathcal{C}) \arrow[r] & \mathcal{Q}_n(\mathrm{Sp}(\mathcal{C})) \end{tikzcd}$$
Since $\mathrm{Sp}(\mathcal{C})$ is stable it is in particular weakly n-excisive, and hence the bottom horizontal functor is fully-faithful. We then define $\widetilde{\mathcal{Q}}_n(\mathcal{C}) = \mathcal{Q}_n(\mathcal{C}) \times_{\mathcal{Q}(\mathrm{Sp}(\mathcal{C}))} \mathrm{Sp}(\mathcal{C})$, the fibre product being computed in Cat_*^ω, that is, we first pass to compact objects, then take the fibre product, and then take Ind completions again). Then $\widetilde{\mathcal{Q}}_n(\mathcal{C})$ embeds fully-faithfully in $\mathcal{Q}_n(\mathcal{C})$ and the map $F_n\colon \mathcal{C} \to \mathcal{Q}_n(\mathcal{C})$ uniquely factors through a compactly generated functor $\widetilde{L}_n^{(\infty)}\colon \mathcal{C} \to \widetilde{\mathcal{Q}}_n(\mathcal{C})$, with right adjoint $\widetilde{R}_n^{(\infty)}$ such that $\widetilde{R}_n^{(\infty)} \widetilde{L}_n^{(\infty)} \simeq R_n^{(\infty)} L_n^{(\infty)}$.

In particular, the functor $\widetilde{L}_n^{(\infty)}$ is fully-faithful if and only if $L_n^{(\infty)}$ is, i.e., if and only if $\mathcal{C}$ is weakly n-excisive. We also note that there is a natural equivalence $\mathcal{Q}_n(\mathrm{Sp}(\mathcal{C})) \simeq \mathrm{Sp}(\mathcal{Q}_n(\mathcal{C}))$, so that the right vertical arrow in the above square exhibit its target as the stabilization of its source, and since $\mathrm{Sp}(-)$ preserves finite limits we have that the map $\widetilde{L}_n^{(\infty)}: \mathcal{C} \to \widetilde{\mathcal{Q}}_n$ induces an equivalence on stabilizations. Finally, let us note that the constructions $\mathcal{Q}_n(-)$ and $\widetilde{\mathcal{Q}}_n(-)$ are functorial in compactly generated functors.

Lemma 15. *$\widetilde{\mathcal{Q}}_n(\mathcal{C})$ is weakly n-excisive.*

Proof. Since $\widetilde{\mathcal{Q}}_n(\mathcal{C})$ is a full subcategory of $\mathcal{Q}_n(\mathcal{C})$ closed under colimits we may instead show that $\mathcal{Q}_n(\mathcal{C})$ is weakly n-excisive. We need to show that any strongly cocartesian $(n+1)$-cube ρ in $\mathcal{Q}_n(\mathcal{C})$ is cartesian.

Let us write $\mathcal{N}_n^{(m)}(\mathcal{C}) := \mathcal{N}_n^{\geq 1}....\mathcal{N}_n^{\geq 1}(\mathcal{C})$ as shorthand for the m-fold iterated application of $\mathcal{N}_n^{\geq 1}$, and denote by

$$L_n^{(\infty-m)}: \mathcal{N}_n^{(m)}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{Q}_n(\mathcal{C}) : R_n^{(\infty-m)}$$

the associated adjunction. Now any $\mathbf{P}^{\geq 1}(n+1)$-indexed diagram in $\mathcal{Q}_n(\mathcal{C})$ is a filtered colimit of diagrams valued in $\mathcal{Q}_n(\mathcal{C})^c$, and any $\mathbf{P}^{\geq 1}(n+1)$-indexed diagram valued in $\mathcal{Q}_n(\mathcal{C})^c$ is equivalent to the image of a $\mathbf{P}^{\geq 1}(n+1)$-indexed diagram in $\mathcal{N}_n^{(m)}(\mathcal{C})^c$ for some m. We conclude that any strongly cocartesian $(n+1)$ in $\mathcal{Q}_n(\mathcal{C})$ is a filtered colimit of strongly cocartesian $(n+1)$ of the form $L_n^{(\infty-m)}(\rho')$ for some strongly cocartesian $(n+1)$-cocartesian cube in $\mathcal{N}_n^{(m)}(\mathcal{C})^c$. It will hence suffice to show that the latter type of $(n+1)$-cubes are cartesian. For a given strongly cocartesian $(n+1)$-cube ρ' in $\mathcal{N}_n^{(m)}(\mathcal{C})$, to show that $L_n^{(\infty-m)}\rho'$ is cartesian, it will suffice to check that it induces a cartesian square of spaces when mapping into it compact objects, and hence enough to show that $R_n^{(\infty-k)}F_n^{(m)}\rho'$ is cartesian in $\mathcal{N}_n^{(k)}(\mathcal{C})$ for every k. In other words, we need to show that $R_n^{(\infty-k)}F_n^{(m)}$ is n-excisive. Since the property of being n-excisive is preserved under pre-composing with a colimit preserving functor and post-composition by a limit preserving functor we have that

$$R_n^{(\infty-k)}L_n^{(\infty-m)} \text{ is } n \text{ excisive } \Rightarrow R_n^{(\infty-k')}L_n^{(\infty-m')} \text{ is } n \text{ excisive for all } k' \leq k \text{ and } m' \leq m$$

It will hence suffice to show that $R_n^{(\infty-m)}L_n^{(\infty-m)}$ is n-excisive. Indeed, unwinding the definitions, this functor is exactly the n-excisive approximation of $\mathrm{id}_{\mathcal{N}_n^{(m)}(\mathcal{C})}$. □

Lemma 16. *Let $\mathcal{C}$ be a pointed compactly generated ∞-category. Then the map $\mathcal{C} \to \widetilde{\mathcal{Q}}_n(\mathcal{C})$ is an n-excisive equivalence.*

Proof. It has already been established above that, essentially by construction, the map $\mathcal{C} \to \widetilde{\mathcal{Q}}_n(\mathcal{C})$ induces an equivalence on stabilizations and the unit $\mathrm{id}_{\mathcal{C}} \Rightarrow \widetilde{G}_n\widetilde{F}_n$ exhibits $\widetilde{G}_n\widetilde{F}_n$ as the n-excisive approximation of $\mathrm{id}_{\mathcal{C}}$. We are hence done by the third characterization of Proposition 9. □

Corollary 17. *Let $f: \mathcal{C} \to \mathcal{D}$ be a compactly generated functor between pointed compactly generated ∞-categories. Then f is an n-excisive equivalence if and only if it induces an equivalence on stabilizations and the induced functor $\widetilde{\mathcal{Q}}_n(\mathcal{C}) \to \widetilde{\mathcal{Q}}_n(\mathcal{D})$ is fully-faithful.*

Proof. By Lemma 16 we have that f is an n-excisive equivalence if and only if $\widetilde{\mathcal{Q}}_n(\mathcal{C}) \to \widetilde{\mathcal{Q}}_n(\mathcal{D})$ is an n-excisive equivalence. On the other hand, by Proposition 14 and Lemma 15 the latter is equivalent to $\widetilde{\mathcal{Q}}_n(\mathcal{C}) \to \widetilde{\mathcal{Q}}_n(\mathcal{D})$ being fully-faithful and inducing an equivalence on stabilizations. The last property is equivalent to f itself inducing an equivalence on stabilizations.

□

Corollary 18. *The collection of n-excisive equivalences is closed under base change in $\mathcal{C}\mathrm{at}_*^{\omega}$.*

Proof. Consider a pullback square

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{E} \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}^{\omega}_{*}$. Recall that pullbacks in $\mathcal{C}\mathrm{at}^{\omega}$ are calculated by first passing to compact objects, then taking the pullback, and finally taking ind completions. Since ind completions preserve fully-faithful functors we observe that fully-faithful compactly generated functors are closed under base change. Since equivalences are also closed under base change, the desired result follow from the characterization of Corollary 17 since the functors $\mathrm{Sp}(-)$ and $\widetilde{\mathcal{Q}}_n(-)$ preserve fibre products in $\mathcal{C}\mathrm{at}^{\omega}_{*}$. □

Proposition 19. *For $n \geq 2$ let*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{E} \end{tikzcd} \tag{3}$$

be a commutative diagram of pointed compactly generated ∞-categories and compactly generated functors whose vertical arrows are $(n-1)$-excisive equivalences and whose horizontal arrows are 1-excisive equivalences. Then the square

$$\begin{tikzcd} \mathcal{P}_n(K_{\mathcal{A}}) \arrow[r] \arrow[d] & \mathcal{P}_n(K_{\mathcal{B}}) \arrow[d] \\ \mathcal{P}_n(K_{\mathcal{D}}) \arrow[r] & \mathcal{P}_n(K_{\mathcal{E}}) \end{tikzcd} \tag{4}$$

is cartesian in $\mathrm{Fun}^{\omega}_{*}(\mathrm{Sp}(\mathcal{E}), \mathrm{Sp}(\mathcal{E}))$*, where $K_{\mathcal{A}}, K_{\mathcal{B}}, K_{\mathcal{D}}$ and $K_{\mathcal{E}}$ denote the respective comonads on* $\mathrm{Sp}(\mathcal{E})$ *associated to the composite left adjoint from that corner to* $\mathrm{Sp}(\mathcal{E})$.

Proof. By Proposition 9 the maps $K_{\mathcal{A}} \to K_{\mathcal{D}}$ and $K_{\mathcal{B}} \to K_{\mathcal{E}}$ are $\mathcal{P}_{n-1}$-equivalences, and hence the square (4) becomes cartesian after applying $\mathcal{P}_{n-1}$. To show that it is cartesian also on the level of $\mathcal{P}_{n-1}$ it will hence suffice to show that the square

$$\begin{tikzcd} \partial_n K_{\mathcal{A}} \arrow[r] \arrow[d] & \partial_n K_{\mathcal{B}} \arrow[d] \\ \partial_n K_{\mathcal{D}} \arrow[r] & \partial_n K_{\mathcal{E}} \end{tikzcd}$$

is cartesian. Now for every k we have a coaugmented cosimplicial object

$$\partial_k \,\mathrm{id}_{\mathcal{E}} \longrightarrow \partial_k(\Omega^{\infty}_{\mathcal{E}}\Sigma^{\infty}_{\mathcal{E}}) \rightrightarrows \partial_k(\Omega^{\infty}_{\mathcal{E}}\Sigma^{\infty}_{\mathcal{E}}\Omega^{\infty}_{\mathcal{E}}\Sigma^{\infty}_{\mathcal{E}}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \partial_k(\Omega^{\infty}_{\mathcal{E}}\Sigma^{\infty}_{\mathcal{E}}\Omega^{\infty}_{\mathcal{E}}\Sigma^{\infty}_{\mathcal{E}}\Omega^{\infty}_{\mathcal{E}}\Sigma^{\infty}_{\mathcal{E}}) \cdots$$

which exhibits its coaugmentation as its totalization by [Heu21, Corollary B.5]. We may compute these derivatives via the chain rule (see [Lur14, §6.3.2]). We note that the functors $\Sigma^{\infty}_{\mathcal{E}}$ and $\Omega^{\infty}_{\mathcal{E}}$ are 1-excisive and their first derivative is the identity on $\mathrm{Sp}(\mathcal{E})$. The above diagram then becomes

$$\partial_k \,\mathrm{id}_{\mathcal{E}} \longrightarrow \partial_k \,\mathrm{id}_{\mathrm{Sp}(\mathcal{E})} \rightrightarrows \partial_k K_{\mathcal{E}} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \partial_k(K_{\mathcal{E}}K_{\mathcal{E}}) \cdots$$

The counit map $K_{\mathcal{E}} \to \mathrm{Id}_{\mathrm{Sp}(\mathcal{E})}$ induces an equivalence $\mathcal{P}_1(K) \simeq \mathcal{P}_1(K_{\mathcal{E}}) \simeq \mathrm{Id}_{\mathrm{Sp}(\mathcal{E})}$. In particular, for $k = 1$ both this augmented cosimplicial objects are constant with the value $\mathrm{Id}_{\mathrm{Sp}(\mathcal{E})}$. For a given $k \geq 1$, let us consider the induced map from this cosimplcial object to the right Kan extension of its restriction to $\Delta_{\leq 1}$, which we can also coaugment with its totalization, which is the same as the limit of $\Delta_{\leq 1}$. We obtain a map of coaugmented cosimplicial objects

$$\begin{tikzcd}
\partial_k \mathrm{id}_{\mathcal{E}} \arrow[r] \arrow[d] & \partial_k \mathrm{id}_{\mathrm{Sp}(\mathcal{E})} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \partial_k K_{\mathcal{E}} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, shift left=2] \arrow[d] & \partial_k (K_{\mathcal{E}} K_{\mathcal{E}}) \; \cdots \arrow[d] \\
\partial_k \mathrm{id}_{\mathrm{Sp}(\mathcal{E})} \times_{\partial_k K} \partial_k \mathrm{id}_{\mathrm{Sp}(\mathcal{E})} \arrow[r] & \partial_k \mathrm{id}_{\mathrm{Sp}(\mathcal{E})} \arrow[r, shift left] \arrow[r, shift right] & \partial_k K_{\mathcal{E}} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, shift left=2] & \partial_k K_{\mathcal{E}} \times_{\partial_k \mathrm{id}_{\mathrm{Sp}(\mathcal{E})}} \partial_k K_{\mathcal{E}} \; \cdots
\end{tikzcd}$$

where the coaugmentation witness the respective totalizations. Let us write $\mathcal{Z}^{\bullet}_{k,\mathcal{E}}$ for the coaugmented simplicial object corresponding to fibre of the above vertical map. Since all the functors in the original square (3) induce an equivalence on stabilizations by Proposition 9 we may identify the stabilizations of all the four corners compatibly with $\mathrm{Sp}(\mathcal{E})$. Performing the above construction in all four cases yields the following commutative diagram of coaugmented cosimplicial objects (valued in the stable ∞-category of entry-wise exact functors $\mathrm{Sp}(\mathcal{E})^k \to \mathrm{Sp}(\mathcal{E})$)

