Polyhedral Groups

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1 The Lobachevski Model, Polyhedra and Reflections

In this lecture we will define the polyhedral groups which are particular examples of cocompact Kleinian groups and calculate some examples. Recall that on $\mathbb{H}^3$ every two points are connected by a unique geodesic. Hence it makes sense to talk about **convex sets** in $\mathbb{H}^3$, which are sets containing the geodesic between each two points in them, or the convex hall of a set.

The case we are interested is that of a **polyhedron**, which are the convex hall of a finite set of points. It turns out that polyhedra are most convenient to handle in the Lobachevski model for $\mathbb{H}^3$. In this model we consider the quadratic form

$$q(x) = -x_1^2 + x_2^2 + x_3^2 + x_4^4$$

where we use the notation $x = (x_1, x_2, x_3, x_4)$. We shall denote by $b(x, y)$ the corresponding biliear form

$$b(x, y) = \frac{q(x + y) - q(x) - q(y)}{2}$$

Now consider

$$\widetilde{\mathbb{H}}^3 = \{x \in \mathbb{R}^4 | q(x) = -1\}$$

Note that $\widetilde{\mathbb{H}}^3$ can't contain vectors x with $x_1 = 0$. Sense $\widetilde{\mathbb{H}}^3$ is also closed under sign we see that it has two connected components: one component has $x_1 > 0$ and the other $x_1 < 0$.

The pseudo-riemannian metric on $\mathbb{R}^4$ induces a metric on $\widetilde{\mathbb{H}}^3$ which is actually riemannian. Now the map $x \mapsto -x$ induces an isometry between the two components of $\widetilde{\mathbb{H}}^3$ and the resulting riemannian manifold

$$\mathbb{H}^3 = \widetilde{\mathbb{H}}^3 / \pm 1$$

is called the Lobachevski model of the hyperbolic space.

The group $\mathrm{O}_q(\mathbb{R})$ of linear transformations over $\mathbb{R}$ that preserve q acts naturally by isometries on $\mathbb{H}^3$. Note that $-1 \in \mathrm{O}_q(\mathbb{R})$ acts trivially so this is actually an action of $\mathrm{PO}_q(\mathbb{R})$. It turns out that $\mathrm{PO}_q(\mathbb{R})$ is the full group of isometries of

$\mathbb{H}^3$. It has two connected components: one of orientation preserving isometries (denoted by $\mathrm{PSO}_q(\mathbb{R})$) and the other with orientation reversing isometries.

It can be shown that in the Lobachevski model geodesics are given simply by intersecting linear 2-plains in $\mathbb{R}^4$ with $\widetilde{\mathbb{H}}^3$ (if the intersection is non-empty) and projecting to $\mathbb{H}^3$. Similarly the analogue of a hyperplane is given by intersecting a linear 3-plain in $\mathbb{R}^4$ with $\widetilde{\mathbb{H}}^3$ and projecting to $\mathbb{H}^3$.

It is convenient to describe a hyperplane H by

$$H = \{x \in \mathbb{R}^4 | b(x,v) = 0\}$$

where $v \in \mathbb{R}^4$ is some vector with $q(v) > 0$ (if $q(v) < 0$ then the intersection of this hyperplane with $\mathbb{H}^3$ will be empty). Note that such a hyperplane divides $\mathbb{H}^3$ into two half-spaces: one is given by $b(x,v) < 0$ and the other by $b(x,v) > 0$.

Just like in the Euclidean case, if the intersection of finitely many half spaces is bounded then its closure is a polyhedron. Further more any polyhedron can be constructed in that way. Hence in order to describe a polyhedron we shall give a list of vectors $v_1, ..., v_n$, normalized so that $q(v_i) = 1$, such that our the polyhedron is the intersection of the half spaces given by $b(x, v_i) < 0$.

The intersection of the polyhedron with each of the hyperplane (if non-empty) are called the **faces** of the polyhedron. The intersection of two faces (if non-empty) is an **edge** and the intersection of two edges (if non-empty) is a **vertex**. Each edge is the geodesic between two vertices and the polyhedron is the convex hall of its vertices.

As in Euclidean geometry, every two adjacent faces have a well defined **angle** between them, called the **dihedral angle** of the corresponding edge. This angle can be defined by choosing a point x on the edge and taking in T_x two vectors which are normal to the two hyperplane respectively. The dihedral angle is then defined as π minus the angle between these vectors (think of the analogous Euclidean situation). This does not depend on the choice of x on the edge.

