∞-Sheaves

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June 13, 2011

1 Simplicial Presheaves

Let C be a category. We wish to model the ∞-category of ∞-functors from C to topological spaces. We will do this by putting a model structure on the category $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})^{\Delta^{\mathrm{op}}}$. We will refer to objects in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ as **presheaves** and objects in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$ as **simplicial presheaves**.

Note that for every object X we have a representable presheaf $rX \in \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$. By abuse of notation we will also sometimes consider rX as a (constant) simplicial presheaf. It should be clear from the context which use is taken each time.

The various options to put model structures $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$ contain in some sense two extreme choices:

Definition 1.1.
1. The **projective** model structure on $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$. Here weak equivalences and fibrations are object-wise and cofibrations are defined via the left lifting property.
2. The **injective** model structure on $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$. Here weak equivalences and cofibrations are object-wise and fibrations are defined via the right lifting property.

The advantage of the projective model structure is that fibrations are easy to describe, and the advantage of the injective model structure is that cofibrations are easy to described. However the projective model structure has another advantage, which is not met by the injective model structure:

Theorem 1.2. *The family of maps*

$$\partial\Delta^n \otimes rX \longrightarrow \Delta^n \otimes rX$$

(where X ranges over all objects of C) generate the set of all cofibrations in the projective model structure.

Proof. This is equivalent to saying that a map of simplicial presheaves is $F \longrightarrow G$ is an object-wise fibration if and only if it satisfies the right lifting property with respect to all the maps $\partial\Delta^n \otimes X \longrightarrow \Delta^n \otimes X$ where X is representable. This is clear once we recall Yoneda's fundamental lemma saying that

$$\mathrm{Map}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Set}_\Delta)}(X, F) \cong F(X)$$

□

In particular if F is a projective-cofibrant simplicial presheaf then the pushout along a diagram of the form

$$(*) \quad \begin{array}{ccc} X \otimes \partial\Delta^n & \longrightarrow & F \\ \downarrow & & \\ X \otimes \Delta^n & & \end{array}$$

is also a cofibrant object. In this way one can construct many projective-cofibrant objects as sequential colimits of skeletons

$$F = \mathrm{colim}\, F_n$$

where F_0 is coproduct of representables and F_{n+1} is obtained from F_n via a sequence of pushouts of the form $(*)$.

Example: If C is the one object category associated with a group G then a simplicial presheaf is just a simplicial set with an action of G. The construction above will gives us simplicial sets with a **free** G-action.

Note that in the group action example every simplicial G-set which is free level wise will give us a cofibrant object. This example is slightly misleading because in general not every simplicial presheaf which is representable level-wise has to be cofibrant.

The technical problem is due to the fact that a simplicial object has degeneracies, and so the realization might involves also "collapsing" colimits, or pushouts which are not along cofibrations. When C is a group category this can't happen but if C is a bit more complicated it is not always easy to "separate out" the degeneracies.

The technical condition that a simplicial object which is representable levelwise needs to satisfy in order to be cofibrant is to be **Reedy cofibrant**, which is also sometimes called having **split degeneracies**. In that case the realization will coincide with the homotopy colimit and will be cofibrant. More technically we will be able to describe the realization using only pushouts of the form $(*)$.

Now given a simplicial presheaf which is levelwise representable things are still not that bad. We can apply to it the Reedy cofibrant replacement functor and obtain a simplicial obejct in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$ whose realization can serve as a cofibrant replacement of our object. Another way to think of the fibrant replacement of such a U as the homotopy colimit of the diagram $\{U_n\}$.

2 Grothendieck Topologies and ∞-Sheaves

We will use the "covering sieve" approach to Grothendieck topologies:

Definition 2.1. Let $X \in C$ be an object and $rX \in \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ the corresponding representable **presheaf**. A **sieve** on X is a subfunctor $R \hookrightarrow rX$, i.e. a map of presheaves $R \longrightarrow rX$ such that for each $Y \in C$ the map

$$R(Y) \longrightarrow rX(Y) = \mathrm{Hom}_C(Y, X)$$

is an injective map of sets.

Note that the data of a sieve R on X gives for each object $Y \in C$ a collection of maps $R(Y) = \{f : Y \longrightarrow X\}$ such that if $f \in R(Y)$ and $g : Z \longrightarrow Y$ is any map then

$$f \circ g \in R(Z)$$

Examples:

1. Let $X \in C$ be an object and $f : Y \longrightarrow X$ a morphism. Then we can define a sieve R_f by setting $R_Y(Z)$ to be the set of all morphisms $Z \longrightarrow X$ which factor through Y.

2. Let X is a topological space and $C = O(X)$ is the category of open subsets of X and inclusions. Let $U \in O(X)$ and $\mathcal{F} = \{U_\alpha\}$ an open covering of U. Then we can define a sieve $R_{\mathcal{F}}$ by letting $R_{\mathcal{F}}(V)$ contain the inclusion $V \hookrightarrow U$ if and only V is contained in U_α for some α. This is the way in which sieves can encode the information of coverings.

