Le complexe cotangent abstrait

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Let us fix a field k of characteristic 0 and let A be a commutative algebra over k. One may then consider its **cotangent module** Ω_A, also known as the module of Kähler differentials. We may construct Ω_A explicitly as the quotient I_A/I_A^2, where $I_A \subseteq A \otimes A$ is the kernel of the multiplication map $A \otimes A \longrightarrow A$. If A is finitely generated then we consider it as the algebra of functions on an affine variety $X = \mathrm{spec}(A)$ over k. If X is smooth then Ω_A can be identified with the module of (globally defined) differentials 1-forms on X, or sections of the **cotangent bundle** of X. If X is not smooth then Ω_A is not such a well-behaved object. It turns out that one way to "fix" Ω_A is to consider A as a (non-negatively graded) commutative **differential graded algebra** over k (or CDGA's for short). One may then observe that the formation of cotangent modules, when extended naively to the setting of CDGA's, does not preserve quasi-isomorphisms. This can be fixed by considering a suitable **derived version** of Ω_A, which is called the **cotangent complex** of A, and often denoted by L_A. The object L_A will not be an ordinary A-module, but a **complex of A-modules**. One may interpret this phenomenon as saying that if X is a singular variety then its cotangent bundle whould be replaced by a suitable sheaf of complexes (or complex of sheaves). The cotangent complex plays a fundamental role in the theory of commutative algebras and CDGA's. For example:

1. L_A plays a key role in the classification of **deformations** of A. This can also be extended to the case where we replace $X = \mathrm{spec}(A)$ by a non-affine scheme.

2. L_A can be used to set up an **obstruction theory** for CDGA's. Such a theory can be used, for example, to produce results of the following type (see [Lu09, 7.4.3.4]):

   Theorem 1 (The cotagent complex Whitehead theorem)**.** *Let $f : A \longrightarrow B$ be a map of non-negatively graded CDGA's over k such that the induced map $H_0(A) \longrightarrow H_0(B)$ is an isomoprhism of k-algebras. Then f is a quasi-isomorphism if and only if the induced map $f_* : A \otimes_B^L L_B \longrightarrow L_A$ is a quasi-isomorphism of A-modules.*

The **abstract cotangent complex formalism** is an attempt to understand the contangent complex from an abstract point of view. For this we will need to understand how Ω_A is related to the theory of **modules** of A. We first recall the following general construction. For an associative algebra A and an A-bimodule M, one can form what is called the **square-zero extension** $M \rtimes A$ of A by M. This is an associative ring whose underling abelian group is $M \oplus A$ and where the multiplication is given by $(m,a)(n,b) = (mb + an, ab)$. The algebra $M \rtimes A$ is equipped with a natural homomorphism $M \rtimes A \longrightarrow A$ whose kernel is M. If A is a commutative algebra and M is an ordinary A-module then one can think of M as an A-bimodule in a natural way and form the same square-zero extension, which will in this case be a commutative algebra.

We consider the square-zero extension as an object not of the category CAlg of commutative k-algebras, but of the category $\mathrm{CAlg}_{/A}$ whose objects are k-algebras B equipped with a map $B \longrightarrow A$ and whose morphisms are commutative triangles

$$\begin{array}{ccc} B & \longrightarrow & B' \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

A fundamental observation is that $M \rtimes A$ is not an arbitrary object of $\mathrm{CAlg}_{/A}$. For starters, it admits a map $A \longrightarrow M \rtimes A$ given by $a \mapsto (0,a)$, where we consider A as an object of $\mathrm{CAlg}_{/A}$ via the identity map $A \longrightarrow A$. We note that this is the terminal object of $\mathrm{CAlg}_{/A}$. Second, it is not hard to check that the fiber product $(M \rtimes A) \times_A (M \rtimes A)$ (which is the categorical product in the $\mathrm{CAlg}_{/A}$) is naturally isomorphic to $(M \oplus M) \rtimes A$. The sum map of A-modules $M \oplus M \longrightarrow M$ induces a natural map $(M \rtimes A) \times_A (M \rtimes A) \longrightarrow M \rtimes A$. It can then be checked that these maps satisfy (diagrammatically) all the axioms of an **abelian group**. We say that $M \rtimes A$ is an **abelian group object** of $\mathrm{CAlg}_{/A}$. We then have the following classical fact:

Claim 2. *The formation of square-zero extensions induces an equivalence*

$$\mathrm{Mod}(A) \xrightarrow{\cong} \mathrm{Ab}(\mathrm{CAlg}_{/A})$$

If $\mathcal{C}$ is a nice enough category, then the forgetful functor $\mathrm{Ab}(\mathcal{C}) \longrightarrow \mathcal{C}$ will admit a left adjoint $\mathbb{Z} : \mathcal{C} \longrightarrow \mathrm{Ab}(\mathcal{C})$. If $X \in \mathcal{C}$ is an object then we think of $\mathbb{Z}X$ as the free abelian group generated from X. For example, if $\mathcal{C}$ is the category of sets then $\mathrm{Ab}(\mathcal{C})$ is the category of abelian group and $\mathbb{Z}$ is the usual free abelian group functor. The category $\mathrm{CAlg}_{/A}$ is nice, and consequently the forgetful functor $\mathrm{Ab}(\mathrm{CAlg}_{/A}) \longrightarrow \mathrm{CAlg}_{/A}$ admits a left adjoint $\mathbb{Z}_{/A} : \mathrm{CAlg}_{/A} \longrightarrow \mathrm{Ab}(\mathrm{CAlg}_{/A})$. Under the identification $\mathrm{Mod}(A) \cong \mathrm{Ab}(\mathrm{CAlg}_{/A})$ we may write this left adjoint as a functor $\mathbb{Z}_{/A} : \mathrm{CAlg}_{/A} \longrightarrow \mathrm{Mod}(A)$, which is given explicitly by

$$\mathbb{Z}_{/A}(B) = \Omega_B \otimes_A A.$$

In particular, the cotangent module Ω_A can be identified with $\mathbb{Z}_{/A}(\mathrm{Id}_A)$. In order to interpret the contangent module abstractly we are hence led to the following definition:

