Mehina 2009 lecture 2 - Surfaces

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1 Rational Surfaces

We now wish to study some examples of surfaces, i.e. algebraic varieties of dimension 2. We will restrict our selves only to smooth complete varieties. The theory of surfaces is more complicated then the theory of curves (though it is still manageable) so we would like to start by constructing the simplest kind of surfaces.

An important equivalence relation in algebraic geometry is that of birationality: Two varieties X, Y are called birational if there exist open dense subsets $U \subseteq X, V \subseteq Y$ such that $U \cong V$. We say that X and Y are birational over a field k if $V \cong U$ over k. Note that it is possible, for example, that two varieties would be birational over $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$.

The concept of birationality is connected to the concept of the function field. Recall that the function field of an n-dimensional irreducible variety is some algebraic extension of the field $k(x_1, ..., x_n)$ of rational functions in n variables over the base field k (we will usually consider the function field over an algebraically closed field k, so if our variety is defined over $\mathbb{Q}$ we will consider rational functions with coefficients in $\overline{\mathbb{Q}}$). It turns out that two irreducible varieties are birational if and only if their functions fields are isomorphic.

Hence the simplest kind of function field for an n-dimensional variety is $k(x_1, ..., x_n)$ itself i.e. varieties which are birational to $\mathbb{P}^n$. Such varieties are called **rational**.

It turns out that for $n = 1$ the following is true: if a curve C has $k(x)$ as a function field then C is actually isomorphic (over k) to $\mathbb{P}^1$. This is not the case, however, for higher dimensions, and in particular for surface. Hence we want to start by studying surfaces (usually defined over $\mathbb{Q}$) whose function field over $\overline{\mathbb{Q}}$ is isomorphic to $\overline{\mathbb{Q}}(x, y)$.

First examples are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ which both have $\mathbb{A}^2$ as a dense open set and hence posses $\overline{\mathbb{Q}}(x, y)$ as their function field. Note that these two varieties are not isomorphic over $\overline{\mathbb{Q}}$. $\mathbb{P}^2$ satisfies Bezout's theorem which implies in particular that every two curves in $\mathbb{P}^2$ intersect. In contrast we havethe curves $\mathbb{P}^1 \times \{p_1\}, \mathbb{P}^1 \times \{p_2\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ which don't intersect for $p_1 \neq p_2 \in \mathbb{P}^1$.

2 Blow Ups

In the classification of rational surface the basic construction tool is called **blow up**. Let us explain how this works. Let X be a variety of dimension n and $p \in X$ a point. It can be shown that there always exist an open affine subset U containing x and regular functions $f_1, ..., f_n$ on U such such that the ideal generated by $f_1, ..., f_n$ is exactly the maximal ideal of functions vanishing at p. In particular all the f_i's vanish at p and do not vanish at any other point on U.

Now consider the projective space $\mathbb{P}^{n-1}$ with projective coordinates $x_1, ..., x_n$. Define the variety $\widetilde{U} \subseteq U \times \mathbb{P}^{n-1}$ by the equations

$$f_i x_j = f_j x_i$$

for $1 \leq i < j \leq n$. These are well defined equations because their dependence on the projective coordinates $x_1, ..., x_n$ is homogeneous. Note that at each $q \in U$ such that $q \neq p$ the functions $f_1, ..., f_n$ don't all vanish and hence the equations become

$$(x_1 : x_2 : ... : x_n) = (f_1(q) : f_2(q) : ... : f_n(q))$$

which means that $(x_1 : x_2 : ... : x_n)$ is determined by q. At the point $p \in U$, however, all the f_i's vanish and hence the equations don't give any constraint on the x_i's. In particular we see that the natural map $\pi : \widetilde{U} \longrightarrow U$ which forgets the $(x_1 : ... : x_n)$ coordinate induces an isomorphism

$$\widetilde{U} \setminus \pi^{-1}(p) \cong U \setminus \{p\}$$

and the fiber above p is isomorphic to $\mathbb{P}^{n-1}$. Define $\widetilde{X}$ to be the gluing of $\widetilde{U}$ and $X \setminus \{p\}$ along $U \setminus \{p\}$. $\widetilde{X}$ is called the **blow up** of X at the point p. Note that this construction required a choice of an appropriate neighborhood U and functions $f_1, ..., f_n$ but it can be shown that different choices (as long as the condition on the ideal generated by $f_1, ..., f_n$ is satisfied) would give isomorphic blow ups, so the blow up only depends on the point we chose.

