Quasi-unital ∞-Categories

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Inspired by Lurie's theory of quasi-unital algebras we prove an analogous result for ∞-categories. By constructing a suitable model category of non-unital complete Segal spaces, we show that the unital structure of an ∞-category can be uniquely recovered from the underlying non-unital structure once suitable candidates for units have been identified. The main result of this paper can be used to produce a proof for the 1-dimensional cobordism hypothesis, as described in a forthcoming paper of the author.

Introduction

The notion of units in higher category theory carries considerably more structure than the corresponding discrete notion. Informally speaking, given an ∞-category $\mathcal{C}$ we are provided not only with a unit morphism $I_C : C \longrightarrow C$ for every object $C \in \mathcal{C}$, but also with the precise way in which these are units, i.e., with explicit homotopies of the form $I_D \circ f \simeq f$ and $f \circ I_C \simeq f$ for every morphism $f : C \longrightarrow D$. Furthermore, we are provided with higher homotopies exhibiting the inner coherence of the above data as well as its compatibility with composition of morphisms (along with all of its higher structure).

This bundle of information is encoded differently in different **models** for the theory of ∞-categories. In this paper we will take up the point of view developed by Rezk in his foundational paper [14]. In [14], Rezk constructs a model for the theory of ∞-categories in the form of **complete Segal spaces**, which are simplicial spaces satisfying certain conditions. Such a simplicial space X determines an ∞-category $\mathcal{C}$ which can be described informally as follows:

(1) The space of 0-simplices of X corresponds to the **objects** of $\mathcal{C}$.

(2) The space of 1-simplices of X corresponds to the **morphisms** of $\mathcal{C}$, where the target and source of a given morphism are provided by the face maps $d_0, d_1 : X_1 \longrightarrow X_0$.

(3) The space of 2-simplices of X encodes the composition in $\mathcal{C}$. In particular, we can think of a triangle $\sigma \in X_2$ as encoding a homotopy from $d_0(\sigma) \circ d_2(\sigma)$ to $d_1(\sigma)$.

(4) The space of 3-simplices of X provides us with **associativity** homotopies. Similarly, the spaces of n-simplices of X for $n > 3$ provide us with higher coherence homotopies for the associativity structure.

In this setting it natural to ask what do the structure maps $\rho^* : X_n \longrightarrow X_m$ for the various $\rho : [m] \longrightarrow [n]$ encode. This can be described as follows. The higher face maps give information which is analogous to the source and target maps encoded by the face maps $[0] \longrightarrow [1]$, i.e. they tells us to which objects, morphisms, etc. a specific piece of structure applies to. The degeneracy maps, on the other hand, have a different interpretation - they encode the **unital structure** of $\mathcal{C}$.

The 0'th degeneracy map $s_0 : X_0 \longrightarrow X_1$ tells us for each object what its identity morphism is. Similarly, the two degeneracy maps $s_0, s_1 : X_1 \longrightarrow X_2$ provides us with homotopies of the form $I_D \circ f \simeq f$ and $f \circ I_C \simeq f$ for each morphism $f : C \longrightarrow D$. The higher degeneracy maps can be interpreted as exhibiting the coherence of the unital structure with the composition and associativity structure.

The fact that the unital structure is encoded in the collection of degeneracies shows that it contains a somewhat intricate web of data. However, it also tells what we need to do in order to **forget it**: we should simply consider X without the degeneracy maps, i.e. consider only the underlying **semi-simplicial space**.

A first motivation for forgetting this data comes from situations in which there are no natural choices for this vast unital structure. Such a case occurs, for example, when one is attempting to construct various **cobordism ∞-categories**. Suppose that we want to describe the ∞-category whose objects are closed n-manifolds and morphisms are cobordisms between them. Since cobordisms have their own automorphisms we can't simply take them as a set, but rather as the space classifying the corresponding topological automorphism groupoid. Gluing of cobordisms induces a weak composition operation on these classifying spaces.

As explained in [12, § 2.2], this composition structure naturally leads to a **semi-simplicial space** $\mathrm{semiCob}_n$ satisfying the Segal condition. Such objects are referred to there as **semiSegal spaces**. In order to promote $\mathrm{semiCob}_n$ to a full simplicial space, one needs to understand the behaviour of **units** in these cobordism categories.

Now given an n-manifold M there will certainly be an equivalence class of cobordisms $M \longrightarrow M$ which are candidates for being the "identity" - all cobordisms which are diffeomorphic to $M \times I$. However it is a bit unnatural to choose any **specific one** of them. Note that even if we choose a specific identity cobordism $M \times I$ we will still have to arbitrarily choose diffeomorphisms of the form $[M \times I] \coprod_M W \cong W$ for each cobordism W out of M as well as many other coherence homotopies.

These choice problems can be overcome in various ways, some more ad-hoc than others, and in the end a unital structure can be obtained. In other words, $\mathrm{semiCob}_n$ can be promoted to a **Segal space** Cob_n. However, there is great convenience in **not having to make these choices**. As claimed (but not proved) in [12], this unital structure is actually uniquely determined once we verify that suitable candidates for units exist.

Exploring this issue further, we see that an obvious necessary condition for a semiSegal space to come from a Segal space is that each object admits an endomorphism which is neutral with respect to composition (up to homotopy). Following Lurie [10, § 5.4.3], we will call such morphisms **quasi-units**. Informally, one is led to consider the following questions:

(1) Given a non-unital ∞-category $\overline{\mathcal{C}}$ in which every object admits a quasi-unit, can $\overline{\mathcal{C}}$ be promoted to an ∞-category $\mathcal{C}$?

(2) If such a $\mathcal{C}$ exists, is it essentially unique?

(3) Given two ∞-categories $\mathcal{C}, \mathcal{D}$ with underlying non-unital ∞-categories $\overline{\mathcal{C}}, \overline{\mathcal{D}}$, can the functor category $\mathcal{C}^{\mathcal{D}}$ be reconstructed from a suitable functor category $\overline{\mathcal{C}}^{\overline{\mathcal{D}}}$?

In this paper we give a positive answer to the above questions. More precisely, we will construct a monoidal model category Comp_s which is a quasi-unital analouge of Rezk's complete Segal space model category Comp. We will then show that the forgetful functor $\mathrm{Comp} \longrightarrow \mathrm{Comp}_s$ fits into a Quillen equivalence between Comp and Comp_s. Furthermore, we will show that this Quillen equivalence preserves suitable symmetric monoidal structures. This will yield an affirmative answer to all three questions above.

Before outlining our construction, let us explain our motivation for considering this question. As explained in [12], a result of this kind can be used to facilitate the construction of the cobordism categories. However, the relationship between such questions and the cobordism hypothesis goes beyond this mere added efficiency. In particular, one can actually use the result above in order to **prove** the $n = 1$ case of the cobordism hypothesis (in the setting of ∞-categories). This application is described in [3].

Let us now describe our approach to constructing Comp_s. As explained above, when we encode the structure of an ∞-category in a simplicial space, what we need to do in order to remove the unital structure is to remove the degeneracy maps. This leads to the notion of a **semiSegal space**, which is defined formally in § 1.4. The data of a semiSegal space X describes a non-unital ∞-category with space of objects X_0.

The next step is to understand what it means for a morphism $f \in X_1$ such that $d_0(f) = d_1(f)$ to be a **quasi-unit**, i.e. to be neutral with respect to composition. This is defined formally in § 1.4. We shall say that a semiSegal space is **quasi-unital** if it admits quasi-units for every object. One can then phrase question (1) above in terms of quasi-unital semiSegal spaces. However, in order to get any intelligent answer one should work not only with such semiSegal spaces themselves, but also with a correct notion of mappings between them. In particular, questions (1) and (2) should be considered together for an entire suitable ∞-category of quasi-unital semiSegal spaces.

A first discouraging observation is that maps of semiSegal spaces need not in general send quasi-units to quasi-units. This statement should be interpreted as follows: the structure of units is **not a mere condition**. Indeed, if this was the case one would expect the forget-the-units functor to be fully-faithful. Instead, we see that if X, Y are two Segal spaces with underlying semiSegal spaces $\overline{X}, \overline{Y}$, then a map $\overline{f} : \overline{X} \longrightarrow \overline{Y}$ has a chance of coming from a map $f : X \longrightarrow Y$ only if it sends quasi-units to quasi-units. Hence we conclude that the collection of quasi-units should be **marked** as part of the data. The main result of this paper says that this is in fact all one needs to specify - all the additional unital structure is then essentially **uniquely determined**.

Our second observation is that instead of marking the quasi-units, one can mark the slightly larger collection of **invertible morphisms**. These are the morphisms composition with which

induces weak equivalences on mapping spaces. A simple lemma (which we prove in § 1.5) says that a map of semiSegal spaces sends quasi-units to quasi-units if and only if it sends invertible edges to invertible edges. Furthermore, the condition that an object admits a quasi-unit is equivalent to the condition that this object admits an invertible morphism out of it.

From this point of view we see that marking the invertible edges is essentially the same as marking the quasi-units. Furthermore, this alternative is much more convenient in practice. This is due to the fact that invertibility is a considerably more robust notion - for example, one does not need to check that a morphism has equal source and target before considering its invertibility. Furthermore, a morphism will stay invertible if we "deform it a little bit", i.e. the space of invertible 1-simplices is a union of connected components of X_1.

Now in order to prove a result such as the one we are interested in here one might like a convenient model category in which one can consider semi-simplicial spaces for which certain 1-simplices have been **marked**. This naturally leads to the category of **marked semi-simplicial spaces** in which we will work from § 1.7 onward.

In [14] Rezk constructs two successive left Bousfield localizations of the Reedy model structure on the category of $\mathcal{S}^{\Delta^{\mathrm{op}}}$ of simplicial spaces. Our strategy in this paper will be to mimic Rezk's constructions in the category $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ of marked semi-simplicial spaces. We will start in § 1.7 where we will establish the existence of a (monoidal) model structure on $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ which is analogous to the **Reedy model structure** on the category $\mathcal{S}^{\Delta^{\mathrm{op}}}$. We will refer to this structure as the **marked model structure**. We will then construct a Quillen adjunction (which is not an equivalence)

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathcal{RK}^+}{\overset{\mathcal{F}^+}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

between the Reedy model structure on $\mathcal{S}^{\Delta^{\mathrm{op}}}$ and the marked model structure on $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$. This Quillen adjunction will be the basis of comparison between the model category Comp_s that will be constructed in this paper and the model category Comp of complete Segal spaces.

We will continue our strategy in § 2 where we will localize the marked model structure in order to obtain the **semiSegal model category** Seg_s. This model structure is analogous to the Segal model structure of [14]. The fibrant objects of Seg_s will be called **marked semiSegal spaces**. We will then say that a marked semiSegal space is **quasi-unital** if each object admits an invertible edge out of it and if all invertible edges are marked. This will formalize the intuition described above regarding how to describe quasi-unital ∞-categories. We will denote by $\mathrm{QsS} \subseteq \mathrm{Seg}_s^{\mathrm{fib}}$ the full simplicial subcategory spanned by quasi-unital marked semiSegal spaces.

Following the footsteps of Rezk we observe that QsS itself is still not a model for the correct ∞-category of quasi-unital ∞-categories. As in the analogous case of Segal spaces, the problem is that equivalences in QsS are far too strict. To obtain the correct notion one needs to **localize** the ∞-category associated to QsS with respect to a certain natural family of **Dwyer-Kan**

equivalences which will be studied in §2.2. We will consider this ∞-localization (see §1.2) of QsS as the **∞-category of (small) quasi-unital ∞-categories**.

In order to obtain a model category that models this ∞-category we will construct in §3 a final left Bousfield localization of Seg_s, which we denote by Comp_s. The fibrant objects of Comp_s will be referred to as **complete marked semiSegal spaces**. The purpose of §3 is to prove the following (see Theorems 3.1.3 and 3.2.1):

Theorem 0.0.1

(1) The underlying ∞-category of Comp_s (see §1.2) is equivalent to the ∞-category of quasi-unital ∞-categories (in the above sense).

(2) The adjunction $\mathcal{F}^+ \vdash \mathcal{RK}^+$ above descends to a (suitably monoidal) **Quillen equivalence** *between* Comp *and* Comp_s*. Passing to underlying ∞-categories we obtain the desired equivalence between quasi-unital and unital ∞-categories.*

Theorem 0.0.1 will then give us the desired positive answers to all three questions appearing above.

Relation to other work

The theory developed here is closely related and much inspired by the theory of quasi-unital algebras introduced by Lurie in [10, §5.4.3]. There he considers non-unital algebra objects in a general monoidal ∞-category $\mathcal{D}$. Enforcing an existence condition for quasi-units and an appropriate unitality condition for morphisms one obtains the ∞-category of quasi-unital algebra objects in $\mathcal{D}$. It is then proven that

Theorem 0.0.2 ([10]) *The forgetful functor from the ∞-category of algebra objects in $\mathcal{D}$ to the ∞-category of quasi-unital algebra objects in $\mathcal{D}$ is an equivalence of ∞-categories.*

Note that if $\mathcal{D}$ is the monoidal ∞-category of spaces (with the Cartesian product) then algebra objects in $\mathcal{D}$ can be identified with pointed ∞-categories with one object. Similarly, quasi-unital algebra objects in $\mathcal{D}$ can be considered as pointed quasi-unital ∞-categories with one object. Hence we see that there is a strong link between the main result of this paper and Theorem 0.0.2. However, even when restricting attention to quasi-unital ∞-categories with one object, our result is not a particular case of Theorem 0.0.2. This is due to the fact that the mapping space between quasi-unital ∞-categories with one object does not coincide, in general, with the corresponding space of **pointed** maps.

In the context of **strict n-categories** the notion of **quasi-units** has enjoyed a fair amount of interest as well. In [9], Kock defines the notion of a **fair n-category**, which in our terms can be called a **strict quasi-unital n-category**. For $n = 2$ and for a variation of the $n = 3$ case Kock and Joyal have shown that a (non-strict) unital structure can be uniquely recovered

(see [7]). In [8] Kock and Joyal further show that every simply connected homotopy 3-type can be modelled by a fair 3-groupoid (see [8]). The main difference between their work and the present paper is that we address the (manifestly non-strict) case of quasi-unital ∞-categories (or $(\infty,1)$-categories, as opposed to (n,n)-categories). Furthermore, our results are framed in terms of a complete equivalence between the notions of unital and quasi-unital ∞-categories.

1 Preliminaries and overview

Let Δ denote the simplex category, i.e., the category whose objects are the finite ordered sets $[n] = \{0,...,n\}$ and whose morphisms are non-decreasing maps. Let $\mathcal{S} = \mathrm{Set}^{\Delta^{\mathrm{op}}}$ denote the category of **simplicial sets**. The category $\mathcal{S}$ can be endowed with the **Kan model structure**, making it a model for the homotopy theory of spaces. When working with the Kan model structure we will often refer to objects in $\mathcal{S}$ as **spaces**. We will say that two maps $f,g : K \longrightarrow L$ in $\mathcal{S}$ are **homotopic** (denoted $f \sim g$) if they induce the same map in the homotopy category associated to the Kan model structure. A **point** in a space K will mean a 0-simplex and a **path** in K will mean a 1-simplex.

The category $\mathcal{S}$ can also be endowed with the **Joyal model structure**, making it a model for the homotopy theory of ∞-categories. We will refer to Joyal-fibrant simplicial sets as **∞-categories**.

A **simplicial category** is a category enriched in $\mathcal{S}$. The category $\mathrm{Cat}_{\mathcal{S}}$ of small simplicial categories carries a model structure in which the weak equivalences are the Dwyer-Kan equivalences (see [0]). This model category admits a Quillen equivalence

$$\mathcal{S} \underset{\mathcal{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathrm{Cat}_{\mathcal{S}}$$

to the Joyal model category, where $\mathcal{N}$ is the **simplicial nerve functor** (see [11, Theorem 2.2.5.1]). In particular, $\mathrm{Cat}_{\mathcal{S}}$ is an equivalent model for the homotopy theory of ∞-categories.

1.1 Symmetric monoidal and simplicial model categories

In this paper we will work a lot with **symmetric monoidal simplicial model categories**. Let us briefly review the relevant definitions:

Definition 1.1.1 Let $\mathcal{M}$ be a category and $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ a symmetric monoidal product. We say that $\otimes$ is **closed** if there exists an internal mapping functor $\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathcal{M}$, typically denoted by $(X,Y) \mapsto Y^X$, together with natural maps $\nu_{X,Y} : Y^X \otimes X \longrightarrow Y$ which induce isomorphisms

$$\mathrm{Hom}\left(Z, Y^X\right) \xrightarrow{\simeq} \mathrm{Hom}(Z \otimes X, Y)$$

for every X,Y,Z. These isomorphisms are sometimes referred to as the **exponential law**.

Remark 1.1.2 If $\mathcal{M}$ is presentable and $\otimes$ is a symmetric monoidal product then $\otimes$ is closed if and only if it preserves colimits separately in each variable. This follows from the adjoint functor theorem.

Definition 1.1.3 Let $(\mathcal{C},\otimes),(\mathcal{D},\boxtimes)$ be two symmetric monoidal categories and $\mathcal{C} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathcal{D}$ an adjunction. Let

$$\alpha_{X,Y} : \mathcal{R}(X)\otimes\mathcal{R}(Y) \longrightarrow \mathcal{R}(X\boxtimes Y) \qquad u : 1_{\mathcal{C}} \longrightarrow \mathcal{R}(1_{\mathcal{D}})$$

be a lax structure on $\mathcal{R}$ and

$$\beta_{Z,W} : \mathcal{L}(Z\otimes W) \longrightarrow \mathcal{L}(Z)\boxtimes\mathcal{L}(W) \qquad v : \mathcal{L}(1_{\mathcal{C}}) \longrightarrow 1_{\mathcal{D}}$$

a colax structure on $\mathcal{L}$. We will say that the lax-colax pair $(\alpha_{X,Y},u),(\beta_{Z,W},v)$ is **compatible** with the adjunction if the diagrams

$$\begin{array}{ccc} Z\otimes W & \longrightarrow & \mathcal{R}(\mathcal{L}(Z))\otimes\mathcal{R}(\mathcal{L}(W)) \\ \downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{L}(Z),\mathcal{L}(W)}} \\ \mathcal{R}(\mathcal{L}(Z\otimes W)) & \xrightarrow{\mathcal{R}(\beta_{Z,W})} & \mathcal{R}(\mathcal{L}(Z)\boxtimes\mathcal{L}(W)) \end{array} \qquad \begin{array}{ccc} \mathcal{L}(\mathcal{R}(X)\otimes\mathcal{R}(Y)) & \xrightarrow{\mathcal{L}(\alpha_{X,Y})} & \mathcal{L}(\mathcal{R}(X\boxtimes Y)) \\ \downarrow{\scriptstyle \beta_{\mathcal{R}(X),\mathcal{R}(Y)}} & & \downarrow \\ \mathcal{L}(\mathcal{R}(X))\boxtimes\mathcal{L}(\mathcal{R}(Y)) & \longrightarrow & X\boxtimes Y \end{array}$$

commute (where the unnamed maps are given by the unit/counit of the adjunction $\mathcal{L}\dashv\mathcal{R}$). An adjunction together with a compatible lax-colax pair is called a **lax-monoidal adjunction**. We will say that a lax-monoidal adjunction is **strongly monoidal** if $\mathcal{L}$ is **monoidal**, i.e. if $\beta_{Z,W}$ and v are natural isomorphisms. We refer the reader to [17] for more details.

Remark 1.1.4 Let $\mathcal{L}\dashv\mathcal{R}$ be an adjunction. Then for any lax structure $(\alpha_{X,Y},u)$ on $\mathcal{R}$ there is a unique colax structure $(\beta_{Z,W},v)$ on $\mathcal{L}$ such that the pair $(\alpha_{X,Y},u),(\beta_{Z,W},v)$ is compatible with the adjunction. Similarly, any colax structure on $\mathcal{L}$ can be extended to a unique compatible lax-colax pair.

Definition 1.1.5 Let $\mathcal{M}$ be a model category with a closed symmetric monoidal product $\otimes$ such that the unit of $\otimes$ is cofibrant. We say that $\mathcal{M}$ is **compatible** with $\otimes$ if for every pair of cofibrations $f : X' \longrightarrow X, g : Y' \longrightarrow Y$ the induced map

$$h : \left[X'\otimes Y\right] \coprod_{X'\otimes Y'} \left[X\otimes Y'\right] \longrightarrow X\otimes Y$$

is a cofibration, and is further a trivial cofibration if at least one of f,g is trivial. This condition is commonly referred to as the **pushout-product axiom**. In this case we say that $\mathcal{M}$ is a **symmetric monoidal model category**.

Remark 1.1.6 The definition above can be extended to the case where the unit of $\otimes$ is not necessarily cofibrant (see [6] Definition 4.2.6). However, since in our case the units will always be cofibrant it will simplify matters for us to assume this from now on.

Example 1.1.7 The Kan model structure on $\mathcal{S}$ is compatible with the Cartesian monoidal structure.

Definition 1.1.8 Let $(\mathcal{M},\otimes),(\mathcal{N},\boxtimes)$ be two symmetric monoidal model categories. A **lax-monoidal Quillen adjunction** $\mathcal{L}\dashv\mathcal{R}$ between $\mathcal{M}$ and $\mathcal{N}$ is a Quillen adjuntion equipped with a lax-colax structure compatible with the adjunction (see Definition 1.1.3). We note that there is no compatibility requirement between the lax-colax pair and the model structure. We will say that a lax-monoidal Quillen adjunction is **strongly monoidal** if $\mathcal{L}$ is monoidal, i.e., if the structure maps of the colax structure are isomorphisms. We will say that a lax-monoidal adjunction is **weakly monoidal** if the structure maps of the colax structure are weak equivalences.

Definition 1.1.9 Let $\mathcal{M}$ be a symmetric monoidal model category. A **simplicial sturcture** on $\mathcal{M}$ is a strongly monoidal Quillen adjunction

$$\mathcal{S}\underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}}\mathcal{M}\ .$$

where $\mathcal{S}$ is endowed with the Kan model structure. In this case we say that $\mathcal{M}$ is a **symmetric monoidal simplicial model category**. $\mathcal{M}$ then acquires a natural enrichment over $\mathcal{S}$ given by

$$\mathrm{Map}_{\mathcal{M}}(X,Y)\overset{\mathrm{def}}{=}\mathcal{R}\left(Y^X\right)$$

and one has natural isomorphisms

$$\mathrm{Map}_{\mathcal{S}}\left(K,\mathrm{Map}_{\mathcal{M}}(X,Y)\right)\cong\mathrm{Map}_{\mathcal{M}}\left(\mathcal{L}(K),Y^X\right)\cong\mathrm{Map}_{\mathcal{M}}(\mathcal{L}(K)\otimes X,Y)$$

for $K\in\mathcal{S}^{\mathrm{op}},X\in\mathcal{M}^{\mathrm{op}}$ and $Y\in\mathcal{M}$. When there is no room for confusion we will usually abuse notation and denote $\mathcal{L}(K)$ simply by K.

Remark 1.1.10 A simplicial structure can be defined also for $\mathcal{M}$ which do not posses a symmetric monoidal structure but instead carry an action of $\mathcal{S}$ which satisfies analogous conditions to those of definition 1.1.5.

1.2 Underlying ∞-categories and ∞-localizations

Though working mostly with model categories, we will always consider them as strict incarnations of their **underlying ∞-categories**. To give a precise definition let us follow the approach of [4] via a universal property. We begin by recalling the notion of ∞-localization (see [4]).

Definition 1.2.1 Let $f:\mathcal{C}\longrightarrow\mathcal{D}$ be a map of ∞-categories (i.e., fibrant objects in the Joyal model structure, see §1) and let $\mathcal{W}$ be a collection of morphisms (edges) in $\mathcal{C}$. We will say that the map f exhibits $\mathcal{D}$ as the **∞-localization** of $\mathcal{C}$ with respect $\mathcal{W}$ if for every ∞-category $\mathcal{E}$ the restriction map

$$f^*:\mathrm{Fun}(\mathcal{D},\mathcal{E})\longrightarrow\mathrm{Fun}(\mathcal{C},\mathcal{E})$$

is fully-faithful, and its essential image is spanned by those functors $g:\mathcal{C}\longrightarrow\mathcal{E}$ which send every morphism in $\mathcal{W}$ to an equivalence. In this case the ∞-category $\mathcal{D}$ is essentially determined by this condition and will often be denoted simply by $\mathcal{C}[\mathcal{W}^{-1}]$.

Now when $\mathcal{W}$ is a set the ∞-localization $\mathcal{C}[\mathcal{W}^{-1}]$ always exists (as a locally small ∞-category). Furthermore, if $\mathcal{M}$ is a model category with a collection of weak equivalences $\mathcal{W}$ then it can be shown that $\mathcal{M}_\infty \stackrel{\text{def}}{=} \mathcal{M}[\mathcal{W}^{-1}]$ exists as a locally small ∞-category without requiring that $\mathcal{W}$ is a set. We will refer to $\mathcal{M}_\infty$ as the **underlying ∞-category of** $\mathcal{M}$.

The ∞-category $\mathcal{M}_\infty$ can be explicitly constructed in various ways. By taking the **Rezk nerve** of $\mathcal{M}$ with respect to $\mathcal{W}$ one can construct a simplicial space which is a model for $\mathcal{M}_\infty$ in the model category of complete Segal spaces (see [14]). Alternatively, one can construct the **Dwyer-Kan simplicial localization** of $\mathcal{M}$ with respect to $\mathcal{W}$ (see [1]) to obtain a simplicial category which is a model for $\mathcal{M}_\infty$ via the (derived) simplicial nerve functor ([4, Proposition 2.2.1]). Thirdly, and most relevant to us, is the last option, which exists when $\mathcal{M}$ carries the structure of a **simplicial model category**. In this case, the full simplicial subcategory $\mathcal{M}^{\text{fc}} \subseteq \mathcal{M}$ spanned by the fibrant-cofibrant objects is a model for $\mathcal{M}_\infty$ (see [2]). Note that in this case $\mathcal{M}^{\text{fc}}$ will be a fibrant simplicial category, and so we can write directly $\mathcal{N}(\mathcal{M}^{\text{fc}}) \simeq \mathcal{M}_\infty$.