If the two hyperplane are given by two normalized vectors v, u as described above, one can take the vectors $v_x, u_x \in T_x$ which correspond to v and u under the usual Euclidean translation as normal vectors. Hence we see that if the dihedral angle is α then $b(v,u) = -\cos(\alpha)$.

Just like in the Euclidean case we can define the **reflection through a hyperplane**. Given a hyperplane in $\mathbb{H}^3$ given (as above) by the normalized vector v, the following linear transformation on $\mathbb{R}^4$ is called a (hyperbolic) **reflection** through the hyperplane defined by v:

$$R_v(x) = x - 2b(x,v)v$$

This linear transformation preserves q because

$$q(x - 2b(x,v)v) = q(x) - 4b(x,v)b(x,v) + 4b(x,v)^2 = q(x)$$

Hence $R_v \in \mathrm{O}_q(\mathbb{R})$ and we write $\widetilde{R_v}$ for its image in $\mathrm{PO}_q(\mathbb{R})$. It is also easy to verify that $\det(R_v) = -1$ so that $\widetilde{R_v}$ is in the non-trivial connected component of $\mathrm{PO}_q(\mathbb{R})$.

Now consider a polyhedron P with defining normalized vectors $v_1, ..., v_n$. The **polyhedral group** associated to P is the subgroup $\Gamma(P) \subseteq \mathrm{PO}_q(\mathbb{R})$ generated by the $\widetilde{R}_{v_i}$'s.

Theorem 1.1. *$\Gamma(P)$ is discrete in $\mathrm{PO}_q(\mathbb{R})$ if and only if the angle between each two adjacent faces is a submultiple of π, i.e. of the form $\frac{\pi}{n}$ for some natural $n \geq 2$. Further more in that case P is a fundamental domain for the action of $\Gamma(P)$ on $\mathbb{H}^3$.*

A polyhedron satisfying the conditions above will be called **good**. Now the group $\Gamma(P)$ has an index two subgroup $\Gamma^+(P) \subseteq \Gamma(P)$ of orientation preserving isometries. When P is good then $\Gamma^+(P)$ is a discrete subgroup of $\mathrm{PSO}_q(\mathbb{R})$.

In the more familiar upper half plain model of the hyperbolic plain one can identify the group of orientation preserving isometries with $\mathrm{PSL}_2(\mathbb{C})$. Hence by theorem 1.1 we get a cocompact Kleinian group. Our mission now is to calculate the invariant trace field and quaternion algebra of this group from the vectors $v_1, ..., v_n$.

2 Clifford Algebras

In order to preform the mission above we need to understand the isomorphism $\mathrm{PSO}_q(\mathbb{R}) \cong \mathrm{PSL}_2(\mathbb{C})$ a bit better. For this we will use the theory of Clifford algebras.

Let U be a vector space of dimension m over a field K and q a non-degenerate quadratic form on U. Let $\mathrm{O}_q(K)$ be the group of linear transformations from U to U which preserve q. Each element in $\mathrm{O}_q(K)$ has determinant ± 1 and we denote by $\mathrm{SO}_q(K)$ the subgroup of elements of determinant 1.

Let $u_1, ..., u_m$ be a basis for U which is orthogonal with respect to q. We define the determinant of q to be

$$\det(q) = \prod_{i=1}^{m} q(u_i) \in K^*/(K^*)^2$$

If we change to another orthogonal basis the value of $\prod_{i=1}^{m} q(u_i)$ will change by a square in K so the determinant is a well defined element in $K^*/(K^*)^2$.

Now Define $C_q(K)$ to be the (non-commutative) K-algebra generated over K by $u_1, ..., u_m$ modulu the ideal generated by

$$\{u^2 - q(u) | u \in U\}$$

We think of U as the sub vector space of $C_q(K)$ spanned by $u_1, ..., u_m$. First note that for each $v, u \in U$ we get

$$b(u, v) = q(u + v) - q(u) - q(v) = (v + u)^2 - v^2 - u^2 = vu + uv$$

In particular if $i \neq j$ then $u_i u_j = -u_i u_j$ so that $C_q(K)$ is spanned over K by monomials of the form $u_1^{n_1} u_2^{n_2} ... u_m^{n_m}$. Further more sense $u_i^2 \in K$ we can restrict only to monomials of the above form where each $n_i \in \{0,1\}$. It is an easy exercise to show that the set