(5)

$$\begin{tikzcd}[column sep=small]
\mathcal{Z}_{k,\mathcal{A}} \arrow[rr] \arrow[dr] \arrow[dd] & & \partial_n(K_{\mathcal{A}} \circ \ldots \circ K_{\mathcal{A}}) \arrow[rr] \arrow[dr] \arrow[dd] & & \partial_n(K_{\mathcal{A}}) \times \ldots \times \partial_n(K_{\mathcal{A}}) \arrow[dr] \arrow[dd] & \\
& \mathcal{Z}_{k,\mathcal{B}} \arrow[rr, crossing over] & & \partial_n(K_{\mathcal{B}} \circ \ldots \circ K_{\mathcal{B}}) \arrow[rr, crossing over] & & \partial_n(K_{\mathcal{B}}) \times \ldots \times \partial_n(K_{\mathcal{B}}) \arrow[dd] \\
\mathcal{Z}_{k,\mathcal{D}} \arrow[rr] \arrow[dr] & & \partial_n(K_{\mathcal{D}} \circ \ldots \circ K_{\mathcal{D}}) \arrow[rr] \arrow[dr] & & \partial_n(K_{\mathcal{D}}) \times \ldots \times \partial_n(K_{\mathcal{D}}) \arrow[dr] & \\
& \mathcal{Z}_{k,\mathcal{E}} \arrow[rr] \arrow[from=uu, crossing over] & & \partial_n(K_{\mathcal{E}} \circ \ldots \circ K_{\mathcal{E}}) \arrow[rr] \arrow[from=uu, crossing over] & & \partial_n(K_{\mathcal{E}}) \times \ldots \times \partial_n(K_{\mathcal{E}})
\end{tikzcd}$$

By the chain rule we may compute $\partial_k(K_{(-)} \circ \ldots \circ K_{(-)}) \simeq \partial_*(K_{(-)}) \circ \ldots \circ \partial_*(K_{(-)})$ using the composition of symmetric sequences. For m-fold compositions this is a direct sum of objects parameterized by chains of length $m+1$ of equivalence relations on $\{1, \ldots, k\}$ starting from the finest one and ending in the coarsets one. The vertical map at place m then projects to the summand involving chains with a single non-identity step. In particular, the remaining summands in $\mathcal{Z}^m_{k,(-)}$ correspond to such chains where each step has degree less then k. In particular, since the maps $K_{\mathcal{B}} \to K_{\mathcal{E}}$ and $K_{\mathcal{A}} \to K_{\mathcal{D}}$ are $\mathcal{P}_{n-1}$-equivalences by Proposition 9 they also induce equivalences on the respective $\mathcal{Z}^{\bullet}_{n,(-)}$'s, that is, the two vertical maps in the left most face of (5) are equivalence. It then follows that the front and back faces of the right cube in (5) are cartesian. As $n \geq 2$ we have $\partial_n \mathrm{id}_{\mathrm{Sp}(\mathcal{E})} = 0$ and so passing to the totalizations we obtain a cube

(6)

$$\begin{tikzcd}
\partial_n \mathrm{id}_{\mathcal{A}} \arrow[rr] \arrow[dr] \arrow[dd] & & \Omega(\partial_n K_{\mathcal{A}}) \arrow[dr] \arrow[dd] & \\
& \partial_n \mathrm{id}_{\mathcal{B}} \arrow[rr, crossing over] & & \Omega(\partial_n K_{\mathcal{B}}) \arrow[dd] \\
\partial_n \mathrm{id}_{\mathcal{D}} \arrow[rr] \arrow[dr] & & \Omega(\partial_n K_{\mathcal{D}}) \arrow[dr] & \\
& \partial_n \mathrm{id}_{\mathcal{E}} \arrow[rr] \arrow[from=uu, crossing over] & & \Omega(\partial_n K_{\mathcal{E}})
\end{tikzcd}$$

in which the front and back faces are cartesian. Since the left face is is also cartesian by virtue of (3) being cartesian, and since the ∞-category of entry-wise exact functors $\mathrm{Sp}(\mathcal{E})^k \to \mathrm{Sp}(\mathcal{E})$ is stable, we also conclude from this that the right face is cartesian, from which the desired result follows.

$\square$

3. Linear extensions

Let $\mathcal{C}$ be a compactly generated stable ∞-category. In this section we will discuss a way of constructing n-excisive ∞-categories whose stabilization is $\mathcal{C}$. We start with a finitary functor $Q\colon \mathcal{C} \to \mathcal{C}$ and construct a compactly generated ∞-category $\mathrm{LaxEq}^{\omega}(\mathcal{C}, Q)$ which we call the lax equalizer ∞-category. We first define the full subcategory of compact objects $\mathrm{LaxEq}^{c}(\mathcal{C}, Q)$ to be the fibre product, computed simply in ∞-categories, in the square

$$\begin{tikzcd} \mathrm{LaxEq}^{c}(\mathcal{C}, Q) \arrow[r] \arrow[d, "\pi^c"] & \mathrm{Ar}(\mathcal{C}) \arrow[d, "(t,s)"] \\ \mathcal{C}^c \arrow[r, "(\mathcal{C},Q)"] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

Explicitly, objects in LaxEq^{c} are given by a compact object $X \in \mathcal{C}^c$ together with a map $f\colon X \to Q(X)$ in $\mathcal{C}$. We note that $Q(X)$ does not have to be compact.

Lemma 20.

(1) $\mathrm{LaxEq}^{c}(\mathcal{C}, Q)$ *admits finite colimits and the functor* $\pi^c\colon \mathrm{LaxEq}^{c}(\mathcal{C}, Q) \to \mathcal{C}^c$ *preserves and detects finite colimits. In addition,* $\mathrm{LaxEq}^{c}(\mathcal{C}, Q)$ *is idempotent complete.*
(2) Let $\overline{\rho}\colon \mathcal{I}^{\triangleleft} \to \mathrm{LaxEq}^{c}(\mathcal{C}, Q)$ *be a cone diagram for some* ∞*-category* $\mathcal{I}$*. If* $\pi\overline{\rho}$ *and* $Q\pi\overline{\rho}$ *are both limit cones the* $\overline{\rho}$ *itself is a limit cone.*
(3) Let $\rho\colon \mathcal{I} \to \mathrm{LaxEq}^{c}(\mathcal{C}, Q)$ *be a diagram indexed by some* ∞*-category* $\mathcal{I}$*. If* $\pi^c\rho$ *admits a limit in* $\mathcal{C}^c$ *and this limit is preserved by* Q *then* ρ *admits a limit in* $\mathrm{LaxEq}^{c}(\mathcal{C}, Q)$*, and this limit is preserved by* π*.*

To prove Lemma 20, let us a consider a slightly more general, but also more symmetric setup. Suppose given two ∞-categories $\mathcal{A}, \mathcal{B}$ and two functors $F, G\colon \mathcal{A} \to \mathcal{B}$. Then we may consider the lax equalizer ∞-category

$$\mathrm{LaxEq}(F, G) \coloneqq \mathcal{A} \times_{\mathcal{B}\times\mathcal{B}} \mathrm{Ar}(\mathcal{B})$$

where the fibre product is along the functor $(F, G)\colon \mathcal{A} \to \mathcal{B}$. We then write $\pi\colon \mathrm{LaxEq}(F, G) \to \mathcal{A}$ for the projection on the first coordinate.

Lemma 21.

(1) Let $\rho\colon \mathcal{I} \to \mathrm{LaxEq}(F, G)$ *be a diagram indexed by some* ∞*-category* $\mathcal{I}$*. If* $\pi\rho$ *admits a colimit in* $\mathcal{A}$ *and this colimit is preserved by* F *then* ρ *admits a colimit in* $\mathrm{LaxEq}(F, G)$*. Furthermore, in this case an arbitrary extension* $\overline{\rho}\colon \mathcal{I}^{\triangleright} \to \mathrm{LaxEq}(F, G)$ *is a colimit cone if and only if* $\pi\rho$ *and* $F\pi\rho$ *are colimit cones.*
(2) Let $\rho\colon \mathcal{I} \to \mathrm{LaxEq}(F, G)$ *be a diagram indexed by some* ∞*-category* $\mathcal{I}$*. If* $\pi\rho$ *admits a limit in* $\mathcal{A}$ *and this limit is preserved by* G *then* ρ *admits a limit in* $\mathrm{LaxEq}(F, G)$*. Furthermore, in this case an arbitrary extension* $\overline{\rho}\colon \mathcal{I}^{\triangleleft} \to \mathrm{LaxEq}(F, G)$ *is a limit cone if and only if* $\pi\rho$ *and* $F\pi\rho$ *are limit cones.*

Proof of Lemma 20 assuming Lemma 21. Apply Lemma 21 in the case where $\mathcal{A} = \mathcal{C}^c$, $\mathcal{B} = \mathcal{C}$, F is the embedding $\mathcal{C}^c \subseteq \mathcal{C}$ and G is the restriction of Q to $\mathcal{C}^c$, and use the fact that $\mathcal{C}^c$ admits finite colimits and the embedding $\mathcal{C}^c \subseteq \mathcal{C}$ preserves finite colimits. ☐

Proof of Lemma 21. Statement (2) is dual to Statement (1), and can be deduced from it by replacing $\mathcal{A}$ and $\mathcal{B}$ by the opposites and using the identification

$$\mathrm{LaxEq}(G^{\mathrm{op}}, F^{\mathrm{op}}) \simeq \mathrm{LaxEq}(F, G)^{\mathrm{op}}.$$

We hence just prove (1). Let $\rho\colon \mathcal{I} \to \mathrm{LaxEq}(F,G)$ be a diagram and suppose that $\pi\rho$ extends to a colimit cone $\phi\colon \mathcal{I}^{\triangleright} \to \mathcal{A}$ such that $F\phi$ is a colimit cone in $\mathcal{B}$. Consider the lifting problem

$$\begin{tikzcd} \mathcal{I} \arrow[r, "\rho"] \arrow[d] & \mathrm{LaxEq}(F,G) \arrow[r] \arrow[d] & \mathrm{Ar}(\mathcal{B}) \arrow[d] \\ \mathcal{I}^{\triangleright} \arrow[r, "\phi"] \arrow[ur, dashed] \arrow[urr, dotted] & \mathcal{A} \arrow[r, "(F,G)"] & \mathcal{B} \times \mathcal{B} \end{tikzcd} \tag{7}$$

Since the right square is pullback, dashed lifts in the right square are in bijection with dotted lifts in the external rectangle. Now the external rectangle is encoded by a natural transformation $\eta\colon F\pi\rho \Rightarrow G\pi\rho$, and dotted lifts in the external rectangle correspond to extensions of η to a natural transformation $F\phi \Rightarrow G\phi$. The condition that $F\pi\phi$ is a colimit diagram implies that $F\phi$ is a left Kan extension of its restriction to $\mathcal{I}$, and hence a dotted lift in the external rectangle exists and is essentially unique. We conclude that a dotted lift in the left square exists an is essentially unique as well. Call this lift $\overline{\rho}\colon \mathcal{I}^{\triangleright} \to \mathrm{LaxEq}(F,G)$. We claim that $\overline{\rho}$ is a colimit cone. Indeed, for every object $(X, \tau\colon F(X) \to G(X))$ we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{LaxEq}(F,G)}(\overline{\rho}(*), (X,\tau)) &= \mathrm{Eq}[\mathrm{Map}_{\mathcal{A}}(\phi(*), X) \rightrightarrows \mathrm{Map}_{\mathcal{B}}(F(\phi(*)), G(X))] \\ &= \mathrm{Eq}[\lim_{i \in \mathcal{J}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{A}}(\phi(i), X) \rightrightarrows \lim_{i \in \mathcal{J}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{B}}(F(\phi(*)), G(X))] \\ &= \lim_{i \in \mathcal{J}^{\mathrm{op}}} \mathrm{Eq}[\mathrm{Map}_{\mathcal{A}}(\phi(*), X) \rightrightarrows \mathrm{Map}_{\mathcal{B}}(F(\phi(*)), G(X))] \\ &= \lim_{i \in \mathcal{J}^{\mathrm{op}}} \mathrm{Map}_{\mathrm{LaxEq}(F,G)}(\rho(i), (X,\tau)), \end{aligned}$$

and so $\overline{\rho}$ is a colimit cone. To prove the last claim, we need to show that an arbitrary extension $\overline{\rho}'$ of ρ is a colimit cone if and only if $\pi\overline{\rho}'$ and $F\pi\overline{\rho}'$ are colimit cones. Indeed, by the uniqueness of colimit cones, this is equivalent to saying that $\overline{\rho}'$ is equivalent to ρ if and only if $\pi\overline{\rho}'$ is equivalent to ϕ (in which case automatically $F\pi\overline{\rho}'$ is equivalent to $F\phi$). Indeed, this is exactly the uniqueness property of dashed lifts in the left square of (7). ☐

We now define $\mathrm{LaxEq}^{\omega}(\mathcal{C}, Q) = \mathrm{Ind}(\mathrm{LaxEq}^{c}(\mathcal{C}, Q))$ to be the Ind-completion of $\mathrm{LaxEq}^{c}(\mathcal{C}, Q)$ and write $\pi\colon \mathrm{LaxEq}(\mathcal{C}, Q) \to \mathcal{C}$ for the colimit preserving extension of π^{c}. By Lemma 20 we have that $\mathrm{LaxEq}^{\omega}(\mathcal{C}, Q)$ is compactly generated and π is a compactly generated functor.

Proposition 22.

(1) If Q is reduced then $\mathrm{LaxEq}(\mathcal{C}, Q)$ *is pointed.*

(2) For $n \geq 1$, if Q is n-excisive then $\mathrm{LaxEq}(\mathcal{C}, Q)$ *is n-excisive.*

(3) For $n \geq 1$, if $\mathcal{P}_n(Q) = 0$ then the functor $\mathrm{LaxEq}(\mathcal{C}, Q) \to \mathcal{C}$ *is an n-excisive equivalence.*

Proof. For (1), apply Lemma 20(2) with $I = \varnothing$.

