Periodicity in Stable Homotopy Groups of Spheres

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1 Introduction

The basic question in homotopy theory is to understand the set of (pointed) homotopy classes of maps between two finite complexes, $[X, Y]_*$. This is notoriously hard to compute, and in general this pointed set has no additional structure.

A first step is to reduce the problem to computing the homotopy groups of finite complexes:

$$\pi_n(Y) = [S^n, Y]_*$$

Since finite complexes are hocolimits of spheres and since maps out of hocolimits depend in a reasonable way on the maps from each of the components, we get that in principle if we understand homotopy groups of finite complexes then we understand maps between finite complexes.

Can we reduce the problem to computing only homotopy groups of **spheres**? The answer is unfortunately no, and the reason is that maps **into** hocolimits are not controlled in any reasonable way by the maps into the components.

A basic example is the following: if $n \geq 3$ then $\pi_3(\mathbb{C}P^n) = 0$ but

$$\pi_3(\mathbb{C}P^n \vee \mathbb{C}P^n) = \mathbb{Z}$$

The one exception here is the fundamental group π_1 for which we have Van-Kampan theorem - i.e. it behaves nicely under homotopy pushouts (which are the building blocks of the homotopy colimits constructing CW complexes) which means that in principle it is relatively computable.

1.1 Stable Maps

A much nicer object to deal with is the set of **stable maps** between two complexes

$$\{X, Y\} = \lim_{k \to \infty} [\Sigma^k X, \Sigma^k Y]$$

By Freudenthal's suspension we know that this limit stabilizes as soon as the connectivity of Y is at least half the dimension of X. This set is not exactly the set of maps from X to Y, but it is a good approximation of it and it posses some very nice features:

1. $\{X, Y\}$ has a natural structure of an abelian group. In fact, we can define stable maps of degree $n \in \mathbb{Z}$ by
$$\{X, Y\}_n = \lim_{k \to \infty} [\Sigma^{n+k} X, \Sigma^k Y]$$
and obtain a graded group
$$\{X, Y\}_* = \sum_{n \in \mathbb{Z}} \{X, Y\}_n$$
$\{X, X\}_*$ has thus a natural structure of a graded ring (with respect to composition).

2. The set $\{X, Y\}_*$ behaves well with respect to hocolimits in both direction. In fact, the functor $Y \mapsto \{X, Y\}_*$ is a generalized homology theory and $X \mapsto \{X, Y\}$ is a generalized cohomology theory. Thus if we understand $\{S^0, S^0\}_*$ we can in principle understand $\{X, Y\}_*$ for every two finite complexes.

3. $\{S^0, S^0\}_n \otimes \mathbb{Q} = \mathbb{Q}$ if $n = 0$ and 0 otherwise. This implies that for every finite complex Y we have
$$\pi_n^s(Y) \otimes \mathbb{Q} = \{S^0, Y\}_n \otimes \mathbb{Q} = H^n(Y, \mathbb{Q})$$
which is well understood. It is thus left to understand the torsion part of $\pi_n^s(S^0)$. It is also easy to show that $\pi_n^s(S^0) = 0$ for $n < 0$ so we are left with $n > 0$.

2 Periodic Patterns

Here is a table of the first 19 stable homotopy groups of S^0:

n	1	2	3	4	5	6	7	8	9	10
$\pi_n^s(S^0)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$

n	11	12	13	14	15	16
$\pi_n^s(S^0)$	$\mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7$	0	$\mathbb{Z}_3$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$	$(\mathbb{Z}_2)^2$

n	17	18	19
$\pi_n^s(S^0)$	$(\mathbb{Z}_2)^4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{11}$

Although these are only the first portion of the homotopy groups of spheres, we already see here the a basic periodic pattern.

Note that when 4 divides $n+1$ then we get a factor of the form $\mathbb{Z}_{2^k}$ in $\pi_n^s(S^0)$ where k is the largest number such that 2^{k-1} divides $n + 1$. For $p = 3, 5, 7, 11$ we see that if $2(p - 1)$ divides $n + 1$ then we get a factor of the form $\mathbb{Z}_{p^k}$ in $\pi_n^s(S^0)$ where k is the largest number such that p^{k-1} divides $n + 1$.