$$\{u_1^{n_1}u_2^{n_2}...u_m^{n_m}|n_i \in \{0,1\}\}$$

is actually a basis for $C_q(K)$ over K, so the dimension of $C_q(K)$ over K is 2^n. We $C_q(K)$ has a sub algebra $B_q(K)$ which is spanned by

$$\{u_1^{n_1}u_2^{n_2}...u_m^{n_m}|n_i \in \{0,1\}, \sum_i n_i = 0 \pmod 2\}$$

This is an 2^{n-1}-dimensional algebra over K. If n is even then the element $\alpha = u_1u_2...u_m$ is in the center of $A_q(K)$. If further more

$$q(\alpha) = \det(q) \in K^*/(K^*)^2$$

is non-trivial then $K(\alpha)$ is a central subfield of $B_q(K)$ so we can think of $B_q(K)$ as an algebra of dimension 2^{n-2} over $K(\alpha)$. We then denote it by $A_q(K(\alpha))$ (and preserve the convention that the field in brackets is the field over which the object is considered).

Now let us restrict to the case where U is 4-dimensional and assume that $\det(q) \in K^*/(K^*)^2$ is non-trivial. Now $C_q(K)$ has dimension 16, $B_q(K)$ has dimension 8, and $A_q(K(\alpha)$ has dimension 4. In fact $A_q(K(\alpha))$ is generated (over $K(\alpha)$) by the anticommuting elements u_1u_2, u_1u_3 which satisfy

$$(u_1u_2)^2 = -q(u_1)q(u_2)$$

$$(u_1u_3)^2 = -q(u_1)q(u_3)$$

So we get that $A_q(K(\alpha))$ is a quaternion algebra over $K(\alpha)$ with an explicit description.

Recall the Skolem-Noehter theorem which says that all automorphisms of $A_q(K(\alpha))$ (over $K(\alpha)$ of course) are inner. It turns out that a similar statement holds for $C_q(K)$: every automoprhism of $C_q(K)$ which preserves U is obtained by conjugating with an element $x \in A_q(K(\alpha))$ whose quaternionic norm lies in K (as apposed to $K(\alpha)$). Further more two such x, y induce the same automorphism if and only if their ratio is a scalar in K (because K is the center of $C_q(K)$). Hence the automorphism group of $C_q(K)$ can be identified with

$$D_q(K) = \{x \in A_q(K(\alpha))|x\overline{x} \in K\}/K^*$$

Now every $T \in \mathrm{O}_q(K)$ induces an automorphism of $C_q(K)$ which preserves U. Hence we get a map

$$SO_q(K) \longrightarrow D_q(K)$$

Which by the discussion above is an isomorphism of groups.

Now let $V \subseteq A_q(K(\alpha))$ be the subspace spanned over K by $1, u_1u_4, u_2u_4, u_3u_4$. Then the map

$$v \mapsto vu_4$$

sends V into U and is invertible with inverse

$$u \mapsto uu_4^{-1} = u\frac{u_4}{q(u_4)}$$

Hence the map induced on U by x-conjugation can be induced on V by

$$v \mapsto xvu_4x^{-1}u_4^{-1}$$

But recall that $A_q(K(\alpha))$ is spanned over $K(\alpha)$ by $1, u_1u_2, u_1u_3, u_2u_3$ which all commute with u_4. α, on the other hand, anticommutes with u_4. This means that conjugation by u_4 induces the action of $\mathrm{Gal}(K(\alpha)/K)$ on $A_q(K(\alpha))$, which we denote by $x \mapsto \sigma(x)$. Hence we can see which element in $\mathrm{SO}_q(K)$ corresponds to x by calculating the action

$$v \mapsto xv\sigma(x^{-1})$$

on V which preserves a form isomorphic to q. Note that this form is also equivalent to the form given by the norm on $A_q(K(\alpha))$.

Now recall that on $A_q(K(\alpha))$ we also have the usual conjugation $x \mapsto \overline{x}$ of quaternion algebras which is the identity on $K(\alpha)$. Note that the basis we took for for V is in fact $\{1, \alpha u_1u_2, \alpha u_1u_3, \alpha u_2u_3\}$ and so the vectors in V are exactly the vectors satisfying $\sigma(v) = \overline{v}$.