Given a sieve R on X and a morphism $f : Y \longrightarrow X$ we define the pullback f^*R as the pullback of the diagram

$$\begin{array}{ccc} & & R \\ & & \downarrow \\ rY & \longrightarrow & rX \end{array}$$

in the category $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ of presheaves (which in turn is just pullback object-wise). Since a pullback of an injective map is injective it follows that the pullback of a sieve is a sieve.

We now come to the definition of a Grothendieck topology.

Definition 2.2. Let C be a category. A **Grothendieck topology** on C consists of the following data: for each object $X \in C$, a collection of sieves $J(X)$ on X which are called **covering sieves**. This data is required to satisfy the following axioms:

1. The sieve rX on X is a covering sieve for every X.

2. If $R \in J(X)$ is a covering sieve on X and $f : Y \longrightarrow X$ is any morphism then the pullback f^*R is a covering sieve on Y.

3. Let R be a covering sieve on X and S any sieve on X. Suppose that for each morphism $f : Y \longrightarrow X$ which belongs to R the pullback f^*S is a covering sieve of Y. Then S is a covering sieve of X.

A category C together with a Grothendieck topology τ is called a **Grothendieck site**.

Examples:

1. Let X is a topological space and $C = O(X)$ is the category of open subsets of X and inclusions. We can define the covering sieves on U to be exactly those of the form $R_{\mathcal{F}}$ for some open covering $\mathcal{F} = \{U_\alpha\}$. This is the standard Grothendieck topology on X.

2. Let S be a base schemes. We can do the same with the category of smooth (affine/general) schemes over S by replacing the notion of an open covering $U = \{U_\alpha\}$ with families of open inclusions/ étale maps/fpqc maps/fppf maps/smooth maps $\{U_\alpha \longrightarrow U\}$ whose images cover U. All these constructions have both "big" and a "small" versions. We will not elaborate on that here.

We now wish to construct a model structure on $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$ which will model the notion of an ∞-sheaf, or a sheaf of topological spaces. For this we will need the notion of a hypercovering. In order to phrase the notion of a hypercovering through covering sieves we will need the following notion:

Definition 2.3. A map $F \longrightarrow G$ of presheaves is called **locally surjective** (or a **generalized cover**) if for every object $X \in C$ and every point $x \in G(X)$ there exists a covering sieve R on X such that for every $f : Y \longrightarrow X$ in R the point $f^*x \in G(Y)$ is in the image of the map $F(Y) \longrightarrow G(Y)$.

Definition 2.4. Let $X \in C$ be an object. We say that a map of **simplicial** presheaves $U \longrightarrow rX$ is a **hypercovering** if

1. Each U_n is a coproduct of representable.

2. The following maps of presheaves

$$U_0 \longrightarrow X$$

$$U_1 \longrightarrow U_0 \times_X U_0$$

$$U_n \longrightarrow U^{\partial \Delta^n}, n \geq 2$$

are locally surjective where $U^{\partial \Delta^n}$ is the presheaf which associates to each object Y the set of maps $\partial \Delta^n \longrightarrow F(Y)$.

Remark 2.5. This can also be phrased using a single condition by saying that for every n the associated map of presheaves

$$U_n \longrightarrow U^{\partial \Delta^n} \times_{rX^{\partial \Delta^n}} rX$$

is locally surjective.

Another way to phrase the second property is that every diagram of the form

$$\begin{array}{ccc} rY \otimes \partial\Delta^n & \longrightarrow & U \\ \downarrow & & \downarrow \\ rY \otimes \Delta^n & \longrightarrow & rX \end{array}$$

admits a lift when restricting to an appropriate covering sieves. Note that if our Grothendieck topology was trivial then this condition is just the right lifting property with respect to cofibrations, so we would get that the second condition is that $U \longrightarrow rX$ is a trivial fibration. In general this property is sometimes called a **local trivial fibration** but one should be careful with that terminology because this will not be the definition of fibrations in the model structure we will construct.

We now return to the issue of defining a model structure capturing the notion of an ∞-sheaf. We will do so by taking the projective model structure on $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$ and do a left Bousfield localization with respect to a certain family of maps. There are essentially two inequivalent to do this, yielding two different notions of ∞-sheaves:

1. We can localize with respect to all maps of the form $R \longrightarrow rX$ when R is a covering sieve (here we consider presheaves as constant simplicial presheaves).
2. We can localize with respect to all maps of the form $U \longrightarrow rX$ which are hypercoverings.