We now prove (2). We first verify that every strongly cocartesian $(n+1)$-cube ρ in $\mathrm{LaxEq}(\mathcal{C}, Q)$ is cartesian. By Remark 6 we may assume that ρ takes values in $\mathrm{LaxEq}^{c}(\mathcal{C}, Q)$. Then $\pi\rho$ is a strongly cocartesian $(n+1)$-cube in $\mathcal{C}^{c}$ and since $\mathcal{C}$ is 1-excisive and Q is n-excisive we have that both $\pi\rho$ and $Q\pi\rho$ are cartesian cubes. By Lemma 20(2) we conclude that ρ itself is cartesian in $\mathrm{LaxEq}^{c}(\mathcal{C}, Q)$ and hence in $\mathrm{LaxEq}(\mathcal{C}, Q)$. Let now $\rho_0\colon \mathbf{P}^{\geq 1}(n) \to \mathrm{LaxEq}^{c}(\mathcal{C}, Q)$ be a face-wise cocartesian punctured $(n+1)$-cube. Then $\pi\rho_0$ is face-wise cocartesian, and since $\mathcal{C}$ is 1-excisive we have that $\pi\rho_0$ extends to an $(n+1)$-cube $\phi\colon \mathbf{P}(n+1) \to \mathcal{C}$ which is both cartesian and strongly cocartesian. Since Q is n-excisive we have that $Q\phi$ is also cartesian, and hence by Lemma 20(2) ρ_0 extends to a cartesian $(n+1)$-cube ρ in $\mathrm{LaxEq}(\mathcal{C}, Q)$ such that $\pi\rho$ is cartesian in $\mathcal{C}$. Thus $\pi\rho$ must coincide with ϕ, so that $\pi\rho$ is strongly cocartesian. By Lemma 20 the functor π^{c} detects finite colimits and hence ρ is strongly cocartesian. We thus conclude that $\mathrm{LaxEq}(\mathcal{C}, Q)$ is n-excisive.

Finally, let us assume that $\mathcal{P}_n(Q) = 0$ and show that $\pi\colon \mathrm{LaxEq}(\mathcal{C}, Q) \to \mathcal{C}$ is an n-excisive equivalence. By Lemma 16 we may equivalently show that $\widetilde{\mathcal{Q}}_n(\mathrm{LaxEq}(\mathcal{C}, Q)) \to \widetilde{\mathcal{Q}}_n(\mathcal{C}) \simeq \mathcal{C}$ is an n-excisive equivalence.

To begin, since the the formation of reduced punctured cubes commutes the pullback squares and arrow categories we obtain pullback square

$$\begin{tikzcd} \mathcal{N}_n(\mathrm{LaxEq}^c(\mathcal{C},Q)) \arrow[r] \arrow[d, "\pi^c"] & \mathrm{Ar}(\mathcal{N}_n(\mathcal{C})) \arrow[d, "(t,s)"] \\ \mathcal{N}_n(\mathcal{C}^c) \arrow[r, "(\mathcal{N}_n(\mathcal{C}),Q_*)"] & \mathcal{N}_n(\mathcal{C}) \times \mathcal{N}_n(\mathcal{C}) \end{tikzcd}$$

where $Q_*\colon \mathcal{N}_n(\mathcal{C}^c) \to \mathcal{N}_n(\mathcal{C})$ is the functor obtained by applying Q entry-wise. We consequently deduce that

$$\mathcal{N}_n(\mathrm{LaxEq}(\mathcal{C},Q)) \simeq \mathrm{LaxEq}(\mathcal{N}_n(\mathcal{C}),Q_*),$$

where by abuse of notation we write Q_* also for its filtered colimit preserving extension from $\mathcal{N}_n(\mathcal{C}^c) = \mathcal{N}_n(\mathcal{C})^c$ to $\mathcal{N}_n(\mathcal{C})$. Similarly, for each $m \geq 1$ we may consider the functor $Q_*^{(m)}\colon \mathcal{N}_n^{(m)}(\mathcal{C}^c) \to \mathcal{N}_n^{(m)}(\mathcal{C})$, where $Q_*^{(1)} = Q_*$ and $Q_*^{(m)}$ is given by applying $Q_*^{(m-1)}$ entry-wise. We then similarly obtain that

$$\mathcal{N}_n^{(m)}(\mathrm{LaxEq}(\mathcal{C},Q)) \simeq \mathrm{LaxEq}(\mathcal{N}_n^{(m)}(\mathcal{C}),Q_*^{(m)}).$$

Since sequential colimits commute with fibre products and arrow categories we may pass to $m = \infty$ to obtain that

$$\mathcal{Q}_n(\mathrm{LaxEq}(\mathcal{C},Q)) \simeq \mathrm{LaxEq}(\mathcal{Q}_n(\mathcal{C}),Q_*^{(\infty)}),$$

where $Q_*^{(\infty)}$ is the filtered colimit preserving functor whose value on each $\mathcal{N}_n^{(m)}(\mathcal{C}^c)$ is given by composing $Q_*^{(m)}$ with the map $\mathcal{N}_n^{(m)}(\mathcal{C}) \to \mathcal{P}_n(\mathcal{C})$. Finally, using the equivalence

$$\mathrm{Map}_{\mathcal{Q}_n(\mathcal{C})}(L_n^{(\infty)}X, Q_*^{(\infty)}L_n^{(\infty)}(X)) \simeq \mathrm{Map}_{\mathcal{C}}(X, R_n^{(\infty)}Q_*^{(\infty)}L_n^{(\infty)}(X))$$

associated to the adjunction $L_n^{(\infty)} \dashv R_n^{(\infty)}$ we see that there is a natural equivalence

$$\widetilde{\mathcal{Q}}_n(\mathrm{LaxEq}(\mathcal{C},Q)) = \mathcal{Q}_n(\mathrm{LaxEq}(\mathcal{C},Q)) \times_{\mathcal{Q}_n(\mathcal{C})} \mathcal{C} \simeq \mathrm{LaxEq}(\mathcal{C}, R_n^{(\infty)}Q_*^{(\infty)}L_n^{(\infty)}) \simeq \mathrm{LaxEq}(\mathcal{C},\mathcal{P}_n(Q)).$$

Since $\mathcal{P}_n(Q) = 0$ we conclude that the functor $\widetilde{\mathcal{Q}}_n(\mathrm{LaxEq}(\mathcal{C},Q)) \to \mathcal{C}$ is an equivalence. □

The ∞-category $\mathrm{LaxEq}(\mathcal{C},Q)$ enjoys the following universal property:

Proposition 23. *For a given compactly generated ∞-category $\mathcal{E}$, compactly generated functors $\mathcal{E} \to \mathrm{LaxEq}(\mathcal{C},Q)$ are in natural bijection with pairs (f,η), where $f\colon \mathcal{E} \to \mathcal{C}$ is a compactly generated functor and $\eta\colon f \Rightarrow Qf$ is a natural transformation.*

Proof. Since $\mathrm{LaxEq}(\mathcal{C},Q)$ is idempotent complete such functors correspond to finite colimit preserving functors $\mathcal{E}^c \to \mathrm{LaxEq}^c(\mathcal{C},Q)$. By Lemma 20 we have that these are exactly the functors $\mathcal{E}^c \to \mathrm{LaxEq}^c(\mathcal{C},Q)$ such that the composite $\mathcal{E} \to \mathrm{LaxEq}^c(\mathcal{C},Q) \to \mathcal{C}$ preserves finite colimits. In addition, for a given finite colimit preserving functor $f\colon \mathcal{E}^c \to \mathcal{C}^c$, the data of a lift of f to $\mathrm{LaxEq}^c(\mathcal{C},Q)$ is equivalent to the data of a lift of $(\mathcal{C},Q)f\colon \mathcal{E}^c \to \mathcal{C}^c \times \mathcal{C}$ to $\mathrm{Ar}(\mathcal{C})$, which is the same as the data of a natural transformation $f \Rightarrow Qf$. □

The universal property of Proposition 23 can also be described as follows. Any compactly generated functor $f\colon \mathcal{E} \to \mathcal{C}$ admits a finitary right adjoint $g\colon \mathcal{C} \to \mathcal{E}$. Then pre-composing with g is left adjoint to pre-composing with f, which means that the natural transformations $f \Rightarrow Qf$ are in bijection with natural transformations $fg \Rightarrow Q$ of endo-functors on $\mathcal{C}$. This holds in particular for the universal case of $\mathcal{E} = \mathrm{LaxEq}(\mathcal{C},Q)$ and $f = \pi$, that is, the universal natural transformation $\eta\colon \pi \Rightarrow Q\pi$ corresponds to some natural transformation $\eta^{\mathrm{ad}}\colon \pi\phi \Rightarrow Q$, where $\phi\colon \mathcal{C} \to \mathrm{LaxEq}(\mathcal{C},Q)$ is the finitary right adjoint of π. We note that the construction $Q \mapsto \mathrm{LaxEq}(\mathcal{C},Q)$ is visibly functorial in Q, so that we can assemble it to a functor

$$\mathrm{LaxEq}(\mathcal{C},-)\colon \mathrm{Fun}_*^{\omega}(\mathcal{C},\mathcal{C}) \to (\mathcal{C}\mathrm{at}_*^{\omega})_{/\mathcal{C}}.$$

On the other hand, the association that takes a compactly generated functor $f\colon \mathcal{E} \to \mathcal{C}$ to the corresponding endo-functor $fg\colon \mathcal{C} \to \mathcal{C}$, where g is the right adjoint of f, forms a functor in the other direction

$$K(-)\colon (\mathcal{C}\mathrm{at}^{\omega}_{*})_{/\mathcal{C}} \to \mathrm{Fun}^{\omega}_{*}(\mathcal{C},\mathcal{C}) \qquad K(\mathcal{E} \xrightarrow{f} \mathcal{C}) = fg.$$

Corollary 24. *The natural transformation $\eta^{\mathrm{ad}}\colon K(\mathrm{LaxEq}(\mathcal{C},Q)) \Rightarrow Q$ is a unit exhibiting the functor $\mathrm{LaxEq}(\mathcal{C},-)$ as right adjoint to the functor $K(-)$.*

Proposition 25. *Let $n \geq 2$ and suppose that Q is n-homogeneous. Write $K' := K(\mathrm{LaxEq}(\mathcal{C},Q))$. Then the map $K' \to Q \oplus \mathrm{id}_{\mathcal{C}}$ determines by the counit $K' \to Q$ of the above adjunction and the canonical augmentation $K' \to \mathrm{id}_{\mathcal{C}}$ induces an equivalence*

$$\mathcal{P}_n(K') \simeq Q \oplus \mathrm{id}_{\mathcal{C}}\,.$$

Proof. By Proposition 22 the functor $\mathcal{E} \to \mathcal{C}$ is an $(n-1)$-excisive approximation, and hence the map $K' \to \mathrm{id}_{\mathcal{C}} = K(\mathcal{C})$ is a $\mathcal{P}_{n-1}$-equivalence. On the other hand, since Q is n-homogeneous we have that $\mathcal{P}_{n-1}(Q) = 0$. The map $K' \to \mathrm{id}_{\mathcal{C}} \oplus Q$ is hence a $\mathcal{P}_{n-1}$-equivalence. To show that it is a $\mathcal{P}_n$-equivalence it will suffice to show that the induced map

$$\partial_n K' \to \partial_n Q$$

is an equivalence.

To simplify notation let us write $\mathcal{E} := \mathrm{LaxEq}(\mathcal{C},Q)$, and let $\phi\colon \mathcal{C} \to \mathcal{E}$ be the finitary right adjoint of $\pi\colon \mathcal{E} \to \mathcal{C}$. The canonical cosimplicial resolution

$$\mathrm{id}_{\mathcal{E}} \longrightarrow \phi\pi \rightleftarrows \phi\pi\phi\pi \rightleftarrows \phi\pi\phi\pi\phi\pi \ \cdots$$

which we can also write as

$$\mathrm{id}_{\mathcal{E}} \longrightarrow \phi\pi \rightleftarrows \phi K'\pi \rightleftarrows \phi K'K'\pi \ \cdots$$

The induced composite map

$$\mathrm{id}_{\mathcal{E}} \to \mathrm{Tot}[\phi K^{\bullet}\pi] \to \mathrm{Eq}[\ \phi\pi \rightrightarrows \phi K\pi\ ] \to \mathrm{Eq}[\ \phi\pi \rightrightarrows \phi Q\pi\ ].$$

is an equivalence; indeed, for $X, Y \in \mathcal{E}$ the map

$$\mathrm{Map}_{\mathcal{E}}(X,Y) \to \mathrm{Eq}[\mathrm{Map}_{\mathcal{E}}(\pi(X),\pi(Y)) \to \mathrm{Map}_{\mathcal{E}}(\pi(X),Q(\pi(Y)))$$

is an equivalence of spaces by the construction of $\mathcal{E}$. Passing to n-derivatives and using the fact that ϕ and ψ are 1-excisive we obtain an equivalence

$$\partial_n \mathrm{id}_{\mathcal{E}} \xrightarrow{\simeq} \mathrm{Eq}[\ \partial_n(\phi\pi) \rightrightarrows \partial_n(\phi Q\pi)\ ] \simeq \mathrm{Eq}[\ \partial_n(\mathrm{id}_{\mathcal{C}}) \rightrightarrows \partial_n(Q)\ ] \simeq \Omega(\partial_n Q).$$

At the same time, by [Heu21, Corollary B.5] the cosimplicial object

$$\partial_n \mathrm{id}_{\mathcal{E}} \longrightarrow \partial_n \mathrm{id}_{\mathcal{C}} \rightleftarrows \partial_n K' \rightleftarrows \partial_n(K'K') \ \cdots$$

induced by passing to derivatives exhibits its coaugmentation as its totalization. The triviality of the derivatives of K' in the range $1 < k < n$ implies that $\partial_n(K')^{\circ m}$ is a direct sum of m copies of $\partial_n K'$, or, more precisely, that the map $\partial_n(K' \circ \ldots \circ K') \to \partial_n K' \times \ldots \times \partial_n K'$ induced by plugging the map $K' \to \mathrm{id}_{\mathcal{C}}$ each time in all entries but one is an equivalence. This, in turn, implies that the above cosimplicial object is right Kan extended from $\Delta_{\leq 1}$, and so degenerates to an equivalence

$$\partial_n \mathrm{id}_{\mathcal{E}} \to \Omega(\partial_n K').$$

We conclude that the map $\Omega(\partial_n K') \to \Omega(\partial_n Q)$ is an equivalence, and hence the map $\partial_n K' \to \partial_n Q$ is an equivalence is as well. $\square$

By Proposition 22 the functor $\mathrm{LaxEq}(\mathcal{C},-)$ sends n-excisive functors in $\mathrm{Fun}^{\omega}_*(\mathcal{C},\mathcal{C})$ to n-excisive compactly generated ∞-categories over $\mathcal{C}$. It then follows that the adjunction $K(-) \dashv \mathrm{LaxEq}(\mathcal{C},-)$ induces an adjunction

$$\mathcal{P}_n K(-)\colon (\mathcal{C}\mathrm{at}^{\le n}_*)_{/\mathcal{C}} \rightleftarrows \mathrm{Fun}^{\le n}_*(\mathcal{C},\mathcal{C})\colon \mathrm{LaxEq}(\mathcal{C},-),$$

where the right adjoint is still $\mathrm{LaxEq}(\mathcal{C},-)$ and the left adjoint is sends $\mathcal{E} \to \mathcal{C}$ to $\mathcal{P}_n(K(\mathcal{E}))$, that is, to the n-excisive approximation of the associated finitary endo-functor $K(\mathcal{E})\colon \mathcal{C} \to \mathcal{C}$.