A particularly common case of ∞-localization is the following (see [11, Definition 5.2.7.2]):

Definition 1.2.2 Let $\mathcal{C}$ be an ∞-category and $\mathcal{W}$ a collection of morphisms in $\mathcal{C}$. We will say that a map of ∞-categories $f : \mathcal{C} \longrightarrow \mathcal{D}$ exhibits $\mathcal{D}$ as a **left ∞-localization** of $\mathcal{C}$ with respect to $\mathcal{W}$ if the following conditions are satisfied:

(1) The map f admits a fully-faitful right adjoint.

(2) A morphism α is in $\mathcal{W}$ if and only if $f(\alpha)$ is an equivalence.

It is not hard to verify that any left ∞-localization is in particular an ∞-localization in the sense above (see [11, Warning 5.2.7.3]). Note that if $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a left ∞-localization with respect to some $\mathcal{W}$ then this $\mathcal{W}$ is completely determined by condition (2) above. One may hence simply say that f is a left ∞-localization (without referring to $\mathcal{W}$) whenever f has a fully-faithful right adjoint. The most common source of left ∞-localizations is the following:

Lemma 1.2.3 *Let $\mathcal{M}$ be a model category and let $\overline{\mathcal{M}}$ be a left Bousfield localization of $\mathcal{M}$. Let $\mathcal{W}$ be the collection of weak equivalences of $\overline{\mathcal{M}}$. Then the natural left Quillen functor $\mathcal{M} \longrightarrow \overline{\mathcal{M}}$ (given by the identity) induces a left ∞-localization $\mathcal{M}_\infty \longrightarrow \overline{\mathcal{M}}_\infty$ with respect to the image of $\mathcal{W}$ in $\mathcal{M}_\infty$.*

Proof According to [4] the Quillen adjunction $\text{Id} : \mathcal{M} \dashv \overline{\mathcal{M}}$ induces an adjunction $\mathcal{M}_\infty \dashv \overline{\mathcal{M}}_\infty$ in the ∞-categorical sense. Furthermore, by computing derived mapping spaces using cosimplicial resolutions of the source one may easily verify that the right Quillen functor $\text{Id} : \overline{\mathcal{M}} \longrightarrow \mathcal{M}$ induces a weak equivalence on derived mapping spaces, i.e., that the induced functor

$$\text{Id}_\infty : \overline{\mathcal{M}}_\infty \longrightarrow \mathcal{M}_\infty$$

is fully-faithful. Finally, property (2) of Definition 1.2.2 is satisfied since model categories are saturated as relative categories. □

1.3 Semi-simplicial spaces and the Reedy model structure

Let $\Delta_s \subseteq \Delta$ denote the subcategory consisting only of **injective** maps. A **semi-simplicial set** is a functor $\Delta_s^{\mathrm{op}} \longrightarrow \mathrm{Set}$. Similarly, a **semi-simplicial spaces** is a functor $\Delta_s^{\mathrm{op}} \longrightarrow \mathcal{S}$. The category of semi-simplicial spaces will be denoted by $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$.

We will denote by Δ^n the standard n-simplex considered as a **semi-simplicial set** (it is given by the functor $\Delta_s^{\mathrm{op}} \longrightarrow \mathrm{Set}$ represented by $[n]$). When we want to refer to the standard simplex as a **simplicial set** we will denote it as $|\Delta^n| \in \mathcal{S}$. This notation is consistent with our notation for the **geometric realization** functor which we consider as the functor

$$|\bullet| : \mathcal{S}^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}$$

given by the coend $|X| = \int^{\Delta_s} X_\bullet \times |\Delta^\bullet|$.

For a subset $I \subseteq [n]$ we will denote by $\Delta^I \subseteq \Delta^n$ the semi-simplicial subset corresponding to the sub-simplex spanned by I. We will denote by

$$\mathrm{Sp}^n = \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} \ldots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subseteq \Delta^n \tag{1–1}$$

the **spine** of Δ^n, i.e., the semi-simplicial subset consisting of all the vertices and all the edges between consecutive vertices.

We will occasionally abuse notation and consider Δ^n as a semi-simplicial space as well (levelwise discrete). Orthogonally, we will sometimes consider a space $K \in \mathcal{S}$ as a semi-simplicial space which is concentrated in degree zero, i.e., as the semi-simplicial space given by $K_0 = K$ and $K_n = \emptyset$ for $n > 0$.

The category $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ carries the **Reedy model structure** with respect to the Kan model structure on $\mathcal{S}$ and the obvious Reedy structure on Δ_s. Since Δ_s is a Reedy category in which all non-trivial morphisms are increasing the Reedy model structure coincides with the **injective model structure**. This is a particularly nice situation because we have a concrete description for all three classes of maps. In particular, the weak equivalences and cofibrations are defined levelwise, and fibrations are defined in terms of **matching objects**. We refer the reader to [5, § 15] for more details.

Consider the adjunction

$$\mathcal{S}^{\Delta_s^{\mathrm{op}}} \underset{\mathcal{F}}{\overset{\mathcal{LK}}{\rightleftarrows}} \mathcal{S}^{\Delta^{\mathrm{op}}}$$

where $\mathcal{F}$ is the forgetful functor (or the pullback along the inclusion $\Delta_s \hookrightarrow \Delta$) and $\mathcal{LK}$ is the left Kan extension functor. Given a simplicial space Z we shall denote by $\mathrm{Core}(Z) \subseteq \mathcal{F}(Z)$ the minimal semi-simplicial subspace of $\mathcal{F}(Z)$ containing all the non-degenerate simplices. Now recall the standard (non-Cartesian) symmetric monoidal product $X, Y \mapsto X \otimes Y$ on $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ defined, as in [15] § 3, by the formula

$$X \otimes Y \overset{\mathrm{def}}{=} \mathrm{Core}(\mathcal{LK}(X) \times \mathcal{LK}(Y))$$

The unit of $\otimes$ is Δ^0.

Remark 1.3.1 One can obtain an explicit description of the space of k-simplices in $X \otimes Y$ as follows: let $P_k^{n,m}$ denote the set of injective order preserving maps

$$\rho : [k] \longrightarrow [n] \times [m]$$

such that $p_{[n]} \circ \rho : [k] \longrightarrow [n]$ and $p_{[m]} \circ \rho : [k] \longrightarrow [m]$ are **surjective** (such maps are sometimes called **shuffles**). Then one has

$$(X \otimes Y)_k = \coprod_{n,m \leq k} P_k^{n,m} \times X_n \times Y_m.$$

In particular, the set of k-simplices of $\Delta^n \otimes \Delta^m$ can be identified with the set of **all** injective order preserving maps $[k] \longrightarrow [n] \times [m]$.

Remark 1.3.2 The natural transformations $\mathrm{Core}(\bullet) \longrightarrow \mathcal{F}(\bullet)$ and $\mathcal{LK}(\mathcal{F}(\bullet)) \longrightarrow \bullet$ may be combined to produce a natural map

$$\mathcal{LK}(X \otimes Y) = \mathcal{LK}(\mathrm{Core}(\mathcal{LK}(X) \times \mathcal{LK}(Y))) \xrightarrow{\simeq} \mathcal{LK}(X) \times \mathcal{LK}(Y)$$

which is in fact an **isomorphism** (to see this it is enough to verify the case $X = \Delta^n, Y = \Delta^m$ by a direct computation). Combined with the identity map $\mathcal{LK}(\Delta^0) \longrightarrow \mathcal{LK}(\Delta^0)$ we see that $\mathcal{LK}$ carries the structure of a **monoidal functor**. This implies, in particular, that the geometric realization functor $|\bullet| : \mathcal{S}^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}$ is monoidal as well.

The symmetric monoidal product $\otimes$ is **closed** and the corresponding internal mapping object can be described explicitly as follows: if X, Y are two semi-simplicial spaces then the mapping object Y^X is given by

$$(Y^X)_n = \mathrm{Map}(\Delta^n \otimes X, Y).$$

The Reedy model structure on $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is **compatible** with $\otimes$. This can be easily verified using the explicit formula in Remark 1.3.1. Furthermore, $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ admits a natural **simplicial structure** (see Definition 1.1.9) given by the adjunction

$$\mathcal{S} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

where $\mathcal{L}(K)$ is given by K concentrated in degree 0 and $\mathcal{R}(X) = X_0$. In particular, the Reedy model category $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is a symmetric monoidal simplicial model category with respect to $\otimes$.

1.4 SemiSegal spaces and quasi-units

Definition 1.4.1 Let X be a semi-simplicial space. Let $[n], [m] \in \Delta_s$ be two objects and consider the commutative (pushout) diagram

$$\begin{array}{ccc} [0] & \xrightarrow{0} & [m] \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle g_{n,m}} \\ [n] & \xrightarrow[f_{n,m}]{} & [n+m] \end{array}$$

where $f_{n,m}(i) = i$ and $g_{n,m}(i) = i + n$. We will say that X satisfies the **Segal condition** if for each $[n], [m]$ as above the induced commutative diagram

$$\begin{array}{ccc} X_{m+n} & \xrightarrow{g^*_{n,m}} & X_m \\ {\scriptstyle f^*_{n,m}}\downarrow & & \downarrow{\scriptstyle 0^*} \\ X_n & \xrightarrow{n^*} & X_0 \end{array}$$

is **homotopy Cartesian**. We will say that X is a **semiSegal space** if it is **Reedy fibrant** and satisfies the Segal condition. Note that in that case the above square will induce a homotopy equivalence

$$X_{m+n} \simeq X_m \times_{X_0} X_n.$$

Remark 1.4.2 A simple induction argument shows that if X is a semiSegal space then the natural map

$$X_n = \mathrm{Map}\left(\Delta^n, X\right) \longrightarrow \mathrm{Map}\left(\mathrm{Sp}^n, X\right) \simeq X_1 \times_{X_0} X_1 \times_{X_0} \ldots \times_{X_0} X_1$$

is a weak equivalence (where $\mathrm{Sp}^n \subseteq \Delta^n$ is the n'th spine, see 1–1). For X Reedy fibrant the above property is equivalent to the Segal condition.

Example 1.4.3 Let $\mathcal{D}$ be a small non-unital simplicial category in which the mapping spaces are Kan simplicial sets. We can associate with $\mathcal{D}$ a semiSegal space via a non-unital analogue of the nerve construction as follows. For each n, let $\mathcal{C}^s([n])$ denote the non-unital category whose objects are the numbers $0, \ldots, n$ and whose mapping spaces are

$$\mathrm{Map}_{\mathcal{C}^s([n])}(i,j) = \begin{cases} \emptyset & i \geq j \\ * & i < j \end{cases}$$

As $\mathcal{C}^s([n])$ depends functorially on $[n] \in \Delta_s$ we can get a semi-simplicial space $N(\mathcal{D})$ by setting

$$N(\mathcal{D})_n = \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}^s([n]), \mathcal{D}).$$

Note that $N(\mathcal{D})$ will generally not be Reedy fibrant, but after applying the Reedy fibrant replacement functor (which is a levelwise equivalence) one indeed obtains a **semiSegal space**.

We think of a general semiSegal space as encoding a relaxed version of Example 1.4.3, i.e. a non-unital ∞-category. This can be described as follows: the objects of this non-unital ∞-category are the points of X_0. Given two points $x, y \in X_0$ we define the **mapping space** between them by

$$\mathrm{Map}_X(x,y) = \{x\} \times_{X_0} X_1 \times_{X_0} \{y\},$$

i.e., as the fiber of the (Kan) fibration

$$X_1 \xrightarrow{(d_1, d_0)} X_0 \times X_0$$

over the point (x,y). The space X_2 of triangles then induces a "weak composition" operation on these mapping spaces, which can be described as follows. Given three points $x,y,z \in X_0$, let us denote by

$$\mathrm{Comp}(x,y,z) = \left\{\sigma \in X_2 \mid \sigma|_{\Delta^{\{0\}}} = x, \sigma|_{\Delta^{\{1\}}} = y, \sigma|_{\Delta^{\{2\}}} = z\right\}$$

The three restriction maps $\sigma \mapsto \sigma|_{\Delta^{\{i,j\}}}$ yield a diagram of the form

$$\begin{array}{ccc} & \mathrm{Comp}(x,y,z) & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ \mathrm{Map}_X(x,y)\times\mathrm{Map}_X(y,z) & & \mathrm{Map}_X(x,z) \end{array}$$

Since X is a semiSegal space it follows that φ is a trivial Kan fibration. One can then construct a candidate for the composition of $f \in \mathrm{Map}_X(x,y)$ and $g \in \mathrm{Map}_X(y,z)$ by choosing a triangle $\sigma \in \mathrm{Comp}(x,y,z)$ lying above (f,g) and considering $\sigma|_{\Delta^{\{0,2\}}}$ as the composition $g \circ f$. It should be noted that the choice of σ (and hence of $g \circ f$) is not unique. However, since ϕ is a trivial Kan fibration the space of choices for σ is **contractible**. In this sense the operation of composition is **essentially** well-defined. In a similar way one can use the Segal condition in dimension 3 in order to show that this weak composition is associative up to coherent homotopy, in a suitable sense. We will refer the reader to [14] for a more detailed description in the unital case.

Remark 1.4.4 As in the unital case, a semiSegal space carries more information than the non-unital ∞-categorical structure on X_0 described above. One aspect of this is that X_0 itself is not a set, but a space, and the homotopy type of this space is not determined by the non-unital ∞-categorical structure. In the unital case (as well as the quasi-unital case, as we will see in §3) this issue can be resolved via the notion of **completeness**.

Example 1.4.5 Let Z be a Kan simplicial set. Applying the 0'th coskeleton functor one obtains a semi-simplicial space

$$X = \mathrm{cosk}_0(Z)$$

which is given by $X_n = \mathrm{Map}\left(\mathrm{sk}_0(\Delta^n), Z\right) = Z^{n+1}$. It is then easy to verify that X is a **semiSegal space**. This semiSegal space encodes a trivial non-unital structure in which all the mapping spaces are contractible. However, it can admit arbitrary homotopy types for the space of objects X_0 (see Remark 1.4.4).

Definition 1.4.6 Let X be a semiSegal space. We define its **non-unital homotopy category** $\mathrm{Ho}(X)$ to be the non-unital category whose objects are the points of X_0 and whose morphism sets are given by

$$\mathrm{Hom}_{\mathrm{Ho}(X)}(x,y) \overset{\mathrm{def}}{=} \pi_0(\mathrm{Map}_X(x,y)).$$

The weak composition described above induces an honest composition in $\mathrm{Ho}(X)$ which can be described as follows. A component $[h] \in \pi_0(\mathrm{Map}_X(x,z))$ is the composition of $[f] \in \pi_0(\mathrm{Map}_X(x,y))$ and $[g] \in \pi_0(\mathrm{Map}_X(y,z))$ if and only if there exists a triangle $\sigma \in X_2$ such that $\sigma|_{\Delta^{\{0,1\}}} = f, \sigma|_{\Delta^{\{1,2\}}} = g$ and $\sigma|_{\Delta^{\{0,2\}}} = h$. The Segal condition in dimension 3 ensures that the composition on $\mathrm{Ho}(X)$ is **associative**. However, $\mathrm{Ho}(X)$ is not unital in general.

Our first goal when dealing with semiSegal spaces is to understand when a morphism $f : x \longrightarrow y$ is **neutral** with respect to composition. For this we need to extract in some way the action of f on mapping spaces.

Definition 1.4.7 Let $x, y, z \in X_0$ be points and $f : x \longrightarrow y$ a morphism in X (i.e. a point $f \in \mathrm{Map}_X(x,y)$). Consider the space

$$C^R_{f,z} = \left\{ \sigma \in X_2 \mid \sigma|_{\Delta^{\{1,2\}}} = f, \sigma|_{\Delta^{\{0\}}} = z \right\} \subseteq \mathrm{Comp}(x,y,z)$$

together with the two restriction maps

(1–2)
$$\begin{array}{ccc} & C^R_{f,z} & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ \mathrm{Map}_X(z,x) & & \mathrm{Map}_X(z,y) \end{array}$$

By the Segal condition we see that φ is a weak equivalence. We then define a homotopy class

$$[f]_* \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}\left(\mathrm{Map}_X(z,x), \mathrm{Map}_X(z,y)\right)$$

by setting $[f]_* \overset{\mathrm{def}}{=} [\psi]\circ[\varphi]^{-1}$. This can be considered as the homotopy class of the almost-defined map f_* obtained by composition with f. Similarly, one can define a homotopy class

$$[f]^* : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}\left(\mathrm{Map}_X(y,z), \mathrm{Map}_X(x,z)\right)$$

describing the homotopy class of pre-composition with f.

Warning 1.4.8 The notations $[f]_*$ and $[f]^*$ are a bit abusive, as they implicitly depend on the point $z \in X_0$. We hope that this will not result in any confusion.

Remark 1.4.9 Given a point $z \in X_0$ the definition above yields a **functor** of non-unital categories $R_z : \mathrm{Ho}(X) \longrightarrow \mathrm{Ho}(\mathcal{S})$ given by

$$R_z(x) = \mathrm{Map}_X(z,x), \quad R_z([f]) = [f]_*.$$

Similarly, we can construct a functor $R^z : \mathrm{Ho}(X)^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathcal{S})$ by setting

$$R^z(x) = \mathrm{Map}_X(x,z), \quad R^z([f]) = [f]^*.$$

Warning 1.4.10 The functors described in Remark 1.4.9 can be considered as the representable and corepresentable functors of X after descending to the (non-unital) homotopy category. However, they are not the representable and corepresentable functors of $\mathrm{Ho}(X)$ itself as they take values in $\mathrm{Ho}(\mathcal{S})$ and not in Set. Note that in the non-unital setting there is no Yoneda lemma, and so the associations $z \mapsto R_z$ and $z \mapsto R^z$ can lose information in general.

The above construction can be used to determine when a morphism is neutral with respect to composition:

Definition 1.4.11 Let $x \in X_0$ be an object and $f : x \longrightarrow x$ a morphism in X. We will say that f is a **quasi-unit** if for each $z \in X_0$ the homotopy classes

$$[f]_* \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}\left(\mathrm{Map}_X(z,x), \mathrm{Map}_X(z,x)\right)$$

and

$$[f]^* \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}\left(\mathrm{Map}_X(x,z), \mathrm{Map}_X(x,z)\right)$$

of Definition 1.4.7 are both the **identity** in $\mathrm{Ho}(\mathcal{S})$.

Definition 1.4.12 Let X be a semiSegal space. We will say that X is **quasi-unital** if every object $x \in X_0$ admits a quasi-unit $q : x \longrightarrow x$. We will informally say that X models a **quasi-unital ∞-category**.

Warning 1.4.13 Even if X and Y are quasi-unital, a general map $X \longrightarrow Y$ need not send quasi-units to quasi-units. For example, let $\mathcal{C}$ be the non-unital category with one object $* \in \mathcal{C}$ and one morphism $f : * \longrightarrow *$ such that $f \circ f = f$. Let $\mathcal{D}$ be the non-unital category with one object $* \in \mathcal{D}$ and two morphisms $g, h : * \longrightarrow *$ such that $g \circ g = g \circ h = h \circ g = g$ and $h \circ h = h$. Let X, Y be the non-unital nerves of $\mathcal{C}$ and $\mathcal{D}$ (see Example 1.4.3), so that X and Y are (levelwise discrete) quasi-unital semiSegal spaces (where the edges corresponding to f and h are the quasi-units). Then there is a map $X \longrightarrow Y$ which sends f to g, even though f is a quasi-unit and g is not.

Example 1.4.14 The semiSegal spaces $\mathrm{semiCob}_n$ constructed in [12] § 2.2 (which model the underlying non-unital ∞-category of the n'th cobordism category) are easily seen to be quasi-unital. Informally speaking, any trivial cobordism from an n-manifold M to itself corresponds to a quasi-unit in $\mathrm{semiCob}_n$.

Remark 1.4.15 If X is a quasi-unital semiSegal space then $\mathrm{Ho}(X)$ acquires a unique structure of a **unital category**. However, this structure is not **natural**: given a map $f : X \longrightarrow Y$ between quasi-unital semiSegal spaces, the induced map $\mathrm{Ho}(f) : \mathrm{Ho}(X) \longrightarrow \mathrm{Ho}(Y)$ need not send identity maps to identity maps, see Warning 1.4.13.

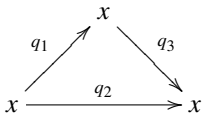
For each $x \in X_0$, we will denote by $X_x^{\mathrm{qu}} \subseteq \mathrm{Map}_X(x,x)$ the maximal subspace spanned by the quasi-units $f \in \left(\mathrm{Map}_X(x,x)\right)_0$. As $[f]_*$ and $[f]^*$ depend only on the path component of f in $\mathrm{Map}_X(x,x)$, we see that X_x^{qu} is a union of connected components of $\mathrm{Map}_X(x,x)$.

Lemma 1.4.16 *Let X be a semiSegal space and $x \in X_0$ a point. If X_x^{qu} is not empty then it is connected.*

Proof Let $q_1, q_2 : x \longrightarrow x$ be two quasi-units. We need to show that q_1, q_2 are in the same connected component of X_x^{qu}. Since X_x^{qu} is a union of components of $\mathrm{Map}_X(x,x)$ it is enough to show that q_1, q_2 are in the same connected component of $\mathrm{Map}_X(x,x)$. Now since q_1 is a quasi-unit the homotopy class

$$[q_1]^* : \mathrm{Map}_X(x,x) \longrightarrow \mathrm{Map}_X(x,x)$$

is the identity. Unwinding the definitions, this implies that there exists a triangle $\sigma \in X_2$ of the form



for some $q_3 : x \longrightarrow x$ which is in the same connected component in $\mathrm{Map}_X(x,x)$ as q_2. Then q_3 is necessarily a quasi-unit, and so the homotopy class

$$[q_3]_* : \mathrm{Map}_X(x,x) \longrightarrow \mathrm{Map}_X(x,x)$$

is the identity. This implies q_1 and q_2 are in the same connected component of $\mathrm{Map}_X(x,x)$. □

1.5 From quasi-units to invertible edges

As explained above, a general map of quasi-unital semiSegal spaces need not preserve quasi-units. However, our interest in this paper is to study quasi-unital ∞-categories and only functors which respect quasi-units. With this in mind, it will be useful to weaken the definition of quasi-units and consider the more robust notion of **invertible edges**:

Definition 1.5.1 Let $x, y \in X_0$ be two objects and $f : x \longrightarrow y$ a morphism in X. We will say that f is **invertible** if for every $z \in X_0$ the homotopy classes

$$[f]_* \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}\left(\mathrm{Map}_X(z,x), \mathrm{Map}_X(z,y)\right)$$

and

$$[f]^* \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}\left(\mathrm{Map}_X(y,z), \mathrm{Map}_X(x,z)\right)$$

of Definition 1.4.7 are **isomorphisms** in $\mathrm{Ho}(\mathcal{S})$.

Remark 1.5.2 Let $f : x \longrightarrow y$ be a map and let $z \in X_0$ be a point. Recall the maps φ, ψ of diagram 1–2. Since ψ is a Kan fibration we see that $[f]_* = [\psi] \circ [\varphi]^{-1}$ is an isomorphism if and only if ψ is a trivial fibration. We hence conclude the following: $[f]_*$ is an isomorphism for every $z \in X_0$ if and only if each map of the form $\sigma : \Lambda^2_2 \longrightarrow X$ such that $\sigma\left(\Delta^{\{1,2\}}\right) = f$ has a contractible space of extensions $\overline{\sigma} : \Delta^2 \longrightarrow X$. A completely analogous argument shows that $[f]^*$ is an isomorphism for every z if and only if each map of the form $\sigma : \Lambda^2_0 \longrightarrow X$ such that $\sigma\left(\Delta^{\{0,1\}}\right) = f$ has a contractible space of extensions $\overline{\sigma} : \Delta^2 \longrightarrow X$.

Invertible morphisms can be described informally as morphisms composition with which (on either side) induces a weak equivalence on mapping spaces. Note that the notion of invertibility does not presuppose the existence of identity morphisms, i.e., it makes sense in the non-unital setting as well.

We will denote by $X_1^{\mathrm{inv}} \subseteq X_1$ the maximal subspace spanned by the invertible vertices $f \in (X_1)_0$. Observe that since X is Reedy fibrant the two restriction maps

$$\mathrm{Map}\left(\Delta^2, X\right) \longrightarrow \mathrm{Map}\left(\Lambda^2_2, X\right) \longrightarrow \mathrm{Map}\left(\Delta^{\{1,2\}}, X\right)$$

are fibrations. It is then an immediate corollary of Remark 1.5.2 that the subspace of X_1 spanned by those f's for which $[f]_*$ is an isomorphism (for every $z \in X_0$) is a union of connected components. Applying the analogous argument to $[f]^*$ we see that $X_1^{\mathrm{inv}} \subseteq X_1$ is a union of connected components.

Given a morphism $[f] \in \mathrm{Map}_{\mathrm{Ho}(X)}(x,y) = \pi_0(\mathrm{Map}_X(x,y))$ in the homotopy (non-unital) category of X, we will say that $[f]$ is invertible if the image of $[f]$ in $\pi_0(X_1)$ belongs to $\pi_0(X^{\mathrm{inv}})$.

