Quasi-Categories

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1 Simplicial Sets and Categories

The category of simplicial sets is very reach and can be used to model many things. The most common use of simplicial sets is to model topological space. We have a pair of adjoint functors

$$|\cdot| : \mathrm{Set}_\Delta \leftrightarrows \mathrm{Top} : \mathrm{Sing}$$

and we can do one of two things. We can either replace each simplicial set S with $\mathrm{Sing}(|S|)$ (which basically forgets all the "non-geometric" information) or we can restrict ourselves to simplicial sets S which are simplicialy homotopic to $\mathrm{Sing}(|S|)$. Either way this allows us to use simplicial tools on S to study the geometry of $|S|$. Note that $|\mathrm{Sing}(X)|$ is in general only weakly equivalent to X, so using simplicial sets we can only study spaces up to weak equivalence.

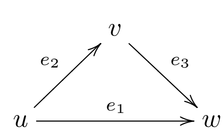
It turns out that this mysterious looking property $S \sim \mathrm{Sing}(|S|)$ is actually equivalent to being a **Kan** simplicial set. This concept is defined as follows:

Definition 1.1. Let $\Lambda^n_k \subseteq \Delta^n$ denote the subcomplex obtained by from $\partial\Delta^n$ by removing the k'th n-simplex. Then we say that a simplicial set S is **Kan** if every map $f : \Lambda^n_k \longrightarrow S$ extends to all of Δ^n.

If we are willing to remember more combinatorial information on the simplicial set (or equivalently, work with more general simplicial sets then just the Kan simplicial sets) then we can use them in order to model categories, and even simplicial categories. Let us start with regular categories. We have a pair of adjoint functors

$$C : \mathrm{Set}_\Delta \leftrightarrows \mathrm{Cat} : N$$

The functor C is defined as follows. The objects of $C(S)$ are the vertices of S. Given two vertices $v, u \in S_0$ then morphism set $\mathrm{Hom}_{C(S)}(v, u)$ is given by the set of all **directed paths** in S from v to u modulu the equivalence relation generated by the identifications $e_1 \sim e_2 * e_3$ for all triangles of the form



In particular we see that $C(\Delta^n)$ is the category whose objects are $\{0, ..., n\}$ and with a single morphism from i to j whenever $i < j$.

Since N and C are adjoint we see that there is no choice but to define N by

$$N(\mathcal{C})_n = \text{Fun}(C(\Delta^n), \mathcal{C})$$

As above if want to use simplicial sets in order to study categories we can either replace each simplicial set S with $N(C(S))$ (which remembers only the categorial aspects of S) or restrict our attention to simplicial sets S which are isomorphic to $N(C(S))$. Note that $C(N((C)))$ is actually isomorphic to $\mathcal{C}$ so using simplicial sets we can capture all the information in Cat.

We this have a natural question: which simplicial sets S are isomorphic to $N(C(S))$?

Theorem 1.2. *Let S be a simplicial set. Then S is isomorphic to $N(C(S))$ is and only if every map $\Lambda^n_k \longrightarrow S$ for $0 < k < n$ has a **unique** extension to Δ^n (call such simplicial sets **uniquely inner Kan**).*

Proof. First we will show that this condition is necessary by showing that every simplicial set of the form $N(\mathcal{C})$ is uniquely inner Kan.Let $\mathcal{C}$ be a category, $n \geq 2$ an integer and $0 < k < n$. We need to show that the map of sets

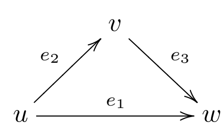
$$\text{Hom}_{\text{Set}_\Delta}(\Delta^n, N(\mathcal{C})) \longrightarrow \text{Hom}_{\text{Set}_\Delta}(\Delta^n_k, N(\mathcal{C}))$$

is an isomorphism. From adjunction it is enough to show to that the map of sets

$$\text{Hom}_{\text{Set}_\Delta}(C(\Delta^n), \mathcal{C}) \longrightarrow \text{Hom}_{\text{Set}_\Delta}(C(\Delta^n_k), \mathcal{C})$$

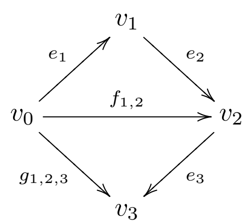
But by definition of $C(-)$ we see that $C(\Delta^n)$ and $C(\Delta^n_k)$ are isomorphic categories so the result is clear.

In the other direction suppose that S is uniquely inner Kan. Then in particular if e_2 and e_3 are two consecutive edges in S then there exists a unique triangle of the form

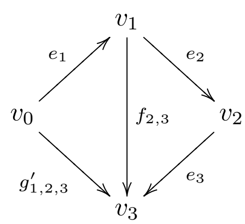


Hence for such simplicial sets we have a well defined notion of **composition** of edges. We claim that this operation is associative. Let v_0, v_1, v_2, v_3 be four vertices and e_1, e_2, e_3 edges from v_0 to v_1, v_1 to v_2 and v_2 to v_3 respectively.

One way to compose them is to take the unique pair of triangles of the form



The second way to compose them is to take the unique pair of triangles of the form



We need to show that $g_{1,2,3} = g'_{1,2,3}$. Note that since S is inner Kan we get that both pairs of triangles can be completed to 3-splices σ^3, τ^3 on $\{v_0, v_1, v_2, v_3\}$. Note that $\sigma^3_{v_i,v_j} = \tau^3_{v_i,v_j}$ for all $i < j$ except maybe for $i = 0, j = 3$, in which case

$$\sigma^3_{v_0,v_3} = g_{1,2,3}$$
$$\tau^3_{v_0,v_3} = g'_{1,2,3}$$

Then by the uniqueness part the two skeletons of σ^3 and τ^3 must be equal and so $g_{1,2,3} = g'_{1,2,3}$.

Hence we have a well defined operation of composition from directed paths to edges. Clearly two equivalent directed paths compose to the same edge and every directed path is equivalent to its composition. This means that the morphisms in $C(S)$ from v to u are exactly the edges from v to u. Hence it is clear that the 2-skeleton of $N(C(S))$ is isomorphic to the 2-skeleton of S. In order to complete the proof show that a uniquely inner Kan simplicial set is determined by its 2-skeleton (which would complete the proof since we saw that $N(C(S))$ is uniquely inner Kan).

This is done by showing inductively that if S is uniquely inner Kan then every map $\partial\Delta^n \longrightarrow S$ for $n \geq 3$ has a unique extension to Δ^n. We leave the details to the reader.

□

We now wish to generalize this situation from categories to simplicial categories. We wish to construct an analogous adjunction

$$\mathfrak{C} : \mathrm{Set}_\Delta \longrightarrow \mathrm{Cat}_\Delta$$

We begin by defining $\mathfrak{C}$. We will do this by defining $\mathfrak{d}^{\mathfrak{n}}$ and then extend to general simplicial sets by declaring $\mathfrak{C}$ to commute with colimits. The objects of $\mathfrak{d}^{\mathfrak{n}}$ are $\{0, ..., n\}$ and the morphism simplicial set from i to j is empty if $i > j$ and equal to

$$\underline{\mathrm{Hom}}_{\mathfrak{C}(S)}(\Delta^1)^{\{i+1,...,j-1\}}$$

As before we define N by

$$N(\mathcal{C})_n = \mathrm{Fun}_{\mathrm{Cat}_\Delta}(\mathfrak{C}(\Delta^n), \mathcal{C})$$

Definition 1.3. A **quasi-category** is a simplicial set S such that every map $\Lambda^n_k \longrightarrow S$ for $0 < k < n$ extends to all of Δ^n.