The First periodic Layer

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In this lecture we will describe the connection between the image of the J-homomorphism described by Ilan and the what we called v_1**-periodic elements** in $\pi_s^*(S^0)$. These periodic elements were constructed as follows. Consider the moore spectrum $M(p)$ given by the cone on the map:

$$S^0 \xrightarrow{p} S^0 \xrightarrow{f} M(p)$$

From the above Pupe sequence we get a long exact sequence in stable homotopy groups

$$... \longrightarrow \pi_n^s(S^0) \xrightarrow{p} \pi_n^s(S^0) \xrightarrow{f_*} \pi_n^s(M(p)) \xrightarrow{\partial}$$

$$\pi_{n-1}^s(S^0) \xrightarrow{p} \pi_{n-1}^s(S^0) \longrightarrow ...$$

This means that $\pi_n^s(M(p))$ sits in the short exact sequence

$$0 \longrightarrow \pi_n^s(S^0)/p \xrightarrow{f_*} \pi_n^s(M(p)) \xrightarrow{\partial_*} \pi_{n-1}^s(S^0)[p] \longrightarrow 0$$

where for an abelian group A we denote by A/p the cokernel of the multiplication by p map by $A[p]$ the kernel of this map (i.e. the p-torsion part of A). In particular the homotopy groups of $M(p)$ are vector spaces over $\mathbb{F}_p$.

Now $M(p)$ admits a self map of degree $2(p-1)$:

$$v_1 : \Sigma^{2(p-1)}M(p) \longrightarrow M(p)$$

This map induces a structure of a graded $\mathbb{F}_p[v_1]$-module on the graded group $\pi_*^s(M(p))$. Now consider the localized graded module $v_1^{-1}\pi_*^s(M(p))$ which is a module over the localized ring $\mathbb{F}_p[v_1, v_1^{-1}]$. An element $\alpha \in \pi_n^s(M(p))$ survives to this localization iff $v_1^n\alpha \neq 0$ for all n. Such an element is called v_1**-periodic**. It gives an infinite family of elements in $\pi_*^s(M(p))$ in dimensions which form an arithmetic sequence with jump $2(p-1)$.

The ring $\mathbb{F}_p[v_1, v_1^{-1}]$ has the property that every graded module over it is free. Hence in order to understand the v_1-periodic elements all we need is to find a set $\alpha_1, ..., \alpha_n$ of generators for $v_1^{-1}\pi_*^s(M(p))$ which are minimal in the sense that $\alpha_i \in \pi_*^s(M(p))$ and $v_1^{-1} \notin \pi_*^s(M(p))$.

Now we have a map $\partial_*\pi_n^s(M(p)) \longrightarrow \pi_{n-1}^s(S^0)[p]$. An element $\alpha \in \pi_{n-1}^s(S^0)[p]$ is called v_1**-periodic** if it is an image by ∂_* of a v_1-periodic element in $\pi_n^s(M(p))$.

In the previous lectures Ilan has describe a map $J : BU \longrightarrow S^1$ from the spectrum of complex K-theory to the sphere spectrum. The homotopy groups of BU are $\mathbb{Z}$ at even places and 0 at odd places (they form the coefficient ring $\mathbb{Z}[t,t^{-1}]$ where $|t| = 2$). Hence the image of $J_* : \pi^s_{n+1}(BU) \longrightarrow \pi^s_n(S^0)$ is some cyclic group in every odd n.

Ilan has shown us that for $n = 1 \mod 4$ the image is $\mathbb{Z}/2$ (this is because this map factors through the spectrum BO of real K-theory) and for $n = 3 \mod 4$ the order of the image is the denominator of $\frac{B_n}{2n}$. It can be shown that p divides this denominator if and only if $2(p-1)|n+1$.