Definition 1.5.3 Let X be a semiSegal space. We will say that X is a **semiKan space** if every edge in X_1 is invertible.

We frame the following lemma for future use:

Lemma 1.5.4 *Let X be a semiSegal space. The collection of invertible morphisms in* $\mathrm{Ho}(X)$ *satisfies the following closure properties:*

(1) (2-out-of-3) If two of $[f],[g]$ and $[f \circ g]$ are invertible then the third is invertible as well.

(2) (2-out-of-6) If $[f] \circ [g]$ is invertible and $[g] \circ [h]$ is invertible then $[f],[g]$ and $[h]$ are invertible.

Proof This follows from the fact that invertible edges are created by the collection of functors

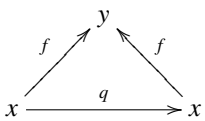
$$R_z, R^z : \mathrm{Ho}(X) \longrightarrow \mathrm{Ho}(\mathcal{S})$$

for all $z \in X_0$ (see Remark 1.4.9) and the class of isomorphisms in $\mathrm{Ho}(\mathcal{S})$ satisfies 2-out-of-3 and 2-out-of-6. □

Our next goal is to verify that for the purpose of studying quasi-unital ∞-categories one can replace the notion of quasi-units with that of invertible edges. We begin with the following observation:

Lemma 1.5.5 *Let X be a semiSegal space and $x \in X_0$ a point. Then x admits a quasi-unit if and only if there exists an invertible edge with source x.*

Proof If x has a quasi-unit then this quasi-unit is in particular an invertible edge with source x. On the other hand, if $f : x \longrightarrow y$ is an invertible edge then according to Remark 1.5.2 there exist a triangle $\sigma : \Delta^2 \longrightarrow X$ of the form



In light of Remark 1.4.9 we conclude that for every $z \in X_0$ the equality

$$[f]_* = [f]_* \circ [q]_*$$

holds in $\mathrm{Hom}_{\mathrm{Ho}(\mathbb{S})}(\mathrm{Map}_X(z,x),\mathrm{Map}_X(z,y))$ and the equality

$$[f]^* = [q]^* \circ [f]^*$$

holds in $\mathrm{Hom}_{\mathrm{Ho}(\mathbb{S})}(\mathrm{Map}_X(y,z),\mathrm{Map}_X(x,z))$. Since f is invertible this implies that q is a quasi-unit. □

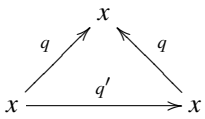
This means that the existence condition for quasi-units can be phrased equivalently in terms of invertible edges. Our next proposition verifies that the associated restrictions on functors are equivalent as well:

Proposition 1.5.6 *Let $\varphi : X \longrightarrow Y$ be a map between quasi-unital semiSegal spaces. The following are equivalent:*

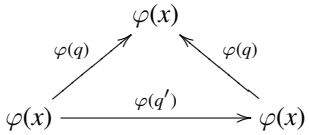
(1) φ sends quasi-units to quasi-units.

(2) φ sends invertible edges to invertible edges.

Proof First assume that φ sends invertible edges to invertible edges and let $x \in X_0$ a point. Since X is quasi-unital there exists a quasi-unit $q : x \longrightarrow x$. Then q is invertible, and hence according to Remark 1.5.2 there exists a triangle of the form



Arguing as in the proof of Lemma 1.5.5 we see that q' is a quasi-unit as well. The map φ then sends this triangle to a triangle of the form



where $\varphi(q)$ is invertible, and hence $\varphi(q')$ is a quasi-unit by the same argument. From Lemma 1.4.16 we get that φ maps all quasi-units of x to quasi-units of $\varphi(x)$.

Now assume that φ sends quasi-units to quasi-units and let $g : x \longrightarrow y$ be an invertible edge. Since X is quasi-unital there exist quasi-units $q : x \longrightarrow x$ and $r : y \longrightarrow y$. Since g is invertible we get from Remark 1.5.2 that there exist triangles of the form

(1–3)

Applying φ to these triangles and using the fact that $\varphi(q), \varphi(r)$ are quasi-units we get from Lemma 1.5.4 (2) that $\varphi(g)$ is invertible. This finishes the proof of Proposition 1.5.6. □

Lemma 1.5.5 and Proposition 1.5.6 suggest that the notion of a quasi-unital ∞-category can be encoded as semiSegal spaces in which every object admits an invertible edge out of it. Furthermore, in order to consider only functors which respect quasi-units one can instead study maps of semiSegal space which preserve invertible edges.

At this point it is worthwhile to consider the particular case of **quasi-unital semiKan spaces** (see Definition 1.5.3). In this case every map automatically respects invertible edges (and hence quasi-units) and so we can study it without any additional technicality. This analysis, which will be explained in the next subsection, will be used in subsequent parts in order to establish the main results of this paper.

1.6 Quasi-unital semiKan spaces and their classifying spaces

Recall that a semiSegal space is called a **semiKan space** if every morphism in X is invertible. In this subsection we will study quasi-unital semiKan spaces via the **geometric realization functor**

$$|\bullet| : \mathcal{S}^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}$$

given by

$$|X| = \int^{\Delta} X \times \Delta^n \cong \mathrm{diag}(X)$$

where $\mathrm{diag}(X)$ is the diagonal simplicial set of X (when X is considered as a bisimplicial set). When X is a quasi-unital semiKan space we will also refer to $|X|$ as the **classifying space of** X.

We will say that X is **connected** if for each $x, y \in X_0$ the mapping space $\mathrm{Map}_X(x,y)$ is **non-empty**. Every quasi-unital semiKan space is a coproduct (in the category of simplicial spaces) of connected quasi-unital semiKan spaces. We then observe that the geometric realization functor commutes with coproducts (in fact with all colimits) and sends connected quasi-unital semiKan spaces to connected simplicial sets. We will consequently refer to $\pi_0(|X|)$ as the **set of connected components of** X.

Definition 1.6.1 We will say that a map $f : X \longrightarrow Y$ of quasi-unital semiKan spaces is a **DK-equivalence** if it induces a weak equivalence

$$\mathrm{Map}_X(x,y) \xrightarrow{\simeq} \mathrm{Map}_Y(f_0(x), f_0(y))$$

for each $x, y \in X_0$ and a surjective map on connected components.

The functor $|\bullet|$ has a right adjoint $\Pi : \mathcal{S} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}}$ given by

$$\Pi(Z)_n = \mathrm{Map}_{\mathcal{S}}(|\Delta^n|, Z)$$

for $Z \in \mathcal{S}$. When Z is a Kan simplicial set, the semi-simplicial space $\Pi(Z)$ is Reedy fibrant, and one can easily verify that it is a quasi-unital semiKan space. Furthermore, the semiSegal space $\Pi(Z)$ clearly extends to a full **Segal space** (given by the same formula), which is a model for

the fundamental ∞-gropoid of Z. In this section we will show that every quasi-unital semiKan X space is DK-equivalent to $\Pi\left(\widehat{|X|}\right)$ where $\widehat{\bullet}$ denotes the functorial Kan replacement functor (see Corollary 1.6.6 below). In particular, every quasi-unital ∞-groupoid can be promoted to a unital one, which is given explicitly as the fundamental ∞-groupoid of its classifying space. This is a first step in proving the equivalence of quasi-unital and unital ∞-categories, and will be used in subsequent parts in order to establish the main results of this paper.

We begin with the following lemmas:

Lemma 1.6.2 *Let X be a semiSegal space. Let $T_n \subseteq \Delta^n$ be the simplicial subset given by the union of all triangles $\Delta^{\{0,i,i+1\}}$ for $i = 1, ..., n-1$. Then the restriction map*

$$X_n = \mathrm{Map}(\Delta^n, X) \longrightarrow \mathrm{Map}\,(T_n, X) \tag{1–4}$$

is a trivial Kan fibration.

Proof Since X is Reedy fibrant we already know that 1–5 is a fibration, and so it is left to prove that it is a weak equivalence. The simplicial subset T_n contains the spine $\mathrm{Sp}^n \subseteq \Delta^n$ (see § 1.3) and may be obtained from Sp^n by performing pushouts along inclusions of the form $\Lambda^2_1 \subseteq \Delta^2$. From the Segal condition we may then conclude that the restriction map

$$\mathrm{Map}\,(T_n, X) \longrightarrow \mathrm{Map}\left(\mathrm{Sp}^n, X\right)$$

is a weak equivalence. Furthermore, the restriction map $X_n \longrightarrow \mathrm{Map}\left(\mathrm{Sp}^n, X\right)$ is a weak equivalence as well (see Remark 1.4.2) and so the map $X_n \longrightarrow \mathrm{Map}\,(T_n, X)$ is a weak equivalence. □

Lemma 1.6.3 *Let X be a semiKan space. For $0 < i \leq n$ let $S^i_n \subseteq \Delta^n$ be the simplicial subset given by the union of the edge $\Delta^{\{0,i\}}$ and the edges $\Delta^{\{j,j+1\}}$ for $j = 1, ..., n-1$. Then the restriction map*

$$X_n = \mathrm{Map}(\Delta^n, X) \longrightarrow \mathrm{Map}\left(S^i_n, X\right) \tag{1–5}$$

is a trivial Kan fibration.

Proof Since X is Reedy fibrant we already know that 1–5 is a fibration, and so it is left to prove that it is a weak equivalence. Let $T_n \subseteq \Delta^n$ be as in Lemma 1.6.2. Then T_n contains S^i_n and may be obtained from S^i_n by performing pushouts alongs inclusions of the form $\Lambda^2_1 \subseteq \Delta^2$ and $\Lambda^2_2 \subseteq \Delta^2$. In light of Remark 1.5.2 we may conclude that the restriction map

$$\mathrm{Map}\,(T_n, X) \longrightarrow \mathrm{Map}\left(S^i_n, X\right)$$

is a weak equivalence. The desired result now follows from Lemma 1.6.2. □

Given a Kan simplicial set Z and points $x, y \in Z$ we will denote by $\Omega(Z, x, y)$ the space of paths in Z from x to y and by $P(Z, x)$ the space of paths in Z which start at x.

Theorem 1.6.4 *Let X be a semiKan space and $x, y \in X_0$ two points. Consider x and y as points in $\widehat{|X|}$ via the natural inclusion $X_0 \hookrightarrow \widehat{|X|}$. Then the natural map*

$$\mathrm{Map}_X(x,y) \longrightarrow \Omega\left(\widehat{|X|}, x, y\right)$$

is a weak equivalence.

Proof We will rely on the main result of [13] which can be stated as follows:

Theorem 1.6.5 (Puppe) *Let X, Y be two semi-simplicial spaces and let $\varphi : X \longrightarrow Y$ be a map such that for each $f : [k] \longrightarrow [n]$ in Δ_s the square*

$$\begin{array}{ccc} X_n & \xrightarrow{f^*} & X_k \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_k} \\ Y_n & \xrightarrow{f^*} & Y_k \end{array}$$

is homotopy Cartesian. Then the square

$$\begin{array}{ccc} X_0 & \longrightarrow & |X| \\ \downarrow & & \downarrow \\ Y_0 & \longrightarrow & |Y| \end{array}$$

is homotopy Cartesian as well.

Let us now prove Theorem 1.6.4. Let $x \in X_0$ be a point and define the semi-simplicial space $P(X,x)$ by:

$$P(X,x)_n = \{\sigma \in X_{n+1} \mid \sigma|_{\Delta^{\{0\}}} = x\} \subseteq X_{n+1}.$$

Fix an $i = 1, ..., n+1$ and let $S^i_{n+1} \subseteq \Delta^{n+1}$ be as in Lemma 1.6.3. Let $Q_n \subseteq \mathrm{Map}\left(S^i_{n+1}, X\right)$ be the subspace of those maps $f : S^i_{n+1} \longrightarrow X$ which send $\Delta^{\{0\}}$ to x. In particular, the natural map $P(X,x)_n \longrightarrow Q_n$ is a pullback of the restriction map $X_{n+1} \longrightarrow \mathrm{Map}\left(S^i_{n+1}, X\right)$ which is a trivial Kan fibration by Lemma 1.6.3. We hence get that $P(X,x)_n \longrightarrow Q_n$ is a trivial Kan fibration as well.

Now consider the map $p : P(X,x) \longrightarrow X$ which sends $\sigma \in P(X,x)_n$ to $\sigma|_{\Delta^{\{1,\ldots,n+1\}}} \in X_n$. Let $q_n : Q_n \longrightarrow \mathrm{Map}\left(\mathrm{Sp}^n, X\right)$ be the map induced by the restriction along the map $\iota : \mathrm{Sp}^n \hookrightarrow S^i_{n+1}$ which sends the vertices $0, ..., n$ of Sp^n to the vertices $1, ..., n+1$ of S^i_{n+1}. Then for each $i = 1, ..., n+1$ we obtain a commutative diagram of the form

(1–6)
$$\begin{array}{ccccc} P(X,x)_n & \xrightarrow{\simeq} & Q_n & \xrightarrow{r_i} & P(X,x)_0 \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle p_0} \\ X_n & \xrightarrow{\simeq} & \mathrm{Map}(\mathrm{Sp}^n, X) & \xrightarrow{s} & X_0 \end{array}$$

where r_i is induced by restriction along $\Delta^{\{0,i\}} \subseteq S^i_{n+1}$ and s is the restriction along $\Delta^{\{i-1\}} \subseteq \mathrm{Sp}^n$. In particular, the horizontal maps of the external rectangle are the simplicial structure maps of the inclusion $\Delta^{\{i-1\}} \subseteq \Delta^n$. Since the right square is homotopy Cartesian we conclude that the external rectangle is homotopy Cartesian as well. This, in turn, implies that for every map $\rho : [k] \longrightarrow [n]$ in Δ_s the square

$$\begin{array}{ccc} P(X,x)_n & \xrightarrow{f^*} & P(X,x)_k \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_k} \\ X_n & \xrightarrow{f^*} & X_k \end{array}$$

is homotopy Cartesian. Hence by Puppe's theorem the left square in the diagram

$$\begin{array}{ccccc} P(X,x)_0 & \longrightarrow & |P(X,x)| & \longrightarrow & P\left(\widehat{|X|}, x\right) \\ \downarrow{\scriptstyle p_0} & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ X_0 & \longrightarrow & |X| & = & |X| \end{array}$$

is a homotopy Cartesian (where ev_1 is the function which associates to a path γ its value $\gamma(1)$). Now note that p_0 and ev_1 are both fibrations. Identifying the fibers of these fibrations we see that the desired result is equivalent to the exterior rectangle being homotopy Cartesian, or equivalently, that the right square is homotopy Cartesian. As $P\left(\widehat{|X|}, x\right)$ is contractible, in order to finish the proof of Theorem 1.6.4 it will suffice to show that the space $|P(X,x)|$ is contractible.

For $n \geq 0$ let $T_{n+1} \subseteq \Delta^{n+1}$ be as in Lemma 1.6.2. Let $R_{n+1} \subseteq T_{n+1}$ be the simplicial subset given by the union of all edges of the form $\Delta^{\{0,i\}}$ for $i = 1, \ldots, n+1$. Then T_{n+1} may be obtained from R_{n+1} by performing pushouts alongs inclusions of the form $\Lambda^2_0 \subseteq \Delta^2$. From Remark 1.5.2 we then conclude that the restriction map

$$\mathrm{Map}\left(T_{n+1}, X\right) \longrightarrow \mathrm{Map}\left(R_{n+1}, X\right)$$

is a trivial Kan fibration, and hence by Lemma 1.6.2 the map

$$X_{n+1} \longrightarrow \mathrm{Map}\left(R_{n+1}, X\right)$$

is a trivial Kan fibration as well. Since the square

$$\begin{array}{ccc} P(X,x)_n & \hookrightarrow & X_{n+1} \\ \downarrow & & \downarrow \\ (P(X,x)_0)^{n+1} & \longrightarrow & \mathrm{Map}\left(R_{n+1}, X\right) \end{array}$$

is Cartesian we see that the map $P(X,x)_n \longrightarrow (P(X,x)_0)^{n+1}$ is a trivial Kan fibration. This means that the natural map (see Example 1.4.5)

$$P(X,x) \longrightarrow \mathrm{cosk}_0(P(X,x)_0)$$

is a levelwise weak equivalence. It will hence suffice to show that $|\mathrm{cosk}_0(P(X,x)_0)|$ is contractible. This, in turn, is due to the fact that any semi-simplicial space of the form $\mathrm{cosk}_0(Z)$ for $Z \neq \emptyset$ admits a canonical semi-simplicial null-homotopy $\Delta^1 \otimes \mathrm{cosk}_0(Z) \longrightarrow \mathrm{cosk}_0(Z)$. □

Corollary 1.6.6 *Let X be a semiKan space. Then the counit map $X \longrightarrow \Pi\left(\widehat{|X|}\right)$ is a DK-equivalence.*

Proof By Theorem 1.6.4 the counit map induces weak equivalences on mapping spaces. Since the map $X_0 \longrightarrow \widehat{|X|}$ is surjective on connected components we see that the map f is in fact a DK-equivalence. □

Corollary 1.6.7 *Let $f : X \longrightarrow Y$ be a map between quasi-unital semiKan spaces. Then f is a DK-equivalence if and only if the induced map $f_* : \widehat{|X|} \longrightarrow \widehat{|Y|}$ is a weak equivalence.*

Proof First note that the connected components of X as a semiKan space are in bijection with the connected components of $\widehat{|X|}$ as a space. Hence Theorem 1.6.4 tells us that $f : X \longrightarrow Y$ is a DK-equivalence if and only if it induces a bijection

$$\pi_0\left(\widehat{|X|}\right) \longrightarrow \pi_0\left(\widehat{|Y|}\right)$$

and for each $x, y \in \widehat{|X|}$ the induced map

$$\Omega\left(\widehat{|X|}, x, y\right) \longrightarrow \Omega\left(\widehat{|Y|}, f(x), f(y)\right)$$

is a weak equivalence. But this is equivalent to $f_* : \widehat{|X|} \longrightarrow \widehat{|Y|}$ being a weak equivalence and we are done. □

We finish this subsection with an application which we record for future use. Recall that in general geometric realization does not commute with **Cartesian products** of semi-simplicial spaces (i.e., levelwise products). The following corollary shows that in the specific case of semiKan spaces, geometric realization **does** commute with Cartesian products:

Corollary 1.6.8 *Let X, Y be two quasi-unital semiKan spaces. Then the natural map $|X \times Y| \longrightarrow |X| \times |Y|$ is a weak equivalence.*

Proof First note that if X, Y are semiKan spaces then $X \times Y$ is a semiKan space as well. Furthermore, it is clear that the natural map

$$\pi_0(|X \times Y|) \longrightarrow \pi_0(|X|) \times \pi_0(|Y|)$$

is an isomorphism (as both sides can be identified with the set of connected components of the semiKan space $X \times Y$).

Now let $x_1, x_2 \in X, y_1, y_2 \in Y$ be points and consider the natural map

$$\Omega\left(|X \times Y|, (x_1, y_1), (x_2, y_2)\right) \longrightarrow \Omega\left(|X| \times |Y|, (x_1, y_1), (x_2, y_2)\right).$$

Theorem 1.6.4 shows that this map is weakly equivalent to the isomorphism

$$\mathrm{Map}_{X \times Y}((x,y),(x,y)) \xrightarrow{\simeq} \mathrm{Map}_X(x,x) \times \mathrm{Map}_Y(y,y)$$

and so is itself a weak equivalence. The desired result now follows. □

1.7 Marked semi-simplicial spaces

The purpose of this paper is to generalize the analysis made in the previous section in order to obtain our desired comparison between quasi-unital and unital ∞-categories. For this we will first need to construct the correct ∞-category of quasi-unital ∞-categories. This will require some modifications to the constructions reviewed so far in order to guarantee that we consider only functors which send invertible edges to invertible edges.

In order to keep track of invertible edges it will be useful to work in a variant of the category of semiSegal spaces where the invertible edges can be somehow **marked**. For this one needs to replace the notion of a semi-simplicial space with that of a **marked semi-simplicial space**. Let us open with the basic definition:

Definition 1.7.1 A **marked semi-simplicial space** is a pair (X, A) where X is a semi-simplicial space and $A \subseteq X_1$ is a subspace. In order to keep the notation clean we will often denote a marked semi-simplicial space (X, A) simply by X. Given two marked semi-simplicial spaces $(X, A), (Y, B)$ we denote by

$$\mathrm{Map}^{+}(X, Y) \subseteq \mathrm{Map}(X, Y)$$

the subspace of maps which send A to B. We will refer to this kind of maps as **marked** maps. We denote by $\mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}$ the $\mathcal{S}$-enriched category of marked semi-simplicial spaces and marked maps between them.

Remark 1.7.2 The analogous notion of marked simplicial sets plays an essential role in the theory of ∞-categories as developed in [11]. Our definition above as well as many of the associated notations follow their analogues in [11].

Definition 1.7.3 Given a semi-simplicial space X we will denote by $X^{\sharp}$ the marked semi-simplicial space (X, X_1) in which **all** edges are marked. The association $X \mapsto X^{\sharp}$ is right adjoint to the forgetful functor $(X, A) \mapsto X$.

Definition 1.7.4 Given a semi-simplicial space X we will denote by $X^{\flat}$ the marked semi-simplicial space $(X, \emptyset)$ in which **no** edges are marked. The association $X \mapsto X^{\flat}$ is left adjoint to the forgetful functor $(X, A) \mapsto X$.

Definition 1.7.5 Let (X, A) be a marked semi-simplicial space. We will denote by $\overline{A} \subseteq \pi_0(X_1)$ the **image** of the map

$$\pi_0(A) \longrightarrow \pi_0(X),$$

i.e., the set of connected components of X_1 which meet A. We refer to $\overline{A}$ as the set of **marked connected components** of X_1.

Definition 1.7.6 We will say that a map $f : (X, A) \longrightarrow (Y, B)$ of marked semi-simplicial spaces is a **marked equivalence** if

(1) The underlying map $f : X \longrightarrow Y$ is a levelwise equivalence.

(2) The induced map $f_* : \overline{A} \longrightarrow \overline{B}$ is an isomorphism of sets.

Theorem 1.7.7 *There exists a left proper combinatorial model category structure on $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ such that*

(1) The weak equivalences are the marked equivalences.

(2) The cofibrations are the maps $f : (X,A) \longrightarrow (Y,B)$ for which the underlying map $X \longrightarrow Y$ is a cofibration (i.e., levelwise injective).

(3) A map is a fibration if and only if it satisfies the right lifting property with respect to all morphisms which are both cofibrations and weak equivalences.

Proof We will use a general existence theorem which is a slightly weaker version of Proposition A.2.6.13 of [11] (which in turn is based on work of Smith). In the following the term **presentable** is used as in [11] (which in classical terminology is often called **locally presentable**).

Theorem 1.7.8 (Lurie, Smith) *Let $\mathcal{M}$ be a presentable category. Let C, W be two classes of morphisms in $\mathcal{M}$ such that*

(1) C is weakly saturated and is generated (as a weakly saturated class of morphisms) by a **set** *of morphisms C_0.*

(2) W is **perfect** *(see Definition A.2.6.10 of [11]).*

(3) W is stable under pushouts along C, i.e., if

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Z & \longrightarrow & W \end{array}$$

is a pushout square such that $f \in C$ and $g \in W$ then $g' \in W$ as well.

(4) If a morphism f in $\mathcal{M}$ has the right lifting property with respect to every morphism in C (or equivalently in C_0) then $f \in W$.

Then there exists a left proper combinatorial model structure on $\mathcal{M}$ such that the weak equivalences are W and the cofibrations are C.

Let us first prove that $\mathcal{S}_+^{\Delta^{\mathrm{op}}}$ is presentable. Consider the category $\mathcal{D}$ whose objects are pairs (X,A,f) where X is a semi-simplicial space, A is a space and $f : A \longrightarrow X_1$ is a map. Then $\mathcal{D}$ can be identified with a suitable category of pre-sheaves (of sets), and is hence presentable. Furthermore, there is an evident functor $\mathcal{R} : \mathcal{S}_+^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{D}$ sending (X,A) to (X,A,ι) where $\iota : A \hookrightarrow X_1$ denotes the inclusion. The functor $\mathcal{F}$ is clearly fully-faithful, and admits a left adjoint $\mathcal{L} : \mathcal{D} \longrightarrow \mathcal{S}_+^{\Delta^{\mathrm{op}}}$ sending (X,A,f) to $(X, \mathrm{Im}(f))$ where $\mathrm{Im}(f) \subseteq X_1$ denotes the image of f. Finally, since filtered colimits preserve injective maps of sets we see that $\mathcal{R}$ preserves filtered colimits. It follows that $\mathcal{S}_+^{\Delta^{\mathrm{op}}}$ is an accessible localization of $\mathcal{D}$ and is hence presentable.

Now let W be the class of marked equivalences and C the class of marked maps which are levelwise injective. We need to show that the classes (W, C) meet the requirements of Theorem 1.7.8. We start by finding a set of morphisms which generates C as a weakly saturated class.