A final consequence of this calculation is that if $p(t) = (t-\beta)(t-\gamma)$ is the minimal polynomial of x over $A_q(K(\alpha))$ (so that $\beta\gamma = Norm(x) \in K$) then the minimal polynomial of $\sigma(x^{-1})$ is $t^2\sigma(p(t^{-1}))$. Hence the minimal polynomial of the corresponding element in $\mathrm{SO}_q(K)$ is

$$q(t) = p(t) \otimes t^2\sigma(p(t^{-1}))$$

where $\otimes$ denotes tensor product of polynomials, i.e. the roots of q are all the products of a roots of $p(t)$ and a root of $t^2\sigma(p(t^{-1}))$. In particular if the norm of x is 1 then the roots of the minimal polynomial of the associated element of $\mathrm{SO}_q(K)$ are

$$\beta\sigma(\beta), \frac{\beta}{\sigma(\beta)}, \frac{\sigma(\beta)}{\beta}, \frac{1}{\beta\sigma(\beta)}$$

In particular if x has norm 1 then the corresponding element $T \in \mathrm{SO}(q,k)$ satisfies

$$\mathrm{Tr}(T) = \mathrm{Tr}(x)\sigma(\mathrm{Tr}(x))$$

Note that in that case if T has at least one eigenvalue which is 1 then the trace of x is in K so

$$(*) \quad \mathrm{Tr}(T) = \mathrm{Tr}(x)^2$$

When $K = \mathbb{R}$ and q has signature $(3,1)$ we get that $\det(q) = -1$ so $K(\alpha) = \mathbb{C}$ and $A_q(K(\alpha)) \cong M_2(\mathbb{C})$. $D_q(\mathbb{R})$ is the group of matrices in $M_2(\mathbb{C})$ of real determinant modulu real scalars. We have a natural map $D_q(\mathbb{R}) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$

which is surjective with kernel isomorphic to $\mathbb{Z}/2$. The non-trivial element in the kernel is

$$\mathbb{R}^* \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

Hence this map induces an isomorphism of groups

$$\mathrm{PSO}_q(\mathbb{R}) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$$

3 The Invariant Trace Field and Quaternion Algebra of $\Gamma^+(P)$

Let us now go back to our reflection group $\Gamma^+(P)$ generated by $\widetilde{R}_{v_1}, ..., \widetilde{R}_{v_m}$. Define $a_{i,j} = b(v_i, v_j)$ and for any sequence $\bar{i} = (i_1, ..., i_m)$ define the cyclic product

$$b_{\bar{i}} = a_{i_1,i_2} a_{i_2,i_3} ... a_{i_m,i_1}$$

Let

$$K = \mathbb{Q}(\{b_{\bar{i}}\}) \subseteq \mathbb{R}$$

to be the subfield of $\mathbb{R}$ generated by all the cyclic products. Define the K-vector space $V \subseteq \mathbb{R}^4$ to be the vector space spanned over K by the vectors of the form

$$v_{\bar{i}} = a_{i_1,i_2} a_{i_2,i_3} ... a_{i_{m-1},i_m} v_{i_m}$$

Then clearly V has dimension 4 over K. Further more

$$q(v_{\bar{i}}) = q(a_{i_1,i_2} a_{i_2,i_3} ... a_{i_{m-1},i_m} v_{i_m}) =$$

$$(a_{i_1,i_2} a_{i_2,i_3} ... a_{i_{m-1},i_m})^2 = b_{(i_1,...,i_{m-1},i_m,i_{m-1},...,i_2)} \in K$$

And so we can restrict q to V. Let $\widetilde{q}$ denote the form q restricted to V and d the determinant of $\widetilde{q}$. We claim that the invariant trace field of $\Gamma^+(P)$ is $K(\alpha)$ (where α as before is a root of f) and the invariant quaternion algebra is $A_q(K(\alpha))$.

We claim that $\Gamma_+(P)$ preserves V. This is because

$$R_{v_j}(v_{(i_1,...,i_m)}) = v_{(i_1,...,i_m)} - 2a_{i_1,i_2} a_{i_2,i_3} ... a_{i_{m-1},i_m} b(v_{i_m}, v_j) v_j =$$

$$v_{(i_1,...,i_m)} - 2v_{(i_1,...,i_m,j)} \in V$$

This means that the isomorphism $\rho : \mathrm{PSO}_q(\mathbb{R}) \xrightarrow{\cong} \mathrm{PGL}_2(\mathbb{C})$ will send $\Gamma^+(P)$ into the sub quaternion algebra $A_{\widetilde{q}}(K)$ of $\mathrm{PGL}_2(\mathbb{C})$ defined over $K(\alpha)$ where $\alpha \in K^*/(K^*)^2$ is the determinant of $\widetilde{q}$.