We see here that we get some sort of periodic phenomenon, but the period depends on the prime. It is $2(p-1)$ for $p > 2$ and 4 for $p = 2$. Very mysterious! This suggests the two main themes of the study of stable homotopy groups:

1. Study $\pi_n^s(S^0)$ one prime at a time.
2. Look for periodic phenomena and try to explain it.

Let us try to give a general overview of where this periodicity comes from. Let $p \in \{S^0, S^0\}_0$ be the stable map of degree p and $M(p^r)$ the cofiber of the map p^r:

$$S^n \xrightarrow{p^r} S^n \xrightarrow{\pi} \Sigma^n M(p^r)$$

in particular $M(r)$ is stably the CW complex with one cell at dimension n and one cell at dimension $n+1$ which is attached by the map p^r. Then it turns out that there is a stable map of degree d

$$f \in \{M(p^r), M(p^r)\}_d$$

for $d = 2(p-1)p^{r-1}$ if $p > 2$ and $d = 4 \cdot 2^{r-1}$ for $p = 2$. Now we have a boundary map from the Pupe sequence $\partial \in \{M(r), S^0\}_{-1}$. Assume that f is represented by a map

$$f : \Sigma^{d+m} M(p^r) \longrightarrow \Sigma^m M(p^r)$$

The map f induces the zero map on homology, but it is detected by complex K-theory. Recall that

$$\widetilde{K}^*(S^0) = \mathbb{Z}[t, t^{-1}]$$

where $|t| = 2$. Hence from the Pupe sequence it is easy to see that

$$\widetilde{K}^*(M(p^r)) = \mathbb{Z}_{p^r}[t, t^{-1}]$$

It turns out that f induces multiplication by $t^{\frac{d}{2}}$ on $\widetilde{K}^*$, and in particular an isomorphism. Thus for each k the map

$$\Sigma^{kd+m} M(p^r) \xrightarrow{f^k} \Sigma^m M(p^r)$$

is not null-homotopic. By composing with π and ∂ we obtain an element $\alpha_k^r \in \pi_{kd-1}^s(S^0)$ for some $m' \geq m$:

$$S^{kd+m'} \xrightarrow{\pi} \Sigma^{kd+m} M(p^r) \xrightarrow{f^k} \Sigma^m M(p^r) \xrightarrow{\partial} S^{m'+1}$$

It turns out that these elements are all non-trivial, and account exactly for the periodicity we saw above. This is part of a much more general phenomenon.

3 Morava K-theories and the Periodicity Theorem

Before we continue we wish to localize our attention at a specific prime p. For this we give the following definition

Definition 3.1. A space X is called p**-local** in the stable category if every map $f : Y \longrightarrow Z$ which is a mod p homology isomorphism induces an isomorphism

$$\{Z, X\}_* \longrightarrow \{Y, X\}_*$$

Using the Pupe sequence we see that this is equivalent to saying that $\{W, X\}_* = 0$ whenever W has trivial reduced mod-p homology.

There exists an augmented localization functor $X \longrightarrow X_p$ such that X_p is p-local and the augmentation map is a mod-p equivalence. This localization functor preserves all the p-part of the homotopy groups, and in particular:

$$\pi_n^s(X_p) = \pi_n^s(X) \otimes \mathbb{Z}_p$$

where $\mathbb{Z}_p$ is now the p-adic integers.

We now return to our periodicity phenomenon. The following cohomology theories are the key to the question:

Definition 3.2. For each p and n there exists a generalized (multiplicative) cohomology theory $K(n)^*$, with reduced version $\widetilde{K}(n)^*$ (we suppress the prime p from the notation), such that

1. $$K(0)^*(X) = H^*(X, \mathbb{Q})$$

2. For $n > 0$ we have
$$K(n)^*(pt) = \mathbb{Z}_p[v_n, v_n^{-1}]$$
where $|v_n| = 2(p^n - 1)$.

3. If X is a p-local finite CW-complex then
$$\widetilde{K}(n+1)^*(X) = 0 \Longrightarrow \widetilde{K}(n)^*(X) = 0$$

4. If X is a p-local finite then for large enough n we have
$$\widetilde{K}^*(n)(X) = K^*(n)(\mathrm{pt}) \otimes \widetilde{H}^*(X, \mathbb{Z}_p)$$

The first Morava K-theory $K^*(1)$ is a mod-p variant of complex K-theory.

Definition 3.3. A topological space is said to have **type** n if n is the smallest number such that $\widetilde{K}(n)^*(X) \neq 0$. If there is no such n then we say that X has type ∞.