Let C_0 be the **set** containing all the morphisms

$$\left[|\partial\Delta^k| \otimes \left(\Delta^n\right)^\flat\right] \coprod_{|\partial\Delta^k|\otimes(\partial\Delta^n)^\flat} \left[|\Delta^k| \otimes \left(\partial\Delta^n\right)^\flat\right] \hookrightarrow |\Delta^k| \otimes (\Delta^n)^\flat \tag{1–7}$$

and all the morphisms

$$\left[|\partial\Delta^k| \otimes \left(\Delta^1\right)^\sharp\right] \coprod_{|\partial\Delta^k|\otimes(\Delta^1)^\flat} \left[|\Delta^k| \otimes \left(\Delta^1\right)^\flat\right] \hookrightarrow |\Delta^k| \otimes \left(\Delta^1\right)^\sharp. \tag{1–8}$$

Applying the standard theory of Reedy categories to the category Δ_s we observe that the morphisms of type 1–7 generate in particular all maps of the form $X^\flat \hookrightarrow Y^\flat$ where $X \hookrightarrow Y$ is a cofibration in $\mathcal{S}^{\Delta_s}$, and hence all injective maps of the form $(X, A) \hookrightarrow (Y, A)$ where the marked subspace does not change. Furthermore, it is clear that the morphisms of type 1–8 generate in particular all morphisms of the form $(X, A_1) \hookrightarrow (X, A_2)$ where the underlying semi-simplicial space does not change. Since any injective map is a composition of these two types of maps it follows that C_0 generates C.

We will now show that the pair (W, C) satisfies the assumptions 2 and 3 of Theorem 1.7.8. Consider the category Set with its trivial model structure (i.e., the weak equivalences are the isomorphisms and all maps are fibrations and cofibrations). We endow $\mathcal{S}^{\Delta_s^{\mathrm{op}}} \times \mathrm{Set}$ with the **product model structure** (i.e., weak equivalences, fibrations and cofibrations are defined coordinate-wise, where on the left we use the Reedy model structure). Let W', C' be the classes of weak equivalences and cofibrations in $\mathcal{S}^{\Delta_s^{\mathrm{op}}} \times \mathrm{Set}$ respectively.

Since both $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ and Set are left proper combinatorial model categories it follows that $\mathcal{S}^{\Delta_s^{\mathrm{op}}} \times \mathrm{Set}$ is a left proper combinatorial model category. This means that W' is stable under pushouts along C' and that W' is perfect (this is part of Smith's theory of combinatorial model categories, cited for example in [11] A.2.6.6).

Now let $F : \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}} \times \mathrm{Set}$ be the functor given by $F(X, A) = (X, \overline{A})$. Then it is clear that F preserves colimits. Since $W = F^{-1}(W')$ and $C = F^{-1}(C')$ we get that W is stable under pushouts along C and that W is perfect (see [11] A.2.6.12). It is then left to check the last assumption of Theorem 1.7.8.

Let $f : (X, A) \longrightarrow (Y, B)$ be a morphism which has the right lifting property with respect to all maps in C_0. Since C_0 contains all maps of the form

$$\left[|\partial\Delta^k| \otimes \left(\Delta^n\right)^\flat\right] \coprod_{|\partial\Delta^k|\otimes\partial(\Delta^n)^\flat} \left[|\Delta^k| \otimes \left(\partial\Delta^n\right)^\flat\right] \hookrightarrow |\Delta^k| \otimes (\Delta^n)^\flat$$

it follows that f is a levelwise equivalence. It is left to show that f induces an isomorphism $\overline{A} \longrightarrow \overline{B}$. Note that since f is a levelwise equivalence it induces an isomorphism $\pi_0(X_1) \longrightarrow$

$\pi_0(Y_1)$ and so the map $\overline{A} \longrightarrow \overline{B}$ is injective. The fact that it is surjective follows from having the right lifting property with respect to $\left(\Delta^1\right)^\flat \hookrightarrow \left(\Delta^1\right)^\sharp$ which is one of the maps in C_0. This completes the proof of Theorem 1.7.7. □

Definition 1.7.9 We will use the terms **marked fibrations** and **marked cofibrations** to denote fibrations and cofibrations in the marked model structure. We will use the term **marked-fibrant semi-simplicial spaces** to denote fibrant objects in the marked model structure.

Remark 1.7.10 The forget-the-marking functor $\overline{\bullet} : \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is both a left and a right Quillen functor. As mentioned above, it has a right adjoint $X \mapsto X^\sharp$ and a left adjoint $X \mapsto X^\flat$. Furthermore, it is easy to verify that both the forgetful functor and its left adjoint preserve cofibrations and weak equivalences.

Lemma 1.7.11 *A marked semi-simplicial space (X, A) is* **marked-fibrant** *if and only if*

(1) X is Reedy fibrant.

(2) A is a union of connected components of X.

Proof Let (X, A) be a marked-fibrant object. From Remark 1.7.10 we see that X is Reedy fibrant. Now consider the maps

$$\left[|\Lambda_i^k| \otimes \left(\Delta^1\right)^\sharp\right] \coprod_{|\Lambda_i^k| \otimes (\Delta^1)^\flat} \left[|\Delta^k| \otimes \left(\Delta^1\right)^\flat\right] \hookrightarrow |\Delta^k| \otimes \left(\Delta^1\right)^\sharp$$

for $k \geq 1$ and $0 \leq i \leq k$. By definition we see that these maps are trivial marked cofibrations. Since (X, A) is Reedy fibrant it satisfies the right lifting property with respect to such maps, which in turn means that the inclusion $A \hookrightarrow X_1$ satisfies the right lifting property with respect to the inclusion of spaces $|\Lambda_i^k| \hookrightarrow |\Delta^k|$ for $k \geq 1$. This means that the inclusion $A \hookrightarrow X_1$ is Kan fibration and hence a union of components of X_1.

In the other direction assume that X is Reedy fibrant and $A \subseteq X_1$ is a union of components. Consider an extension problem

$$\begin{array}{ccc} (Y, B) & \xrightarrow{f} & (X, A) \\ \downarrow & & \\ (Z, C) & & \end{array}$$

such that $(Y, B) \hookrightarrow (Z, C)$ is a trivial marked cofibration. In this case $Y \hookrightarrow Z$ will be a trivial Reedy cofibration and so there will exist an extension $\overline{f} : Z \longrightarrow X$ in the category of semi-simplicial spaces. We claim that $\overline{f}$ will necessarily send C to A. In fact, let $W \subseteq Z_1$ be a connected component which meets C. Since $(Y, B) \hookrightarrow (Z, C)$ is a marked equivalences it follows that W also meets the image of B. Since A is a union of components of X_1 we get that $\overline{f}$ sends all of W to A. This means that $\overline{f}$ sends C to A and we are done. □

Corollary 1.7.12 *A map $f : X \longrightarrow Y$ between marked-fibrant semi-simplicial spaces is a marked equivalence if and only if it is a levelwise equivalence which induces a weak equivalence on the corresponding spaces of marked edges.*

We shall now show that $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ can be endowed with a structure of symmetric monoidal simplicial model category. Let $(X,A),(Y,B)$ be two marked semi-simplicial spaces. According to Remark 1.3.1 one has

$$(X \otimes Y)_1 = (X_1 \times Y_0) \coprod (X_0 \times Y_1) \coprod (X_1 \times Y_1) .$$

We will extend the monoidal product $\otimes$ to marked semi-simplicial spaces by defining $(X,A) \otimes (Y,B)$ to be the marked semi-simplicial space $(X \otimes Y, C)$ where the marking C is given by

$$C = (A \times Y_0) \coprod (X_0 \times B) \coprod (A \times B) \subseteq (X \otimes Y)_1 .$$

We shall abuse notation and denote the resulting monoidal product on $\mathcal{S}_+^{\Delta_s}$ by $\otimes$ as well. This can be justified by noting that the product of two marked semi-simplicial spaces with empty markings again has an empty marking. Furthermore, the unit of the monoidal product $\otimes$ is $\left(\Delta^0\right)^\flat$, which has an empty marking. We can hence consider the definition above as an extension of the monoidal product $\otimes$ from the full subcategory of marked semi-simplicial spaces with empty markings to all of $\mathcal{S}_+^{\Delta_s}$. This can be encoded by saying that the functor $\flat : \mathcal{S}^{\Delta_s} \longrightarrow \mathcal{S}_+^{\Delta_s}$ has a natural colax structure

$$(X \otimes Y)^\flat \xrightarrow{\cong} X^\flat \otimes Y^\flat$$
$$\left(\Delta^0\right)^\flat \xrightarrow{\cong} \left(\Delta^0\right)^\flat$$

making it into a **monoidal functor**. As explained in Remark 1.1.4, the right adjoint $\overline{\bullet} : \mathcal{S}_+^{\Delta_s} \longrightarrow \mathcal{S}^{\Delta_s}$ (see Remark 1.7.10) then admits a unique compatible lax structure, and the adjunction

$$\mathcal{S}^{\Delta_s^{\mathrm{op}}} \underset{\overline{\bullet}}{\overset{(\bullet)^\flat}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

becomes a strongly monoidal Quillen adjunction (see Definition 1.1.8).

The monoidal product $\otimes$ on $\mathcal{S}_+^{\Delta_s}$ is again **closed** and the corresponding internal mapping object is defined as follows:

Definition 1.7.13 Let X, Y be two marked semi-simplicial spaces. The **marked mapping object** from X to Y is the marked semi-simplicial space $\left(Y^X, H\right)$ given by

$$\left(Y^X\right)_n = \mathrm{Map}^+\left(X \times \left(\Delta^n\right)^\flat, Y\right)$$

where the marking H is given by

$$H = \mathrm{Map}^+\left(X \times \left(\Delta^1\right)^\sharp, Y\right) \subseteq \mathrm{Map}^+\left(X \times \left(\Delta^1\right)^\flat, Y\right) = \left(Y^X\right)_1 .$$

Lemma 1.7.14 *The marked model structure on $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ is* **compatible** *with the symmetric monoidal structure $\otimes$ (see Definition 1.1.5).*

Proof Since the Reedy model structure on $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is compatible with the unmarked version of $\otimes$ we only need to verify the following: if $X' \longrightarrow X$ is a marked cofibration and $Y' \longrightarrow Y$ is a trivial marked cofibration then the map

$$h : \left(X \otimes Y'\right)_1 \coprod_{(X' \otimes Y')_1} \left(X' \otimes Y\right)_1 \longrightarrow (X \otimes Y)_1$$

induces an isomorphism on the set of **marked connected components**. Since this map is already a weak equivalence it is enough to check that it is surjective on marked components. But this is a direct consequence of the fact that the map $Y'_1 \longrightarrow Y_1$ induces an isomorphism on the set of marked connected components. □

Finally, the simplicial structure of $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ can be composed with the strongly monoidal Quillen adjunction $\mathcal{S}^{\Delta_s} \dashv \mathcal{S}_+^{\Delta_s}$ to obtain a simplicial structure on the model category $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$. To conclude $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ is a symmetric monoidal simplicial model category.

We finish this subsection with the following definition which we frame for future use:

Definition 1.7.15 Let W be a marked semi-simplicial space with marking $M \subseteq W_1$. We will denote by $\widetilde{W} \subseteq W$ the marked semi-simplicial space such that

$$\widetilde{W}_n = \{\sigma \in W_n \mid f^*\sigma \in M, \forall f : [1] \longrightarrow [n]\}.$$

In particular, all the edges of $\widetilde{W}$ are marked.

1.8 The marked right Kan extension

The Reedy model structures on simplicial and semi-simplicial spaces may be related via a Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathcal{RK}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

where $\mathcal{F}$ is the forgetful functor and $\mathcal{RK}$ is the right Kan extension. The purpose of this section is to construct an analogous Quillen adjunction between $\mathcal{S}^{\Delta^{\mathrm{op}}}$ and the marked model structure on $\mathcal{S}_+^{\Delta^{\mathrm{op}}}$.

We begin by defining the **marked forgetful functor**

$$\mathcal{F}^+ : \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

as follows: given a simplicial space X we will define $\mathcal{F}^+(X)$ to be the marked semi-simplicial space $(\mathcal{F}(X), D)$ where $D \subseteq X_1$ is the subspace of **degenerate 1-simplices**, i.e., the image of $s_0 : X_0 \longrightarrow X_1$.

Remark 1.8.1 The motivation for this definition comes from the fact that when working with simplicial spaces as models for ∞-categories (via Rezk's model category of complete Segal spaces) the degenerate edges encode the identity morphisms. On the other hand, in the model categories of marked semi-simplicial spaces we shall consider later in the paper, the marked edges are going to be the invertible morphisms. Since identity morphisms are inherently invertible it is natural to mark them when forgetting the degeneracy maps. We will later see that this construction leads to the correct equivalence between unital and quasi-unital ∞-categories.

The functor $\mathcal{F}^+$ preserves all colimits and hence admits a right adjoint

$$\mathcal{RK}^+ : \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}.$$

In order to obtain an explicit formula for $\mathcal{RK}^+$ we will need a bit of terminology.

Definition 1.8.2 Let $f : [m] \longrightarrow [n]$ be a map in Δ. We will say that an edge $e \in (\Delta^m)_1$ is **f-degenerate** if f maps both its vertices to the same element of $[n]$. We will denote by $(\Delta^m)^f$ the marked semi-simplicial space (Δ^m, A_f) where Δ^m is considered as a semi-simplicial space which is levelwise discrete and $A_f \subseteq (\Delta^m)_1$ is the set of f-degenerate edges. Now given a marked semi-simplicial space (X, A) we define

$$X_m^f = \mathrm{Map}^+\left((\Delta^m)^f, (X, A)\right).$$

Note that we have a natural inclusion $X_m^f \subseteq X_m$.

We will now construct the functor $\mathcal{RK}^+$ as follows. For each $[n] \in \Delta$ consider the fiber product category

$$\mathcal{C}_n = \Delta_s^{op} \times_{\Delta^{op}} \Delta_{[n]/}^{op}.$$

The objects of $\mathcal{C}_n$ can be identified with maps $f : [m] \longrightarrow [n]$ in Δ and a morphism from $f : [m] \longrightarrow [n]$ to $g : [k] \longrightarrow [n]$ in $\mathcal{C}_n$ can then be described as a commutative triangle

$$\begin{array}{ccc} [k] & \xrightarrow{\quad h \quad} & [m] \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & [n] & \end{array}$$

such that h is **injective**. Now let (X, A) be a marked semi-simplicial space and let $\mathcal{G}_n : \mathcal{C}_n \longrightarrow \mathcal{S}$ be the functor which associates to each $f : [m] \longrightarrow [n]$ the space $\mathcal{G}_n(f) = X_m^f$. Note that for each map $[n] \longrightarrow [n']$ in Δ one has a functor $\mathcal{F}_n : \mathcal{C}_n \longrightarrow \mathcal{C}_{n'}$ and a natural transformation $\mathcal{F}_n^* \mathcal{G}_{n'} \longrightarrow \mathcal{G}_n$. We can then define $\mathcal{RK}^+(X, A)$ by setting

$$\mathcal{RK}^+(X, A)_n = \lim_{\mathcal{C}_n} \mathcal{G}_n$$

which is functorial in $[n] \in \Delta$.

Remark 1.8.3 The category $\mathcal{C}_n$ carries a Reedy structure which is induced from that of Δ_s. If (X, A) is marked-fibrant then the functor $f \mapsto X_m^f$ will be a Reedy fibrant functor from $\mathcal{C}_n$ to $\mathcal{S}$. This means that in this case the limit above will coincide with the respective **homotopy limit**.

Remark 1.8.4 One has natural maps

$$\mathcal{RK}^+(X,A)_n = \lim_{\mathcal{C}_n} X^f_m \longrightarrow \lim_{\mathcal{C}_n} X_m = \mathcal{RK}(X)_n$$

which assemble together to form a natural transformation

$$\mathcal{RK}^+(X,A) \longrightarrow \mathcal{RK}(X).$$

From Lemma 1.7.11 we see that when (X,A) is marked-fibrant the map above identifies $\mathcal{RK}^+(X,A)_n$ with a union of connected components of $\mathcal{RK}(X)_n$.

Evaluating at the object $\mathrm{Id} : \Delta^n \longrightarrow \Delta^n$ of $\mathcal{C}_n$ we obtain maps

$$\mathcal{RK}^+(X,A)_n = \lim_{\mathcal{C}_n} X^f_m \longrightarrow X^{\mathrm{Id}}_n = X_n$$

which fit together to form a natural map of semi-simplicial spaces

$$\mathcal{F}\left(\mathcal{RK}^+(X,A)\right) \longrightarrow X \tag{1–9}$$

We then observe that composition

$$\mathcal{RK}^+(X,A)_0 \xrightarrow{s^*} \mathcal{RK}^+(X,A)_1 \longrightarrow X_1$$

is given by evaluating the limit $\lim_{\mathcal{C}_0} X^f_m$ at the object of $\mathcal{C}_0$ corresponding to the map $s : [1] \longrightarrow [0]$. By the definition of X^f_m it follows that the map 1–9 sends degenerate edges to marked edges, and so induces a natural map

$$v_{X,A} : \mathcal{F}^+\left(\mathcal{RK}^+(X,A)\right) \longrightarrow (X,A)$$

It is then straightforward to verify that $v_{X,A}$ determines a counit map which exhibits $\mathcal{RK}^+$ as right adjoint to $\mathcal{F}^+$. The resulting adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathcal{RK}^+}{\overset{\mathcal{F}^+}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}.$$

is easily seen to be a Quillen adjunction: since any Reedy cofibration in $\mathcal{S}^{\Delta^{\mathrm{op}}}$ is a levelwise injection it follows that $\mathcal{F}^+$ preserves cofibrations. Furthermore, it is not hard to check that $\mathcal{F}^+$ maps levelwise equivalences to marked equivalences, and hence trivial cofibrations to trivial marked cofibrations.

Now the forgetful functor $\mathcal{F} : \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}}$ factors through $\mathcal{F}^+$. This means that the Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathcal{RK}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

factors through $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ as the composition

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathcal{RK}^+}{\overset{\mathcal{F}^+}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \underset{(\bullet)^\sharp}{\overset{\bar{\bullet}}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

where $\bar{\bullet}$ denotes the forgetful functor $(X,A) \mapsto X$.

2 Marked semiSegal spaces

In his paper [14], Rezk constructs a left Bousfield localization of the Reedy model structure on $\mathcal{S}^{\Delta^{\mathrm{op}}}$ in which the new fibrant objects are the Segal spaces. The purpose of this section is to construct a similar working environment for a suitable notion of a **marked semiSegal spaces**. Let us introduce the basic definitions:

Definition 2.0.5 Let $(W,M) \in \mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}$ be a **marked-fibrant** semi-simplicial space. We will say that (W,M) is a **marked semiSegal space** if the following conditions are satisfied:

(1) W is a semiSegal space.

(2) Every marked edge is invertible, i.e., $M \subseteq W_1^{\mathrm{inv}}$.

(3) M is closed under 2-out-of-3, i.e., if there exists a triangle $\sigma \in W_2$ with two marked edges then the third edge is marked as well.

Example 2.0.6 Let X be a **semiSegal space**. We will denote by $X^{\natural}$ the marked semi-simplicial space whose underlying semi-simplicial space is X and whose marking is given by X_1^{inv}. Then it is easy to verify that $X^{\natural}$ is a marked semiSegal space.

Example 2.0.7 Let X be a **semiSegal space**. Then $X^{\flat}$ is also a marked semiSegal space: since the space of marked edges is empty, conditions (2) and (3) are satisfied vacuously.

Example 2.0.8 Let Z be a Kan simplicial set. The 0'th coskeleton functor (see Example 1.4.5) has a marked analogue $\mathrm{cosk}_0^+ : \mathcal{S} \longrightarrow \mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}$ which is given by $\mathrm{cosk}_0^+(Z) = (\mathrm{cosk}_0(Z))^{\sharp}$ (this marked analogue is again the right adjoint to the functor $X \mapsto X_0$). Since $\mathrm{cosk}_0(Z)$ is a semiSegal space in which all edges are invertible we get that $\mathrm{cosk}_0^+(Z)$ is a marked semiSegal space.

We consider a marked semiSegal space as encoding a non-unital ∞-category in which a certain (suitably closed) subspace of the invertible morphisms has been marked. Given a marked semiSegal space (W,M) we will denote by

$$\mathrm{Map}_W^+(x,y) \subseteq \mathrm{Map}_W(x,y)$$

the subspace of marked edges from x to y, i.e. the fiber of the map

$$M \longrightarrow X_0 \times X_0$$

over (x,y).

Definition 2.0.9 We will say that W is a **marked semiKan space** if W is a marked semiSegal space in which all edges are marked. Note that in this case the underlying semi-simplicial space of W is a semiKan space.

Definition 2.0.10 We will say that a marked semiSegal space (W, M) is **quasi-unital** if the following additional conditions are satisfied:

(1) The restricted map $d_1 : M \longrightarrow X_0$ is surjective (i.e., every object admits a marked morphism out of it).

(2) Every invertible edge in W is marked.

We will denote by $\mathrm{QsS} \subseteq \mathrm{Seg}_s^{\mathrm{fib}}$ the full simplicial subcategory spanned by quasi-unital marked semiSegal spaces.

Remark 2.0.11 If (X, M) is a quasi-unital marked semiSegal space then $M = X_1^{\mathrm{inv}}$, i.e., the marking coincides with the subspace of invertible edges.

As explained in §1.5 we can consider a quasi-unital marked semiSegal space as encoding a quasi-unital ∞-category in which the invertible morphisms have been marked. As a result (see Proposition 1.5.6) maps in QsS are the "unital" ones - they are maps of the underlying semiSegal spaces which send quasi-units to quasi-units.

As in the Segal space case, the ∞-category $\mathcal{N}(\mathrm{QsS})$ is a good starting point for understanding the homotopy theory of quasi-unital ∞-categories. However, it is still not the correct model for it. The reason for this is that equivalences in QsS are too strict. The correct notion will be obtained after suitably localizing $\mathcal{N}(\mathrm{QsS})$ with respect to a natural notion of Dwyer-Kan equivalences (see §2.2).

It will be useful to describe the property of being quasi-unital in terms of a suitable right lifting property.

Definition 2.0.12 Consider the following three maps of marked semi-simplicial sets (considered as levelwise discrete semi-simplicial spaces):

(1) The inclusion $\theta_0 : \Delta^{\{0\}} \hookrightarrow (\Delta^1)^\sharp$.

(2) The inclusion $\theta_1 : \Delta^{\{1\}} \hookrightarrow (\Delta^1)^\sharp$.

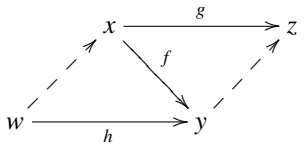
(3) The inclusion $\theta_2 : \left(\Delta^3, M\right) \hookrightarrow \left(\Delta^3\right)^\sharp$ where $M = \left\{\Delta^{\{0,2\}}, \Delta^{\{1,3\}}\right\}$.

We then have the following simple observation:

Lemma 2.0.13 *Let (X, M) be a marked semiSegal space and let $*$ be the terminal marked semi-simplicial space (i.e. $*_n$ is a point for every n and the edge in $*_1$ is marked). Then X is quasi-unital if and only if the terminal map $X \longrightarrow *$ has the right lifting property with respect to $\theta_0, \theta_1, \theta_2$ above.*

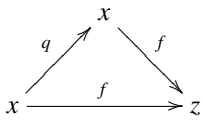
Proof First assume that $X \longrightarrow *$ has the right lifting property with respect to θ_0, θ_1 and θ_2. For θ_0, θ_1 this lifting property implies that every object $x \in X_0$ has a marked edge out of. In particular, Condition (1) of Definition 2.0.10 holds. Let us now prove that Condition (2) holds

as well, i.e., that every invertible edge in X is marked. Let $f : x \longrightarrow y$ be an equivalence in X. In light of the above there exist marked edges of the form $g : x \longrightarrow z$ and $h : w \longrightarrow y$. Since f is invertible we can embed these edges in a diagram of the form



This diagram can in turn be extended to a map $\varphi : \Delta^3 \longrightarrow X$ which sends $\Delta^{\{1,2\}}$ to f, $\Delta^{\{0,2\}}$ to h and $\Delta^{\{1,3\}}$ to g. Since $X \longrightarrow *$ has the right lifting property with respect to θ_2 we see that φ must factor through a map $(\Delta^3)^\sharp \longrightarrow X$. In particular, f is marked.

Now assume that X is a quasi-unital marked semiSegal space. Condition (1) of Definition 2.0.10 implies that $X \longrightarrow *$ has the right lifting property with respect to θ_0. To show the lifting property with respect to θ_1 we need to show that every object $x \in X_0$ has a marked edge into it. According to the above, there exists a marked edge $f : x \longrightarrow y$ out of x. From condition (2) of Definition 2.0.5 we know that f is invertible, and hence there exists a triangle of the form



for some $q : x \longrightarrow x$. From condition (3) of Definition 2.0.5 we may now deduce that q is a marked edge into x, and so X has the right lifting property with respect to θ_1. Finally, since X is quasi-unital we have $M = X_1^{\mathrm{inv}}$ and so by Lemma 1.5.4 this subspace of edges satisfies the 2-out-of-6 rule. This implies that X has the right lifting property with respect to θ_2. □

The remainder of this section is organized as follows. To begin, we will localize the marked model structure on $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ so that marked semiSegal spaces will coincide with the new fibrant objects. The construction of this localization as well as the verification of its compatibility with the monoidal structure will be taken up in §2.1. In particular, we will obtain a notion of internal mapping objects for marked semiSegal spaces.

In §2.2 we will study the notions of fully-faithful maps and Dwyer-Kan equivalences (DK-equivalences for short) between marked semiSegal spaces. Our suggested model for the homotopy theory of quasi-unital ∞-categories is the ∞-localization of $\mathcal{N}(\mathrm{QsS})$ with respect to DK-equivalences. In §2.3 we will attempt to study the notion of quasi-unitality from a model categorical point of view. This will allow us to establish some useful results which will be exploited in the final section, e.g., we will prove that the full subcategory QsS is closed under taking mapping objects. This can be considered as a step towards the main theorem as well as it essentially says that when quasi-units exist then they can be chosen coherently over arbitrary families.

