The descent-fibration method for integral points - St petersburg lecture

Yonatan Harpaz

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1 Introduction

Let k be a number field and S a finite set of places of k. By an $\mathcal{O}_S$**-variety** we understand a separable scheme of finite type over the ring $\mathcal{O}_S \subseteq k$ of S-**integers**. We will always denote by $X = \mathcal{X} \otimes_{\mathcal{O}_S} k$ the base change of $\mathcal{X}$ to k. A fundamental problem in Diophantine geometry is to understand the set $\mathcal{X}(\mathcal{O}_S)$ of S-integral points, and in particular to determine when it is **non-empty**. A typical starting point for such questions is to embed the set $\mathcal{X}(\mathcal{O}_S)$ of S-integral points in the the set of S-integral **adelic points**

$$\mathcal{X}(\mathbb{A}_k) \stackrel{\text{def}}{=} \prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v).$$

If $\mathcal{X}(\mathbb{A}_k) = \emptyset$ one may immediately deduce that $\mathcal{X}$ has no S-integral points. In general, it can certainly happen that $\mathcal{X}(\mathbb{A}_k) \neq \emptyset$ but $\mathcal{X}(\mathcal{O}_S)$ is still empty. One way to account for this phenomenon is given by the integral version of the **Brauer-Manin obstruction**, introduced in [CTX09]. This is done by considering the set

$$\mathcal{X}(\mathbb{A}_k)^{\mathrm{Br}(X)} \stackrel{\text{def}}{=} \mathcal{X}(\mathbb{A}_k) \cap X(\mathbb{A}_k)^{\mathrm{Br}}.$$

When $\mathcal{X}(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ one says that there is a Brauer-Manin obstruction to the existence of S-integral points. Our motivation then leads to the following natural question:

question 1.1. *Given a family $\mathcal{F}$ of $\mathcal{O}_S$-varieties, does the property $\mathcal{X}(\mathbb{A}_k)^{\mathrm{Br}(X)} \neq \emptyset$ implies $\mathcal{X}(\mathcal{O}_S) \neq \emptyset$ for every $\mathcal{X} \in \mathcal{F}$?*

When the answer to Question 1.1 is yes one says that the Brauer-Manin obstruction is the only obstruction to the existence of S-integral points for the family $\mathcal{F}$. In [CTX09] Colliot-Thélène and Xu show that if $\mathcal{X}$ is such that $X = \mathcal{X} \otimes_{\mathcal{O}_S} k$ is a homogeneous space under a simply-connected semi-simple algebraic group G with connected geometric stabilizers, and G satisfies a certain non-compactness condition over S, then the Brauer-Manin obstruction is the only obstruction to the existence of S-integral points on $\mathcal{X}$. Similar results hold when X is a principal homogeneous space of an algebraic group of multiplicative type (Wei, Xu [WX12], [WX13]). On the other hand, there are several known types of counter-examples, i.e., families for which the answer to Question 1.1 is negative. One way to construct such counter-example is to consider varieties which are not simply-connected. In this case, one can sometimes refine the Brauer-Manin obstruction by applying it to various étale coverings of X (Colliot-Thélène, Wittenberg, [CTW12, Example 5.10]). Other types of counter-examples occur when X lacks a sufficient supply of local S-points "at infinity". In [CTW12, Example 5.9] it is shown that the affine surface $\mathcal{X}$ over $\mathbb{Z}$ given by the equation