The above gives us an explicit way to construct n-excisive ∞-categories of a certain specific type, but it does not cover all n-excisive ∞-categories. To obtain a more comprehensive account we will need to slightly elaborate the construction as follows.

Suppose given a pointed compactly generated ∞-category $\mathcal{D}$ equipped with a compactly generated functor $f\colon \mathcal{D} \to \mathcal{C}$ with finitary right adjoint $g\colon \mathcal{C} \to \mathcal{D}$, and associated finitary endo-functor $K(\mathcal{D}) = fg\colon \mathcal{C} \to \mathcal{C}$. The association $[\widetilde{\mathcal{D}} \to \mathcal{D}] \mapsto [K(\widetilde{\mathcal{D}}) \to K(\mathcal{D})]$ then determines a functor

$$(\mathcal{C}\mathrm{at}^{\omega}_*)_{/\mathcal{D}} \to \mathrm{Fun}^{\omega}_*(\mathcal{C},\mathcal{C})_{/K(\mathcal{D})}.$$

We construct a right adjoint to this functor as follows. Given a natural transformation $\widetilde{K} \to K(\mathcal{D})$ let us set

$$Q := \mathrm{cof}[\widetilde{K} \to K(\mathcal{D})].$$

The map $K(\mathcal{D}) \to Q$ then corresponds to a natural transformation $\alpha\colon f \Rightarrow Qf$, which classifies a compactly generated functor $\alpha_*\colon \mathcal{D} \to \mathrm{LaxEq}(\mathcal{C},Q)$. Let us then define $\mathrm{Ext}(\mathcal{D},\widetilde{K})$ to be the compactly generated ∞-category sitting in the pullback square (computed in $\mathcal{C}\mathrm{at}^{\omega}_*$):

(8)
$$\begin{tikzcd} \mathrm{Ext}(\mathcal{D},\widetilde{K}) \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "\sigma_0"] \\ \mathcal{D} \arrow[r, "\alpha_*"] & \mathrm{LaxEq}(\mathcal{C},Q) \end{tikzcd}$$

where the functor $\mathcal{C} \to \mathrm{LaxEq}(\mathcal{C},Q)$ is the "zero section" of $\mathrm{LaxEq}(\mathcal{C},Q)$, classified by the zero natural transformation $\mathrm{id}_{\mathcal{C}} \Rightarrow Q$.

Remark 26. Explicitly, the compact objects of $\mathrm{Ext}(\mathcal{D},\widetilde{K})$ are given by pairs (X,η), where X is an object of $\mathcal{D}$ and η is a null-homotopy of the map $\alpha_X\colon f(X) \to Q(f(X))$. Equivalently, we may describe this data as an object X equipped with together with a lift

$$\begin{tikzcd} & \widetilde{K}(f(X)) \arrow[r] \arrow[d] & 0 \arrow[d] \\ f(X) \arrow[r] \arrow[ur, dashed] & K(\mathcal{D})(f(X)) \arrow[r] \arrow[d, equal] & Qf(X) \\ & fgf(X) & \end{tikzcd}$$

By construction, for a compactly generated ∞-category $\widetilde{\mathcal{D}}$, compactly generated functors $\widetilde{\mathcal{D}} \to \mathrm{Ext}(\mathcal{D},\widetilde{K})$ correspond to pairs (h,τ) where $h\colon \widetilde{\mathcal{D}} \to \mathcal{D}$ is a compactly generated functor and τ is a null-homotopy of the natural transformation $\alpha h\colon fh \Rightarrow Qfh$. Equivalently, if we write $K(\widetilde{\mathcal{D}})\colon \mathcal{C} \to \mathcal{C}$ for the associated endo-functor on $\mathcal{C}$ (given by composing fh with its right adjoint) then the null-homotopy τ can equivalently be thought of as a null-homotopy of the composite $K(\widetilde{\mathcal{D}}) \to K(\mathcal{D}) \to Q$, or equivalently, a lift

$$\begin{tikzcd} & \widetilde{K} \arrow[r] \arrow[d] & 0 \arrow[d] \\ K(\widetilde{\mathcal{D}}) \arrow[r] \arrow[ur, dashed] & K(\mathcal{D}) \arrow[r] & Q. \end{tikzcd}$$

We may organize these observations into the following statement:

Corollary 27. *The association* $[\widetilde{K} \to K(\mathcal{D})] \mapsto \mathrm{Ext}(\mathcal{D}, \widetilde{K})$*, considered as a functor*

$$\mathrm{Fun}^{\omega}_{*}(\mathcal{C}, \mathcal{C})_{/K(\mathcal{D})} \to (\mathcal{C}\mathrm{at}^{\omega}_{*})_{/D},$$

is right adjoint to the functor $[\widetilde{\mathcal{D}} \to \mathcal{D}] \mapsto [K(\widetilde{\mathcal{D}}) \to K(\mathcal{D})]$.

Remark 28. In the construction of $\mathrm{Ext}(\mathcal{D}, \widetilde{K})$, assume that the map $K \to Q$ is null-homotopic, so that $\widetilde{K} = K \oplus \Omega Q$. Then the associated functor $\mathcal{D} \to \mathrm{LaxEq}(\mathcal{C}, Q)$ is homotopic to the composite $\mathcal{D} \xrightarrow{f} \mathcal{C} \xrightarrow{\sigma_0} \mathrm{LaxEq}(\mathcal{C}, Q)$ and we get that a natural equivalence

$$\begin{aligned} \mathrm{Ext}(\mathcal{D}, \widetilde{K}) &= \mathcal{D} \times_{\mathrm{LaxEq}(\mathcal{C}, Q)} \mathcal{C} \\ &= \mathcal{D} \times_{\mathcal{C}} \mathcal{C} \times_{\mathrm{LaxEq}(\mathcal{C}, Q)} \mathcal{C} \\ &= \mathcal{D} \times_{\mathcal{C}} \mathrm{LaxEq}(\mathcal{C}, \Omega Q), \end{aligned}$$

where we have used the fact that the functor $\mathrm{LaxEq}(\mathcal{C}, -) \colon \mathrm{Fun}^{\omega}_{*}(\mathcal{C}, \mathcal{C}) \to (\mathcal{C}\mathrm{at}^{\omega}_{*})_{/\mathcal{C}}$ preserves loops (being a right adjoint).

Let us now consider the adjunction

$$(\mathcal{C}\mathrm{at}^{\omega}_{*})_{/D} \rightleftarrows \mathrm{Fun}^{\omega}_{*}(\mathcal{C}, \mathcal{C})_{/K(\mathcal{D})} \tag{9}$$

of Corollary 27 in the case where $f \colon \mathcal{D} \to \mathcal{C}$ is a 1-excisive equivalence, that is, exhibits $\mathcal{C}$ as the stabilization of $\mathcal{D}$. By the second criterion of Proposition 9 we have that the the left adjoint of this adjunction sends i-excisive equivalences $\widetilde{\mathcal{D}} \to \mathcal{D}$ to $\mathcal{P}_i$-equivalence $K(\widetilde{\mathcal{D}}) \to K(\mathcal{D})$ for every $i \geq 1$. On the other hand, if $\widetilde{K} \to K(\mathcal{D})$ is a $\mathcal{P}_i$-equivalence then $Q := \mathrm{cof}[\widetilde{K} \to K(\mathcal{D})]$ satisfies $\mathcal{P}_i(Q) = 0$, and so by Proposition 22 and Corollary 18 we have that $\mathrm{Ext}(\mathcal{D}, \widetilde{K}) \to \mathcal{D}$ is an i-excisive equivalence. In other words, the right adjoint sends $\mathcal{P}_i$-excisive equivalences to i-excisive equivalences. For every $i \geq 1$ the above adjunction hence restricts to an adjunction

$$(\mathcal{C}\mathrm{at}^{\omega}_{*})_{\sim i/D} \rightleftarrows \mathrm{Fun}^{\omega}_{*}(\mathcal{C}, \mathcal{C})_{\sim i/K(\mathcal{D})} \tag{10}$$

between the full subcategory of $(\mathcal{C}\mathrm{at}^{\omega}_{*})_{/\mathcal{D}}$ spanned by the i-excisive equivalences $\widetilde{\mathcal{D}} \to \mathcal{D}$ and the full subcategory of $\mathrm{Fun}^{\omega}_{*}(\mathcal{C}, \mathcal{C})_{/K(\mathcal{D})}$ spanned by the $\mathcal{P}_i$-equivalences. Let us now consider the composite adjunction

$$(\mathcal{C}\mathrm{at}^{\omega}_{*})_{\sim i/D} \rightleftarrows \mathrm{Fun}^{\omega}_{*}(\mathcal{C}, \mathcal{C})_{\sim i/K(\mathcal{D})} \rightleftarrows \mathrm{Fun}^{\leq i+1}_{*}(\mathcal{C}, \mathcal{C})_{\sim i/K(\mathcal{D})}$$

where in the second adjunction the left adjoint sends $\widetilde{K} \to K(\mathcal{D})$ to $\mathcal{P}_{i+1}(\widetilde{K}) \to \mathcal{P}_{i+1}(K(\mathcal{D}))$ and the right adjoint sends $\widetilde{K}' \to \mathcal{P}_{i+1}(K)$ to $\widetilde{K}' \times_{\mathcal{P}_{i+1}(K)} K \to K$. The composite right adjoint

$$\mathrm{Fun}^{\leq i+1}_{*}(\mathcal{C}, \mathcal{C})_{\sim i/K(\mathcal{D})} \to (\mathcal{C}\mathrm{at}^{\omega}_{*})_{\sim i/D} \tag{11}$$

then takes $\widetilde{K}' \to \mathcal{P}_{i+1}K(\mathcal{D})$ to $\mathrm{Ext}(\mathcal{D}, \widetilde{K})$, with $\widetilde{K} = \widetilde{K}' \times_{\mathcal{P}_{i+1}K(\mathcal{D})} K(\mathcal{D})$.

Proposition 29. *If* $\mathcal{D}$ *is* i*-excisive then the composite right adjoint* (11) *is fully-faithful, and its essential image consists of those* i*-excisive equivalences* $\widetilde{\mathcal{D}} \to \mathcal{D}$ *such that* $\widetilde{\mathcal{D}}$ *is* $(i+1)$*-excisive.*

Corollary 30. *If* $\mathcal{D}$ *if* i*-excisive then the functor* (11) *determined an equivalence between the* ∞*-category of reduced, finitary,* $(i+1)$*-excisive functors* $\widetilde{K}'$ *equipped with a* $\mathcal{P}_i$*-equivalence* $\widetilde{K}' \to \mathcal{P}_n(K)$*, and pointed, compactly generated,* $(i+1)$*-excisive* ∞*-categories* $\widetilde{D}$*, equipped with an* i*-excisive equivalence* $\widetilde{D} \to D$*.*

Proof of Proposition 29. Let $\widetilde{K}' \to \mathcal{P}_{i+1}(K(\mathcal{D}))$ be a $\mathcal{P}_i$-equivalence from an $(i+1)$-excisive functor $\widetilde{K}'$, and let $Q = \mathrm{cof}[\widetilde{K}' \to \mathcal{P}_{i+1}(K(\mathcal{D}))]$. Set $\widetilde{K} := \widetilde{K}' \times_{\mathcal{P}_{i+1}K(\mathcal{D})} K(\mathcal{D})$. Applying Proposition 19 to the square (8) we obtain that the square

$$\begin{tikzcd} \mathcal{P}_{i+1}K(\mathrm{Ext}(\mathcal{D},\widetilde{K})) \arrow[r] \arrow[d] & \mathrm{id}_{\mathcal{C}} \arrow[d] \\ \mathcal{P}_{i+1}K(\mathcal{D}) \arrow[r] & \mathcal{P}_{i+1}K(\mathrm{LaxEq}(\mathcal{C},Q)) \end{tikzcd}$$

is cartesian. Using Proposition 25 we may rewrite the bottom corner as

$$\begin{tikzcd} \mathcal{P}_{i+1}K(\mathrm{Ext}(\mathcal{D},\widetilde{K})) \arrow[r] \arrow[d] & \mathrm{id}_{\mathcal{C}} \arrow[d] \\ \mathcal{P}_{i+1}K(\mathcal{D}) \arrow[r] & \mathrm{id}_{\mathcal{C}} \oplus Q \end{tikzcd}$$

where the right vertical map is the summand inclusion and the bottom horizontal map is induced by the defining map $\mathcal{P}_{i+1}K(\mathcal{D}) \to Q$ and the canonical coaugmentation $K(D) \to \mathrm{id}_{\mathcal{C}}$ of $K(D)$. The fact that this square is cartesian exactly means that the induced map

$$\mathcal{P}_{i+1}K(\mathrm{Ext}(\mathcal{D},\widetilde{K})) \to \widetilde{K}' = \mathrm{fib}[\mathcal{P}_{i+1}(K(\mathcal{D})) \to Q]$$

is an equivalence, which means that the counit of the adjunction is an equivalence on $\widetilde{K}'$. We conclude that $\mathrm{Ext}(\mathcal{D},-)$ is fully-faithful. We now with to identify its essential image. By Proposition 22 and the closure of $(i+1)$-excisive ∞-categories under pullback we have that this essential image consists of $(i+1)$-excisive ∞-categories, so we just need to show that all of them are covered. This is equivalent to saying that if $\widetilde{\mathcal{D}} \to \mathcal{D}$ is a i-excisive equivalence and $\widetilde{\mathcal{D}}$ is $(i+1)$-excisive then the unit

$$\widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}' := \mathrm{Ext}(\mathcal{D}, \mathcal{P}_{i+1}K(\widetilde{\mathcal{D}}) \times_{\mathcal{P}_{i+1}K(\mathcal{D})} K(\mathcal{D}))$$

is an equivalence. Indeed, this is a-priori only an i-excisive equivalence with $(i+1)$-excisive target, but since we already know that the counit is an equivalence, the triangle identities imply that this map induces an equivalence $\mathcal{P}_{i+1}K(\widetilde{\mathcal{D}}) \to \mathcal{P}_{i+1}K(\mathcal{D}')$, and hence the unit is actually an $(i+1)$-excisive equivalence. It is hence an equivalence whenever its domain is also $(i+1)$-excisive by Lemma 11. □

4. Goodwillie towers of ∞-categories

Definition 31. Let $f\colon \mathcal{D} \to \mathcal{E}$ be a compactly generated functor between pointed, compactly generated ∞-categories. We will say that f exhibits $\mathcal{E}$ as the *n-excisive approximation* of $\mathcal{D}$ if $\mathcal{E}$ is n-excisive and f is an n-excisive equivalence.