2.1 The marked semiSegal model structure

The purpose of this subsection is to show that one can identify the full subcategory of marked semiSegal spaces with the subcategory of fibrant objects in a suitable left Bousfield localization of the marked model category $\mathbb{S}_+^{\Delta_s^{\mathrm{op}}}$. In order to do this we will need to identify a set of maps such that the condition of being a marked semiSegal space can be expressed as a locality condition with respect to these maps. To describe this conveniently we will need a bit of terminology.

We will use the phrase **marked horn inclusion** to describe an inclusion of marked semi-simplicial sets of the form

$$(\Lambda^n_i, A) \subseteq (\Delta^n, B)$$

such that $A = B \cap (\Lambda^n_i)_1$. We will be interested in the following kind of marked horn inclusions:

Definition 2.1.1 We will say that a marked horn inclusion

$$(\Lambda^n_i, A) \subseteq (\Delta^n, B)$$

is **admissible** if $B = A$, $n \geq 2$ and in addition one of the following (mutually exclusive) conditions is satisfied:

(1) $0 < i < n$ and $A = \emptyset$.

(2) $i = 0$ and $A = \left\{\Delta^{\{0,1\}}\right\}$.

(3) $i = n$ and $A = \left\{\Delta^{\{n-1,n\}}\right\}$.

Our purpose is to show that conditions (1) and (2) of Definition 2.0.5 can be formulated in terms of locality with respect to admissible marked horn inclusions. We begin with the following technical lemma:

Lemma 2.1.2 *Let $n \geq 2$ and $0 \leq i,j \leq n$. Define $A = \{0,...,j-1\} \cup \{i\}$ and $B = \{j,...,n\} \cup \{i\}$. Let*

$$X = \Delta^A \coprod_{\Delta^{\{i\}}} \Delta^B \subseteq \Delta^n$$

be the corresponding subcomplex of Δ^n and let $M \subseteq X_1$ be the set of all edges of the form $\Delta^{\{i,x\}} \subseteq X$ for $x \in B, x < i$ and all edges of the form $\Delta^{\{x,i\}}$ for $x \in A, x > i$. Then the marked semi-simplicial set $\left(\Lambda^n_i, M\right)$ can be obtained from (X,M) be performing pushouts along admissible marked horn inclusions.

Proof We will say that a simplex $\Delta^J \subseteq \Delta^n$ for $|J| \geq 3$ is **spread** if J contains i and J has a non-empty intersection with both $A \setminus \{i\}$ and $B \setminus \{i\}$. Now define

$$X = X_1 \subseteq X_2 \subseteq ... \subseteq X_{n-1}$$

inductively by letting X_k be the union of X_{k-1} and all the spread simplices of dimension k. Note that if $\Delta^J \subseteq \Delta^n$ is spread of dimension $|J| = k$ then $\Delta^J \cap X_j$ is the horn of Δ^J which contains

all the $(k-1)$-faces except the face opposite the vertex $i \in J$. Furthermore, if i is a maximal element of J then the second biggest element of J is in B and so the last edge of Δ^J is marked. Similarly if i is a minimal element of J then the second smallest element of J is in A and so the first edge of Δ^J is marked. In either case the addition of Δ^J can be performed by a pushout along an admissible marked horn inclusion. To finish the proof note that $X_{n-1} = \left(\Lambda^n_i, M\right)$. □

Lemma 2.1.2 has the following corollary:

Corollary 2.1.3 *Let (X, M) be a fibrant marked semi-simplicial space. Then*

(1) X satisfies condition (1) of Definition 2.0.5 (i.e. the Segal condition) if and only if X is local with respect to inner admissible horn inclusions.

(2) Assume that X satisfies the Segal condition. Then X satisfies condition (2) of Definition 2.0.5 if and only if X is local with respect to non-inner admissible horn inclusions.

Proof Part (1) follows from Lemma 2.1.2 with $0 < i = j < n$ (in which case no marking is involved) together with a simple inductive argument. Now assume that X satisfies the Segal condition. If X is local with respect to admissible horn inclusions then in particular X is local with respect to the inclusions

$$\left(\Lambda^2_0, \left\{\Delta^{\{0,1\}}\right\}\right) \hookrightarrow \left(\Delta^2, \left\{\Delta^{\{0,1\}}\right\}\right)$$

and

$$\left(\Lambda^2_2, \left\{\Delta^{\{1,2\}}\right\}\right) \hookrightarrow \left(\Delta^2, \left\{\Delta^{\{1,2\}}\right\}\right)$$

and so by Remark 1.5.2 every marked edge in X is invertible.

Now assume that every marked edge is invertible. Then X is local with respect to admissible horn inclusions of dimension 2. Assume by induction that X is local with respect to all admissible horn inclusions of dimension k for some $k \geq 2$.

Consider the diagram of marked simplicial sets

$$\begin{array}{ccc}
\Delta^{\{0,\ldots,k-1\}} \underset{\Delta^{\{k-1\}}}{\coprod} \Delta^{\{k-1,k+1\}} \underset{\Delta^{\{k+1\}}}{\coprod} \left(\Delta^{\{k,k+1\}}\right)^\sharp & \longrightarrow & \Delta^{\{0,\ldots,k-1,k+1\}} \underset{\Delta^{\{k+1\}}}{\coprod} \left(\Delta^{\{k,k+1\}}\right)^\sharp \\
\downarrow & & \downarrow \\
\Delta^{\{0,\ldots,k-1\}} \underset{\Delta^{\{k-1\}}}{\coprod} \left(\Delta^{\{k-1,k,k+1\}}, \left\{\Delta^{\{k,k+1\}}\right\}\right) & \longrightarrow & \left(\Delta^{k+1}, \left\{\Delta^{\{k,k+1\}}\right\}\right)
\end{array}$$

From the Segal condition it follows that X is local with respect to both horizontal maps. Furthermore, since X is local with respect to admissible horn inclusions of dimension 2 we see that X is local with respect to the left vertical map. It follows that X is local with respect to the right vertical map as well. Finally, applying Lemma 2.1.2 for the case $j = k, i = n = k+1$ we get that X is local with respect to the inclusion

$$\Delta^{\{0,\ldots,k-1,k+1\}} \underset{\Delta^{\{k+1\}}}{\coprod} \left(\Delta^{\{k,k+1\}}\right)^\sharp \hookrightarrow \left(\Lambda^{k+1}_{k+1}, \left\{\Delta^{\{k,k+1\}}\right\}\right).$$

It hence follows that X is local with respect to

$$\left(\Lambda^{k+1}_{k+1}, \left\{\Delta^{\{k,k+1\}}\right\}\right) \hookrightarrow \left(\Delta^{k+1}, \left\{\Delta^{\{k,k+1\}}\right\}\right).$$

A similar argument establishes the case of

$$\left(\Lambda^{k+1}_{0}, \left\{\Delta^{\{0,1\}}\right\}\right) \hookrightarrow \left(\Delta^{k+1}, \left\{\Delta^{\{0,1\}}\right\}\right).$$

□

Now let $W \in \mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}$ be a marked-fibrant object. In light of Corollary 2.1.3 we see that W will be a semiSegal space if and only if W is local with respect to the set $\mathbf{S}$ defined as follows:

Definition 2.1.4 Let $\mathbf{S}$ be the set which contains:

(1) All admissible marked horn inclusions.

(2) All the maps of the form

$$\left(\Delta^2, A\right) \hookrightarrow \left(\Delta^2\right)^{\sharp}$$

where $A \subseteq \left(\Delta^2\right)_1$ is a set of size 2.

We are now in a position to define our desired model category. Since the marked model structure is combinatorial and left proper, the left Bousfield localization of $\mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}$ with respect to $\mathbf{S}$ exists. In particular, there exists a (combinatorial, left proper) model category Seg_s whose underlying category is $\mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}$ such that

(1) Weak equivalences in Seg_s are maps $f : X \longrightarrow Y$ such that for every marked semiSegal space W the induced map

$$\mathrm{Map}^{+}(Y, W) \longrightarrow \mathrm{Map}(X, W)$$

is a weak equivalence.

(2) Cofibrations in Seg_s are the cofibrations of the marked model structure (i.e., levelwise injective maps).

(3) The fibrant objects in Seg_s are precisely the marked semiSegal spaces.

Definition 2.1.5 We will refer to Seg_s as the **marked semiSegal model structure**. We will denote by **MS-equivalences**, **MS-fibrations** and **MS-cofibrations** the weak equivalences, fibrations and cofibrations in Seg_s respectively (to avoid confusion compare to the terminology in Definition 1.7.9). Note that the notions of an MS-cofibration and a marked cofibration coincide.

The following kind of trivial MS-cofibration will be useful to note:

Definition 2.1.6 Let X be a marked semi-simplicial space and $B \subseteq C \subseteq X_1$ two subspaces. We will say that the map

$$(X, B) \hookrightarrow (X, C)$$

is a **triangle remarking** if (X, C) can be obtained from (X, B) by a sequence of pushouts along maps of the form

$$[L \otimes (\Delta^2, A)] \coprod_{K \otimes (\Delta^2, A)} \left[K \otimes (\Delta^2)^\sharp\right] \hookrightarrow L \otimes (\Delta^2)^\sharp \tag{2–1}$$

for $f : K \hookrightarrow L$ an inclusion of **spaces** and $|A| = 2$ (here we consider spaces as marked semi-simplicial spaces which are concentrated in degree 0).

Lemma 2.1.7 *Any triangle remarking is a trivial MS-cofibration.*

Proof Note that the claim is clearly true if $K = \emptyset$ and L is discrete. Hence the claim is also true for maps of the form $K \hookrightarrow K \coprod L'$ where L' is discrete. Orthogonally, if the inclusion $K \hookrightarrow L$ is surjective on connected components then the map 2–1 is a marked equivalence and so in particular an MS-equivalence. The result now follows by factoring a general inclusion $K \hookrightarrow L$ as $K \hookrightarrow K \coprod L' \longrightarrow L$ where L' is discrete and the map $K \coprod L' \longrightarrow L$ is surjective on connected components. □

This notion is exemplified in the following lemma:

Lemma 2.1.8 *For every $i = 0, \ldots, n$ the map*

$$(\Lambda^n_i)^\sharp \longrightarrow (\Delta^n)^\sharp$$

is a trivial MS-cofibration.

Proof Let $M \subseteq (\Delta^n)_1$ be the set of edges that are contained in Λ^n_i. Then (Δ^n, M) is obtained from $(\Lambda^n_i)^\sharp$ by performing a pushout along an admissible marked horn inclusion. The desired result now follows from the fact that the map

$$(\Delta^n, M) \hookrightarrow (\Delta^n)^\sharp$$

is a triangle remarking. □

Corollary 2.1.9 *If W is a marked semiSegal space then $\widetilde{W}$ (see Definition 1.7.15) is a marked semiSegal space as well. Furthermore, since all the edges in $\widetilde{W}$ are marked we see that $\widetilde{W}$ is in fact a* **marked semiKan space**.

Proof First of all it is clear that $\widetilde{W}$ is marked-fibrant (see Lemma 1.7.11). From Lemma 2.1.8 it follows that $\widetilde{W}$ is local with respect to all admissible marked horn inclusions and so by Proposition 2.1.3 $\widetilde{W}$ satisfies properties (1) and (2) of Definition 2.0.5. Since clearly the marked edges in $\widetilde{W}$ are closed under 2-out-of-3 we get that $\widetilde{W}$ is a marked semiSegal space. □

Remark 2.1.10 If W is a quasi-unital marked semiSegal space then $\widetilde{W}$ is quasi-unital as well. Furthermore, in this case $\widetilde{W}$ contains all invertible edges of W.

Now recall that $\mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}$ is a symmetric monoidal model category with respect to the marked monoidal product $\otimes$. We would like to show that this monoidality survives the localization:

Theorem 2.1.11 *The marked Segal model structure is compatible with the marked monoidal product $\otimes$. In particular, the localization Quillen adjunction*

$$\mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathrm{Seg}_s$$

is strongly monoidal and Seg_s *inherits the simplicial structure of* $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$.

Proof Arguing as in [14] Proposition 9.2, we see that it will be enough to show that for every marked semiSegal space W the objects $W^{(\Delta^n)^\flat}$ and $W^{(\Delta^1)^\sharp}$ are marked semiSegal spaces for every $n \geq 0$. This, in turn, can be easily reduced to checking that for every $f : Y \longrightarrow Z$ in $\mathbf{S}$ (see Definition 2.1.4) the inclusions

$$\left[(\partial\Delta^m)^\flat \otimes Z\right] \coprod_{(\partial\Delta^m)^\flat \otimes Y} \left[(\Delta^m)^\flat \otimes Y\right] \hookrightarrow (\Delta^m)^\flat \otimes Z$$

and

$$\left[(\Delta^1)^\flat \otimes Z\right] \coprod_{(\Delta^1)^\flat \otimes Y} \left[(\Delta^1)^\sharp \otimes Y\right] \hookrightarrow (\Delta^1)^\sharp \otimes Z$$

are trivial MS-cofibrations. Note that the case $m = 0$ above is trivial so we can assume $m \geq 1$.

We begin by clearing up some trivial cases. Observe that for a pair of inclusions of the form

$$f : (X, A) \hookrightarrow (Y, A)$$

$$g : (Z, B) \hookrightarrow (Z, C)$$

such that $f_0 : X_0 \longrightarrow Y_0$ is **surjective** the resulting map

$$[(Z, C) \otimes (X, A)] \coprod_{(Z,B)\otimes(X,A)} [(Z, B) \otimes (Y, A)] \longrightarrow (Z, C) \otimes (Y, A)$$

is in fact an **isomorphism** (and in particular a trivial MS-cofibration). Considering the various types of maps in $\mathbf{S}$ one sees that the only cases which are not covered by the above argument are the following:

(1) The maps of the form

$$\left[(\Delta^1)^\sharp \otimes (\Delta^2, A)\right] \coprod_{(\Delta^1)^\flat \otimes (\Delta^2, A)} \left[(\Delta^1)^\flat \otimes (\Delta^2)^\sharp\right] \longrightarrow (\Delta^1)^\sharp \otimes (\Delta^2)^\sharp = (\Delta^1 \otimes \Delta^2)^\sharp$$

where $|A| = 2$.

(2) The maps of the form

$$\left(\partial\Delta^m\right)^\flat \otimes \left(\Delta^n, A\right) \coprod_{(\partial\Delta^m)^\flat \otimes \left(\Lambda^n_l, A\right)} \left(\Delta^m\right)^\flat \otimes \left(\Lambda^n_l, A\right) \hookrightarrow \left(\Delta^m\right)^\flat \otimes \left(\Delta^n, A\right)$$

where $\left(\Lambda^n_l, A\right) \hookrightarrow (\Delta^n, A)$ is an admissible marked horn inclusion.

For case (1), note that this map induces an isomorphism on the underlying semi-simplicial sets. Furthermore, the marking on the left hand side contains all edges except exactly **one** edge $e \in \left(\Delta^1 \otimes \Delta^2\right)_1$.

Note that each triangle in $\Delta^1 \otimes \Delta^2$ has three distinct edges. Furthermore, every edge in $\Delta^1 \otimes \Delta^2$ lies in some triangle. Hence one can find a triangle which contains e such that its other two edges are not e. This means that there exists a pushout diagram of marked semi-simplicial sets of the form

$$\begin{array}{ccc} \left(\Delta^2, A\right) & \longrightarrow & \left(\Delta^2\right)^\sharp \\ \downarrow & & \downarrow \\ \left[\left(\Delta^1\right)^\sharp \otimes \left(\Delta^2, A\right)\right] \coprod_{(\Delta^1)^\flat \otimes (\Delta^2, A)} \left[\left(\Delta^1\right)^\flat \otimes \left(\Delta^2\right)^\sharp\right] & \longrightarrow & \left(\Delta^1 \otimes \Delta^2\right)^\sharp \end{array}$$

Since the upper horizontal row is a trivial MS-cofibration we get that the lower horizontal map is an MS-cofibration as well.

The proof of case (2) is considerably more technical and is taken on in lemma 2.1.12 below. □

Lemma 2.1.12 *Let $\left(\Lambda^n_l, A\right) \hookrightarrow (\Delta^n, A)$ be an admissible marked horn inclusion. Then the marked semi-simplicial set $(\Delta^m)^\flat \otimes (\Delta^n, A)$ can be obtained from the marked semi-simplicial set*

$$X = \left(\partial\Delta^m\right)^\flat \otimes \left(\Delta^n, A\right) \coprod_{(\partial\Delta^m)^\flat \otimes \left(\Lambda^n_l, A\right)} \left(\Delta^m\right)^\flat \otimes \left(\Lambda^n_l, A\right)$$

by successively performing pushouts along admissible marked horn inclusions. In particular, the inclusion

$$X \subseteq \left(\Delta^m\right)^\flat \otimes \left(\Delta^n, A\right)$$

is a trivial MS-cofibration.

Proof If $m = 0$ then the claim is immediate, so we can assume $m > 0$. In this case the marking of $(\Delta^m)^\flat \otimes (\Delta^n, A)$ is the same as the marking of X, so that we don't need to worry about adding marked edges in the course of performing the desired pushouts. Note that we can harmlessly assume that $0 < l \leq n$ (as the case $0 \leq l < n$ follows from symmetry).

According to Remark 1.3.1 the k-simplices of $\Delta^m \otimes \Delta^n$ are in one-to-one correspondence with injective order preserving maps

$$\sigma = (f, g) : [k] \longrightarrow [m] \times [n].$$

We will consider a k-simplex $\sigma = (f,g)$ as above as an injective marked map

$$\sigma : (\Delta^k, B) \hookrightarrow \left(\Delta^m\right)^\flat \otimes \left(\Delta^n, A\right)$$

where B is defined to be $\left\{\Delta^{\{k-1,k\}}\right\}$ if $\sigma\left(\Delta^{\{k-1,k\}}\right)$ is marked and $\emptyset$ otherwise.

We will say that a k-simplex of $\Delta^m \otimes \Delta^n$ is **full** if it is **not** contained in X. If we describe our k-simplex by a map $\sigma = (f,g)$ as above this translates to the condition that f is surjective and that the image of g contains $\{0,...,n\} \setminus \{l\}$ (so that g is either surjective or misses l). Our purpose is to add all the full simplices to X in a way that involves only pushouts along admissible horn inclusions. For this we distinguish between two kinds of k-simplices of $\Delta^m \otimes \Delta^n$:

Definition 2.1.13 Let

$$\sigma = (f,g) : [k] \longrightarrow [m] \times [n]$$

be a k-simplex of $\Delta^m \otimes \Delta^n$. We will say that σ is **special** if

(1) σ is full.

(2) $g^{-1}(l) \neq \emptyset$.

(3) $f\left(\min g^{-1}(l)\right) = f\left(\max g^{-1}(l-1)\right)$.

If σ is full but not special then we will say that σ is **regular**.

Now for $i = 0,...,m+1$ let X_i denote the union of X and all **special** $(i+n-1)$-simplices of $\Delta^m \otimes \Delta^n$. We now claim the following:

(1) $X_0 = X$.

(2) For $i = 0,...,m$ the semi-simplicial set X_{i+1} is obtained from X_i by a sequence of pushouts along admissible horn inclusions of dimension $i+n$.

(3) $X_{m+1} = \Delta^m \otimes \Delta^n$.

The first claim just follows from the fact that there are no special simplices of dimension less than n. Now X_{i+1} is the union of X_i and all special $(i+n)$-simplices. Hence in order to prove the second claim we will need to find the right **order** in which to add these special $(i+n)$-simplices to X_i. We will do this by sorting them according to the following quantity:

Definition 2.1.14 Let

$$\sigma = (f,g) : [k] \longrightarrow [m] \times [n]$$

be a **full** k-simplex of $\Delta^m \otimes \Delta^n$. We define the **index** of σ to be the quantity

$$\mathrm{ind}(\sigma) = k + 1 - n - |g^{-1}(l)|.$$

Note that for a general full simplex the index is a number between 0 and $k+1-n$. By definition we see that for a **special** k-simplex the index is a number between 0 and $k-n$. In particular, the index of a special $(i+n)$-simplex is a number between 0 and i.

Now fix an $i = 0, ..., m$ and for each $j = 0, ..., i+1$ define $X_{i,j}$ to be the union of X_i and all special $(i+n)$-simplices σ whose index is strictly less than j. We obtain a filtration of the form

$$X_i = X_{i,0} \subseteq X_{i,1} \subseteq ... \subseteq X_{i,i+1} = X_{i+1}.$$

We will show that if σ is a special $(i+n)$-simplex of index j then the intersection

$$\sigma \cap X_{i,j}$$

is an admissible horn of σ (with respect to the marking induced from σ). This means that $X_{i,j+1}$ can be obtained from $X_{i,j}$ by performing pushouts along admissible horn inclusions of dimension $m+i$, implying the second claim above. We start by noting that if $\tau = (f,g)$ is a **regular** k-simplex then τ is a face of the special $(k+1)$-simplex

$$\sigma = (f \circ s_{\max g^{-1}(l-1)}, g \circ s_{\min g^{-1}(l)})$$

where $s_r : [k+1] \longrightarrow [k]$ is the degeneracy map hitting r twice. Furthermore, we see that $\mathrm{ind}(\sigma) = \mathrm{ind}(\tau)$. This means that $X_{i,j}$ contains in particular all regular $(i+n-1)$-simplices whose index is $< j$. Since taking faces cannot increase the index we see that an $(i+n-1)$-simplex τ is contained in $X_{i,j}$ exactly when τ is not regular of index $\geq j$.

Now let $\sigma = (f,g)$ be a **special** $(i+n)$-simplex of index j and let τ be the $(i+n-1)$-face of σ which is opposed to the v'th vertex for $v = 0, ..., i+n$. Then we see that τ will be regular of index $\geq j$ if and only if $v = \min g^{-1}(l)$, in which case $\mathrm{ind}(\tau) = \mathrm{ind}(\sigma) = j$. Since g is surjective we get that

$$0 < \min g^{-1}(l) \leq i + l \leq i + n$$

and so $X_{i,j} \cap \sigma$ is a right horn of σ which is inner if $l < n$. In fact, the only case where this right horn inclusion is not inner is when $\min g^{-1}(l) = k$. By the definition of special we then have

$$f(k) = f(k-1)$$

and so the $\{k-1, k\}$-edge of σ is mapped to a **marked** edge in $(\Delta^m)^\flat \otimes (\Delta^n, A)$. This means that indeed the addition of σ can be done by a pushout along an **admissible** horn inclusion.

It is left to prove the third claim, i.e., that $X_{m+1} = \Delta^m \otimes \Delta^n$. From the considerations above we see that X_{i+1} contains all full k-simplices for $k < n+i$ (as well as all special $(n+i)$-simplices). Since all the full $(m+n)$-simplices are special we get that X_{m+1} contains all full simplices of $\Delta^m \otimes \Delta^n$ of dimension up to $m+n$, yielding the desired result. □

Corollary 2.1.15 *Let W be a marked semiSegal space and X a marked semi-simplicial space. Then W^X is a marked semiSegal space and $\widetilde{W^X}$ is a marked semiKan space.*

Proof The first claim follows from the fact that the fibrant objects in Seg_+ are exactly the marked semiSegal spaces, and all object are cofibrant. It is then a standard fact about monoidal model categories that raising a fibrant object to the power of a cofibrant object yields a fibrant object. Then second claim follows from Corollary 2.1.9. □

2.2 Fully-faithful maps and Dwyer-Kan equivalences

The purpose of this section is to study the notion of **fully-faithful maps** and **Dwyer-Kan equivalences** in the setting of marked semi-simplicial spaces. We begin with the basic definition:

Definition 2.2.1 Let $f : (W, M) \longrightarrow (Z, N)$ be a map of semi-simplicial spaces. We will say that f is **fully-faithful** if the squares

(2–2)
$$\begin{array}{ccc} W_n & \longrightarrow & Z_n \\ \downarrow & & \downarrow \\ (W_0)^{n+1} & \longrightarrow & (Z_0)^{n+1} \end{array} \qquad \begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ W_0 \times W_0 & \longrightarrow & Z_0 \times Z_0 \end{array}$$

are homotopy Cartesian for every $n \geq 1$.

In this paper we will often encounter maps $f : W \longrightarrow Z$ of marked semiSegal spaces which are simultaneously fully-faithful and a marked fibration. This case admits a particularly nice description. It will be convenient to employ the following terminology (which makes sense in any simplicial category):

Definition 2.2.2 Let $g : X \longrightarrow Y, f : W \longrightarrow Z$ be two maps in $\mathbb{S}_+^{\Delta_s^{\mathrm{op}}}$. We will say that f has the **contractible** right lifting property with respect to g if the map

(2–3)
$$\mathrm{Map}_+(Y, W) \longrightarrow \mathrm{Map}_+(Y, Z) \times_{\mathrm{Map}_+(X,Z)} \mathrm{Map}_+(X, W)$$

is a trivial Kan fibration.

Remark 2.2.3 Definition 2.2.2 is equivalent to saying that f has the right lifting property with respect to the maps

$$|\partial\Delta^m| \otimes Y \coprod_{|\partial\Delta^m| \otimes X} |\Delta^m| \otimes X \hookrightarrow |\Delta^m| \otimes Y$$

for every $m \geq 0$.