$$2x^2 + 3y^2 + 4z^2 = 1$$

has a non-empty integral Brauer set, but evidently no integral points. We note that the surface $X = \mathcal{X} \otimes_{\mathcal{O}_S} k$ is geometrically very nice: it can be compactified $X \subset \overline{X}$ such that the complement $D = \overline{X} \setminus X$ is smooth and geometrically irreducible and such that the divisor class $-[D] - K(\overline{X})$ is **ample**. In other words, X is a **log del Pezzo surface**. However, D has no **real points**, and as a result the space of adelic point $\mathcal{X}(\mathbb{A})$ is compact. Since $\mathcal{X}$ is affine this implies that $\mathcal{X}$ could a priori only have finitely many integral points, and it just so happens that it has none. More generally, if $\mathcal{X}$ is an $\mathcal{O}_S$-variety and $X \subseteq \overline{X}$ is a smooth compactification such that $D = \overline{X} \setminus X$ is geometrically irreducible and has no k_v-points for any $v \in S$ then we should not expect $\mathcal{X}$ to be as well-behaved as its geometric features might indicate. When D is not irreducible the situation is even more delicate, since each component (and each intersection of components) may or may not have a k_v-point for each $v \in S$. In this case it is not even clear under what circumstances should we expect $\mathcal{X}$ to match the behavior predicted by its geometry. We hence see that Question 1.1 for integral points is quite a bit more subtle than its rational points counterpart. In order to obtain a better understanding of it it is important to have good tools to establish the existence of integral points, when possible.

The descent-fibration method we wish to adapt first appeared in Swinnerton-Dyer's paper [SD95], where it was applied to the intersection of two diagonal quadrics in $\mathbb{P}^4$ (i.e., diagonal Del-Pezzo surfaces of degree 4). It was later expanded and generalized by authors such as Swinnerton-Dyer, Collot-Thélène, Skorobogatov, Wittenberg and Bender (see [BSD01],[CT01],[Wit07][SD01], [SDS05], [HS]). There are two important things to keep in mind when considering this method. The first is that the method typically requires assuming two hard conjectures. Schinzel's hypothesis, a number theory conjecture concerning polynomials taking simultaneously prime values, and the Tate-Shafarevich conjecture, stating that the Tate-Shafarevich group of ellitic curves is finite (sometimes a statement concerning more general abelian varieties is needed. On the other hand, sometimes it is enough to know the conjecture only for a certain class of elliptic curves). The second conjecture is considered more legitimate than the first. Recent work of Bhargava, Skinner and Zhang shows that this conjecture holds for a 100% of elliptic curves over $\mathbb{Q}$. Schinzel's hypothesis, on the other hand, contains as a particular case the twin prime conjecture. Only one special case of the conjecture is known, the one involving a single linear polynomial, in which case the conjecture reduces to Dirichlet's theorem on primes in arithmetic progressions. In some applications of Swinnerton-Dyer's method this is the only case needed, and hence Schinzel's hypothesis can be removed. The second thing to keep in mind is that if one admits the required conjectures, the domain of applicability of this method includes varieties which are not accessible in any other way, such as **K3 surfaces**. It is hence the only source of information towards the rational point variant of Question 1.1 for K3 surfaces.

In a typical setup for this method one is studying a variety X which is fibered over $\mathbb{P}^1_k$ into genus 1 curves with an associated Jacobian fibration $E \longrightarrow \mathbb{P}^1_k$. The first step is to apply the **fibration method** in order to find a $t \in \mathbb{P}^1(k)$ such that the fiber X_t has points everywhere locally (this part typically uses the vanishing of the Brauer-Manin obstruction, and often requires Schinzel's hypothesis). The second step then consists of modifying t until the Tate-Shafarevich group $Ш^1(E_t)$ (or a suitable part of it) vanishes, implying the existence of a k-rational point on X_t. This part usually assumes, in additional to a possible Schinzel hypothesis, the finiteness of the Tate-Shafarevich group for all relevant elliptic curves, and crucially relies on the properties of the Cassels-Tate pairing.

The goal of this talk is to describe such an adaptation, where one replace torsors under elliptic curves with **torsors under algebraic tori**. This adaptation can applied, in particular, to certain **log K3 surfaces**.

For reasons that will become clear soon it will be convenient to call our initial set of finite places S_0 (instead of S). Let $d \in \mathcal{O}_{S_0}$ be a non-zero S_0-integer satisfying the following condition

Assumption 1.2. *For every $v \notin S_0$ we have $\mathrm{val}_v(d) \leq 1$ and $\mathrm{val}_v(d) = 1$ if v lies above* 2.