It turns out that the second localization factors through the first localization - i.e. if we localize with respect to all hypercoverings then the maps of the form $R \longrightarrow rX$ where R is a covering sieve will automatically becomes equivalences, but the converse is not true in general. We refer the reader to Lurie's book "higher topos theory" where he defines the notion of a hypercomplete ∞-topos. In that language the second localization results in the ∞-topos which is the hypercompletion of the first ∞-topos.

We call the second model structure the **local projective model structure** on $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$. The projective model structure from before is also sometimes referred to as the **global** projective model structure.

Remark 2.6. We can of course localize the injective model structure instead. This will be equivalent, but just less convenient for us.

Since we are doing a left localization the cofibrant objects in the local projective model structure will be the same as in the global projective model structure. The fibrant objects, however, will encode the "sheafness" essence:

Theorem 2.7. *Let $F \in \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta)$ be a simplicial presheaf. Then F is fibrant in the local projective model structure if and only if*

1. *F is fibrant in the global projective model structure, i.e. $F(X)$ is a Kan simplicial set for every X.*

2. *For each hypercovering $U \longrightarrow rX$ such that $U_n = \coprod_{\alpha \in I_n} U_\alpha$. Then the induced map*

$$F(X) \longrightarrow \mathrm{holim}_\Delta \prod_{\alpha \in I_n} F(U_\alpha)$$

is a weak equivalence.

The second property is also called **descent with respect to hypercoverings**.

Proof. In a left Bousfield localization the new fibrant objects are the objects which a fibrant in the old model structure and are local. This means that F is fibrant if and only if each $F(X)$ is Kan and for each hypercovering $U \longrightarrow rX$ the induced map

$$\mathrm{Hom}(rX^{cof}, F) \longrightarrow \mathrm{Hom}(U^{cof}, F)$$

is a weak equivalence.

Since rX is representable it is cofibrant so $rX^{cof} = rX$. Since each U_n is representable we the homotopy colimit $U' = \mathrm{hocolim}_{\Delta^{op}} U_n$ can serve as a cofibrant replacement of U. We then have that

$$\mathrm{Hom}(rX^{cof}, F) = \mathrm{Hom}(rX, F) = F(X)$$

and

$$\mathrm{Hom}(U^{cof}, F) = \mathrm{Hom}(U', F) \simeq \mathrm{holim}_\Delta(\mathrm{Hom}(U_n, F)) = \mathrm{holim}_\Delta \left(\prod_{\alpha \in I_n} F(U_n) \right)$$

Which finishes the proof. □

Remark 2.8. An alternative description of the weak equivalences in the local projective model structure are given by notion of **sheaves of homotopy groups**. Let F be a simplicial presheaf and $X \in C$ be an object. Given any base point $x_0 \in F(X)$ we define the sheaf $\pi_n(F, x_0)$ on the induced Grothendieck site $C_{/X}$ as ths sheafification of the presheaf which associates to each $f : Y \longrightarrow X$ the (pointed set/group/abelian group) $\pi_n(Y, f^*x)$. It can be shown that the weak equivalences in the local model structure are exactly the maps which induce an isomorphism on all sheaves of homotopy groups.

3 Application

Here is a nice application of the notion of ∞-sheaf which is connected to work in progress with T.Schlank. Let S be a base scheme and $f : X \longrightarrow S$ a nice scheme over S. Consider the étale sites $S_{ét}$ and $X_{ét}$ of S and X respectively.

Suppose that the pullback functor $f^* : S_{ét} \longrightarrow X_{ét}$ has a left adjoint $f_! : X_{ét} \longrightarrow S_{ét}$. For example if $S = \mathrm{spec}(k)$ then $f_!(Y) = \pi_0(Y)$ considered as a 0-dimensional scheme over k. If $S = \mathrm{spec}(O_k)$ for a number ring then O_k

then $\pi_0(Y)$ can be given an integral structure so $f_!$ exists as well (at least for sufficiently nice X's).

Given a hypercovering $U \longrightarrow rX$ in $X_{\acute{e}t}$ we can apply the functor $f_!$ to it level-wise and obtain a simplicial presheaf in $S_{\acute{e}t}$. Applying the fibrant replacement functor (∞-sheafification) one gets an ∞-sheaf $\mathcal{F}_U$ over S. This gives us an inverse system of ∞-sheaves $\{\mathcal{F}_U\}$ indexed by the filtered simplicial category of hypercoverings. The inverse system of sheaves of homotopy groups can be thought of as the **relative homotopy groups** of X over S.

If X has an S-rational point then all the ∞-sheaves $\mathcal{F}_U$ will admits a section. In other words if any one of $\mathcal{F}_U$ does not have a section then X can't have an S-rational point. Using standard tools from homotopy theory one can construct obstructions to the existence of a section which take values in certain cohomology groups which can informally be written as

$$H^{n+1}(S, \pi_n(\mathcal{F}_U))$$

These obstructions then give obstructions to the existence of S-rational points on X.