Note that the map π extends to a map $\pi : \widetilde{X} \longrightarrow X$ which induces an isomorphism

$$\widetilde{X} \setminus \pi^{-1}(p) \cong X \setminus \{p\}$$

and the fiber over p os $\mathbb{P}^{n-1}$. This is why this is called a blow up: we replace a point by an $n-1$ dimensional variety, so its like the point blows into something bigger.

Since $\widetilde{X}$ and X have isomorphic open dense subsets they have the same function field. Hence we can use this idea in order to construct new rational surfaces. We can take the rational surfaces we know, like $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ and start blowing them up at points of our desire (note that after blowing up at a point we can choose some other point and blow up there, etc.). As a first non-trivial observation we claim that

Proposition 2.1. *The blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at any point is isomorphic to the blow up of $\mathbb{P}^2$ at any two different points.*

Proof. First of all we can use automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ to move any point to any point and automorphisms of $\mathbb{P}^2$ to move any two different points to any two different points. Hence the relevant blow ups don't depend on specific choices of points (complete the details in the exercise).

Now consider the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ with projective coordinates $(x_1 : x_2), (y_1 : y_2), (z_1 : z_2 : z_3)$ respectively. Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ be defined by the equation

$$(1) \quad z_3 x_2 = z_2 x_1$$

$$(2) \quad z_3 y_2 = z_1 y_1$$

We claim that X is isomorphic to both the blow up of $\mathbb{P}^2$ at the points $\{(1:0:0), (0:1:0)\}$ and to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at $((0:1),(0:1))$. First note that if $(z_1 : z_2 : z_3) \neq (1:0:0), (0:1:0)$ then the first equation determines $(x_1 : x_2)$ from $(z_1 : z_2 : z_3)$ and the second determines $(y_1 : y_2)$ from $(z_1 : z_2 : z_3)$. Hence the projection from X to $\mathbb{P}^2$ identifies this open set with $\mathbb{P}^2 \setminus \{(1:0:0),(0:1:0)\}$.

Now in the neighborhood $U \subseteq \mathbb{P}^2$ given by $z_1 \neq 0$ the second equation still determines $(y_1 : y_2)$ from $(z_1 : z_2 : z_3)$, but the first equation fails to determine $(x_1 : x_2)$ at $(1:0:0)$. In fact we see that we get exactly the equations for blow up at $(1:0:0)$ using the functions $f_1 = \frac{z_3}{z_1}, f_2 = \frac{z_2}{z_1}$ defined on U. Similarly at the neighborhood $V \subseteq \mathbb{P}^2$ given by $z_2 \neq 0$ the second equation becomes the equation for blowing up at $(0:1:0)$. Hence we see that X is isomorphic to the blow up of $\mathbb{P}^2$ at $(1:0:0)$ and $(0:1:0)$.

We now want to show that X is isomorphic to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $((0:1),(0:1))$. First note that on the subset of X where x_1, y_1 aren't both 0 then equations (1) and (2) determine the $(z_1 : z_2 : z_3)$ coordinate from $(x_1 : x_2), (y_1 : y_2)$. Hence this open subset is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{((0:1),(0:1))\}$