Definition 3. Let $\mathcal{C}$ be a category which admits finite limits and let $X \in \mathcal{C}$ an object. A **Beck module** over X is an abelian group object in the category of $\mathcal{C}_{/X}$ of objects equipped with a map to X. We denote by $\mathrm{Ab}(\mathcal{C}_{/X})$ the category of Beck modules. We will denote by $\mathbb{Z}_{/X} : \mathbb{C} \longrightarrow \mathcal{C}_{/X}$ a left adjoint to the forgetful functor (when exists), and will call the object $\mathbb{Z}_{/X}(\mathrm{Id}_X)$ the **abstract cotangent module** of X..

We already saw that if $\mathcal{C} = \mathrm{CAlg}$ then the category of Beck modules $\mathrm{Ab}(\mathrm{CAlg}_{/A})$ is equivalent to the category of A-modules, and under this identification $\mathbb{Z}_{/A}(\mathrm{Id}_A)$ is isomorphic to Ω_A. If $\mathcal{C} = \mathrm{Alg}$ is the category of **associative algebras** then the same square-zero construction induces an equivalence between $\mathrm{Ab}(\mathrm{Alg}_{/A})$ and the category of A-bimodules. Under this equivalence $\mathbb{Z}_{/A}(\mathrm{Id}_A)$ is the A-bimodule I_A which is the kernel of the multiplcation map $A \otimes A \longrightarrow A$ (considered as a map of A-bimodules). If $\mathcal{C} = \mathrm{Grp}$ is the category of groups then the category of Beck modules $\mathrm{Ab}(\mathrm{Grp}_{/G})$ is naturally equivalent to the category of G-modules (via the formation of semi-direct product). The abstract contangent module $\mathbb{Z}_{/G}(\mathrm{Id}_G) \in \mathrm{Mod}(G)$ of G can then be identified with the kernel of the canonical G-module map $\mathbb{Z}G \longrightarrow \mathbb{Z}$.

Definition 3 allows one to give a unified treatment of the notion of a module from an abstract point of view. However, our real goal is to understand the derived counter-part, the **cotangent complex** from an abstract point of view. For such purposes one has to understand what happens to Definition 3 when one goes from ordinary categories to the setting of **higher category theory**. By a higher category we mean a type of category which can accommodate objects of a homotopical nature, such as spaces, chain complexes, differential graded algebras, and more. There are many approaches to this topic, most of which yield theories which are essentially equivalent to a suitable sense. One possibility is to talk about categories which are enriched in spaces (or something that is just as good, simplicial sets). Another is to consider categories which are endowed with a collection of morphisms which are deemed to be some kind of **weak equivalences**. For example, we may take the category of spaces and say that a map $f : X \longrightarrow Y$ is a weak equivalence if it induces an isomorphism on all homotopy groups. Similarly, we can consider the category of chain complexes and say that a map is a weak equivalence if it is a quasi-isomorphism (the latter example is underlying many notions and constructions in homological algebra, and most importantly the notion of derived functors). Other equivalent approaches include quasi-categories, complete Segal spaces and Segal categories. From the perspective of this talk the relevant question is how to generalize the notion of a Beck module when $\mathcal{C}$ is not an ordinary category, but some kind of a higher category, where the main issue is what should be the correct analogue of an abelian group objects. Looking at the example of topological spaces, a direct application of the definition (which just treats spaces as an ordinary category) leads to the notion of a topological abelian group. However, as algebraic topology developed it became apparent that this notion is to rigid, and fails to include many type of objects which are not topological abelian groups, but should be considered as such from a homotopical point of view. For example, spaces which satisfy the axioms of an abelian group but only up to a suitably prescribed homotopy (to make this definition precise is not a trivial manner). This lead to the notion of an **infinite loop space**, and later to that of an Ω-spectrum. The latter is no longer a space, but a square of pointed spaces $X_0, X_1, ...$, together with weak equivalences of the form $f_n : X_n \xrightarrow{\simeq} \Omega X_{n+1}$. The strong role played by the notion of a spectrum in modern algebraic topology and its relations to generalized homology theories have lead to the understanding that the correct analogue of an abelian group object in a higher category $\mathcal{C}$ is a suitable notion of an Ω-spectrum in $\mathcal{C}$. By replacing the notion of an abelian group object by that of an Ω-spectrum object in Definition 3 one arrives at the definition of a **spectral Beck module**. There is an analogue of the free abelian group functor $\mathbb{Z}_{/X}$, known as the **suspension-infinity** functor $\Sigma^{\infty}_{/X}$. Applying this functor to the identity $\mathrm{Id}_X \in \mathcal{C}_{/X}$ one arrives at the definition of the **abstract cotangent complex** of X. Understanding categories of spectral Beck modules in various cases can lead to obtaining suitable obstruction theories, and suitable variants of results such as Theorem 1. More generally, this approach can be used to reduce homotopy theoretical problems to a combination of a non-abelian low dimensional problem, followed by a homotopical but essentially abelian problem, and can be very useful in practice. Current work in progress allows one to obtain new computations of these categories of spectral Beck modules when $\mathcal{C}$ is the category of simplicial monoids, simplicial categories and simplicial operads.

References

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