Remark 2.2.4 When $f : W \longrightarrow Z$ is a marked fibration and $g : X \longrightarrow Y$ is a marked cofibration then the map 2–3 is a fibration. Furthermore, in this case the fiber product on the right hand side coincides with the homotopy fiber product. From this observation we see that a marked fibration $f : W \longrightarrow Z$ is fully-faithful if and only if is satisfies the contractible right lifting property with respect to the maps

$$\overbrace{\Delta^0 \coprod \ldots \coprod \Delta^0}^{n+1} \hookrightarrow (\Delta^n)^\flat$$

for every $n \geq 1$ and the map

$$\Delta^0 \coprod \Delta^0 \hookrightarrow (\Delta^1)^\sharp .$$

We are now ready to prove our main characterization theorem concerning fully-faithful marked fibrations:

Proposition 2.2.5 *Let $f : W \longrightarrow Z$ be a map of semi-simplicial spaces. Then the following assertions are equivalent:*

(1) f is a fully-faithful marked fibration.

(2) f is a marked fibration and satisfies the contractible right lifting property with respect to the maps $(\partial\Delta^n)^\flat \hookrightarrow (\Delta^n)^\flat$ for every $n \geq 1$ and the map $\partial\Delta^1 \hookrightarrow \left(\Delta^1\right)^\sharp$

(3) f satisfies the right lifting property with respect to every marked cofibration $g : X \hookrightarrow Y$ such that $g_0 : X_0 \hookrightarrow Y_0$ is a weak equivalence.

(4) f satisfies the contractible right lifting property with respect to every marked cofibration $g : X \hookrightarrow Y$ such that $g_0 : X_0 \hookrightarrow Y_0$ is a weak equivalence.

(5) f is a fully-faithful MS-fibration.

(6) f is an MS-fibration and satisfies the contractible right lifting property with respect to the maps $\partial\Delta^1 \hookrightarrow \left(\Delta^1\right)^\flat$ and $\partial\Delta^1 \hookrightarrow \left(\Delta^1\right)^\sharp$

Proof

(1) $\Rightarrow$ (2) Invoking Remark 2.2.4 we note that the semi-simplicial set $(\partial\Delta^n)^\flat$ can be obtained from $\overbrace{\Delta^0 \coprod \ldots \coprod \Delta^0}^{n+1}$ by successively performing pushouts along maps of the form

$$\left(\partial\Delta^k\right)^\flat \hookrightarrow \left(\Delta^k\right)^\flat$$

for $k < n$. Hence the claim follows by induction on n.

(2) $\Rightarrow$ (3) Assume f satisfies (2) and let $g : X \hookrightarrow Y$ be a marked cofibration such that $g_0 : X_0 \longrightarrow Y_0$ is a weak equivalence. Then one can factor g as

$$X \xrightarrow{g'} X' \xrightarrow{g''} Y$$

such that g' is a trivial marked cofibration and g'' induces an isomorphism $g''_0 : X'_0 \longrightarrow Y_0$. Since f is a marked fibration it will suffice to show that f satisfies the right lifting property with respect to g''. But this follows by Remark 2.2.3 from the fact that g'' can be written as a (transfinite) composition of pushouts along maps of the form

$$\left[|\Delta^m| \otimes \left(\partial\Delta^n\right)^\flat\right] \coprod_{|\partial\Delta^m| \otimes (\partial\Delta^n)^\flat} \left[|\partial\Delta^m| \otimes \left(\Delta^n\right)^\flat\right] \hookrightarrow |\Delta^m| \otimes \left(\Delta^n\right)^\flat$$

and

$$\left[|\Delta^m| \otimes \partial\Delta^1\right] \coprod_{|\partial\Delta^m| \otimes \partial\Delta^1} \left[|\partial\Delta^m| \otimes \left(\Delta^1\right)^\sharp\right] \hookrightarrow |\Delta^m| \otimes \left(\Delta^1\right)^\sharp$$

for $m \geq 0$.

(3) $\Rightarrow$ (4) This implication follows from the fact that the family of cofibrations $g : X \longrightarrow Y$ such that $g_0 : X_0 \longrightarrow Y_0$ is a weak equivalence is stable under replacing g with

$$|\partial\Delta^m| \otimes Y \coprod_{|\partial\Delta^m|\otimes X} |\Delta^m| \otimes X \hookrightarrow |\Delta^m| \otimes Y.$$

(4) $\Rightarrow$ (5) Assume that f satisfies (4). Then it is straightforward (using Remark 2.2.4) to deduce that f is fully-faithful. To show that f is an MS-fibration it will be enough to show that if $g : X \longrightarrow Y$ is a trivial MS-cofibration then $g_0 : X_0 \hookrightarrow Y_0$ is a weak equivalence. This in turn follows from the fact that if Z is a Kan simplicial set then $\mathrm{cosk}_0^+(Z)$ is a marked semiSegal space (see Example 2.0.8) which means that $\mathrm{Map}_S(Y_0, Z) \longrightarrow \mathrm{Map}_S(X_0, Z)$ is a weak equivalence for every Z.

(5) $\Rightarrow$ (6) Follows from Remark 2.2.4.

(6) $\Rightarrow$ (1) First note that if f satisfies (6) then it has the contractible right lifting property with respect to the map

$$\overbrace{\Delta^0 \coprod \ldots \coprod \Delta^0}^{n+1} \hookrightarrow \left(\mathrm{Sp}^n\right)^\flat$$

where $\mathrm{Sp}^n \subseteq \Delta^n$ is the n-spine (see 1–1). The desired result now follows from the fact that the inclusion

$$\left(\mathrm{Sp}^n\right)^\flat \hookrightarrow \left(\Delta^n\right)^\flat$$

is a trivial MS-cofibration and that f is an MS-fibration.

□

Corollary 2.2.6 *Let $f : W \longrightarrow Z$ be a fully-faithful marked fibration. Let X be a marked semi-simplicial space and $g : X \longrightarrow Z$ a map. Then every lift $\widetilde{g_0} : X_0 \longrightarrow W_0$ of g_0 extends to a lift $\widetilde{g} : X \longrightarrow W$ of g.*

Proposition 2.2.5 allows us in particular to obtain the following description of fully-faithful maps between marked semiSegal spaces, relating them to the classical meaning of the notion:

Corollary 2.2.7 *Let $f : W \longrightarrow Z$ be a map of marked semiSegal spaces. Then f is fully-faithful if and only if for every $x, y \in W_0$ the map f induces weak equivalences*

(2–4) $$\mathrm{Map}_W(x, y) \xrightarrow{\simeq} \mathrm{Map}_Z(f_0(x), f_0(y))$$

and

(2–5) $$\mathrm{Map}^+_W(x, y) \xrightarrow{\simeq} \mathrm{Map}^+_Z(f_0(x), f_0(y)).$$

Proof Factor f as $W \xrightarrow{f'} W' \xrightarrow{f''} Z$ where f' is a trivial marked cofibration and f'' is a marked fibration. Then W' is marked fibrant and marked equivalent to W, and so W' is a marked semiSegal space. It then follows from the general theory of left Bousfield localizations (see [5, Proposition 3.3.16]) that f'' is in fact an MS-fibration. Now since f' is a marked

equivalence we see that f is fully-faithful if and only if f'' is fully-faithful. We may hence assume without loss of generality that f itself is an MS-fibration. In particular, the maps 2–4 and 2–5 are Kan fibrations, which means that they will be weak equivalences if and only if their fibers are contractible. The desired result now follows from the equivalence of (1) and (6) of Proposition 2.2.5. □

The following corollary of Proposition 2.2.5 will be useful later:

Corollary 2.2.8 *Let $f : X \longrightarrow Y$ be a map of marked semi-simplicial spaces. Then the following assertions are equivalent:*

(1) For every MS-fibration $p : W \longrightarrow Z$ and every MS-cofibration $g : X' \hookrightarrow Y'$ such that $g_0 : X'_0 \hookrightarrow Y'_0$ is a weak equivalence the induced map

$$W^{Y'} \longrightarrow Z^{Y'} \times_{Z^{X'}} W^{X'}$$

satisfies the right lifting property with respect to f.

(2) For every MS-cofibration $g : X' \hookrightarrow Y'$ such that $g_0 : X'_0 \hookrightarrow Y'_0$ is a weak equivalence the induced map

$$\left[X \otimes Y'\right] \coprod_{X \otimes X'} \left[Y \otimes X'\right] \longrightarrow Y \otimes Y'$$

is a trivial MS-cofibration.

(3) For every MS-fibration $p : W \longrightarrow Z$ the induced map

$$W^Y \longrightarrow Z^Y \times_{Z^X} W^X$$

is a fully-faithful MS-fibration.

(4) f is an MS-cofibration and the maps

$$\left[X \otimes \left(\Delta^1\right)^\flat\right] \coprod_{X \otimes \partial\Delta^1} \left[Y \otimes \partial\Delta^1\right] \longrightarrow \left[Y \otimes \left(\Delta^1\right)^\flat\right] \tag{2–6}$$

$$\left[X \otimes \left(\Delta^1\right)^\sharp\right] \coprod_{X \otimes \partial\Delta^1} \left[Y \otimes \partial\Delta^1\right] \longrightarrow \left[Y \otimes \left(\Delta^1\right)^\sharp\right] \tag{2–7}$$

are trivial MS-cofibrations.

Proof The equivalence (1) $\Leftrightarrow$ (2) follows directly from the exponential law. The equivalence (2) $\Leftrightarrow$ (3) follows from the exponential law together with Proposition 2.2.5. We will now prove that (3) $\Leftrightarrow$ (4). The direction (4) $\Rightarrow$ (3) follows from the exponential law and Proposition 2.2.5. Now assume f satisfies (3). Then for every trivial MS-fibration $W \longrightarrow Z$ the induced map

$$W^Y \longrightarrow Z^Y \times_{Z^X} W^X$$

is a trivial MS-fibration. This implies that f has the left lifting property with respect to every trivial MS-fibration and so is an MS-cofibration. The second part of (4) then follows from Proposition 2.2.5. □

Remark 2.2.9 The class of maps $f : X \longrightarrow Y$ satisfying the equivalent conditions of Corollary 2.2.8 is weakly saturated in view of characterization (1) and contains all trivial MS-cofibrations.

Remark 2.2.10 If $f : X \longrightarrow Y$ is a map whose underlying map of unmarked semi-simplicial spaces is an isomorphism then f satisfies condition (4) of Corollary 2.2.8. To see this, note that such an f is automatically an MS-cofibration. In addition, the map 2–6 is an isomorphism and the map 2–7 is a triangle remarking (see Definition 2.1.6).

We now turn to the notion of **Dwyer-Kan equivalences**. This notion will be obtained from the notion of fully-faithful maps by requiring an appropriate analogue of "essential surjectivity". This notion is most well behaved for quasi-unital marked semiSegal spaces (see Lemma 2.2.12), but it will be convenient to have it defined in more generality.

Definition 2.2.11 Let X be a marked semi-simplicial space. Let $x \simeq y$ denote the weakest equivalence relation satisfying the following properties:

(1) If x, y are in the same connected component of X_0 then $x \simeq y$.

(2) If there exists a marked edge $f \in X_1$ such that $d_1(f) = x$ and $d_0(f) = y$ then $x \simeq y$.

Lemma 2.2.12 *Let (X, M) be a quasi-unital marked semiSegal space and $x, y \in X_0$ two points. Then $x \simeq y$ if and only if there exists a marked morphism $x \longrightarrow y$.*

Proof We first observe that the relation of having a marked morphism from x to y is already an equivalence relation when X is a quasi-unital marked semiSegal space. It will hence suffice to prove that this relation contains the relation of being in the same connected component. Let $x, y \in X_0$ be two points which lie in the same connected component. Since X is marked fibrant X_0 is Kan and so there exists a path $\gamma : |\Delta^1| \longrightarrow X_0$ connecting x to y. As X is quasi-unital there exists a marked edge $q \in M$ from x to itself. Since X is marked-fibrant the map

$$d_0 : M \times_{X_0} \{x\} \longrightarrow X_0$$

is a Kan fibration (where the fiber product is taken along d_1) and so we can lift γ to a path $\overline{\gamma} : |\Delta^1| \longrightarrow M$ from q to some edge $f \in M \times_{X_0} \{x\}$ such that $d_0(f) = y$. In other words, f is a marked edge from x to y. □

Definition 2.2.13 Let $f : X \longrightarrow Y$ be a map between semi-simplicial spaces. We will say that f is a **Dwyer-Kan equivalence** (DK for short) if it is fully faithful and induces a surjective map on the set of equivalence-classes of $\simeq$.

We are now ready to formally define our ∞-category of quasi-unital ∞-categories:

Definition 2.2.14 Let DK denote the collection of edges of the ∞-category $\mathcal{N}(\mathrm{QsS})$ corresponding to DK-equivalences. We will refer to the ∞-localization (see Definition 1.2.1)

$$\mathrm{Cat}_\infty^{\mathrm{qu}} \stackrel{\mathrm{def}}{=} \mathcal{N}(\mathrm{QsS})[\mathrm{DK}^{-1}]$$

as the **∞-category of (small) quasi-unital ∞-categories**.

Remark 2.2.15 Even though DK is not a set we will see in § 3 that $\mathcal{N}(\mathrm{QsS})[\mathrm{DK}^{-1}]$ is locally small by constructing an explicit model for it.

We will finish this subsection by showing that under a mild additional hypothesis a marked-fibrant semi-simplicial space which is "close enough" to a quasi-unital marked semiSegal space is itself a quasi-unital marked semiSegal space. This will be useful for us when constructing completions (see §3.1).

Lemma 2.2.16 *Let X be a quasi-unital marked semiSegal space, W a marked-fibrant semi-simplicial space and $f : X \longrightarrow W$ a fully-faithful map such that $f_0 : X_0 \longrightarrow W_0$ is surjective on connected components. Then W is a quasi-unital marked semiSegal space and f is a DK-equivalence.*

Proof The fact that f is a DK-equivalence follows directly from the definition. Hence it will suffice to prove that W is a quasi-unital marked semiSegal space.

We can factor f as $X \xrightarrow{f'} X' \xrightarrow{f''} W$ such that f' is a trivial marked cofibration and f'' is a marked fibration. Then X' is marked-fibrant and marked-equivalent to X, so that X' is necessarily a quasi-unital marked semi-simplicial space and f'' is fully-faithful. Hence we can assume without loss of generality that f itself is a marked fibration and that $f_0 : X_0 \longrightarrow W_0$ is surjective.

We start by showing that W is a marked semiSegal space. Let $f : Y \longrightarrow Z$ be a map in $\mathbf{S}$ (see Definition 2.1.4). We need to show that the map

$$\mathrm{Map}^+(Z, W) \longrightarrow \mathrm{Map}^+(Y, W)$$

is a weak equivalence. Note that in all cases the 0'th level map $f_0 : Y_0 \longrightarrow Z_0$ is an isomorphism. Condition (4) of Proposition 2.2.5 then tells us that

$$\begin{array}{ccc} \mathrm{Map}^+(Z, X) & \longrightarrow & \mathrm{Map}^+(Z, W) \\ \downarrow & & \downarrow \\ \mathrm{Map}^+(Y, X) & \longrightarrow & \mathrm{Map}^+(Y, W) \end{array}$$

is homotopy Cartesian. Furthermore, all maps appearing in this diagrams are fibrations. We wish to show that the right vertical fibration is trivial. Since X is a marked semiSegal space the left vertical Kan fibration is trivial. Now since Kan fibrations are trivial if and only if all their fibers are contractible it will be enough to show that the lower horizontal map is surjective. In light of Corollary 2.2.6 it will be enough to show that the map

$$\mathrm{Map}_{\mathrm{S}}(Y_0, X_0) \longrightarrow \mathrm{Map}_{\mathrm{S}}(Y_0, W_0)$$

is surjective. But this just follows from the fact that Y_0 is discrete and the map $X_0 \longrightarrow W_0$ is surjective.

We now show that W is quasi-unital. According to Lemma 2.0.13 we need to show that the map $W \longrightarrow *$ (where $*$ is the terminal marked semi-simplicial space) has the right lifting property with respect to the maps $\theta_0, \theta_1, \theta_2$. Let D_i be the domain of θ_i. Since the D_i's are levelwise discrete and the map $X_0 \longrightarrow W_0$ is surjective we can use Corollary 2.2.6 in order to lift any map $D_i \longrightarrow W$ to a map $D_i \longrightarrow X$. Since X is quasi-unital it satisfies the right lifting property with respect to each θ_i and so the result follows. □

2.3 Q-fibrations and Q-anodyne maps

In the beginning of § 2 we saw that the property of being quasi-unital can be expressed as a certain right lifting property (see Lemma 2.0.13). This idea leads one to define the following relative version of quasi-unitality:

Definition 2.3.1 Let $f : W \longrightarrow Z$ be a map of marked semi-simplicial spaces. We will say that f is a **Q-fibration** if it is an MS-fibration and satisfies the right lifting property with respect to the maps $\theta_0, \theta_1, \theta_2$ of Definition 2.0.12.

Example 2.3.2 Let W be a quasi-unital marked semiSegal space. Then the terminal map $W \longrightarrow *$ is a Q-fibration.

The notion of Q-fibrations has a left-hand-side counterpart:

Definition 2.3.3 Let $f : X \longrightarrow Y$ be an MS-cofibration. We will say that f is **Q-anodyne** if it satisfies the left lifting property with respect to all Q-fibrations.

Example 2.3.4 Since any Q-fibration is an MS-fibration we see that any **trivial** MS-cofibration is Q-anodyne

Remark 2.3.5 The maps θ_0, θ_1 and θ_2 are Q-anodyne. In fact, since $\mathbb{S}_+^{\Delta_s^{\mathrm{op}}}$ is presentable one can identify the collection of all Q-anodyne maps with the weakly saturated class of maps generated by $\theta_0, \theta_1, \theta_2$.

Remark 2.3.6 Since the class of Q-anodyne maps is weakly saturated and contains all trivial MS-cofibrations it is also closed under certain **homotopy pushouts**. More precisely, if we have a homotopy pushout square

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

in Seg_s such that the left vertical map is Q-anodyne then the right vertical map will be Q-anodyne as well as long as the square is **Reedy cofibrant**, i.e., as long as the induced map $Y \coprod_X Z \longrightarrow W$ is an MS-cofibration.

Remark 2.3.7 Although one cannot identify QsS with the subcategory of fibrant objects in some localization of $\mathcal{S}^{\Delta_s^{\mathrm{op}}}_+$, one can still associate with QsS the **weak factorization system** formed by Q-fibrations and Q-anodyne maps. Although not part of a model category, it enables many model categorical arguments and manipulations. The purpose of this section is to exploit this point of view to obtain results which will be used in the next section.

Lemma 2.3.8 *Let $f : X \longrightarrow Y$ be a Q-anodyne map. Then f satisfies the equivalent conditions of Corollary 2.2.8.*

Proof In view of Remark 2.2.9 it will suffice to prove that the θ_i' satisfy condition (4) of Corollary 2.2.8. For θ_2 this is a special case of Remark 2.2.10. For $\theta_i, i = 0, 1$ we need to check that the maps

$$\left[(\Delta^1)^\sharp \times \partial\Delta^1\right] \coprod_{\Delta^{\{i\}}\times\partial\Delta^1} \left[\Delta^{\{i\}} \times (\Delta^1)^\flat\right] \longrightarrow (\Delta^1)^\sharp \otimes (\Delta^1)^\flat$$

and

$$\left[(\Delta^1)^\sharp \times \partial\Delta^1\right] \coprod_{\Delta^{\{i\}}\times\partial\Delta^1} \left[\Delta^{\{i\}} \times (\Delta^1)^\sharp\right] \longrightarrow (\Delta^1)^\sharp \otimes (\Delta^1)^\sharp$$

are trivial MS-cofibrations. Now in the first map the right hand side can be obtained from the left hand side by performing two pushouts along admissible horn inclusions of dimension 2. In the second map one needs to perform in addition a triangle remarking (see Definition 2.1.6). □

Corollary 2.3.9 *Let $f : X \longrightarrow Y$ be a Q-anodyne map and let $p : W \longrightarrow Z$ be a Q-fibration. Then the induced map*

$$f^p : W^Y \longrightarrow Z^Y \times_{Z^X} W^X$$

is a DK-equivalence.

Proof From Lemma 2.3.8 it follows that f^p is a fully-faithful MS-fibration. Since f satisfies the left lifting property with respect to p we get that the induced map of sets

$$\left(W^Y\right)_{0,0} = \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(Y, Z) \times_{\mathrm{Hom}(X,Z)} \mathrm{Hom}(X, W) = \left(Z^Y \times_{Z^X} W^X\right)_{0,0}$$

is surjective. It hence follows that f is essentially surjective and so a DK-equivalence. □

We will now apply some of the ideas collected so far in order to prove that the full subcategory $\mathrm{QsS} \subseteq \mathrm{Seg}_s^{\mathrm{fib}}$ is closed under mapping objects. In fact, we will prove that for any semi-simplicial space A and any quasi-unital marked semiSegal space W the mapping object W^A is quasi-unital.

Proposition 2.3.10 *Let W be a quasi-unital marked semiSegal space and A a marked semi-simplicial space. Then W^A is quasi-unital.*

Proof Let $f : X \longrightarrow Y$ be a Q-anodyne map. To show that the terminal map $W^A \longrightarrow *$ satisfies the right lifting property with respect to f is equivalent to showing that the map $f \otimes A : X \otimes A \longrightarrow Y \otimes A$ satisfies the left lifting property with respect to $W \longrightarrow *$.

Consider the space A_0 as a marked semi-simplicial space concentrated in degree 0. We have a natural inclusion $g : A_0 \hookrightarrow A$ such that g_0 is an isomorphism. By Lemma 2.3.8 we get that the natural map

$$[X \otimes A] \coprod_{X \otimes A_0} [Y \otimes A_0] \longrightarrow Y \otimes A$$

is a trivial MS-cofibration. Hence it will suffice to prove that the map $f \otimes A_0$ satisfies the left lifting property with respect to $W \longrightarrow *$, and it will suffice to do so for $f = \theta_i$,

Let us start with θ_2. Let $A_{0,0}$ be the set of vertices of A_0. Then $\theta_2 \otimes A_{0,0}$ satisfies the left lifting property with respect to $W \longrightarrow *$. The same claim for $\theta_2 \otimes A_0$ follows from the fact that

$$\left[A_0 \otimes \left(\Delta^3, M\right)\right] \coprod_{A_{0,0} \otimes \left(\Delta^3, M\right)} \left[A_{0,0} \otimes \left(\Delta^3\right)^\sharp\right] \hookrightarrow A_0 \otimes \left(\Delta^3\right)^\sharp$$

is a marked equivalence (where $M = \{\Delta^{\{0,2\}}, \Delta^{\{1,3\}}\}$).

Let us now prove the case θ_i for $i = 0, 1$. Unwinding the definitions we need to show that the map of spaces $d_i : W_1^{\mathrm{inv}} \longrightarrow W_0$ satisfies the right lifting property with respect to any map of the form $\emptyset \hookrightarrow A_0$, or, equivalently, admits a section.

Let

$$W_1^{\mathrm{aut}} = \{f \in W_1^{\mathrm{inv}} \mid d_0(f) = d_1(f)\} \subseteq W_1^{\mathrm{inv}}$$

be the subspace of self equivalences. It will be enough to show that the map $d : W_1^{\mathrm{aut}} \longrightarrow W_0$ (induced by either d_0 or d_1) admits a section. Since this claim involves only marked edges it will be convenient to switch to the maximal semiKan space $Z = \widetilde{W} \subseteq W$. In particular, we want to show that the natural map $Z_1^{\mathrm{aut}} \longrightarrow Z$ admits a section.

Consider the Kan replacement $\widehat{|Z|}$ of the realization of Z. Let $\widehat{|Z|}^{S^1}$ be the space of continuous paths $\gamma : S^1 \longrightarrow \widehat{|Z|}$ and let $p : \widehat{|Z|}^{S^1} \longrightarrow \widehat{|Z|}$ be the map $p(\gamma) = \gamma(1)$. Consider the commutative diagram

$$\begin{array}{ccc} Z_1^{\mathrm{aut}} & \longrightarrow & \widehat{|Z|}^{S^1} \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle p} \\ Z_0 & \longrightarrow & \widehat{|Z|} \end{array}$$

Since $\widehat{|Z|}$ is a Kan simplicial set the right vertical map is a fibration. The left vertical map is pullback of the fibration $(d_0, d_1) : Z_1 \longrightarrow Z_0 \times Z_0$ along the diagonal $Z_0 \hookrightarrow Z_0 \times Z_0$, and is hence a fibration as well. By Theorem 1.6.4 the square is homotopy Cartesian. Now since the right vertical map admits a section (given by choosing for each x the constant path at x) we get that the left vertical map admits a section as well. This finishes the proof of Proposition 2.3.10. $\square$

Corollary 2.3.11 *Let $f : X \longrightarrow Y$ be a Q-anodyne map and W a quasi-unital marked semiSegal space. Then the map $W^Y \longrightarrow W^X$ admits a section.*

Proof It is enough to prove that the map $W^Y \longrightarrow W^X$ satisfies the right lifting property with respect to every map of the form $\emptyset \hookrightarrow A$. Using the exponential law we see that this is equivalent to saying that $W \longrightarrow *$ satisfies the right lifting property with respect to $X \otimes A \longrightarrow Y \otimes A$. Applying the exponential law again we reduce to proving that $W^A \longrightarrow *$ satisfies the right lifting property with respect to $f : X \longrightarrow Y$. The result now follows from Proposition 2.3.10 as W^A is quasi-unital and f is Q-anodyne. □

Our final goal of this subsection is to show that the following types of maps are Q-anodyne. Let $f : [n] \longrightarrow [k]$ be a **surjective** map in Δ and let $h : [k] \longrightarrow [n]$ be a section of f. Let $M \subseteq (\mathrm{Sp}^k)_1$ be a marking on the k-spine and let $\widetilde{M} \subseteq (\Delta^k)_1$ be the marking generated from it, i.e., the smallest set containing M which is closed under 2-out-of-3. Let $M_f \subseteq \left(\mathrm{Sp}^n\right)_1$ be the set of all pairs $\{i, i+1\}$ such that either $f(i) = f(i+1)$ or $\Delta^{\{f(i),f(i+1)\}}$ is in M and let $\widetilde{M}_f \subseteq (\Delta^n)_1$ be the marking generated from it. We wish to prove the following:

Proposition 2.3.12 *In the notation above, the map*

$$h : \left(\Delta^k, \widetilde{M}\right) \longrightarrow \left(\Delta^n, \widetilde{M}_f\right)$$

is Q-anodyne.