Let $K = k(\sqrt{d})$ and let T_0 denote the set of places of K lying above S_0. Assumption 1.2 implies that the ring $\mathcal{O}_{T_0}$ is generated, as an $\mathcal{O}_{S_0}$-module, by 1 and d. Let $\mathcal{T}_0$ denote the algebraic group given the equation

$$x^2 - dy^2 = 1$$

We may identify the S-integral points of $\mathcal{T}_0$ with the set of units in $\mathcal{O}_d$ whose norm is 1 (in which case the group operation is given by multiplication in $\mathcal{O}_{T_0}$). We note that technically speaking the algebraic group $\mathcal{T}_0$ is not an algebraic torus, since it does not split over an étale extension of the base ring. For every divisor $a|d$ we may consider the affine $\mathcal{O}_{S_0}$-scheme $\mathcal{Z}^a_0$ given by the equation

$$ax^2 + by^2 = 1 \tag{1}$$

where $b = -\frac{d}{a}$. Our goal is to construct an adaptation of Swinnerton-Dyer's method where curves of genus 1 are replaced by the schemes $\mathcal{Z}^a_0$, and their correponding Jacobians are replaced by $\mathcal{T}_0$. Let $I_a \subseteq \mathcal{O}_{T_0}$ be the $\mathcal{O}_{T_0}$-ideal generated by a and $\sqrt{d}$. The association $(x, y) \mapsto ax + \sqrt{d}y$ identifies the set of S_0-integral points of $\mathcal{X}$ with the set of elements in I_a whose norm is a. We note that I_a is an ideal of norm (a) (in the sense that $\mathcal{O}_{T_0}/I_a \cong \mathcal{O}_{S_0}/(a)$), and hence we may consider the scheme $\mathcal{Z}_0^a$ above as parameterizing generators for I_a whose norm is exactly a. We have a natural action of the algebraic group $\mathcal{T}_0$ on the scheme $\mathcal{Z}_0^a$ corresponding to multiplying a generator by a unit. Now Assumption 1.2 implies that a and b are coprime in $\mathcal{O}_{S_0}$ (i.e., the ideal $(a, b) \subseteq \mathcal{O}_{S_0}$ generated by a, b is equal to $\mathcal{O}_{S_0}$). It can then be shown that then this action exhibits $\mathcal{Z}_0^a$ as a **torsor** under $\mathcal{T}_0$, locally trivial in the étale topology, and hence classified by an element in the étale cohomology group $\alpha_a \in H^1(\mathcal{O}_{S_0}, \mathcal{T}_0)$. The solubility of $\mathcal{Z}_0^a$ is equivalent to the condition $\alpha_a = 0$. Hence our search of S_0-integral points on $\mathcal{Z}_0^a$ naturally leads to the study of étale cohomology groups as above, analogous to how the study the curves of genus 1 leads to the study of the Galois cohomology of their Jacobians.

We next observe that the torsor $\mathcal{Z}_0^a$ is not an arbitrary torsor of $\mathcal{T}_0$. The condition $(a, b) = \mathcal{O}_{S_0}$ implies that $I_a^2 = (a)$. In particular, if $\beta = ax + \sqrt{d}y \in I_a$ has norm a then $\frac{\beta^2}{a} \in \mathcal{O}_{T_0}$ has norm 1. This operation can be realized as a map of $\mathcal{O}_{S_0}$-varieties

$$q : \mathcal{Z}_0^a \longrightarrow \mathcal{T}_0.$$

The action of $\mathcal{T}_0$ on $\mathcal{Z}_0^a$ is compatible with the action of $\mathcal{T}_0$ on itself via the multiplication-by-2 map $\mathcal{T}_0 \xrightarrow{2} \mathcal{T}_0$. We will say that q is a map of $\mathcal{T}_0$-torsors covering the map $\mathcal{T}_0 \xrightarrow{2} \mathcal{T}_0$. It then follows that the element $\alpha_a \in H^1(\mathcal{O}_{S_0}, \mathcal{T}_0)$ is a **2-torsion element**. We are hence naturally lead to study the 2-torsion group $H^1(\mathcal{O}_{S_0}, \mathcal{T}_0)[2]$.