Proposition 32. *For every pointed, compactly generated ∞-category $\mathcal{D}$ there is a tower*

$$\mathcal{D} \to \ldots \to \mathcal{D}_{n-1} \to \ldots \to \mathcal{D}_1$$

such that for $i \geq 1$ the map $\mathcal{D} \to \mathcal{D}_i$ exhibits $\mathcal{D}_i$ as the n-excisive approximation of $\mathcal{D}$. In addition, for every $i \geq 2$ we have that $\mathcal{D}_i = \mathrm{Ext}(\mathcal{D}_{i-1}, K_i)$ for some reduced, finitary functor $K_i\colon \mathcal{D}_1 \to \mathcal{D}_1$ equipped with a natural transformation $K_i \Rightarrow K(\mathcal{D}_{i-1})$ whose cofibre is i-homogeneous.

Remark 33. In the situation of Proposition 32, if $\mathcal{D}$ is n-excisive then the tower is constant from $\mathcal{D}_n$ onwards; indeed, for $i \geq n$ the maps $\mathcal{D} \to \mathcal{D}_i$ are i-excisive equivalences between i-excisive ∞-categories, and are hence equivalence by Lemma 11. In particular, every n-excisive ∞-category $\mathcal{D}$ can be constructed from its stabilization by performing the above $\mathrm{Ext}(-,-)$ construction finitely many times.

Proof. Construct the ∞-categories $\mathcal{D}_i$ inductively for $i = 1, ..., n$. For $i = 1$ one just sets $\mathcal{D}_1 = \mathrm{Sp}(\mathcal{D})$. Assume that $\mathcal{D} \to \mathcal{D}_i \to .. \to \mathcal{D}_1$ has been defined for some i and such that the required properties hold. In particular, the induced map $\mathcal{D} \to \mathcal{D}_i$ is an i-excisive equivalence, and so the induced map $K(\mathcal{D}) \to K(\mathcal{D}_i)$ is a $\mathcal{P}_i$-equivalence. We now define

$$K_{i+1} := \mathcal{P}_{i+1}(K(\mathcal{D})) \times_{\mathcal{P}_{i+1}K(\mathcal{D}_i)} K(\mathcal{D}_i),$$

equipped with the projection $K_{i+1} \to K(\mathcal{D}_i)$, and set $\mathcal{D}_{i+1} = \mathrm{Ext}(\mathcal{D}_i, K_{i+1})$. By the adjunction (10) the canonical factorization of $K(\mathcal{D}) \to K(\mathcal{D}_i)$ through K_i determines a lift of the compactly generated functor $\mathcal{D} \to \mathcal{D}_i$ to a compactly generated functor $\mathcal{D} \to \mathcal{D}_{i+1}$. This last functor is then an $(i+1)$-excisive equivalence since it induces an equivalence on stabilizations and the induced map on the left

$$\begin{tikzcd} \mathcal{P}_{i+1}K(\mathcal{D}) \arrow[r] \arrow[rr, bend left, "\simeq"] & \mathcal{P}_{i+1}K(\mathcal{D}_{i+1}) \arrow[r, "\simeq"] & \mathcal{P}_{i+1}K_{i+1} \end{tikzcd}$$

is an equivalence by 2-out-of-3 (where the equivalence on the right is by Proposition 29, and the composite by construction). $\square$

Proposition 34. *Suppose that $p\colon \mathcal{D} \to \mathcal{D}_n$ is a functor exhibiting $\mathcal{D}_n$ as the n-excisive approximation of $\mathcal{D}$. Then for every n-excisive ∞-category $\mathcal{E}$, restriction along p_n induces an equivalence*

$$r_n\colon \mathrm{LFun}^{\omega}_{*}(\mathcal{D}_n, \mathcal{E}) \xrightarrow{\simeq} \mathrm{LFun}^{\omega}_{*}(\mathcal{D}, \mathcal{E}).$$

Proof. If $\mathcal{E}$ is n-excisive then the compactly generated ∞-category $\mathrm{Fun}(\Delta^i, \mathcal{E})$ is n-excisive for every i. It will hence suffice to show that r_n induces an equivalence on core ∞-groupoids for every $\mathcal{E}$. Consider the finite tower

$$\mathcal{E} = \mathcal{E}_n \to \mathcal{E}_{n-1} \to ... \to \mathcal{E}_1$$

obtained by applying the construction of Proposition 32 to $\mathcal{E}$ (see Remark 33). We prove by induction on i that the induced functor

$$r_i\colon \mathrm{LFun}^{\omega}_{*}(\mathcal{D}_n, \mathcal{E}_i) \to \mathrm{LFun}^{\omega}_{*}(\mathcal{D}, \mathcal{E}_i)$$

induces an equivalence on core ∞-groupoids. For $i = 1$ we have that $\mathcal{E}_1$ is stable and the map $\mathcal{D} \to \mathcal{D}_n$ is an equivalence on stabilization, and hence r_1 is an equivalence. Now suppose that r_i is an equivalence for some $1 \le i \le n-1$ and prove that it is an equivalence for $i+1$. Consider the square of core ∞-groupoids

$$\begin{tikzcd} \mathrm{LFun}^{\omega}_{*}(\mathcal{D}_n, \mathcal{E}_{i+1})^{\simeq} \arrow[r] \arrow[d] & \mathrm{LFun}^{\omega}_{*}(\mathcal{D}, \mathcal{E}_{i+1})^{\simeq} \arrow[d] \\ \mathrm{LFun}^{\omega}_{*}(\mathcal{D}_n, \mathcal{E}_i)^{\simeq} \arrow[r, "\simeq"] & \mathrm{LFun}^{\omega}_{*}(\mathcal{D}, \mathcal{E}_i)^{\simeq} \end{tikzcd}$$

where the bottom horizontal arrow is an equivalence by the induction hypothesis. To show that the top horizontal arrow is an equivalence we check that the square induces an equivalence on vertical fibres. This is the same as saying that for every fixed compactly generated functor $f\colon \mathcal{D}_n \to \mathcal{E}_i$, restriction along $\mathcal{D} \to \mathcal{D}_n$ induces an equivalence

$$\mathrm{LFun}^{\omega}_{/\mathcal{E}_i}(\mathcal{D}_n, \mathcal{E}_{i+1}) \xrightarrow{\simeq} \mathrm{LFun}^{\omega}_{/\mathcal{E}_i}(\mathcal{D}, \mathcal{E}_{i+1}).$$

Now by construction we have that

$$\mathcal{E}_{i+1} = \mathrm{Ext}(\mathcal{E}_i, K_{i+1})$$

where $K_{i+1} \Rightarrow K(\mathcal{E}_i)$ is a natural transformation of functors $\mathcal{E}_1 \to \mathcal{E}_1$ whose cofibre is $(i+1)$-homogeneous. The compactly generated functors $\mathcal{D}_n \xrightarrow{f} \mathcal{E}_i \to \mathcal{E}_1$ and $\mathcal{D} \to \mathcal{D}_n \xrightarrow{f} \mathcal{E}_i \to \mathcal{E}_1$ composed with their finitary right adjoints yield a map of endo-functors $\alpha\colon K(\mathcal{D}) \Rightarrow K(\mathcal{D}_n)$ on $\mathcal{E}_1$. By the adjunction 9, it will suffice to show that α a $\mathcal{P}_{i+1}$-equivalence. For this, let us write $f_*: \mathrm{Sp}(\mathcal{D}) \to \mathcal{E}_1$ for the induced functor, and let $g_*: \mathcal{E}_1 \to \mathrm{Sp}(\mathcal{D})$ be its right adjoint. Then we may identify α with the induced map

$$f_*\Sigma^\infty_{\mathcal{D}}\Omega^\infty_{\mathcal{D}} g_* \Rightarrow f_*\Sigma^\infty_{\mathcal{D}_n}\Omega^\infty_{\mathcal{D}_n} g_*.$$

Since the natural transformation $\Sigma^\infty_{\mathcal{D}}\Omega^\infty_{\mathcal{D}} \Rightarrow \Sigma^\infty_{\mathcal{D}_n}\Omega^\infty_{\mathcal{D}_n}$ is a $\mathcal{P}_{i+1}$-equivalence by virtue of the assumption that $\mathcal{D} \to \mathcal{D}_n$ is an n-excisive equivalence and $i+1 \le n$, we have that α is a $\mathcal{P}_{i+1}$-equivalence by Lemma 4. □

Corollary 35 ([Heu21]). *For a pointed, compactly generated $\mathcal{C}$, the ∞-category of n-excisive approximations of $\mathcal{C}$ is contractible, and the formation of n-excisive approximations assembles to form a functor*

$$\mathcal{C}\mathrm{at}^{\omega}_* \to \mathcal{C}\mathrm{at}^{\le n}_* \qquad \mathcal{C} \mapsto \mathcal{P}_n(\mathcal{C}),$$

which is left adjoint to the inclusion $\mathcal{C}\mathrm{at}^{\le n}_* \subseteq \mathcal{C}\mathrm{at}^{\omega}_*$.

Given a pointed compactly generated ∞-category $\mathcal{D}$, its Goodwillie tower above $\{\mathcal{D}_n\}_n$ above determines a compactly generated functor

$$f_\infty: \mathcal{D} \to \mathcal{D}_\infty := \lim_n \mathcal{D}_n,$$

with finitary right adjoint $g_\infty: \mathcal{D}_\infty \to \mathcal{D}$ (the limit is computed in $\mathcal{C}\mathrm{at}^{\omega}_*$). If we write

$$f_n: \mathcal{D} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{D}_n : g_n$$

for each of the finite stage adjunctions in the Goodwillie tower then each unit $u_n: \mathrm{id} \Rightarrow g_n f_n$ exhibits $g_n f_n$ as the n-excisive approximation of the identity since the map f_n is an n-excisive equivalence and $\mathcal{D}_n$ is n-excisive. In particular, we may identify each u_n with the map $\mathrm{id} \Rightarrow \mathcal{P}_n(\mathrm{id})$. The total unit of $f_\infty \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} g_\infty$ can then be identified with the limit

$$\mathrm{id} \Rightarrow \lim_n g_n f_n = \lim_n \mathcal{P}_n(\mathrm{id}).$$

In particular, the unit is an equivalence when evaluated on objects on which the Goodwillie tower of the identity converges. In particular, if we consider the full subcategory $\mathcal{D}_{\mathrm{conv}} \subseteq \mathcal{D}$ spanned by those objects $X \in \mathcal{C}$ on which the Goodwillie tower of the identity converges then f_∞ determines a fully-faithful embedding of $\mathcal{D}_{\mathrm{conv}}$ in $\mathcal{D}_\infty$. In particular, a certain full subcategory of $\mathcal{D}$ is equivalent to a certain full subcategory of $\mathcal{D}_\infty$. When these full subcategories are relatively large (as happens in many cases of interest), the Goodwillie tower of $\mathcal{D}$ can be efficiently used to obtain information on $\mathcal{D}$ via the simpler pieces $\mathcal{D}_n$.