Now the embedding $A^1_q(K(\alpha)) \hookrightarrow A^*_q(K(\alpha))$ induces an embedding $PA^1_q(K(\alpha)) \hookrightarrow PA^*_q(K(\alpha))$. Now the norm map $N : A^*_q(K(\alpha)) \longrightarrow K^*(\alpha)$ induces a map

$$\widetilde{N} : PA^*_q(K(\alpha)) \longrightarrow K^*(\alpha)/(K^*(\alpha))^2$$

and we get a short exact sequence

$$1 \longrightarrow PA^1_q(K(\alpha)) \hookrightarrow PA^*_q(K(\alpha)) \longrightarrow K^*(\alpha)/(K^*(\alpha))^2 \longrightarrow 0$$

Since the last group is a 2-torsion group it follows that the square of any element in $PA^*_q(K(\alpha))$ lies in the image of $PA^1_q(K(\alpha))$.

Let $\Gamma^{(2)}_+(P)$ be the subgroup of $\Gamma_+(P)$ generated by squares of elements. Then the above discussion implies that ρ induces an embedding of $\Gamma^{(2)}_+(P)$ in $PA^1_q(K(\alpha))$. This means that the traces of elements in $\Gamma^{(2)}_+(P)$ lie in $K(\alpha)$, so the invariant trace field is contained in $K(\alpha)$. We will now show that the invariant trace field contains K and since it cannot be real we see it has to be $K(\alpha)$. This implies that the invariant quaternion algebra of $\Gamma^+(P)$ is in fact $A_q(K(\alpha))$.

We will now show that the invariant trace field of $\Gamma_+(P)$ contains K. Let $r_i \in \Gamma(P)$ denote the reflection by v_i and $\gamma_{i,j} = r_i r_j \in \Gamma^+(P)$. Let $m_{i,j}$ be elements in $\mathrm{SL}_2(\mathbb{C})$ representing the image of $\gamma_{i,j}$ in $\mathrm{PGL}_2(\mathbb{C})$. Then

$$\gamma_{i,j}(x) = x - 2b(x,v_i)v_i - 2b(x - 2b(x,v_i)v_i, v_j)v_j =$$

$$x - 2b(x,v_i)v_i - 2b(x,v_j)v_j + 4a_{i,j}b(x,v_i)v_j$$

So as a transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ we see that $\gamma_{i,j}$ has trace

$$4 - 2 - 2 + 4a^2_{i,j} = 4a^2_{i,j}$$

From formula $(*)$ we get that $\mathrm{Tr}(m_{i,j}) = \pm a_{i,j}$ and so $a_{i,j}m_{i,j} = m^2_{i,j} + I$. Now for any sequence $i_1, ..., i_m$ the product

$$\prod_{k=1}^{m} m_{i_k, i_{((k+1) \mod m)}} = \pm I$$

So

$$m^{-1}_{i_m,i_1} = \pm \prod_{k=1}^{m-1} m_{i_k,i_{k+1}} = \prod_{k=1}^{m} \frac{m^2_{i_k,i_{k+1}} - I}{a_{i,j}}$$

and by applying the trace we get

$$a_{i_m,i_1} = \prod_{k=1}^{m} \frac{\mathrm{Tr}(m^2_{i_k,i_{k+1}} - I)}{a_{i,j}}$$

so

$$b_{(i_1,\ldots,i_m)} = \prod_{k=1}^{m} a_{i_k,i_{((k+1) \mod m)}} = \prod_{k=1}^{m} \mathrm{Tr}(m^2_{i_k,i_{k+1}} - I)$$

is in the invariant trace field of $\Gamma^+(P)$.