Finally, an obvious necessary condition for the existence of S_0-integral points on $\mathcal{Z}_0^a$ is that $\mathcal{Z}_0^a$ carries an S_0-integral **adelic point**. This condition restricts the possible elements α_a to a suitable subgroup of $H^1(\mathcal{O}_{S_0}, \mathcal{T}_0)$, which we may call $Ш^1(\mathcal{T}_0, S_0)$. We are now interested in studying the 2-torsion subgroup $Ш^1(\mathcal{T}_0, S_0)[2]$. This is in analogy, for example, with the situation one faces when studying curves of genus 1 which are given as the intersection of two quadrics. Such curves always admit a map to their Jacobian E which covers the multiplication by 2 map $E \xrightarrow{2} E$. When the curve has points everywhere locally one is lead to study the group $Ш^1(E)[2]$.

Before we proceed to analyze the solubility in $\mathcal{O}_S$ of $\mathcal{Z}_0^a$, let us note what type of theorems we can expect to get. Let $f(t, s), g(t, s) \in \mathcal{O}_S[t, s]$ be two homogeneous polynomials of even degrees n, m respectively, let $V \longrightarrow \mathbb{P}^1$ be the vector bundle $\mathcal{O}(-n) \oplus \mathcal{O}(-m) \oplus \mathcal{O}(0)$ and let $\overline{Y} \subseteq \mathbb{P}(V)$ be the conic bundle surface given by the equation

$$f(t, s)x^2 + g(t, s)y^2 = z^2 \tag{2}$$

Let $Y \subseteq \overline{Y}$ be the complement of the divisor D given by $z = 0$. The variety Y admits a natural $\mathcal{O}_{S_0}$-model $\mathcal{Y} \subseteq \mathcal{O}(-n) \oplus \mathcal{O}(m)$ given by the equation

$$f(t, s)x^2 + g(t, s)y^2 = 1 \tag{3}$$

Our main theorem is the following:

Theorem 1.3. *Let $f(t,s), g(t,s) \in \mathcal{O}_S[t,s]$ be homogeneous polymoials such that $f(t,s)g(t,s)$ is separable and has only even degree prime factors. Assume that the homogeneous Schinzel's hypothesis holds and that f, g satisfy a certain* ***Condition (D)****. Assume in addition that there exists an S-integral adelic point $(P_v) = (t_v, s_v, x_v, y_v)$ such that*

1. *There exists a $v \in S$ such that $-f(t_v, s_v)g(t_v, s_v)$ is a square in k_v,*
2. *For every $v \notin S$ we have $\mathrm{val}_v\, f(t_v, s_v)g(t_v, s_v) \le 1$.*
3. *(P_v) is orthogonal to the vertical Brauer group of Y over $\mathbb{P}^1_k$.*

Then $\mathcal{Y}$ has an S-integral point.

Remark 1.4. If $\deg(f) = 2$ and $\deg(g) = 0$ then $\mathcal{Y}$ is a log del Pezzo surface. If $\deg(f) = \deg(g) = 2$ then $\mathcal{Y}$ is a log K3 surface.

Remark 1.5. The homogeneous version of Schinzel's hypothesis is known in more cases than the non-homogeneous version. Most importantly, the case where all the polynomials are linear follows from the Hardy-Littlewood conjecture, in the form proved by Green, Tao and Ziegler.

Current work in progress allows one to dispense with the hypothesis that the prime factors of $f(t,s)g(t,s)$ have even degrees. This generalization has the following attractive feature: when all the irreducible factors of $f(t,s)g(t,s)$ are linear one may use the above proven case of the homogeneous Schinzel's hypothesis, rendering Theorem 1.3 completely unconditional.

Remark 1.6. Condition (D) appearing in the formulation of Theorem 1.3 is analogous to Condition (D) appearing in[CTSSD98b]. It is an explicit condition which is straightforward to verify, and implies, in particular, that $\mathrm{Br}(Y)[2]$ is contained in the vertical Brauer group of Y. The conditions on the adelic point are more specific to our integral points setting. The first and third imply together that $\mathcal{Y}(\mathbb{A}_k)^{\mathrm{Br}^{\mathrm{vert}}(Y)}$ is **non-compact**, i.e., $\mathcal{Y}$ does not suffer from the problems at infinity described above. The second condition is a more technical feature of our application, and is meant to guarantee Assumption 1.2 above. It is most likely that this condition can be removed, if the arithmetic duality theory of algebraic tori would be extended to more general group schemes.