5. Examples

5.1. **Divided powers coalgebras and Koszul duality.** Let $\mathcal{C}$ be a stable compactly generated ∞-category equipped with a symmetric monoidal structure which preserves colimits in each variable separately. Let $\mathcal{I}$ be the singled color $\mathcal{C}$-enriched ∞-operad whose unary operation object is the monoidal unit $1 \in \mathcal{C}$ and all other operation objects are the zero object $0 \in \mathcal{C}$. Let $\mathcal{O}$ be a single color $\mathcal{C}$-enriched ∞-operad which is equipped with a map $\mathcal{O} \to \mathcal{I}$ inducing an equivalence on enriched ∞-categories of colors and on nullary operations. Then the compactly generated ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is pointed and the compactly generated functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{I}}(\mathcal{C}) \simeq \mathcal{C}$$

induces by $\mathcal{O} \to \mathcal{I}$ exhibits $\mathcal{C}$ as the stabilization of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. The associated comonad $K := \Sigma^\infty\Omega^\infty$ on $\mathcal{C}$ is then given by

$$K(V) = \mathcal{I} \circ_{\mathcal{O}} V = \mathcal{I} \circ_{\mathcal{O}} \mathcal{I} \circ_{\mathcal{I}} V = \sum_{i \ge 1} (\mathrm{B}\mathcal{O}(i) \otimes V^{\otimes i})_{\mathrm{h}\Sigma_i} = V \oplus \sum_{i \ge 2} (\mathrm{B}\mathcal{O}(i) \otimes V^{\otimes i})_{\mathrm{h}\Sigma_i},$$

where $\mathrm{B}\mathcal{O} := \mathcal{I} \circ_{\mathcal{O}} \mathcal{I}$ is the Bar construction of $\mathcal{O}$, which has the structure of a $\mathcal{C}$-enriched *co-operad* with $\mathcal{O}(0) = 0$ and $\mathcal{O}(1) = 1$. This cooperad is also known as the *Koszul dual* of $\mathcal{O}$. In particular, for every $n \ge 1$

we have $\mathcal{P}_n(K) = \sum_{i=1}^n (\mathcal{O}(i) \otimes V^{\otimes i})_{h\Sigma_i}$, so that $\mathcal{P}_n(K)$ is an n-excisive functor whose n-homogeneous parts are trivially glued to each other. Let us now unwind the definition for the n-excisive approximations, constructed in the proof of Proposition 32. The 1-excisive approximation is $\mathcal{D}_1 = \mathcal{C}$ itself. To construct the 2-excisive approximation, one considers the 2-excisive approximation $\mathcal{P}_2 K(V) = V \oplus (\mathrm{B}\mathcal{O}(2) \otimes V \otimes V)_{h\Sigma_2}$ and sets $Q = \mathrm{cof}[\mathcal{P}_2 K(V) \to V] = \Sigma(\mathrm{B}\mathcal{O}(2) \otimes V \otimes V)_{h\Sigma_2}$. Since the augmentation is a projection from a direct sum the map $\mathcal{P}_2 K(V) \to Q$ is null-homotopic and we are in the situation of Remark 28. In particular, the 2-excisive approximation of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is given by

$$\mathcal{D}_2 := \mathrm{Ext}(\mathcal{C}, K) = \mathrm{LaxEq}(\mathcal{C}, \Omega Q) = \mathrm{LaxEq}(\mathcal{C}, [\mathrm{B}\mathcal{O}(2) \otimes (-) \otimes (-)]_{h\Sigma_2}).$$

The compact objects of $\mathcal{D}_2$ are then given by pairs (X, α) where X is an object of $\mathcal{C}$ and $\alpha\colon X \to [\mathrm{B}\mathcal{O}(2) \otimes X \otimes X]_{h\Sigma_2}$ is a structure map, which is a form of a quadratic co-operation. Such a co-operation is also known as a *divided powers* $\mathrm{B}\mathcal{O}(2)$*-comultiplication* (this is not the same as a cocommutative $\mathrm{B}\mathcal{O}(2)$-comultiplication, which rather corresponds to a map $X \to [\mathrm{B}\mathcal{O}(2) \otimes X \otimes X]^{h\Sigma_2}$). Equivalently, we may identify $\mathcal{D}_2$ with the Ind-completion of the ∞-category of divided powers coalgebras in $\mathcal{C}^c$ over the *cofree cooperad* $F_2 := \mathrm{cofree}(\mathrm{B}\mathcal{O}_{\leq 2})$ generated from the binary co-operation $\mathrm{B}\mathcal{O}(2) \otimes (-) \otimes (-)$; indeed, being cofreely generated means in particular that specifying a divided power coalgebras over it is the same as specifying only the generating co-operation. We write this as

$$\mathcal{D}_2 = \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_2}(\mathcal{C}^c)).$$

Let us now consider the next step, namely, the 3-excisive approximation $\mathcal{D}_3$ of $\mathrm{Alg}_{\mathcal{O}}(V)$. For this, we need to find the cofibre of the map

$$\mathcal{P}_3 K \Rightarrow \mathcal{P}_3 K(\mathcal{D}_2). \tag{12}$$

Now the endo-functor $K(\mathcal{D}_2)\colon \mathcal{C} \xrightarrow{\mathrm{cofree}} \mathcal{D}_2 \xrightarrow{\mathrm{forget}} \mathcal{C}$ sends V to the underlying object of the cofree divided powers F_2-coalgebra generated from V, that is,

$$K(\mathcal{D}_2)(V) = V \oplus \sum_{i \geq 2} [F_2(i) \otimes V^{\otimes i}]_{h\Sigma_i}.$$

The 3-excisive approximation of $K(\mathcal{D}_2)$ is then given by

$$\mathcal{P}_3 K(\mathcal{D}_2)(V) = V \oplus [F_2(i) \otimes V^{\otimes 2}]_{h\Sigma_2} \oplus [F_2(i) \otimes V^{\otimes 3}]_{h\Sigma_3}.$$

If we now unwind the construction of cofree co-operads we see that $F_2(2) = \mathrm{B}\mathcal{O}(2)$ and

$$F_2(3) = (\mathrm{B}\mathcal{O}_{\leq 2} \circ \mathrm{B}\mathcal{O}_{\leq 2})(2) = [\oplus_{\Sigma_3} \mathrm{B}\mathcal{O}(2) \otimes \mathrm{B}\mathcal{O}(2)]_{\Sigma_2} = [\oplus_{\Sigma_3} \mathrm{B}\mathcal{O}(2) \otimes \mathrm{B}\mathcal{O}(2)]^{\Sigma_2}.$$

In words, $F_2(3)$ is the Σ_3-object of $\mathcal{C}$ left/right induced from the Σ_2-object $\mathrm{B}\mathcal{O}(2) \otimes \mathrm{B}\mathcal{O}(2)$, where the Σ_2-action on the latter is by the action on the second factor. The map (12) is then an equivalence on 2-excisive approximations and its cofibre Q is the 3-homogeneous functor given by

$$Q(V) = (\mathrm{cof}[\mathrm{B}\mathcal{O}(3) \to F_2(3)] \otimes V^{\otimes 3})_{h\Sigma_3}.$$

Let $K' = \mathcal{P}_3 K \times_{\mathcal{P}_3 K(\mathcal{D}_2)} K(\mathcal{D}_2)$, so that K' lies in an exact sequence

$$K' \to K(\mathcal{D}_2) \to Q.$$

The 3-excisive approximation of $\mathrm{Alg}_{\mathcal{O}}(C)$ is then given by the ∞-category $\mathcal{D}_3 = \mathrm{Ext}(\mathcal{D}_2, K')$ whose compact objects are triples (X, α, η), where $c_2\colon X \to (\mathrm{B}\mathcal{O}(2) \otimes X \otimes X)_{h\Sigma_2}$ is a quadratic structure map

and η is a null-homotopy of the dotted composed map, or, equivalently, a choice of a dashed lift:
(13)

$$\begin{tikzcd}
 & & (\mathrm{B}\mathcal{O}(3) \otimes X \otimes X \otimes X)_{\mathrm{h}\Sigma_3} \arrow[r] \arrow[d] & 0 \arrow[d] \\
X \arrow[r, "c_2"'] \arrow[urr, dashed, "c_3"] \arrow[rrr, dotted, bend right] & (\mathrm{B}\mathcal{O}(2) \otimes X \otimes X)_{\mathrm{h}\Sigma_2} \arrow[r] & (F_2(3) \otimes X \otimes X \otimes X)_{\mathrm{h}\Sigma_3} \arrow[r] & (\mathrm{cof}[\mathrm{B}\mathcal{O}(3) \to F_2(3)] \otimes X \otimes X \otimes X)_{\mathrm{h}\Sigma_3}
\end{tikzcd}$$

Of course, one needs to understand what is the second horizontal map, which in principle, is determined by the unit of the adjunction $\mathcal{D}_2 \rightleftarrows \mathcal{C}$. For this, note that by definition $F_2(3)$ is Σ_3-induced from the Σ_2-object $\mathrm{B}\mathcal{O}(2) \otimes \mathrm{B}\mathcal{O}(2)$. Since the monoidal product preserves colimits in each variable, this means that the Σ_3-object $F_2(3) \otimes X \otimes X \otimes X$ is induced from the Σ_2-object $\mathrm{B}\mathcal{O}(2) \otimes \mathrm{B}\mathcal{O}(2) \otimes X \otimes X \otimes X = \mathrm{B}\mathcal{O}(2) \otimes (\mathrm{B}\mathcal{O}(2) \otimes X \otimes X) \otimes X$, where Σ_2-acts by its action on the internal $\mathrm{B}\mathcal{O}(2) \otimes X \otimes X$ component. Using again that the tensor product commutes with colimits in each variable we then obtain a natural equivalence

$$(F_2(3) \otimes X \otimes X \otimes X)_{\mathrm{h}\Sigma_3} \simeq \mathrm{B}\mathcal{O}(2) \otimes (\mathrm{B}\mathcal{O}(2) \otimes X \otimes X)_{\mathrm{h}\Sigma_2} \otimes X.$$

The map appearing in (13) in then given by the composite

$$\mathrm{B}\mathcal{O}(2) \otimes X \otimes X)_{\mathrm{h}\Sigma_2} \to (\mathrm{B}\mathcal{O}(2) \otimes X \otimes X)^{\mathrm{h}\Sigma_2} \to \mathrm{B}\mathcal{O}(2) \otimes X \otimes X \xrightarrow{\mathrm{id} \otimes \alpha \otimes \mathrm{id}} \mathrm{B}\mathcal{O}(2) \otimes (\mathrm{B}\mathcal{O}(2) \otimes X \otimes X)_{\mathrm{h}\Sigma_2} \otimes X,$$

where the first map is the trace map and the second forgets the fixed point structure. In particular, we may describe a compact object of $\mathcal{D}_3$ as consists of objects $X \in \mathcal{C}$ equipped with a quadratic structure map $c_2 \colon X \to (\mathrm{B}\mathcal{O}(2) \otimes X \otimes X)_{\mathrm{h}\Sigma_2}$, a trinary structure map $c_3 \colon X \to (\mathrm{B}\mathcal{O}(3) \otimes X \otimes X \otimes X)_{\mathrm{h}\Sigma_3}$, and a homotopy relating the two resulting maps $X \to (F_2(3) \otimes X \otimes X \otimes X)_{\mathrm{h}\Sigma_3}$. This is exactly the structure of a divided powers coalgebra over the co-operad $F_3 = \mathrm{cofree}(\mathrm{B}\mathcal{O}_{\leq 3})$ cofreely generated from the degree ≤ 3 part of $\mathrm{B}\mathcal{O}$, so that we can write

$$\mathcal{D}_3 = \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_3}(\mathcal{C}^c)).$$

More generally, one checks that the n-excisive approximation of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is given by

$$\mathcal{D}_n = \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_n}(\mathcal{C}^c)),$$

where $F_n = \mathrm{cofree}(\mathrm{B}\mathcal{O}_{\leq n})$ is the cooperad cofreely generated from the degree $\leq n$ part of $\mathrm{B}\mathcal{O}$.

Proposition 36 (Koszul duality for truncated operads)**.** *Suppose (in addition to the standing assumptions* $\mathcal{O}(0) = 0, \mathcal{O}(1) = 1$*) that* $\mathcal{O}$ *is n-truncated, that is, has no non-zero operations in degrees $> n$. Then the coooperad* $\mathrm{B}\mathcal{O}$ *is cofreely generated from its $\leq n$ part, and the n-excisive tower described above stabilizes at the n'th stage on* $\mathcal{D}_n = \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_n}(\mathcal{C}^c)) = \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{\mathrm{B}\mathcal{O}}(\mathcal{C}^c))$*. In addition, the ∞-category* $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ *is weakly n-excisive, the n-excisive approximation functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{\mathrm{B}\mathcal{O}}(\mathcal{C}^c))$$

is fully-faithful and its essential image is the minimal colimit-closed full subcategory containing the trivial divided powers coalgebras (that is, those whose structure maps are all zero).

Proof. We first show that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is weakly n-excisive. Since the underlying object functor $U \colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ preserves and detects limits, it will suffice to show that U is n-excisive. For this, we show inductively that the functor $U_i(A) = \tau_i\mathcal{O} \circ_{\mathcal{O}} A$ from $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ to $\mathcal{C}$ is i-excisive by induction on i (where for $i = n$ we have $\tau_n\mathcal{O} = \mathcal{O}$ and so $U_i = U$). For $i = 1$ the functor U_1 coincides with the functor $\Sigma^{\infty} \colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Sp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) = \mathcal{C}$, and is hence 1-excisive. Now let $2 \leq i \leq n$ and suppose the claim is known for $i - 1$.

Then the exact sequence $\tau_i\mathcal{O} \to \tau_{i+1}\mathcal{O} \to \mathcal{O}(i+1)$ (where $\mathcal{O}(i+1)$ is considered as a symmetric sequence concentrated in degree $i+1$) induces an exact sequence

$$U_i(A) \to U_{i+1}(A) \to \mathcal{O}(i+1) \circ_{\mathcal{O}} A.$$

Since $\mathcal{O}(i+1)$ is concentrated in degree $i+1$ the $\mathcal{O}$-action on it factors through the augmentation $\mathcal{O} \to \mathcal{O}(1)$. We then get that

$$\mathcal{O}(i+1) \circ_{\mathcal{O}} A = \mathcal{O}(i+1) \circ_{\mathcal{O}(1)} \mathcal{O}(1) \circ_{\mathcal{O}} A = \mathcal{O}(i+1) \circ_{\mathcal{O}(1)} U_1(A) = (\mathcal{O}(i+1) \otimes U_1(A) \otimes \dots \otimes U_1(A))_{\mathrm{h}\Sigma_n},$$

which is an $(i+1)$-excisive functor in A. We conclude that U_{i+1} sits in an exact sequence between an i-excisive functor and an $(i+1)$-excisive functor, and is hence $(i+1)$-excisive. This finishes the proof that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is weakly n-excisive.