Note that the last property implies that if X is a non-contractible p-local finite CW complex then $K(n)^*(X)$ is non-zero for large enough n. Thus each such space has a type n for some finite n.

Examples:

1. The p-localization of a sphere is a type 0 p-local space.

2. Let S^0_p be the p-localization of S^0. Then

$$\widetilde{K}(0)^n(S^0_p) = \widetilde{H}^n(S^0_p) = \begin{cases} \mathbb{Q}_p & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Let $p^r \in \{S^0_p, S^0_p\}_0$ be the map induced by the degree p^r map. Let $M(p^r)$ be the cofiber of this map. Note that p^r induces isomorphism on $\widetilde{K}(0)^*$ and so from the Pupe sequence we get $\widetilde{K}(0)^*(M(p^r)) = 0$.

Also note that p^r induces the zero map on all higher $K(n)$'s (because their coefficient ring is a $\mathbb{Z}_p$-module). Hence from the Pupe sequence we see that $K(1)^*(M(p^r)) \neq 0$ so $M(p^r)$ is a type 1 space.

In fact, the map f defined above induces a map $M(p^r) \longrightarrow M(p^r)$ which is a $K(1)$-isomorphism: it induces multiplication by an appropriate power of v_n.

Note that both in the case of S^1_p and $M(p^r)$ we has a map which induces isomorphism on the lowest non-zero Morava K-theory. It turns out that this is part of a general phenomenon which is called the **periodicity theorem**:

Theorem 3.4. *1. (Existence): Let X is a p-local finite complex. Then there exists a self map $f \in \{X, X\}_d$ which induces isomorphism on $K(n)^*(X)$ and induces the zero map on $K^*(m)(X)$ for $m \neq n$. In particular f induces multiplication by some power of v_n (where v_0 is just p) and d is the corresponding multiple of $|v_n| = 2(p^n - 1)$. Such a map is called a v_n-map.*

2. (Uniqueness): Let X, Y be p-local finite complexes of type n and $f \in \{X, X\}_d, g \in \{Y, Y\}_e$ v_n-maps on them. Then for each map $h : X \longrightarrow Y$ there exist numbers i, j such that $di = ej$ and $h \circ f^i$ is stably homotopic to $g^j \circ h$. In particular a v_n map is unique up to taking powers and all continuous maps respect them.

Corollary 3.5. *Let X be a finite p-local CW-complex of type n and f a v_n-map. Then the cofiber of f is a space of type $n+1$. In particular there exist spaces of all types.*

Let us now explain how this theorem gives the periodicity phenomenon. Suppose we want to consider the p-part of the stable homotopy groups of spheres, i.e. $\pi^s_*(S^0_p)$. Consider an element $\alpha \in \pi^s_*(S^0_p)$. If for every r we have $p^r\alpha \neq 0$ then we obtain an infinite family of non-trivial elements, all with the same degree as α.

Now suppose that $p^r\alpha = 0$ for some r. Then α reduces to a well defined map in $\{M(p^r), S^0\}_0$. On $M(p^r)$ we have a v_1-map f so we can compose it in the entrance with f many times and obtain elements in $\{M(p^r), S^0\}_*$ which can be sent to elements in $\pi^s_*(S^0)$ via the map $\pi \in \{S^0, M(p^r)\}_0$.

If all these elements are non-trivial then we have obtained an infinite family of elements in $\pi^s_*(S^0)$ whose degrees form an arithmetic sequence with jump divisible by $2(p-1)$. This is called a v_1-periodic family. If one of these elements (and so all the rest) is trivial (say the one obtained by composing with f^{r_1}) then it induces a map in $\{M(p^r, v_1^{r_1}), S^0\}_0$ where $M(p^r, v_1^{r_1})$ is the cofiber of the map f^{r_1}.

Now $M(p^r, v_1^{r_1})$ is a type 2 space so we have a v_2-map and we can repeat the process to obtain a possibly infinite sequence of elements in $\pi_*^s(S^0)$ which is periodic with period a multiple of $2(p^2-1)$ (this is called a v_2-periodic family). If we fail we continue to the appropriate cofiber and so on.

This gives us filtration on the ring $\pi_n^s(S^0)$ into periodic families which is called the **chromatic filtration**. It is a theorem that all elements in $\pi_*^s(S^0)$ fit in one of these infinite families. The theory that studies this phenomenon in order to get a global understanding of stable homotopy groups is what we want to study coming weeks.