Let us now return to our torsors $\mathcal{Z}^a_0$. It turns out that the groups $H^1(\mathcal{O}_{S_0}, \mathcal{T}_0)$ are more well-behaved when the algebraic group $\mathcal{T}_0$ is a **algebraic torus**, i.e, splits in an étale extension of the base ring. Let S be the union of S_0 with all the places which ramify in K and all the places above 2, and let T be the set of places of K which lie above S. Let $\mathcal{O}_T$ denote the ring of T-integers in K. Let $\mathcal{T}$ be the base change of $\mathcal{T}_0$ from $\mathcal{O}_{S_0}$ to $\mathcal{O}_S$. We note that $\mathcal{T}$ becomes isomorphic to $\mathbb{G}_m$ after base changing from $\mathcal{O}_S$ to $\mathcal{O}_T$, and $\mathcal{O}_T/\mathcal{O}_S$ is an étale extension of rings. This means that $\mathcal{T}$ is an **algebraic torus** over $\mathcal{O}_S$. We will denote by $\widehat{\mathcal{T}}$ be the character group of $\mathcal{T}$ considered as an étale sheaf over $\mathrm{spec}(\mathcal{O}_S)$. We will use the notation $H^i(\mathcal{O}_S, \mathcal{F})$ to denote étale cohomology of $\mathrm{spec}(\mathcal{O}_S)$ with coefficients in the sheaf $\mathcal{F}$.

Definition 1.7.

1. We will denote by $Ш^1(\mathcal{T}, S) \subseteq H^1(\mathcal{O}_S, \mathcal{T})$ the kernel of the map

$$H^1(\mathcal{O}_S, \mathcal{T}) \longrightarrow \prod_{v \in S} H^1(k_v, \mathcal{T} \otimes_{\mathcal{O}_S} k_v).$$

2. We will denote by $Ш^2(\widehat{\mathcal{T}}, S) \subseteq H^2(\mathcal{O}_S, \widehat{\mathcal{T}})$ the kernel of the map

$$H^2(\mathcal{O}_S, \widehat{\mathcal{T}}) \longrightarrow \prod_{v \in S} H^2(k_v, \widehat{\mathcal{T}} \otimes_{\mathcal{O}_S} k_v).$$

Since $\mathcal{T}$ is an algebraic torus we may apply [Mil, Theorem 4.6(a), 4.7] and deduce that the groups $Ш^1(\mathcal{T}, S)$ and $Ш^2(\widehat{\mathcal{T}}, S)$ are finite and that the cup product in étale cohomology with compact support induces a perfect pairing

$$Ш^1(\mathcal{T}, S) \times Ш^2(\widehat{\mathcal{T}}, S) \longrightarrow \mathbb{Q}/\mathbb{Z} \tag{4}$$

Since 2 is invertible in $\mathcal{O}_S$ the multiplication by 2 map $\mathcal{T} \longrightarrow \mathcal{T}$ is surjective when considered as a map of étale sheaves on $\mathrm{spec}(\mathcal{O}_S)$. We hence obtain a short exact sequence of étale sheaves

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathcal{T} \xrightarrow{2} \mathcal{T} \longrightarrow 0.$$

We define the **Selmer group** $\mathrm{Sel}(\mathcal{T}, S)$ to be the subgroup $\mathrm{Sel}(\mathcal{T}, S) \subseteq H^1(\mathcal{O}_S, \mathbb{Z}/2)$ consisting of all elements whose image in $H^1(\mathcal{O}_S, \mathcal{T})$ belongs to $Ш^1(\mathcal{T}, S)$. We hence obtain a short exact sequence

$$0 \longrightarrow \mathcal{T}_S(\mathcal{O}_S)/2 \longrightarrow \mathrm{Sel}(\mathcal{T}, S) \longrightarrow Ш^1(\mathcal{T}, S)[2] \longrightarrow 0$$

where $\mathcal{T}_S(\mathcal{O}_S)/2$ denotes the cokernel of the map $\mathcal{T}(\mathcal{O}_S) \xrightarrow{2} \mathcal{T}(\mathcal{O}_S)$. Similarly, we have a short exact sequence of étale sheaves