Now the fact that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is weakly n-excsive means that the n-excisive equivalence

$$\mathrm{B}_n\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_n}(\mathcal{C}^c))$$

is fully-faithful by Proposition 14. It then follows from the third characterization of Proposition 9 that B_n is an m-excisive equivalence for every $m \geq n$. Since $\mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_n}(\mathcal{C}^c))$ is n-excisive it is also m-excisive for every $m \geq n$, and hence B_n is also an m-excisive approximation for every $m \geq n$. We conclude the forgetful functors

$$\mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_m}(\mathcal{C}^c)) \to \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_n}(\mathcal{C}^c))$$

are equivalences for every $m \geq n$, so that the tower of excisive approximations stabilizes at the n'th stage. This also means that the map of endo-functors

$$K(\mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_m}(\mathcal{C}^c))) \to K(\mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{F_n}(\mathcal{C}^c)))$$

on $\mathcal{C}$ is an equivalence for every $m \geq n$, and so the map of cooperads $F_m \to F_n$ are equivalences for every $m \geq n$. The map $\mathrm{B}\mathcal{O} \to \mathrm{cofree}(\mathrm{B}\mathcal{O}_{\leq n})$ is an equivalence on operations of degree $\leq m$ for every $m \geq n$, and is hence an equivalence. In particular, $\mathrm{B}\mathcal{O}$ is cofreely generated from its $\leq n$ part, and so $F_n = \mathrm{B}\mathcal{O}$.

To finish the proof we need to identify the essential image of

$$\mathrm{B}_n\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{coAlg}^{\mathrm{dp}}_{\mathrm{B}\mathcal{O}}(\mathcal{C}^c)).$$

For this, we note that the composite of

$$\mathcal{C}^c \xrightarrow{\text{free}} \mathrm{Alg}^c_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\mathrm{B}_n} \mathrm{coAlg}^{\mathrm{dp}}_{\mathrm{B}\mathcal{O}}(\mathcal{C}^c)$$

coincides with the trivial coalgebra functor $\mathrm{Triv}\colon \mathcal{C}^c \to \mathrm{coAlg}^{\mathrm{dp}}_{\mathrm{B}\mathcal{O}}(\mathcal{C}^c)$ which associates to an object $\mathcal{C}^c$ the divided powers coalgebra all of whose structure maps are zero. It follows that the essential image of B_n contains all trivial divided powers coalgebras. On the other hand, since $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is generated under colimits by free algebras and B_n preserves colimits we conclude that the image of B_n is generated under colimits by the trivial divided powers coalgebras, as desired. □

5.2. **Cartesian categories and Tate algebras.** In this section we will discuss the Goodwillie tower of pointed compactly generated ∞-categories which arise as the pointification of cartesian compactly generated ∞-categories. The principal example to keep in mind is that of pointed spaces $\mathcal{S}_*$, which arise as the pointification of the (cartesian) ∞-category of spaces, but one equally consider diagrams of spaces, any ∞-topoi, or more general cartesian ∞-categories.

recall that the ∞-category $\mathcal{C}\mathrm{at}^\omega$ of (unpointed) compactly generated ∞-categories and compactly generated functors admits a symmetric monoidal structure, where the tensor product is determined by the universal property that compactly generated functors $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ correspond to functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ (the product being computed in $\mathcal{C}\mathrm{at}^\omega$) which preserves compact objects and entry-wise colimits. A commutative monoid object in $\mathcal{C}\mathrm{at}^\omega$ then corresponds to a compactly generated ∞-category $\mathcal{C}$ equipped with a symmetric monoidal structure which preserves compact objects and entry-wise colimits. For example, the ∞-category of spaces is such a monoid with the tensor product being the cartesian product.

More generally, any compactly generated ∞-category whose cartesian product preserves compact objects and entry-wise colimits is a commutative monoid in $\mathcal{C}\mathrm{at}^{\omega}$. We will refer to these as *cartesian compactly generated ∞-categories*.

This tensor product preserves pointed compactly generated ∞-categories, and so the symmetric monoidal structure restricts to one on $\mathcal{C}\mathrm{at}^{\omega}_{*}$. In addition, the pointifcation functor

$$\mathcal{C}\mathrm{at}^{\omega} \to \mathcal{C}\mathrm{at}^{\omega}_{*} \qquad \mathcal{D} \mapsto \mathcal{D}_* := \mathcal{D}_{*/}$$

is symmetric monoidal (in fact, we can identify it with the functor $\mathcal{C} \mapsto \mathcal{D} \otimes \mathcal{S}_*$, and $\mathcal{S}_*$ is an idempotent object of $\mathcal{C}\mathrm{at}^{\omega}$), and hence if $\mathcal{D}$ is a commutative monoid in $\mathcal{C}\mathrm{at}^{\omega}$ then $\mathcal{D}_*$ inherits such a structure as well. In particular, if $\mathcal{D}$ is a cartesian compactly generated ∞-category then $\mathcal{D}_*$ inherits a symmetric monoidal structure which preserves compact objects and entry-wise colimits. In the case of spaces, the resulting symmetric monoidal structure on $\mathcal{S}_*$ is the *smash product*. We will consequently call the tensor product on $\mathcal{D}_*$ induced by the cartesian product on $\mathcal{D}$ the associated smash product, for any cartesian compactly generated $\mathcal{D}$. This terminology is also justified by the fact that it is always given by the same formula

$$X \wedge Y := \mathrm{cof}[X \coprod_{*} Y \to X \times Y] \qquad X, Y \in \mathcal{D}_*.$$

In addition, the canonical compactly generated functor $\mathcal{D} \mapsto \mathcal{D}_*$ sending X to $X_+ := X \coprod *$ is canonically symmetric monoidal. Moving from pointed to stable, we note that the tensor product on compactly generated ∞-categories preserves stable ∞-categories and the stabilization functor $\mathrm{Sp}(-)$ is symmetric monoidal. In particular, if $\mathcal{D}_*$ is the pointification of a cartesian compactly generated ∞-category $\mathcal{D}$ then the smash product on $\mathcal{C}_*$ induces a tensor product on $\mathrm{Sp}(\mathcal{D}_*)$, which we will refer to as smash product of spectrum objects, and denote it also with the symbol $\wedge$. In this case, the compactly generated functor

$$\Sigma^{\infty} \colon \mathcal{D}_* \to \mathrm{Sp}(\mathcal{D}_*)$$

is canonically symmetric monoidal, and its right adjoint is lax symmetric monoidal.

We will now try to describe the excisive approximations of $\mathcal{D}_*$ in this case in terms of a notion called *Tate coalgebras*. To simplify notation, let us write $\mathcal{C} := \mathrm{Sp}(\mathcal{D}_*)$ from now on. Let nC be the non-unital commutative cooperad and $\mathrm{nC}_n := \mathrm{cofree}(\mathrm{nC}_{\leq n})$ the cooperad freely generated from the $\leq n$ part of nC. For brevity will refer to nC_n-coalgebras as n-coalgebras. We define $\mathcal{E}_n := \mathrm{Ind}(\mathrm{coAlg}_{\mathrm{nC}_n}(\mathcal{C}^c))$ to be the ∞-category of compactly generated n-coalgebras in $\mathcal{C}$. For We will need to know the following:

Proposition 37. *There is a natural cartesian square of pointed compactly generated ∞-categories*

$$\begin{tikzcd}
\mathcal{E}_n \arrow[r] \arrow[d] & \mathrm{LaxEq}(\mathcal{C}, ((-) \wedge ... \wedge (-))^{h\Sigma_n}) \arrow[d] \\
\mathcal{E}_{n-1} \arrow[r] & \mathrm{LaxEq}\left(\mathcal{C}, \mathrm{nC}_n(n+1) \wedge (-) \wedge ... \wedge (-))^{h\Sigma_n}\right)
\end{tikzcd}$$

The above pullback square expresses the fact that refining an $(n-1)$-coalgebra to an n-coalgebra involves specifying a degree n symmetric cooperation $X \to (X \wedge ... \wedge X)^{h\Sigma_n}$ which is compactible with all degree n cooperations that can be obtained by composing cooperations of degree $< n$ (and which is encoded in the $n+1$ entry of $\mathrm{nC}_n = \mathrm{cofree}(\mathrm{nC}_{\leq n})$.

Our goal in the present section is to obtain an explicit description of $\mathcal{D}_n$ in terms of n-coalgebras equipped with additional Tate compatibility data.

Let $K := \Sigma^{\infty}\Omega^{\infty} \colon \mathcal{C} \to \mathcal{C}$ be the finitary endo-functor induced by the adjunction $\Sigma^{\infty} \dashv \Omega^{\infty}$.

Lemma 38. *For every $n \geq 1$ the n-linear part $\partial_n K$ of K is given by the formula*

$$\partial_n K(X_1, ..., X_n) = X_1 \wedge ... \wedge X_n.$$

Given that the n-linear parts of K are known, the equivalence type of $\mathcal{P}_n(K)$ (as an endo-functor) is encoded by exact squares

$$\begin{tikzcd} \mathcal{P}_i K(X) \arrow[r] \arrow[d] & \mathcal{P}_{i-1}K(X) \arrow[d, "\delta^i_X"] \\ (X \wedge \dots \wedge X)^{h\Sigma_i} \arrow[r] & (X \wedge \dots \wedge X)^{t\Sigma_i} \end{tikzcd}$$

for $i = 2, \dots, n$, which encode the way the i-excisive approximation is glued from the $(i-1)$-excisive approximation and the i-linear part, where the ladder starts at $\mathcal{P}_1 K = \mathrm{id}$. In particular, the natural transformations δ^i can be considered as the structure constants, which together with the smash product, completely determine the equivalence type of each finite excisive approximation of K. The left vertical map corresponds to a Σ_n-invariant natural transformation $K \Rightarrow (-) \wedge \dots \wedge (-)$ and hence to a Σ_n-invariant natural

$$\Sigma^\infty(-) \Rightarrow \Sigma^\infty(-) \wedge \dots \wedge \Sigma^\infty(-) = \Sigma^\infty((-) \wedge \dots \wedge (-))$$

of functors $\mathcal{D}_* \to \mathcal{C}$. Unwinding the definitions, these are induced by the smash-diagonals $X \to X \wedge \dots \wedge X$. Similarly, the right vertical map δ^i corresponds to the composite natural transformation

$$\delta^i_X \colon \Sigma^\infty(X) \to [\Sigma^\infty(X) \wedge \dots \wedge \Sigma^\infty(X)]^{t\Sigma_i}.$$

We refer to these as the Tate diagonals.

Lemma 39. *For every $m, n \geq 1$ the n-linear part $\partial_n K_m$ of K is given by the formula*

$$\partial_n K_m(X_1, \dots, X_n) = \mathrm{nC}_m(n) \wedge X_1 \wedge \dots \wedge X_n.$$

In particular, this agrees with $\partial_n K$ for $m \geq n$.

Now consider the n-excisive approximation $\mathcal{D}_n$ of $\mathcal{D}_*$. For every $i \geq 1$ write $K_i := K(\mathcal{D}_i) = \Sigma^\infty_{\mathcal{D}_i}\Omega^\infty_{\mathcal{D}_n}$, so that the map $K \to K_i$ is a $\mathcal{P}_i$-equivalence. Then $\mathrm{Sp}(\mathcal{D}_n) \simeq \mathcal{C}$ and the induced map $K \Rightarrow K_n$ is a $\mathcal{P}_n$-equivalence. In particular, the Tate diagonals δ^i_X for $i = 1, \dots, n$ above can be similarly defined for $X \in \mathcal{D}_n$, and the approximation functor $f_n \colon \mathcal{D}_* \to \mathcal{D}_n$ will be compactible with Tate diagonals. For example, the first Tate diagonal

$$\delta^2_X \colon \Sigma^\infty(X) \Rightarrow [\Sigma^\infty(X) \wedge \Sigma^\infty(X)]^{t\Sigma_2}$$

extends to a natural transformation $\delta^2_X \colon X \to (X \wedge X)^{t\Sigma_2}$ of endo-functors of $\mathcal{D}_1 = \mathcal{C}$. The higher Tate diagonals on $\mathcal{D}_*$ will generally not extend to $\mathcal{D}_n$.

We wish to compare this to the situation of n-coalgebras. For this, let us write $f_n \colon \mathcal{E}_n \to \mathcal{C}$ for the forgeful functor, which preserves compact objects by construction, and let $g_n \colon \mathcal{C} \to \mathcal{E}_n$ be its right adjoint. Let $L_n := f_n g_n$ be the resulting endo-functor of $f_n \dashv g_n$. Then $L_n(X)$ is the cofree n-coalgebra generated from X, and one can show that

$$\mathcal{P}_n L_n(X) = \oplus_{i=1}^n (X \wedge \dots \wedge X)^{h\Sigma_n}.$$

In particular, $\partial_n L_n(X_1, \dots, X_n) = X_1 \wedge \dots \wedge X_n$.