Proof For each $i = 0, ..., n$ let $S_i \subseteq \Delta^n$ be the 1-dimensional semi-simplicial subset containing all the vertices and all the edges of the form $\Delta^{\{j,j+1\}}$ such that $f(j) = f(j+1) = i$. Then clearly the inclusion $\Delta^{\{h(i)\}} \subseteq S_i$ is Q-anodyne. Let $S \subseteq \Delta^n$ be the (disjoint) union of all the S_i's.

Let h_1, h_2 be two sections of f. We will define the marked semi-simplicial subset $T(h_1, h_2) \subseteq \left(\Delta^n, \widetilde{M}_f\right)$ to be the (not necessarily disjoint) union of $S^\sharp \subseteq \left(\Delta^n, \widetilde{M}_f\right)$ and all edges of the form $\Delta^{\{h_1(i),h_2(i+1)\}}$ for $i = 0, ..., n$. In particular,

$$T(h_1, h_2) = \left(S \cup \bigcup_i \Delta^{\{h_1(i),h_2(i+1)\}}, N\right) \subseteq \left(\Delta^n, \widetilde{M}_f\right)$$

where N is the marking induced from $\widetilde{M}_f$, i.e., N contains all the edges of S and the edges $\Delta^{\{h_1(i),h_2(i+1)\}}$ for which $\Delta^{\{i,i+1\}}$ is in M. Note that when $h_1 = h_2 = h$ we have

$$T(h, h) = S^\sharp \coprod_{\Delta^0 \times \{0,\dots,k\}} \left(h(\mathrm{Sp}^k), h(M)\right).$$

Now consider the commutative square

$$\begin{array}{ccc} \left(\mathrm{Sp}^k, M\right) & \xrightarrow{h} & \left(\Delta^k, \widetilde{M}\right) \\ \downarrow & & \downarrow \\ T(h, h) & \longrightarrow & \left(\Delta^n, \widetilde{M}_f\right) \end{array}$$

Since $h : \Delta^0 \times \{0, ..., k\} \hookrightarrow S^\sharp$ is Q-anodyne we get that the top horizontal row is Q-anodyne. In light of Remark 2.3.6 it will now be enough to show that this square is a Reedy cofibrant homotopy pushout square in the marked Segal model structure. As Reedy cofibrancy is immediate it will be enough to show that both horizontal maps are trivial MS-cofibrations. The top horizontal map is very easy:

Lemma 2.3.13 *The inclusion $\iota : \left(\mathrm{Sp}^k, M\right) \hookrightarrow \left(\Delta^k, \widetilde{M}\right)$ is a trivial MS-cofibration.*

Proof Factor ι as

$$\left(\mathrm{Sp}^k, M\right) \overset{\iota'}{\hookrightarrow} \left(\Delta^k, M\right) \overset{\iota''}{\hookrightarrow} \left(\Delta^k, \widetilde{M}\right).$$

Then ι' is a pushout along the trivial MS-cofibration $\left(\mathrm{Sp}^k\right)^\flat \hookrightarrow \left(\Delta^k\right)^\flat$ and ι'' is a triangle remarking (Definition 2.1.6). □

To show that bottom horizontal map is a trivial MS-cofibration it will be convenient to prove a slightly stronger lemma:

Lemma 2.3.14 *For any two sections h_1, h_2 of f the inclusion*

$$T(h_1, h_2) \subseteq \left(\Delta^m, \widetilde{M}_f\right)$$

is a trivial MS-cofibration.

Proof We begin by arguing that it is enough to prove the lemma for just one pair of sections h_1, h_2. We say that two pairs $(h_1, h_2), (h_1', h_2')$ are **neighbours** if

$$\sum_{i=0}^{n} |h_1(i) - h_1'(i)| + |h_2(i) - h_2'(i)| = 1.$$

It is not hard to see that the resulting neighbouring graph is connected, i.e., that we can get from any pair (h_1, h_2) to any other pair (h_1', h_2') by a sequence of pairs such that each consecutive couple of pairs are neighbours. Hence it is enough to show that the property of $T(h_1, h_2) \hookrightarrow \left(\Delta^m, \widetilde{M}_f\right)$ being a trivial MS-cofibration respects the neighbourhood relation. To see why this is true observe that if (h_1, h_2) and (h_1', h_2') are neighbours then one can add to $T(h_1, h_2)$ a single triangle $\sigma \subseteq \Delta^m$ such that $R \overset{\text{def}}{=} T(h_1, h_2) \cup \sigma$ contains $T(h_1', h_2')$ and such that R can be obtained from either $T(h_1, h_2), T(h_1', h_2')$ by performing a pushout along a 2-dimensional admissible marked horn inclusion and possibly a remarking. Hence the claim for either $T(h_1, h_2)$ or $T(h_1', h_2')$ is equivalent to $R \hookrightarrow \left(\Delta^m, \widetilde{M}_f\right)$ being a trivial MS-cofibration.

Now that we know that it is enough to prove for a single choice of (h_1, h_2) let us choose the pair $h_{\max}(i) = \max(f^{-1}(i))$ and $h_{\min}(i) = \min(f^{-1}(i))$. Then we see that $T(h_{\max}, h_{\min}) = \left(\mathrm{Sp}^m, M_f\right)$ and the map $\left(\mathrm{Sp}^m, M_f\right) \hookrightarrow \left(\Delta^m, \widetilde{M}_f\right)$ is a trivial MS-cofibration from Lemma 2.3.13. □

This finishes the proof of Proposition 2.3.12. □

3 Complete marked semiSegal spaces

In this section we will further localize the model category Seg_s to obtain our target model category Comp_s. We will then show the underlying ∞-category $(\mathrm{Comp}_s)_\infty$ is equivalent to the ∞-category $\mathrm{Cat}_\infty^{\mathrm{qu}}$ of quasi-unital ∞-categories (see Definition 2.2.14). Finally, we will prove the main theorem of this paper by showing that Comp_s is Quillen equivalent to the Rezk's model category Comp, and that this Quillen equivalence preserves mapping objects.

We begin with a description of the fibrant objects in Comp_s, which are called **complete marked semiSegal spaces**. The notion of completeness, the construction of the completion functor and many of the related proofs are inspired by their respective analogues in [14]. We begin with the basic definition:

Definition 3.0.15 Let (X, M) be a marked semiSegal space. We will say that X is **complete** if the following two conditions are satisfied:

(1) X is quasi-unital.

(2) The restricted maps $d_0 : X_1^{\mathrm{inv}} \longrightarrow X_0$ and $d_1 : X_1^{\mathrm{inv}} \longrightarrow X_0$ are both homotopy equivalences.

Remark 3.0.16 If W is a complete marked semiSegal space then $\widetilde{W}$ is complete as well.

Remark 3.0.17 If W is a marked semiKan space then W is complete if and only if it is homotopy-constant as a semi-simplicial space (i.e., levelwise equivalent to a constant semi-simplicial space). This follows from the Segal condition and the fact that Δ_s is weakly contractible (see Lemma 3.2.7 below).

Let $\mathrm{CsS} \subseteq \mathrm{QsS}$ denote the full simplicial subcategory spanned by complete marked semiSegal spaces. Our next goal is to construct a suitable left Bousfield localization Comp_s of Seg_s such that CsS will coincide with the full subcategory of fibrant (and cofibrant) objects in Comp_s.

We begin with the following characterization of complete marked semiSegal spaces (compare with Lemma 2.0.13):

Lemma 3.0.18 *Let (X, M) be a marked semiSegal space. Then X is complete if and only if X is local with respect to the class of Q-anodyne maps (see Definition 2.3.3).*

Proof The class of Q-anodyne maps is the weakly saturated class of maps generated by $\theta_0, \theta_1, \theta_2$ (see Definition 2.0.12 and Remark 2.3.5). Hence to be local with respect to Q-anodyne maps is equivalent to being local with respect to $\theta_0, \theta_1, \theta_2$. Let us first assume that X is complete. Then in particular X is quasi-unital and so $M = X_1^{\mathrm{inv}}$. It then follows from condition (2) of Definition 3.0.15 that (X, M) is local with respect to θ_0, θ_1. Furthermore, by Lemma 1.5.4 we now get that M satisfies the 2-out-of-6 property and hence (X, M) is local with respect to θ_2.

Now assume that X is local with respect to $\theta_0, \theta_1, \theta_2$, so that $d_0, d_1 : M \longrightarrow X_0$ are equivalences. In light of Lemma 2.0.13 we may deduce that X is quasi-unital unital, and hence in particular $M = X_1^{\mathrm{inv}}$. This shows that X is complete. □

Since Seg_s is combinatorial and left proper the left Bousfield localization of Seg_s with respect to the maps $\theta_0, \theta_1, \theta_2$ exists. We will denote the resulting localization by Comp_s. Then Comp_s is a combinatorial model category satisfying the following properties:

(1) A map $f : X \longrightarrow Y$ of marked semi-simplicial spaces is an equivalence in Comp_s if and only if for every complete marked semiSegal space W the induced map

$$\mathrm{Map}_+(Y, W) \longrightarrow \mathrm{Map}_+(X, W)$$

is a weak equivalence.

(2) A map $f : X \longrightarrow Y$ of marked semi-simplicial spaces is a cofibration in Comp_s if and only if it is a cofibration in Seg_s (i.e. a levelwise inclusion).

(3) A marked semi-simplicial space W is fibrant in Comp_s if and only if it is a complete marked semiSegal space.

Remark 3.0.19 Since CsS can be identified with the full simplicial subcategory of fibrant-cofibrant objects in Comp_s we may conclude that the underlying ∞-category $(\mathrm{Comp}_s)_\infty$ is equivalent to the simplicial nerve $\mathcal{N}(\mathrm{CsS})$.

Theorem 3.0.20 *The complete model structure is compatible with the marked monoidal product $\otimes$. In particular, the localization Quillen adjunction*

$$\mathrm{Seg}_+^{\Delta_s^{\mathrm{op}}} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathrm{Comp}_s$$

is strongly monoidal and Comp_s *inherits the simplicial structure of* Seg_s.

Proof Arguing as in [14] Proposition 9.2, we see that it will be enough to establish the following:

Proposition 3.0.21 *Let X be a marked semi-simplicial space and W a complete marked semiSegal space. Then W^X is complete.*

Proof From Proposition 2.3.10 we get that W^X is quasi-unital. In particular, the marked edges of W^X are exactly the equivalences. Let us now prove that W^X satisfies condition (2) of Definition 3.0.15.

For $i = 0, 1$ consider the restriction map

$$p^i : W^{\left(\Delta^1\right)^\sharp} \longrightarrow W^{\left(\Delta^0\right)} = W.$$

Since W is complete we get by definition that the maps

$$p_0^i : \left(W^{\left(\Delta^1\right)^\sharp}\right)_0 \longrightarrow W_0$$

are weak equivalences. By Corollary 2.3.9 we get that p^i is also a DK-equivalence and hence a marked equivalence. Using the exponential law this implies that the restriction map

$$\mathrm{Map}^+\left(X \otimes \left(\Delta^1\right)^\sharp, W\right) \longrightarrow \mathrm{Map}^+\left(X \otimes \Delta^{\{i\}}, W\right)$$

is a weak equivalence. Applying the exponential law again we get that W^X satisfies condition (2) of Definition 3.0.15. □

□

Remark 3.0.22 In the above notation, since W^X is complete we get that $\widetilde{W^X}$ is complete as well. In particular, $\widetilde{W^X}$ is a homotopy-constant marked semi-simplicial space (see Remark 3.0.17).

3.1 Completion

In this section we will prove that $(\mathrm{Comp}_s)_\infty = \mathcal{N}(\mathrm{CsS})$ is equivalent to the left ∞-localization of $\mathcal{N}(\mathrm{QsS})$ with respect to DK-equivalences (see Definition 1.2.2). In particular, $(\mathrm{Comp}_s)_\infty$ is equivalent to the ∞-category $\mathrm{Cat}_\infty^{\mathrm{qu}}$ of quasi-unital ∞-categories. In order to prove this we will construct a **completion functor**

$$\widehat{\bullet} : \mathrm{QsS} \longrightarrow \mathrm{CsS}$$

such that

(1) The induced map of ∞-categories

$$\mathcal{N}(\widehat{\bullet}) : \mathcal{N}(\mathrm{QsS}) \longrightarrow \mathcal{N}(\mathrm{CsS})$$

is left adjoint to the (fully-faithful) inclusion $\mathcal{N}(\mathrm{CsS}) \hookrightarrow \mathcal{N}(\mathrm{QsS})$.

(2) A map in $\mathcal{N}(\mathrm{QsS})$ is a DK-equivalence if and only if its image under $\mathcal{N}(\widehat{\bullet})$ is an equivalence.

To construct the completion functor we proceed as in [14]. Let X be a quasi-unital marked semiSegal space. Consider the bi-semi-simplicial spaces $X_{\bullet,\bullet}, Y_{\bullet,\bullet} : \Delta_s^{\mathrm{op}} \times \Delta_s^{\mathrm{op}} \longrightarrow \mathcal{S}$ given by

$$X_{n,m} = \mathrm{Map}^+\left(\left(\Delta^n\right)^\flat \otimes \left(\Delta^m\right)^\sharp, X\right)$$

and

$$Y_{n,m} = \mathrm{Map}((\Delta^n)^\sharp \otimes (\Delta^m)^\sharp, X).$$

We define the marked semi-simplicial space $\left(\overline{X}, M\right)$ by setting

$$\overline{X}_n = |X_{n,\bullet}|$$

and

$$M = |Y_{1,\bullet}| \subseteq |X_{1,\bullet}|.$$

We then define the **completion** $\widehat{X}$ of X to be the **marked-fibrant replacement** of $\left(\overline{X}, M\right)$.

Theorem 3.1.1 *Let X be a quasi-unital marked semiSegal space. Then*

(1) $\widehat{X}$ is a complete marked semiSegal space.

(2) The natural map $X \longrightarrow \widehat{X}$ is a DK-equivalence.

Proof Let $n \geq 0$ be an integer. From Proposition 2.3.12 we conclude that for each map $f : [k] \longrightarrow [m]$ in Δ_s the map

$$f^* : X_{\bullet,m} \longrightarrow X_{\bullet,k}$$

is a DK-equivalence and in particular fully-faithful. Hence the induced square

$$\begin{array}{ccc} X_{n,m} & \longrightarrow & (X_{0,m})^{n+1} \\ \downarrow{\scriptstyle f_n^*} & & \downarrow{\scriptstyle (f_0^*)^{n+1}} \\ X_{n,k} & \longrightarrow & (X_{0,k})^{n+1} \end{array}$$

is homotopy Cartesian. From Corollary 1.6.8 the natural map

$$\left| X_{0,\bullet}^{n+1} \right| \longrightarrow |X_{0,\bullet}|^{n+1}$$

is a weak equivalence and so Puppe's Theorem (see Theorem 1.6.5) implies that the square

$$\begin{array}{ccc} X_{n,0} & \longrightarrow & (X_{0,0})^{n+1} \\ \downarrow & & \downarrow \\ |X_{n,\bullet}| & \longrightarrow & |X_{0,\bullet}|^{n+1} \end{array}$$

is homotopy Cartesian. The same argument with $Y_{\bullet,\bullet}$ instead of $X_{\bullet,\bullet}$ shows that the square

$$\begin{array}{ccc} Y_{1,0} & \longrightarrow & (Y_{0,0})^2 \\ \downarrow & & \downarrow \\ |Y_{1,\bullet}| & \longrightarrow & |Y_{0,\bullet}|^2 \end{array}$$

is homotopy Cartesian. This implies that the map $X \longrightarrow \overline{X}$ (and hence also the map $X \longrightarrow \widehat{X}$) is fully-faithful. We now observe that the map $X_0 \longrightarrow \widehat{X}_0$ is surjective on connected components. Since $\widehat{X}$ is marked-fibrant we deduce from Lemma 2.2.16 that $\widehat{X}$ is a quasi-unital marked semiSegal space and the map $X \longrightarrow \widehat{X}$ is a DK-equivalence.

It is left to show that $\widehat{X}$ is complete. Since $\widehat{X}$ is quasi-unital we know that all invertible edges in $\widehat{X}$ are marked. Hence it will suffice to show that the maps

$$|d_0|, |d_1| : |Y_{1,\bullet}| \longrightarrow |Y_{0,\bullet}| = \overline{X}_0$$

are weak equivalences. But this follows from Corollary 1.6.7 since the maps $d_0, d_1 : Y_{1,\bullet} \longrightarrow Y_{0,\bullet}$ are DK-equivalences of marked semiKan spaces. $\square$

Our goal now is to show that the completion functor exhibits $\mathcal{N}(\mathrm{CsS})$ as the left ∞-localization of $\mathcal{N}(\mathrm{QsS})$ with respect to DK-equivalences. We start by showing that the notions of DK-equivalence and marked equivalence coincide in CsS.

Proposition 3.1.2 *Let $f : X \longrightarrow Y$ be a DK-equivalence between complete marked semiSegal spaces. Then f is a marked equivalence.*

Proof Since f is in particular fully-faithful it will be enough to show that the map $f_0 : X_0 \longrightarrow Y_0$ is a weak equivalence. Let $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ be the induced map between the corresponding maximal semiKan subspaces. Then clearly $\widetilde{f}$ is a DK-equivalence as well. From Corollary 1.6.7 it follows that the induced map

$$\left|\widetilde{f}\right| : \left|\widetilde{X}\right| \longrightarrow \left|\widetilde{Y}\right|$$

is a weak equivalence. But since X, Y are complete their corresponding maximal semiKan space are homotopy-constant and so their realization is naturally equivalent to their space of objects. It follows that f_0 is an equivalence and we are done. □

We are now ready to state the main theorem of this subsection:

Theorem 3.1.3 *The induced functor $\mathcal{N}(\widehat{\bullet}) : \mathcal{N}(\mathrm{QsS}) \longrightarrow \mathcal{N}(\mathrm{CsS})$ exhibits $\mathcal{N}(\mathrm{CsS})$ as the left ∞-localization of $\mathcal{N}(\mathrm{QsS})$ with respect to DK-equivalences.*

We will postpone the proof of Theorem 3.1.3 until the end of this subsection. Before that, we will need to establish some way to spot weak equivalences in Comp_s. This will be achieved using a weak notion of cylinder object:

Definition 3.1.4 Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$ and object. We will say that a cofibration of the form

$$X \coprod X \xrightarrow{\iota_0 \coprod \iota_1} IX$$

exhibits IX as a **weak cylinder object** for X if the two maps $\iota_0, \iota_1 : X \longrightarrow IX$ are weak equivalences which become equal in $\mathrm{Ho}(\mathcal{M})$. Given a weak cylinder object as above and two maps $f, g : X \longrightarrow Y$ we will say that f, g are **homotopic** via IX if the corresponding map

$$X \coprod X \xrightarrow{f \coprod g} Y$$

factors through IX. This notion is in general stronger then f, g being equal in $\mathrm{Ho}(\mathcal{M})$.

Our reason for introducing this notion is that in the model category Comp_s we have very natural choices for weak cylinder object, namely:

Lemma 3.1.5 *Let X be a marked semi-simplicial space. Then the natural map*

$$X \coprod X \hookrightarrow X \otimes \left(\Delta^1\right)^\sharp$$

exhibits $X \otimes \left(\Delta^1\right)^\sharp$ as a weak cylinder object of X.

Proof Clearly the map in question is a cofibration. It will hence be enough to show that the two maps $\iota_0, \iota_1 : \Delta^0 \longrightarrow \left(\Delta^1\right)^\sharp$ are equal in $\mathrm{Ho}(\mathrm{Comp}_s)$. Let $\rho : \left(\Delta^1\right)^\sharp \longrightarrow W$ be a fibrant replacement of $\left(\Delta^1\right)^\sharp$ in Comp_s. We need to show that $\rho \circ \iota_0$ and $\rho \circ \iota_1$ are in the same connected component of $\mathrm{Map}_+(\Delta^0, W) = W_0$. But this is clear because the map ρ determines a marked edge from $\rho \circ \iota_0$ to $\rho \circ \iota_1$ and W is complete (so that $\widetilde{W}$ is homotopy-constant). □

The notion of homotopy between maps which is associated to the above choice of weak cylinder objects will be called $\left(\Delta^1\right)^\sharp$**-homotopy**. There is a corresponding notion of a $\left(\Delta^1\right)^\sharp$**-homotopy equivalence**, which in general is stronger then being a weak equivalence in Comp_s. These types of equivalences are analogous to the notion of **categorical equivalences** in [14].

Proposition 3.1.6 *Let $p : W \longrightarrow Z$ be a DK-equivalence between quasi-unital marked semiSegal spaces which admits a section $g : Z \longrightarrow W$. Then f is a weak equivalence in* Comp_s.

Proof We can assume without loss of generality that f is a marked fibration. We claim that g is a homotopy inverse of f. On one direction the composition $f \circ g$ is the identity. We need to show that $g \circ f$ is equivalent to the identity Comp_s. For this it will suffice to produce a $\left(\Delta^1\right)^\sharp$-homotopy from $g \circ f$ to the identity, or in other words a marked edge from $g \circ f$ to the identity in W^W.

Since the mapping object Z^W is quasi-unital (Proposition 2.3.9) there exists a marked edge $h \in (Z^W)_1$ from f to itself. The edge h corresponds to a map

$$h : W \otimes \left(\Delta^1\right)^\sharp \longrightarrow Z$$

whose restriction to each $Z \otimes \Delta^{\{i\}}$ is f. Now consider the commutative square

$$\begin{array}{ccc} W \times \partial\Delta^1 & \longrightarrow & W \\ \downarrow & \overset{\widetilde{h}}{\nearrow} & \downarrow f \\ W \times \left(\Delta^1\right)^\sharp & \xrightarrow{h} & Z \end{array}$$

where the top horizontal map is given by $(g \circ f) \coprod \mathrm{Id}$. Since the right vertical map is a fully-faithful marked fibration and the left vertical map is a cofibration which induces an isomorphism on the 0'th level we get from Proposition 2.2.5 that a lift $\widetilde{h} : W \times \left(\Delta^1\right)^\sharp \longrightarrow W$ indeed exists. Then $\widetilde{h}$ gives an equivalence from $g \circ f$ to the identity in W^W and we are done. □

Corollary 3.1.7 *Let $f : X \hookrightarrow Y$ be a Q-anodyne map and let W be a quasi-unital marked semiSegal space. Then the map $f^* : W^Y \longrightarrow W^X$ is an equivalence in* Comp_s.

Proof From Proposition 2.3.9 and Corollary 2.3.11 we know that if $f : X \longrightarrow Y$ is Q-anodyne then the map $W^Y \longrightarrow W^X$ is a DK-equivalence which admits a section. The desired result now follows from Proposition 3.1.6. □

Proof of Theorem 3.1.3 From the second part of Theorem 3.1.1 we get that a map $f : X \longrightarrow Y$ of quasi-unital semiSegal spaces is a DK-equivalence if and only if $\widehat{f} : \widehat{X} \longrightarrow \widehat{Y}$ is a DK-equivalence. In view of Proposition 3.1.2 we deduce that the collection of maps sent by $\widehat{\bullet}$ to equivalences are precisely the DK-equivalences. Hence it is left to prove that $\mathcal{N}(\widehat{\bullet})$ is indeed a left adjoint to the inclusion $\mathcal{N}(\mathrm{CsS}) \hookrightarrow \mathcal{N}(\mathrm{QsS})$. For this it will be enough to show that the natural map

$$X \longrightarrow \widehat{X}$$

is a weak equivalence in Comp_s. Indeed, in this case the restriction map

$$\mathrm{Map}_{\mathrm{CsS}}\left(\widehat{X}, W\right) \longrightarrow \mathrm{Map}_{\mathrm{QsS}}(X, W)$$

will be a weak equivalence for every $W \in \mathrm{CsS}$ and hence we would be able to consider $X \longrightarrow \widehat{X}$ as a unit transformation (see [11, Proposition 5.2.2.8]). Now the marked semi-simplicial space $\widehat{X}$ is the homotopy colimit of the Δ_s^{op}-diagram $[m] \mapsto X_{\bullet,m}$ where $X_{\bullet,0} = X$. Since Δ_s is weakly contractible it will suffice to show that for each $\rho : [k] \longrightarrow [n]$ in Δ_s the natural map

$$\rho^* : X_{\bullet,n} = X^{(\Delta^n)^\sharp} \longrightarrow X^{\left(\Delta^k\right)^\sharp} = X_{\bullet,k}$$

is a weak equivalence in Comp_s. Now from Proposition 2.3.12 we know that the inclusion

$$\left(\Delta^k\right)^\sharp \hookrightarrow \left(\Delta^n\right)^\sharp$$

is Q-anodyne. The desired result now follows from Corollary 3.1.7. □

3.2 Proof of the main theorem

Let Comp denote Rezk's model category of complete Segal spaces (so that the underlying category of Comp is the category $\mathcal{S}^{\Delta^{\mathrm{op}}}$ of simplicial spaces). Recall the Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathcal{RK}^+}{\overset{\mathcal{F}^+}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

described in § 1.8. The purpose of this section is to prove the following theorem, which is the main result of this paper:

Theorem 3.2.1 *The Quillen adjunction $\mathcal{F}^+ \dashv \mathcal{RK}^+$ descends to a Quillen equivalence*

$$\mathrm{Comp} \underset{\mathcal{RK}^+}{\overset{\mathcal{F}^+}{\rightleftarrows}} \mathrm{Comp}_s \tag{3–1}$$

Note that a-priori it is not even clear that this is a Quillen adjunction. Even though $\mathcal{F}^+$ preserves cofibrations, to show that it preserves trivial cofibrations is equivalent to showing that $\mathcal{RK}^+$ maps complete marked semiSegal spaces to complete Segal spaces. Fortunately, this claim as well as the desired Quillen equivalence will both follow from the following theorem:

Theorem 3.2.2 *Let X be a complete marked semiSegal space. Then the counit map*

$$\nu_X : \mathcal{F}^+\left(\mathcal{RK}^+(X)\right) \longrightarrow X$$

is a marked equivalence.