$$0 \longrightarrow \widehat{\mathcal{T}} \xrightarrow{2} \widehat{\mathcal{T}} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and we define the **dual Selmer group** $\mathrm{Sel}(\widehat{\mathcal{T}}, S) \subseteq H^1(\mathcal{O}_S, \mathbb{Z}/2)$ to be the subgroup consisting of all elements whose image in $H^2(\mathcal{O}_S, \widehat{\mathcal{T}})$ belongs to $Ш^2(\widehat{\mathcal{T}}, S)$. The dual Selmer group then sits in a short exact sequence of the form

$$0 \longrightarrow H^1(\mathcal{O}_S, \widehat{\mathcal{T}})/2 \longrightarrow \mathrm{Sel}(\widehat{\mathcal{T}}, S) \longrightarrow Ш^2(\widehat{\mathcal{T}}, S)[2] \longrightarrow 0$$

The map $H^1(\mathcal{O}_S, \mathbb{Z}/2) \longrightarrow H^1(\mathcal{O}_S, \mathcal{T})$ can be described explicitly as follows. Since S contains all the places above 2 the Kummer sequence associated to the sheaf $\mathbb{G}_m$ yields a short exact sequence

$$0 \longrightarrow \mathcal{O}_S^*/(\mathcal{O}_S^*)^2 \longrightarrow H^1(\mathcal{O}_S, \mathbb{Z}/2) \longrightarrow \mathrm{Pic}(\mathcal{O}_S)[2] \longrightarrow 0$$

More explicitly, an element of $H^1(\mathcal{O}_S, \mathbb{Z}/2)$ may be represented (mod squares) by a non-zero element $a \in \mathcal{O}_S$ such that $\mathrm{val}_v(a)$ is even for every $v \notin S$. The map

$H^1(\mathcal{O}_S, \mathbb{Z}/2) \longrightarrow \mathrm{Pic}(\mathcal{O}_S)[2]$ is then given by sending a to the class of $\frac{\mathrm{div}(a)}{2}$, where $\mathrm{div}(a)$ is the divisor of a when considered as a function on $\mathrm{spec}(\mathcal{O}_S)$. Let $I_a \subseteq \mathcal{O}_T$ be the ideal corresponding to the pullback of $\frac{\mathrm{div}(a)}{2}$ from $\mathcal{O}_S$ to $\mathcal{O}_T$. Then I_a is an ideal of norm (a) and we can form the $\mathcal{O}_S$-scheme $\mathcal{Z}^a$ parameterizing elements of I_a of norm a (such a scheme admits explicit affine equations locally on $\mathrm{spec}(\mathcal{O}_{S_0})$ by choosing locally generators for the ideal I_a). The scheme $\mathcal{Z}^a$ is a torsor under $\mathcal{T}_S$, and the classifying class of $\mathcal{Z}^a$ is the image of a in $H^1(\mathcal{O}_S, \mathcal{T}_S)$. We note that such a scheme automatically has $\mathcal{O}_v$-points for every $v \notin S$. Hence we see that the class $H^1(\mathcal{O}_S, \mathbb{Z}/2)$ represented by a belongs to $\mathrm{Sel}(\mathcal{T}, S)$ if and only if the torsor $\mathcal{Z}^a$ has local points over S, i.e., if and only if it has an S-integral **adelic point**.

We note that if a is such that $\mathrm{div}(a) = 0$ on $\mathrm{spec}(\mathcal{O}_S)$ (i.e., a is an S-unit), then $\mathcal{Z}^a$ is the scheme parameterizing T-units in K whose norm is a, and can be written as

$$x^2 - dy^2 = a \tag{5}$$

Furthermore, if a is a divisor of d (so that in particular a is an S-unit), then the scheme $\mathcal{Z}^a$ coincides with the base change of the our scheme of interest $\mathcal{Z}_0^a$ (see 1) from $\mathcal{O}_{S_0}$ to $\mathcal{O}_S$.