4 Example

We shall now calculate the invariant trace field and quaternion algebra for the simplest good polyhedron, which is the tetrahedron whose dihedral angles are all $\frac{\pi}{2}, \frac{\pi}{3}$ or $\frac{\pi}{4}$. We represent a polyhedron via its coxeter diagram

$$\begin{array}{ccc} \cdot & - & \cdot \\ \| & & \| \\ \cdot & - & \cdot \end{array}$$

where each dot represents a face and the angle between two faces is $\frac{\pi}{n+2}$ where n is the number of lines between the vertexes. Now let $v_1, ..., v_4$ are normalized vectors representing the faces as before (where v_1 correspond to the lower right vertex of the coxeter diagram and the rest follow with counter clock wise order) and set $a_{i,j} = b(v_i, v_j)$. As we saw above $a_{i,j} = -\cos(\alpha_{i,j})$ where $\alpha_{i,j}$ is the dihedral angle. This means that $a_{i,j}$ is either 0, $\frac{1}{2}$ or $\frac{1}{\sqrt{2}}$.

Now note that the cyclic products $b_{\bar{i}}$ correspond to accumulated products taken over closed paths in the coxeter diagram. In our case each such closed path must pass a double edge an even number of times, which means that all the $b_{\bar{i}}$ are rational. Hence $K = \mathbb{Q}$.

Let V be the $\mathbb{Q}$ form of q defined in the previous section and let $\widetilde{q}$ be the form q restricted to V. Then we see that V is spanned over $\mathbb{Q}$ by the basis $B = \{v_1, \sqrt{2}v_2, \sqrt{2}v_3, 2v_4\}$. In this basis the form $\widetilde{q}$ is represented by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 4 \end{pmatrix}$$

In a more familiar notation, if $\{x, y, z, w\}$ is a dual basis to B then $\widetilde{q}$ can be written as

$$x^2 + 2xy + 2xw + 2y^2 + 2yz + 2z^2 + 4zw + 4w^2 =$$

$$(x + y + w)^2 + (y - w + z)^2 + (z + 3w)^2 - 7w^2$$

and so an orthogonal basis $\{u_1, u_2, u_3, u_4\}$ for V is given by the dual basis to $\{x + y + z, y - w + z, z + 3w, w\}$. Then

$$\widetilde{q}(u_1) = 1$$

$$\widetilde{q}(u_2) = 1$$

$$\widetilde{q}(u_3) = 1$$

$$\widetilde{q}(u_4) = -7$$

so the determinant is -7 and the invariant trace field is $\mathbb{Q}(\sqrt{-7})$. Further more we see that the invariant quaternion algebra is $\mathbb{Q}(\sqrt{-7})(i, j)$ such that

$$i^2 = -q(u_1)q(u_2) = -1$$

$$j^2 = -q(u_1)q(u_3) = -1$$

Note that this quaternion algebra is non-trivial. To see this, note that this quaternion algebra is obtained from the (non-trivial) standard quaternion algebra $\mathbb{Q}(i,j)$ by tensoring with $\mathbb{Q}(\sqrt{-7})$.

By the theory of quaternion algebras we know that such a tensoring trivializes the algebra if and only if the field $\mathbb{Q}(\sqrt{-7})$ was a subfield of $\mathbb{Q}(i,j)$, i.e. if there was an element $x \in \mathbb{Q}(i,j)$ satisfying $x^2 = -7$. This element would have to be traceless, so $x = bi + cj + dk$ $(b,c,d \in \mathbb{Q})$. such that

$$-b^2 - c^2 - d^2 = -7$$

This would means that there exist integers a, b, c, d, with $a \neq 0$, such that

$$7a^2 - b^2 - c^2 - d^2 = 0$$

A simple check shown that this equation has no non-zero solution mod 8, and so by decent has no non-zero integer solutions.

This quaternion algebra cannot ramify in an ideal of characteristic $\neq 2$. The reason is that mod every p there exists a solution to

$$a^2 + b^2 + c^2 + d^2$$

and if $p \neq 2$ then this equation is smooth so we can lift the solution to $\mathbb{Q}_p$, resulting in an element of norm zero in $\mathbb{Q}_p(i,j)$ (this actually shows that $\mathbb{Q}(i,j)$ has no finite ramification other then 2).

Since $\mathbb{Q}(\sqrt{-7})$ has no real embeddings we see that $\mathbb{Q}(\sqrt{-7})$ cannot ramify at ∞. Since it must ramify it an even (and non-zero because its non-trivial) number of places we see that it must ramify exactly at the two ideals

$$P_2 = \left(\frac{1+\sqrt{-7}}{2}\right)$$

$$P_2' = \left(\frac{1-\sqrt{-7}}{2}\right)$$

which sit over the prime 2 in $\mathbb{Q}(\sqrt{-7})$.