Now let $U \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be given by $x_2 \neq 0, y_2 \neq 0$ and use the functions $x = \frac{x_1}{x_2}, y = \frac{y_1}{y_2}$ as coordinates on $U \cong \mathbb{A}^2$. The blow up of U at $(0,0)$ is the subspace $\widetilde{U} \subseteq U \times \mathbb{P}^1$ given by

$$x w_2 = y w_1$$

where $(w_1 : w_2)$ are projective coordinates on $\mathbb{P}^1$. We now embed $\widetilde{U}$ as an open subset in X by the map

$$(x, y, (w_1 : w_2)) \mapsto ((x:1),(y:1),(w_1 : w_2 : x w_2))$$

Note that equation (1) is satisfied automatically and equation 2 is satisfied because $x w_2 = y w_1$. The image of this map is the open subset $V \subseteq X$ given by $x_2 \neq 0, y_2 \neq 0$. On V we have an inverse to this map given by

$$((x_1 : x_2),(y_1 : y_2),(z_1 : z_2 : z_3)) \mapsto \left(\frac{x_1}{x_2}, \frac{y_1}{y_2}, (z_1 : z_2) \right)$$

Note that equations (1), (2) imply that z_1 and z_2 can't both be 0 on V (because then z_3 would be 0 as well). Hence we see that $\widetilde{U} \cong V$. Hence we see that X is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $((0:1),(0:1))$. $\square$

3 Rationality Over $\mathbb{Q}$ and Rational Points

Up to now we have dealt with surfaces which are rational over $\overline{\mathbb{Q}}$, i.e. whose rational function field over $\overline{\mathbb{Q}}$ is isomorphic to $\overline{\mathbb{Q}}(x, y)$. If X is a surface defined over $\mathbb{Q}$ then we can also inquire as to the rational functions on X defined over $\mathbb{Q}$ (this can be defined intrinsically as rational functions which are Galois invariant or on affine local neighborhoods as functions which have their coefficients in $\mathbb{Q}$). If the field of rational functions over $\mathbb{Q}$ is isomorphic to $\mathbb{Q}(x, y)$ then we say that X is rational over $\mathbb{Q}$.

If a variety is rational over $\mathbb{Q}$ then we can say a lot about the rational points in it. As in the previous section we get that there exists an open $U \subseteq X$ which is isomorphic to some open set $V \subseteq \mathbb{A}^n$, only this time U is defined over $\mathbb{Q}$ (one simply checks that it is Galois invariant) and the isomorphism is defined over $\mathbb{Q}$ as well. If we can find an explicit isomorphism $V \cong U$ then we can use in as a parametrization of the rational points in U. It is then left to check the closed set $X \setminus U$ which is of a lower dimension.

4 Example : The Cubic Surface

Question: what are all the non-trivial rational solutions to the equation

$$(*)\ y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0$$

First we need to understand the geometry of the surface $X \subseteq \mathbb{P}^3$ defined by this equation over $\overline{\mathbb{Q}}$. We claim that this surface is isomorphic over $\overline{\mathbb{Q}}$ to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 5 points (and hence by the theorem above also to $\mathbb{P}^2$ blown up at 6 points). This is in fact true for every smooth surface which is defined inside $\mathbb{P}^3$ by a single cubic equation (such surfaces are called **cubic surfaces**), but we will show it here only for this surface. In particular we can use this to get a parametrization of the rational points solutions to $(*)$.

Let Y be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at five points. In order to construct a map from Y to X we first find two disjoint lines in X. Recall that a line in $\mathbb{P}^3$ is a subset of the form $\{tv + su | t, s \in \mathbb{Q}\}$ for some fixed $v, u \in \mathbb{P}^3$. The projective geometry is very elegant, and in particular one has a unique line through every two points and every two lines meet at a unique point.

Let ω be a third root of unity. The we have the following 27 lines on X:

$$l^1_{i,j} : \{(y_0, y_1, y_2, y_3) | y_0 + \omega^i y_1 = y_2 + \omega^j y_3 = 0\}$$

$$l^2_{i,j} : \{(y_0, y_1, y_2, y_3) | y_0 + \omega^i y_2 = y_1 + \omega^j y_3 = 0\}$$

$$l^3_{i,j} : \{(y_0, y_1, y_2, y_3) | y_0 + \omega^i y_3 = y_1 + \omega^j y_2 = 0\}$$

It can be shown that these are all the lines which lie on X. In fact, every smooth cubic surface has exactly 27 lies on it. Note that only the three lines $l^1_{0,0}, l^2_{0,0}$ and $l^3_{0,0}$ are defined over $\mathbb{Q}$. The rest are defined over the extension field

$Q(\omega)$. Hence we will start by working over the $\mathbb{Q}(\omega)$ (but we will see in the end that we can actually make everything be defined over $\mathbb{Q}$).