Let us now consider the 2-excisive case. The 2-excisive functor $\mathcal{P}_2 K$ is then given by the fibre product

$$\begin{tikzcd} \mathcal{P}_2 K(X) \arrow[r, "p"] \arrow[d] & X \arrow[d, "\delta^2_X"] \\ (X \wedge X)^{h\Sigma_2} \arrow[r] & (X \wedge X)^{t\Sigma_2} \end{tikzcd}$$

By Remark **??** we may identify $\mathcal{D}_2$ as the ∞-category whose compact objects are objects $X \in \mathcal{C}^c$ equipped with a factorization $X \xrightarrow{\alpha} \mathcal{P}_2 K(X) \xrightarrow{p} X$ of the identity maps $\mathrm{id} \colon X \to X$ through $\mathcal{P}_2 K(X)$. Since $\mathcal{P}_2 K$ sits in the above pullback square, we see that this is equivalent to equipping X with the structure of a cocommutative comultiplication $X \to (X \wedge X)^{h\Sigma_2}$ together with a homotopy between the composite $X \to (X \wedge X)^{h\Sigma_2} \to (X \wedge X)^{t\Sigma_2}$ with the Tate diagonal δ^2_X. In other words, these are 2-coalgebras equipped with a compatibility homotopy between the two resulting Tate diagonals. We may call such objects *Tate* 2*-coalgebras*. In particular, we have a forgetful functor $\mathcal{D}_2 \to \mathcal{E}_2$, which sits in a fibre square (in $\mathcal{C}\mathrm{at}^{\omega}_*$)

$$\begin{tikzcd} \mathcal{D}_2 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{E}_2 \arrow[r, "\delta^2_*"] & \mathrm{LaxEq}(\mathcal{C}, ((-) \otimes (-))^{t\Sigma_n}) \end{tikzcd}$$

where the bottom horizontal map is induced by the Tate diagonal δ^2_*. The vertical arrows in this square are 1-excisive equivalences, and on the level of endo-functors on $\mathcal{C}$, this squares becomes the pullback square

$$\begin{tikzcd} \mathcal{P}_2 K_2(X) \arrow[r] \arrow[d] & X \arrow[d] \\ X \oplus (X \wedge X)^{h\Sigma_n} \arrow[r, "(\mathrm{id},\delta^2)"] & X \oplus (X \wedge X)^{t\Sigma_n} \end{tikzcd}$$

whose horizontal arrows induce equivalences on bilinear parts (which for the bottom row are both trivial). Our goal is to similarly construct functors $\mathcal{D}_n \to \mathcal{E}_n$ over $\mathcal{C}$ which induce equivalences $\partial_n K_n \xrightarrow{\simeq} \partial_n L_n$ and fit together into pullback squares

$$\begin{tikzcd} \mathcal{D}_n \arrow[r] \arrow[d] & \mathcal{D}_{n-1} \arrow[d] \\ \mathcal{E}_n \arrow[r] & \mathcal{E}_{n-1} \times_{\mathrm{LaxEq}(\mathcal{C}, (\mathrm{nC}_{n-1}(n) \wedge (-) \wedge \dots \wedge (-))^{t\Sigma_n})} \mathrm{LaxEq}(\mathcal{C}, ((-) \wedge \dots \wedge (-))^{t\Sigma_n}) \end{tikzcd} \tag{14}$$

whose horizontal arrows are $(n-1)$-equivalences.

Suppose this data has been constructed for $n-1$ and consider the square

$$\begin{tikzcd} \mathcal{D}_{n-1} \arrow[r] \arrow[d] & \mathrm{LaxEq}(\mathcal{C}, ((-) \wedge \dots \wedge (-))^{t\Sigma_n}) \arrow[d] \\ \mathcal{E}_{n-1} \arrow[r] & \mathrm{LaxEq}(\mathcal{C}, (\mathrm{nC}_{n-1}(n) \wedge (-) \wedge \dots \wedge (-))^{t\Sigma_n}) \end{tikzcd}$$

Indeed, from the cube

(15)

$$\begin{tikzcd} \mathcal{P}_n K_n(X) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{P}_{n-1} K_n(X) \arrow[dr, "\simeq"] \arrow[dd] & \\ & \mathcal{P}_n K_{n-1}(X) \arrow[rr] \arrow[dd] & & \mathcal{P}_{n-1} K_{n-1}(X) \arrow[dd] \\ (X \wedge \dots \wedge X)^{h\Sigma_n} \arrow[rr] \arrow[dr] & & (X \wedge \dots \wedge X)^{t\Sigma_n} \arrow[dr] & \\ & (\mathrm{nC}_{n-1}(n) \wedge X \wedge \dots \wedge X)^{h\Sigma_n} \arrow[rr] & & (\mathrm{nC}_{n-1}(n) \wedge X \wedge \dots \wedge X)^{t\Sigma_n} \end{tikzcd}$$

whose front and back faces are cartesian we deduce a cartesian square

$$\begin{tikzcd}
\mathcal{P}_n K_n(X) \arrow[r] \arrow[d] & \mathcal{P}_n K_{n-1}(X) \arrow[d] \\
(X \wedge \dots \wedge X)^{\mathrm{h}\Sigma_n} \arrow[r] & Q
\end{tikzcd}$$

with

$$Q = (\mathrm{nC}_{n-1}(n) \wedge X \wedge \dots \wedge X)^{\mathrm{h}\Sigma_n} \times_{(\mathrm{nC}_{n-1}(n)\wedge X \wedge \dots \wedge X)^{\mathrm{t}\Sigma_n}} (\overline{X} \wedge \dots \wedge \overline{X})^{\mathrm{t}\Sigma_n}.$$

Now compact objects of $\mathcal{D}_n$ can be described as compact objects $X \in \mathcal{D}_{n-1}$, with image $\overline{X} \in \mathcal{C}$, equipped with a dotted lift of the form

$$\begin{tikzcd}
 & \mathcal{P}_n K_n(\overline{X}) \arrow[d] \\
\overline{X} \arrow[r] \arrow[ur, dashed] & \mathcal{P}_n K_{n-1}(\overline{X}) .
\end{tikzcd}$$

By the above fibre square we see that this is the same as objects $X \in \mathcal{D}_{n-1}$ equipped with a dotted lift of the form

$$\begin{tikzcd}
 & (\overline{X} \wedge \dots \wedge \overline{X})^{\mathrm{h}\Sigma_n} \arrow[d] \\
\overline{X} \arrow[r] \arrow[ur, dashed] & Q .
\end{tikzcd}$$

We can informally describe this as specifying a cocommutative n-fold cooperation $\alpha_n \colon \overline{X} \to \overline{X} \wedge \dots \wedge \overline{X}$ which is equipped with two type of compatibility constraints. The compatibility with $\partial_n K_{n-1}^{\mathrm{h}\Sigma_n}$ can be described in terms of the compatibility this n-fold product with the lower order cocommutative cooperations of lower degree already defined on $\overline{X}$ by virtue of its lift to $\mathcal{E}_{n-1}$ (recall that we assume already having a factorization of the functor $\mathcal{D}_{n-1} \to \mathcal{C}$ as $\mathcal{D}_{n-1} \to \mathcal{E}_{n-1} \to \mathcal{C}$ with the first functor inducing an equivalence $\partial_{n-1} K_{n-1} \xrightarrow{\simeq} \partial_{n-1} L_{n-1}$). The compatibility with $(\overline{X} \wedge \dots \wedge \overline{X})^{\mathrm{t}\Sigma_n}$ then says that the composed map

$$\overline{X} \to (\overline{X} \wedge \dots \wedge \overline{X})^{\mathrm{h}\Sigma_n} \to (\overline{X} \wedge \dots \wedge \overline{X})^{\mathrm{t}\Sigma_n}$$

need to agree with the Tate diagonal δ_X^n already defined on $\overline{X}$ by virtue of its lifting to $\mathcal{D}_{n-1}$. All this can be organized into a fibre square

$$\begin{tikzcd}
\mathcal{D}_n \arrow[r] \arrow[d] & \mathcal{D}_{n-1} \arrow[d] \\
\mathrm{LaxEq}(\mathcal{C}, ((-) \wedge \dots \wedge (-))^{\mathrm{h}\Sigma_n}) \arrow[r] & \mathrm{LaxEq}(\mathcal{C}, Q)
\end{tikzcd}$$

Combining the above square with Proposition 37 then yields a cartesian square

$$\begin{tikzcd}
\mathcal{D}_n \arrow[r] \arrow[d] & \mathcal{D}_{n-1} \arrow[d] \\
\mathcal{E}_n \arrow[r] & \mathcal{E}_{n-1} \times_{\mathrm{LaxEq}(\mathcal{C}, (\mathrm{nC}_{n-1}(n) \wedge (-) \wedge \dots \wedge (-))^{\mathrm{h}\Sigma_n})} \mathrm{LaxEq}(\mathcal{C}, Q)
\end{tikzcd}$$

which is equivalent to the desired square (14) since the functor $\mathrm{LaxEq}(\mathcal{C}, -)$ preserves fibre products.

5.3. **Rational spaces and Lie algebras.** If $\mathcal{C}$ is a stable symmetric monoidal ∞-category in which the Tate construction vanishes for all finite permutation groups vanish and $\mathcal{O}$ is some spectral cooperad then the notions of $\mathcal{O}$-coalgebras and divided powers $\mathcal{O}$-coalgebras in $\mathcal{C}$ canonically coincides: indeed, in this case the norm map determines an equivalence

$$(\mathcal{O}(n) \otimes (-) \otimes \ldots \otimes (-))_{\mathrm{h}\Sigma_n} \to (\mathcal{O}(n) \otimes (-) \otimes \ldots \otimes (-))^{\mathrm{h}\Sigma_n}.$$

The same holds if one considers coalgebras or divided powers coalgebras over the cofree cooperad $\mathrm{cofree}(\mathcal{O}_{\leq n})$ generated by the $\leq n$ part of $\mathcal{O}$ (in fact, for that we only need the Tate construction to coincide for Σ_i with $i \leq n$). If, furthermore, $\mathcal{C}$ is the stabilization of (the pointification of) a cartesian compactly generated ∞-category then the resulting notion of Tate coalgebras in $\mathcal{C}$ discussed in §5.2 coincide for the same reason with both the structure of a cocommutative algebra and a divided powers cocommutative algebra.

An example where this happens is if $\mathcal{C}$ is a $\mathbb{Q}$-rational ∞-category, for example, if $\mathcal{C}$ is the ∞-category of chain complexes over $\mathbb{Q}$. In this case $\mathcal{C}$ is also the stabilization of a the cartesian category $\mathcal{S}^{\geq 2}_{\mathbb{Q}}$ of simply connected rational spaces, which is a left exact localization of the ∞-category $\mathcal{S}^{\geq 2}$ of pointed simply connected spaces by the maps which induce an equivalence on rationalized homotopy groups. Then $\mathcal{S}^{\geq 1}_{\mathbb{Q}}$ is a cartesian compactly generated ∞-category and so

$$\mathrm{Ch}(\mathbb{Q}) = \mathrm{Sp}(\mathcal{S}^{\geq 1}_{\mathbb{Q}}) = \mathrm{Sp}(\mathcal{S}^{\geq 1}_{\mathbb{Q},*})$$

inherits a smash product as in §5.2, which simply corresponds to the usual tensor product of chain complexes over $\mathbb{Q}$. Since the Bar construction of the Lie operad is the commutative cooperad (up to a shift, which we ignore at this point), we conclude the following:

Corollary 40. *The notions of n-Tate coalgebras, n-cocommutative coalgebras and divided powers n-cocommutative coalgebras all coincide. In particular, the n-excisive approximations of both $\mathcal{S}^{\geq 2}_{\mathbb{Q},*}$ and $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}(\mathbb{Q}))$ coincide with the ∞-category of (compactly generated) n-cocommutative coalgebras in $\mathrm{Ch}(\mathbb{Q})$.*

Passing to the limit one obtains compactly generated functors

$$\begin{tikzcd} \mathcal{S}^{\geq 2}_{\mathbb{Q},*} \arrow[r] & \mathrm{coAlg}^{\omega}_{\mathrm{nC}}(\mathrm{Ch}(\mathbb{Q})) \arrow[d, equal] & \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}(\mathbb{Q})) \arrow[l] \\ & \lim_n \mathrm{coAlg}^{\omega}_{\mathrm{nC}_n}(\mathrm{Ch}(\mathbb{Q})) & \end{tikzcd}$$

which are fully-faithful when restricted to the full subcategories of $\mathcal{S}^{\geq 2}_{\mathbb{Q},*}$ and $\mathrm{Lie}(\mathrm{Ch}(\mathbb{Q}))$ on which the Goodwillie tower converges. On the side of $\mathcal{S}^{\geq 1}_{\mathbb{Q},*}$ this includes all objects, and on the side of $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}(\mathbb{Q}))$ it includes at least the full subcategory $\mathrm{Alg}^{\geq 1}_{\mathrm{Lie}}(\mathrm{Ch}(\mathbb{Q}))$ of connected Lie algebras. We hence obtain a pair of fully-faithful inclusions

$$\begin{tikzcd} \mathcal{S}^{\geq 2}_{\mathbb{Q},*} \arrow[r, hook] & \mathrm{coAlg}^{\omega}_{\mathrm{nC}}(\mathrm{Ch}(\mathbb{Q})) & \mathrm{Alg}^{\geq 1}_{\mathrm{Lie}}(\mathrm{Ch}(\mathbb{Q})) \arrow[l, hook'] \end{tikzcd}$$

Proposition 41 (Main theorem of rational homotopy theory)**.** *The essential image of both fully-faithful inclusions above coincide, and consists of the closure under filtered colimits of the full subcategory spanned by the commutative coalgebras whose underlying object is 2-connective. This is also the full subcategory of $\mathrm{coAlg}^{\omega}_{\mathrm{nC}}(\mathrm{Ch}(\mathbb{Q}))$ generated by the trivial coalgebra structure on $\mathbb{Q}[2]$.*

5.4. **Simply connected spaces as Tate coalgebras.** One of the new results of [Heu21] is an extension of the classical theory of Quillen from rational simply connected spaces to all simply connected spaces. In particular, the description of the n-excisive approximations of $\mathcal{S}_*$ in terms of n-Tate coalgebras in §5.2 yields a compactly generated functor

$$\mathcal{S}_* \to \mathrm{coAlg}^{\omega}_{\mathrm{tate}} = \lim_n \mathrm{coAlg}^{\omega}_{n-\mathrm{tate}},$$

which is fully-faithful when restricted to spaces on which the Goodwillie tower converges. This includes, for example, all simply connected spaces, and so we obtain a fully-faithful inclusion

$$\mathcal{S}_*^{\geq 2} \hookrightarrow \mathrm{coAlg}^{\omega}_{\mathrm{tate}} .$$

Theorem 42 ([Heu21]). *The essential image of the above inclusion is the full subcategory*

$$\mathrm{coAlg}^{\omega,\geq 2}_{\mathrm{tate}} \subseteq \mathrm{coAlg}^{\omega}_{\mathrm{tate}}$$

generated under filtered colimits by compact Tate coalgebras whose underlying finite spectrum is 2-connective.

The proof involves showing that $\mathrm{coAlg}^{\omega}_{\mathrm{tate}}$ is ind-comonadic over spectra, and that the associated finitray comonad coincides with $\Sigma^\infty\Omega^\infty$ when evaluated on 2-connective spectra. On the other hand, it is also known from the work of Blomquist and Harper [BH16] that $\mathcal{S}_*^{\geq 2}$ is comonadic (and hence ind-comonadic, as it is a compactly generated functor) over 2-connective spectra.

References

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