Now suppose that $2(p-1)|n+1$ and consider an element $\alpha \in \pi^s_n(S^0)[p]$ which is in the image of J. J induces a map

$$BU \wedge M(p) \longrightarrow S^1 \wedge M(p) = \Sigma M(p)$$

which we will call J_p. Recall the map $f : S^0 \longrightarrow M(p)$. The map J_p closes the commutative square

$$\begin{tikzcd}
BU \wedge S^0 \arrow[r, "Id\wedge f"] \arrow[d, "J\wedge Id"] & BU \wedge M(p) \arrow[d, "J\wedge Id"] \\
S^1 \wedge S^0 \arrow[r, "Id\wedge f"] & S^1 \wedge M(p)
\end{tikzcd}$$

Now since α is in the image of J_* it follows that $f_*\alpha \in \pi_n(M(p))$ is in the image of $(J_p)_*$. We have the coefficient ring

$$\pi^s_*(BU \wedge MU) = \mathbb{F}_p[t, t^{-1}]$$

So we can assume that $f_*\alpha = (J_p)_* t^k$ where k is such that $n+1 = 2(p-1)k$. Which can also be identified with $BU_*(M(p))$ - the homology version of complex K-theory of $M(p)$. Now the self map v_1 of $M(p)$ induces multiplication by t^{p-1} on $BU_*(M(p))$ which means that induced self map of $BU \wedge M(p)$ induces multiplication by t^{p-1} on the coefficient ring. This means that

$$v_1^m(f_*\alpha) = (J_p)_*(t^{k+m(p-1)})$$

and so $f_*\alpha$ is a v_1-periodic element in $\pi^s_n(M(p))$. Now it turns out that the degree -1 self map

$$g = f \circ \partial : M(p) \longrightarrow \Sigma M(p)$$

commutes up to stable homotopy with v_1. Since α is an element of order p there exists a $\gamma \in \pi_{n+1}(M(p))$ such that $\partial_*\gamma = \alpha$. Then $g_*\gamma = f_*\alpha$. Since g commutes with v_1 and $g_*\alpha$ is v_1-periodic it follows that γ is v_1-periodic as well, i.e. α is v_1-periodic. Hence the elements in the image of J which Ilan talked about are part of the first periodicity layer and this is the connection between Ilan's talk and the periodicity in $\pi^s_*(S^0)$.

A natural question now arises if there are v_1-periodic elements which are not in the image of J. The key to answering this question is to calculate $v_1^{-1}\pi^s_*(M(p))$. For $p \neq 2$ this was done by Miller in a paper from 78. The theorem is that $v_1^{-1}\pi_*^s(M(p))$ has a basis of size 2 over $\mathbb{F}_p[v_1, v_1^{-1}]$ given by f and $g \circ f$ which lie in degrees 0 and -1 respectively. Now for every m

$$\partial \circ v_1^m \circ g \circ f = \partial \circ f \circ \partial \circ v_1^m \circ f = 0$$

which means that the v_1-periodic elements $v_1^m \circ g \circ f$ don't contribute any v_1-periodic elements in $\pi_*^s(S^0)$.

Since we know that we have at least one family of v_1-periodic elements it has to come from the v_1 family $v_1^m \circ f$ (which works out degree wise). Hence the theorem essentially tells us that the p-torsion part of the image of J coincides with the p-torsion v_1-periodic elements.

Let us try to illustrate the proof of this theorem. It relies heavily on the Adams spectral sequence to compute stable homotopy groups. Recall from previous lectures that for a connective spectrum X of finite type (e.g. a CW-complex consisting of finitely many cells in each dimension) we have a spectral sequence with

$$E^2_{s,t}(X) = \mathrm{Ext}_{s,t}(\widetilde{H}^*(X, \mathbb{F}_p), \mathbb{F}_p) \Longrightarrow \pi^s_{t-s}(X)$$

In particular for $X = M(p)$ we have

$$\widetilde{H}^*(X, \mathbb{F}_p) = \mathbb{F}_p\alpha \otimes \mathbb{F}_p\beta$$

where α is a generator at dimension 1 and β a generator at dimension 0. The first part is to calculate some of the E^2 term. Note that this is a purely algebraic calculation.