Before we proceed to prove Theorem 3.2.2 let us derive two short corollaries of it, which together imply that 3–1 is a Quillen equivalence. We start with the following observation:

Corollary 3.2.3 *Assume Theorem 3.2.2 and let X be a complete marked semiSegal space. Then $\mathcal{RK}^+(X)$ is a complete Segal space.*

Proof First since X is marked-fibrant we get that $\mathcal{RK}^+(X)$ is Reedy fibrant. Since $\mathcal{F}^+\left(\mathcal{RK}^+(X)\right) \simeq X$ we get that $\mathcal{RK}^+(X)$ satisfies the Segal condition and hence is a Segal space. Furthermore, we get that $\mathcal{RK}^+(X)_1^{\mathrm{inv}} \simeq X_1^{\mathrm{inv}}$ and in particular the map

$$d_0 : \mathcal{RK}^+(X)_1^{\mathrm{inv}} \longrightarrow \mathcal{RK}^+(X)_0$$

is a weak equivalence. Since s_0 is a section of d_0 we get that s_0 is a weak equivalence and hence $\mathcal{RK}^+(X)$ is a complete Segal space. □

Now Corollary 3.2.3 implies that

$$\mathrm{Comp} \underset{\mathcal{RK}^+}{\overset{\mathcal{F}^+}{\rightleftarrows}} \mathrm{Comp}_s$$

is indeed a Quillen adjunction. We then get a derived adjunction

$$\mathrm{Ho}(\mathrm{Comp}) \rightleftarrows \mathrm{Ho}\left(\mathrm{Comp}_s\right)$$

between the respective homotopy categories. Since every object in Comp is cofibrant Theorem 3.2.2 tells us that the counit of the derived adjunction is a natural isomorphism. To show that the unit map is an isomorphism it will be enough to show that $\mathcal{F}^+$ **detects equivalences**, i.e. that if $f : X \longrightarrow Y$ is a map of simplicial spaces such that $\mathcal{F}^+(f)$ is a weak equivalence in Comp_s then f is an equivalence in Comp (the unit transformation is then an equivalence by a standard argument).

Corollary 3.2.4 *Assume Theorem 3.2.2. Then the functor $\mathcal{F}^+$ detects equivalences.*

Proof By definition the equivalences in Comp are detected by mapping into complete Segal spaces. Hence the claim that $\mathcal{F}^+$ detects equivalences will follow once we show that every complete Segal space is in the image of $\mathcal{RK}^+$ (up to a levelwise equivalence).

Let Y be a complete Segal space. Note that $\mathcal{F}^+(Y)$ is then **almost** a complete marked semiSegal space in the following sense: let $\mathcal{F}^\natural(Y)$ be the marked simplicial space which has the same underlying semi-simplicial space as $\mathcal{F}^+(Y)$ but whose marking is given by

$M = (\mathcal{F}^+(Y))_1^{\mathrm{inv}} = Y_1^{\mathrm{inv}}$. Since every degenerate edge in Y is an equivalence in $\mathcal{F}^+(Y)$ we have an inclusion

$$\iota : \mathcal{F}^+(Y) \hookrightarrow \mathcal{F}^\natural(Y)$$

which is a levelwise equivalence by definition. Since Y is complete we get that Y_1^{inv} is the union of all connected components which contain degenerate edges. This means that ι is a marked equivalence. Furthermore, the completeness of Y implies the completeness of $\mathcal{F}^\natural(Y)$, and so ι can serve as a **fibrant replacement** of $\mathcal{F}^+(Y)$. The upshot of this is that $\mathcal{F}^+(Y)$ is **marked equivalent** to its fibrant replacement (and not just weakly equivalent in Comp_s).

Now let u_Y be the map given by the composition

$$Y \longrightarrow \mathcal{RK}^+\left(\mathcal{F}^+(Y)\right) \longrightarrow \mathcal{RK}^+\left(\mathcal{F}^\natural(Y)\right).$$

From the discussion above we see that we can consider u_Y as the derived unit in the **pre-localized** Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathcal{RK}^+}{\overset{\mathcal{F}^+}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} .$$

This means that the composition

$$\mathcal{F}^+(Y) \overset{\mathcal{F}^+(u_Y)}{\longrightarrow} \mathcal{F}^+\left(\mathcal{RK}^+\left(\mathcal{F}^\natural(Y)\right)\right) \overset{\nu_{\mathcal{F}^\natural(Y)}}{\longrightarrow} \mathcal{F}^\natural(Y)$$

is a marked equivalence. Since the second map is a marked equivalence by Theorem 3.2.2 we get that $\mathcal{F}^+(u_Y)$ is a marked equivalence. This means that u_Y is a levelwise equivalence and hence Y is in the essential image of $\mathcal{RK}^+$. □

Let $\mathrm{CS} \overset{\mathrm{def}}{=} \mathrm{Comp}^{\mathrm{fib}} \subseteq \mathrm{Comp}$ be the full simplicial subcategory spanned by complete Segal spaces. In light of Corollary 3.2.3 the functors $\mathcal{RK}^+$ and $\mathcal{F}^\natural$ above restrict to an adjunction

$$\mathrm{CS} \underset{\mathcal{RK}^+}{\overset{\mathcal{F}^\natural}{\rightleftarrows}} \mathrm{CsS}$$

which is an equivalence of fibrant simplicial categories. We hence obtain the following version of our main theorem:

Corollary 3.2.5 *The forgetful functor*

$$\mathrm{Cat}_\infty \simeq \mathcal{N}(\mathrm{CS}) \longrightarrow \mathcal{N}(\mathrm{CsS}) \simeq \mathrm{Cat}_\infty^{\mathrm{qu}}$$

is an equivalence of ∞-categories.

Now in order to prove Theorem 3.2.2 we will need a few lemmas which will help us compute $\mathcal{RK}^+(X)$ more easily. The outline of our strategy is inspired by the proof of [10, Proposition 5.4.3.16].

We begin by introducing the following replacement for the category $\mathcal{C}_n$.

Definition 3.2.6 For $n \geq 0$ let us denote by $\mathcal{C}^0_n \subseteq \mathcal{C}_n$ the full subcategory spanned by objects of the form $f : [m] \longrightarrow [n]$ such that f is **surjective**.

The following lemma seems to be well-known to experts, but we were unable to find a short proof in the literature.

Lemma 3.2.7 *The category Δ_s is weakly contractible.*

Proof Consider the simplicial set Z which is the left Kan extension of the terminal semi-simplicial set. Then Z has a unique non-degenerate simplex in each dimension and the face of any non-degenerate simplex of Z is again non-degenerate. This means that the nerve of Δ_s can be identified with the barycentric subdivision of Z, and so it will be enough to show that Z is weakly contractible. This, in turn, can be verified directly: Z has trivial homology groups in each dimension and is simply connected by virtue of Van-Kampen's theorem. □

Corollary 3.2.8 *The category $\mathcal{C}^0_n$ is weakly contractible.*

Proof The category $\mathcal{C}^0_n$ is isomorphic to $(\Delta_s^{op})^n$: the isomorphism is given by sending a surjective map $f : [m] \longrightarrow [n]$ to the vector of linearly ordered sets $(f^{-1}(0), ..., f^{-1}(n))$ considered as an object of $(\Delta_s^{op})^n$. The result now follows from Lemma 3.2.7. □

We will say that a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is **coinitial** if for every $d \in \mathcal{D}$ the nerve of the category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d}$ is contractible. Equivalently, if the functor $f^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$ is cofinal (see [11, Theorem 4.1.3.1]). We recall that restricting a diagram along a coinitial map induces an equivalence on homotopy limits (see [11, Theorem 4.1.1.8] for cofinal maps, or [5, Theorem 19.6.7], where the property coinitial is called "homotopy left cofinal").

Lemma 3.2.9 *The inclusion $\mathcal{C}^0_n \hookrightarrow \mathcal{C}_n$ is coinitial.*

Proof We need to show that for every object $X \in \mathcal{C}_n$ the category $\mathcal{C}^0_n \times_{\mathcal{C}_n} \mathcal{C}_{n/X}$ is weakly contractible. Let X be the object corresponding to a morphism $g : [k] \longrightarrow [n]$. The objects of the category $\mathcal{C}^0_n \times_{\mathcal{C}_n} \mathcal{C}_{n/X}$ can be identified with commutative diagrams of the form

(3–2)
$$\begin{array}{ccc} [k] & \xrightarrow{\quad h \quad} & [m] \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & [n] & \end{array}$$

such that f is surjective and h is injective (and g remains fixed). A morphism $\mathcal{C}^0_n \times_{\mathcal{C}_n} \mathcal{C}_{n/X}$ between two diagrams as above is a morphism of diagrams in the opposite direction which is the identity on $[k]$ and $[n]$. Let us denote

$$\mathcal{E}^X_i = \begin{cases} \Delta_s^{op} & g^{-1}(i) = \emptyset \\ \Delta_s^{op}{}_{/g^{-1}(i)} & g^{-1}(i) \neq \emptyset \end{cases}$$

There is a natural functor

$$T : \mathcal{C}^0_n \times_{\mathcal{C}_n} \mathcal{C}_{n/X} \longrightarrow \prod_{i=0}^{n} \mathcal{E}^X_i$$

which sends the diagram 3–2 to the vector

$$(f^{-1}(0), ..., f^{-1}(n)) \in \prod_{i=0}^{n} \mathcal{E}^X_i$$

where $f^{-1}(i)$ is considered as a finite ordered set equipped with a natural injective map from $g^{-1}(i)$. The functor T admits an inverse

$$S : \prod_{i=0}^{n} \mathcal{E}^X_i \longrightarrow \mathcal{C}^0_n \times_{\mathcal{C}_n} \mathcal{C}_{n/X}$$

which sends a vector $(A_0, ..., A_n)$ of finite ordered sets to their concatenation $A_0 * A_1 * ... * A_n$, considered as an object in Δ_s fitting naturally into a diagram of the form 3–2. We may then conclude that T is an equivalence of categories.

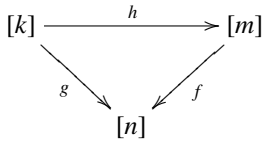
Now when $g^{-1}(i) \neq \emptyset$ the category $\mathcal{E}^X_i$ has a terminal object and so is weakly contractible. When $g^{-1}(i) = \emptyset$ we have $\mathcal{E}^X_i = \Delta^{op}_s$ which is weakly contractible as well (see Lemma 3.2.7). We conclude that $\mathcal{C}^0_n \times_{\mathcal{C}_n} \mathcal{C}_{n/X} \cong \prod_{i=0}^{n} \mathcal{E}^X_i$ is weakly contractible and so the proof is complete. □

In view of Remark 1.8.3 we now have the following corollary:

Corollary 3.2.10

$$\mathcal{RK}^+(X)_n \simeq \mathrm{holim}_{\mathcal{C}^0_n} \mathcal{G}_n.$$

Proof of Theorem 3.2.2 In light of Corollary 3.2.10 and Corollary 3.2.8 the proof of Theorem 3.2.2 will be done once we show that for each n the restricted functor $\mathcal{G}_n|_{\mathcal{C}^0_n}$ is homotopy-constant. Let X be a complete marked semiSegal space and suppose we are given a diagram



such that both f, g are surjective and h is injective. We wish to show that h induces an equivalence

$$h^* : X^f_m \xrightarrow{\simeq} X^g_k.$$

Since g is a surjective map between simplices it admits a section $s : [n] \longrightarrow [k]$. One then obtain a sequence

$$X^f_m \xrightarrow{h^*} X^g_k \xrightarrow{s^*} X^{\mathrm{Id}}_n.$$

From the 2-out-of-3 rule we see that it will be enough to prove the lemma for $k = n$ and $g = \mathrm{Id}$. Note that in this case $X^{\mathrm{Id}}_n = X_n$ and we can consider h as a section of f.

According to Proposition 2.3.12 we get that the map $X^{(\Delta^m, A_f)} \longrightarrow X^{(\Delta^n)^\flat}$ is a DK-equivalence. By Propositions 3.0.21 and 3.1.2 this map is a levelwise equivalence. Evaluating at level 0 we get the desired result. □

3.3 Monoidality

Since both Comp and Comp_s are monoidal model categories, the underlying ∞-categories Comp_∞ and $(\mathrm{Comp}_s)_\infty$ inherit natural structures of symmetric monoidal ∞-categories (see [11, Example 4.1.3.6]). It is then natural to ask whether the Quillen equivalence $\mathcal{F}^+ \dashv \mathcal{RK}^+$ can be promoted to a weakly monoidal one in the sense of Definition 1.1.8, yielding a similar structure on the underlying adjunction of ∞-categories. Unfortunately, this is not exactly the case. However, we seem to be in a somewhat dual situation, in which we have a **lax structure** on the left functor $\mathcal{F}^+$ (as opposed to a colax one), whose structure maps are weak equivalences (this in turn determines no structure on $\mathcal{RK}^+$, and we do not know if $\mathcal{RK}^+$ carries any colax structure).

To avoid this technicality we will work directly with the fibrant simplicial categories CS and CsS whose coherent nerves are equivalent Comp_∞ and $(\mathrm{Comp}_s)_\infty$ respectively. As explained above (see Corollary 3.2.5 and the discussion preceding it), the forgetful functor $\mathrm{Cat}_\infty \longrightarrow \mathrm{Cat}_\infty^{\mathrm{qu}}$ can be modelled directly by a functor of simplicial categories

$$\mathcal{F}^\natural : \mathrm{CS} \longrightarrow \mathrm{CsS}$$

Now the monoidal structure on Comp is the Cartesian one, and as such is inherited by the full subcategory $\mathrm{CS} \subset \mathrm{Comp}$ of fibrant (-cofibrant) objects. Unfortunately, the same statement does not hold for Comp_s. In particular, the tensor product of two object in CsS might not lie in CsS. This difficulty can be overcome as follows. Following [10, §4.1.3] we may regard CS and CsS as **simplicial colored operads** whose multi-mapping spaces are given by

$$\mathrm{Mul}(X_1, .., X_n, Y) \overset{\mathrm{def}}{=} \mathrm{Map}(X_1 \otimes ... \otimes X_n, Y)$$

(where the tensor product is computed inside Comp or Comp_s respectively). According to [10, Proposition 4.1.3.10] and [10, Corollary 4.1.3.16] the ∞-operads $\mathcal{N}^\otimes(\mathrm{CS})$ and $\mathcal{N}^\otimes(\mathrm{CsS})$ are in fact symmetric monoidal ∞-categories which are equivalent to the underlying symmetric monoidal ∞-categories Comp_∞ and $(\mathrm{Comp}_s)_\infty$ respectively (here $\mathcal{N}^\otimes$ is the **operadic nerve**, see [10, Definition 2.1.1.23.])

In this section we shall exhibit a lax structure on the functor $\mathcal{F}^\natural$. Such a structure naturally leads to a map of ∞-operads

$$\mathcal{N}^\otimes\left(\mathcal{F}^\natural\right) : \mathcal{N}^\otimes(\mathrm{CS}) \longrightarrow \mathcal{N}^\otimes(\mathrm{CsS})$$

Furthermore, we will show that the structure maps of this lax structure are equivalences. Since $\mathcal{F}^\natural$ itself is also an equivalence we may deduce that $\mathcal{N}^\otimes\left(\mathcal{F}^\natural\right)$ is an equivalence of ∞-operads, and hence in particular an equivalence of symmetric monoidal ∞-categories.

To begin, consider the adjunction

$$\mathcal{S}^{\Delta_s^{\mathrm{op}}} \underset{\mathcal{F}}{\overset{\mathcal{LK}}{\rightleftarrows}} \mathcal{S}^{\Delta^{\mathrm{op}}}$$

where $\mathcal{F}$ is the forgetful functor and $\mathcal{LK}$ is the left Kan extension functor. This adjunction carries a compatible lax-colax structure $(\alpha_{X,Y}, u), (\beta_{Z,W}, v)$ (see Definition 1.1.3). The colax

structure was described in Remark 1.3.2. Here we shall be mostly concerned with the lax structure $(\alpha_{X,Y}, u)$, and so it will be worth while to give an explicit description. First observe that the map

$$u : \Delta^0 \longrightarrow \mathcal{F}(\mathcal{LK}(\Delta^0))$$

is just the unit of the adjunction. Second, the maps

$$\alpha_{X,Y} : \mathcal{F}(X) \otimes \mathcal{F}(Y) = \mathrm{Core}(\mathcal{LK}(\mathcal{F}(X)) \times \mathcal{LK}(\mathcal{F}(Y))) \longrightarrow \mathcal{F}(X \times Y)$$

are induced by the tensor of counit maps $\mathcal{LK}(\mathcal{F}(X) \times \mathcal{LK}(\mathcal{F}(Y)) \longrightarrow X \times Y$ followed by the natural inclusion $\mathrm{Core}(X \times Y) \hookrightarrow \mathcal{F}(X \times Y)$.

Now let X, Y be two simplicial space let $M \subseteq X_1, N \subseteq Y_1$ be the union of components meeting degenerate edges. By definition (see § 1.7) the marked subspace of $\mathcal{F}^+(X) \otimes \mathcal{F}^+(Y)$ may be identified with

$$\left(s^*(X_0) \times Y_0\right) \coprod \left(X_0 \times s^*(Y_0)\right) \coprod \left(s^*(X_0) \times s^*(X_1)\right) \subseteq (\mathcal{F}(X) \otimes \mathcal{F}(Y))_1$$

which is mapped by $\alpha_{X,Y}$ onto the subspace

$$s^*(X_0) \times s^*(Y_0) \subseteq \mathcal{F}(X \times Y)_1.$$

Similarly, the marked subspace of $\mathcal{F}^\natural(X) \otimes \mathcal{F}^\natural(Y)$ may be identified with

$$(M \times Y_0) \coprod (X_0 \times N) \coprod (M \times N) \subseteq (\mathcal{F}(X) \otimes \mathcal{F}(Y))_1$$

which is mapped by $\alpha_{X,Y}$ onto the subspace

$$M \times N \subseteq \mathcal{F}(X \times Y)_1.$$

We may hence conclude that $\alpha_{X,Y}$ gives rise to natural maps

$$\alpha^+_{X,Y} : \mathcal{F}^+(X) \otimes \mathcal{F}^+(Y) \longrightarrow \mathcal{F}^+(X \otimes Y)$$

and

$$\alpha^\natural_{X,Y} : \mathcal{F}^\natural(X) \otimes \mathcal{F}^\natural(Y) \longrightarrow \mathcal{F}^\natural(X \otimes Y).$$

Combined with the obvious natural map

$$u^+ : \Delta^0 \longrightarrow \mathcal{F}^+\left(\mathcal{LK}\left(\Delta^0\right)\right)$$

and

$$u^\natural : \Delta^0 \longrightarrow \mathcal{F}^\natural\left(\mathcal{LK}\left(\Delta^0\right)\right)$$

we hence obtain lax structures on $\mathcal{F}^+$ and $\mathcal{F}^\natural$ respectively. Our goal in this section is to prove that the structure maps $\alpha^\natural$ and $u^\natural$ are equivalences. Since the natural transformation $\mathcal{F}^+(X) \longrightarrow \mathcal{F}^\natural(X)$ is a weak equivalence in Comp_s it will suffice to show that the natural transformations $\alpha^+_{X,Y}$ and u^+ are weak equivalences in Comp_s.

We start with the following direct corollary of Theorem 3.2.2:

Lemma 3.3.1 *Let Z be an (unmarked) semi-simplicial space. Then the composition of natural maps*

$$Z^\flat \longrightarrow (\mathcal{F}(\mathcal{LK}(Z)))^\flat \longrightarrow \mathcal{F}^+(\mathcal{LK}(Z))$$

is a weak equivalence in Comp_s.

Proof Let (W, M) be a complete marked semi-simplicial space. Mapping the above composition into W yields the map

$$\mathrm{Map}_+\left(\mathcal{F}^+(\mathcal{LK}(Z)), W\right) \longrightarrow \mathrm{Map}_+\left(Z^\flat, W\right).$$

By adjunction the above map can be written as

$$\mathrm{Map}\left(Z, \mathcal{F}(\mathcal{RK}^+(W))\right) \longrightarrow \mathrm{Map}\,(Z, W) \tag{3–3}$$

where the map is induced by the map $\mathcal{F}\left(\mathcal{RK}^+(W)\right) \longrightarrow W$ of semi-simplicial spaces underlying the counit map. By Theorem 3.2.2 this counit map is a marked equivalence and so the underlying map is a levelwise equivalence. Since both $\mathcal{F}\left(\mathcal{RK}^+(W)\right)$ and W are Reedy fibrant we get that the map 3–3 is a weak equivalence. Since this is true for any complete marked semi-simplicial space W we get that the map $Z^\flat \longrightarrow \mathcal{F}^+(\mathcal{LK}(Z))$ is a weak equivalence in Comp_s as desired. □

Now, applying Lemma 3.3.1 for $Z = \Delta^0$ we obtain that u^+ is a weak equivalence in Comp_s. It is left to prove the following:

Proposition 3.3.2 *The natural map*

$$\alpha^+_{X,Y} : \mathcal{F}^+(X) \otimes \mathcal{F}^+(Y) \longrightarrow \mathcal{F}^+(X \times Y)$$

is a weak equivalence in Comp_s *for every two simplicial spaces X, Y.*

Proof Note that for every n, the left Kan extension $\mathcal{LK}(\Delta^n)$ is the standard n-simplex (considered as a levelwise discrete simplicial space). Now any simplicial space is a colimit of such $\mathcal{LK}(\Delta^n)$'s, which is simultaneously a homotopy colimit with respect to the Reedy model structure (and hence also with respect to the model structure of Comp). Since $\mathcal{F}^+$ preserves colimits and sends levelwise equivalences to marked equivalences it will be enough to prove the claim for $X = \mathcal{LK}(\Delta^n), Y = \mathcal{LK}(\Delta^m)$. In particular, we need to show that the lower horizontal map in the diagram

$$\begin{array}{ccc} \Delta^n \otimes \Delta^m & \longrightarrow & \mathcal{F}^+\left(\mathcal{LK}\left(\Delta^n \otimes \Delta^m\right)\right) \\ \downarrow & & \downarrow \simeq \\ \mathcal{F}^+\left(\mathcal{LK}\left(\Delta^n\right)\right) \otimes \mathcal{F}^+\left(\mathcal{LK}\left(\Delta^m\right)\right) & \longrightarrow & \mathcal{F}^+\left(\mathcal{LK}\left(\Delta^n\right) \times \mathcal{LK}\left(\Delta^m\right)\right) \end{array}$$

is a weak equivalence (to check that this diagram commutes note that the underlying diagram of semi-simplicial spaces is one of the compatibility diagram of the lax structure of $\mathcal{F}$ and the colax structure of $\mathcal{LK}$, see Definition 1.1.3). Now from Lemma 3.3.1 we get that upper horizontal

map and the left vertical map are weak equivalences (for the left vertical map one uses the fact that in a symmetric monoidal model category the product of two weak equivalences between cofibrant objects is again a weak equivalence). Since the right vertical map is an isomorphism (see Remark 1.3.2) the result follows from the 2-out-of-3 property. □

Remark 3.3.3 By using adjunction and the exponential law one sees that the lax structure on $\mathcal{F}^+$ induces a natural map

$$\mathcal{F}^+\left(\mathcal{RK}^+(Y)^X\right) \longrightarrow Y^{\mathcal{F}^+(X)}$$

for every $X \in \mathrm{Comp}, Y \in \mathrm{Comp}_s$. Proposition 3.3.2 then implies that this map is a weak equivalence whenever Y is fibrant, i.e. a complete marked semiSegal space. In particular, if $Y = \mathcal{F}^\natural(Y')$ for some complete Segal space Y' then we get a sequence of weak equivalences

$$\mathcal{F}^+\left((Y')^X\right) \xrightarrow{\simeq} \mathcal{F}^+\left(\left(\mathcal{RK}^+\left(\mathcal{F}^\natural(Y')\right)\right)^X\right) \xrightarrow{\simeq} \mathcal{F}^\natural(Y')^{\mathcal{F}^+(X)} \simeq \mathcal{F}^\natural(Y')^{\mathcal{F}^\natural(X)}$$

which induces an equivalence

$$\mathcal{F}^\natural\left(\left(Y'\right)^X\right) \xrightarrow{\simeq} \mathcal{F}^\natural\left(Y'\right)^{\mathcal{F}^\natural(X)}.$$

This equivalence can be interpreted as follows: if $\mathcal{C}, \mathcal{D}$ are two ∞-categories and $\overline{\mathcal{C}}, \overline{\mathcal{D}}$ their respective underlying quasi-unital ∞-categories, then the quasi-unital functor category $\overline{\mathcal{C}}^{\overline{\mathcal{D}}}$ is equivalent to the underlying quasi-unital ∞-category of the functor category $\mathcal{C}^{\mathcal{D}}$.

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