Lemma 1.8. *Assume Condition 1.2 is satisfied and let $a|d$ be an element dividing d. Let $\mathcal{Z}_0^a$ be the $\mathcal{O}_{S_0}$-scheme given by 1 and let $\mathcal{Z}^a = \mathcal{Z}_0^a \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_S$ be the corresponding base change. If $\mathcal{Z}^a$ has an S-integral point then $\mathcal{Z}_0^a$ has an S_0-integral point.*

On the dual side, we may consider the short exact sequence

$$0 \longrightarrow \widehat{\mathcal{T}} \longrightarrow \widehat{\mathcal{T}} \otimes \mathbb{Q} \longrightarrow \widehat{\mathcal{T}} \otimes (\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

Since $\widehat{\mathcal{T}} \otimes \mathbb{Q}$ is a uniquely divisible sheaf we get an identification

$$H^2(\mathcal{O}_S, \widehat{\mathcal{T}}) \cong H^1(\mathcal{O}_S, \widehat{\mathcal{T}} \otimes (\mathbb{Q}/\mathbb{Z}))$$

By the Hochschild-Serre spectral the latter may be identified with the kernel of the corestriction map $\mathrm{Cores} : H^1(\mathcal{O}_T, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(\mathcal{O}_S, \mathbb{Q}/\mathbb{Z})$. The map $H^1(\mathcal{O}_S, \mathbb{Z}/2) \longrightarrow H^2(\mathcal{O}_S, \widehat{\mathcal{T}})$ can then be identified with the restriction map $H^1(\mathcal{O}_S, \mathbb{Z}/2) \xrightarrow{\mathrm{res}} H^1(\mathcal{O}_T, \mathbb{Z}/2) \subseteq H^1(\mathcal{O}_T, \mathbb{Q}/\mathbb{Z})$ Sending a class $[a] \in H^1(\mathcal{O}_S, \mathbb{Z}/2)$ to the class of the quadratic extension $K(\sqrt{a})$. The group $\text{Ш}^2(\widehat{\mathcal{T}}, S)[2]$ is then the group classifying everywhere unramified quadratic extensions of K, splitting over T, whose corestriction to k vanishes. Indeed, it is not hard to show that such extensions must always come from quadratic extensions of k. We then obtain the following explicit description of $\mathrm{Sel}(\widehat{\mathcal{T}})$:

Corollary 1.9. *Let $[a] \in H^1(\mathcal{O}_S, \mathbb{Z}/2)$ be a class represented by an element $a \in \mathcal{O}_S$ (such that $\mathrm{val}_v(a)$ is even for every $v \notin S$). Then $[a] \in \mathrm{Sel}(\widehat{\mathcal{T}}, S)$ if and only if every place in T splits in $K(\sqrt{a})$.*

Corollary 1.10. *The kernel of the map $\mathrm{Sel}(\widehat{\mathcal{T}}, S) \longrightarrow \text{Ш}^2(\widehat{\mathcal{T}}, S)$ has rank 1 and is generated by the class $[d] \in \mathrm{Sel}(\widehat{\mathcal{T}}, S)$.*

Our proposed formalism enables the following Corollary, which is the core point behind the adaptation of Swinnerton-Dyer's method:

Corollary 1.11. *Assume Condition 1.2 is satisfied. If* $\mathrm{Sel}(\widehat{\mathcal{T}})$ *is generated by* $[d]$ *then for every* $a|d$ *the* $\mathcal{O}_{S_0}$*-scheme* $\mathcal{Z}_0^a$ *given by 1 satisfies the* S_0*-integral Hasse principle.*

Proof. Assume that $\mathcal{Z}_0^a$ has an S-integral adelic point. If $\mathrm{Sel}(\widehat{\mathcal{T}})$ is generated by $[d]$ then $Ш^2(\widehat{\mathcal{T}}, S)[2] = 0$ and by the perfect pairing 4 we may deduce that $Ш^1(\mathcal{T}_S)[2] = 0$. Since $\mathcal{Z}_0^a$ has an S_0-integral adelic point the base change $\mathcal{Z}^a = \mathcal{Z}_0^a \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_S$ has an S-integral adelic point. The class classifying $\mathcal{Z}^a$ as a $\mathcal{T}$-torsor hence lies in $Ш^1(\mathcal{T}, S)[2]$, and since the latter group vanishes it follows that $\mathcal{Z}^a$ has an S-integral point. Then result now follows from Lemma 1.8. ☐

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