We start by finding two disjoint lines. The pair $l^1_{1,1}, l^1_{2,2}$ (which we will call l_1, l_2 from now for short) has this property. It also has the property that they are preserved (as an unordered pair) by the Galois group $\Gamma = \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$. We will see that this will come handy later when we'll want to make things work over $\mathbb{Q}$.

Let $Y = l_1 \times l_2$. Now the idea is as follows: for every $(v, u) \in Y$ there exists a unique line which passes through both v and u. By bezout's theorem, if this line is not contained in X then it intersects X in exactly three points: v, u and some third point w. We then get a map $\varphi(v, u) = w$ which defined on some open subset $U \subseteq Y$ where we remove the (finite number of) pairs (v, u) for which the line $\{tv + su\}$ is contained in X.

It turns out that there are exactly 5 such pairs. It is worth while to go over the 27 lines above and check that only five of them meet both l_1 and $l2$. We will instead write an explicit equation and see exactly who these pairs are.

We will use the projective coordinates $(x_0 : x_1), (z_0 : z_1)$ on Y in the following way:

$$v = (x_0 : -\omega x_0 : x_1 : -\omega x_1)$$

$$u = (z_0 : -\omega^2 z_0 : z_1 : -\omega^2 z_1)$$

We want to find the point on the line between u and v that is in X. Now a point on the line between them can be written as

$$P_{t,s} = tv + su = (tx_0 + sz_0 : -\omega tx_0 - \omega^2 sz_0 : tx_1 + sz_1 : -\omega tx_1 - \omega^2 sz_1)$$

If we substitute this expression in the cubic we get

$$(tx_0 + sz_0)^3 + \left(-\omega tx_0 - \omega^2 sz_0\right)^3 + (tx_1 + sz_1)^3 + \left(-\omega tx_1 - \omega^2 sz_1\right)^3 =$$

$$3ts\left[t(1-\omega)(x_0^2 z_0 + x_1^2 z_1) + s(1-\omega^2)(x_0 z_0^2 + x_1 z_1^2)\right]$$

Hence we see that if $x_0^2 z_0 + x_1^2 z_1$ and $x_0 z_0^2 + x_1 z_1^2$ are not both zero then the unique solution (up to rescaling) is

$$(**) \;\; (t, s) = \omega(x_0 z_0^2 + x_1 z_1^2), \omega^2(x_0^2 z_0 + x_1^2 z_1)$$

If $x_0^2 z_0 + x_1^2 z_1$ and $x_0 z_0^2 + x_1 z_1^2$ are both zero then the line going through v and u is contained completely in X. When can this happen? clearly if $(x_0 : x_1) = (1 : 0)$ then $(z_0 : z_1) = (0 : 1)$ and vice-versa, so we found two solutions. The rest of the solutions satisfy $x_1 \neq 0$ and $z_1 \neq 0$ and then we can normalize by them and get

$$\frac{x_0^2 z_0}{x_1^2 z_1} = -1$$

$$\frac{x_0 z_0^2}{x_1 z_1^2} = -1$$

which means that $x_0 = z_0$ and are both a cube root of -1, i.e. $x_0, z_0 \in \{-1, -\omega, -\omega^2\}$. Hence we have found that there are exactly 5 solutions:

$$S = \{((0:1),(1,0)),((1:0),(0,1)),((-1:1),(-1,1)),((-\omega:1),(-\omega,0)),((-\omega^2:1),(-\omega^2,1))\}$$

and hence exactly 5 lines in X that meet both l_1 and l_2.

Returning to the case where $((x_0 : x_1),(z_0 : z_1))$ is not is S, we can substitute in the expression $(**)$ and get our desired map

$$\varphi((x_0 : x_1),(z_0 : z_1)) = tv + su =$$
$$(-x_0^2 z_0^2 + \omega x_0 x_1 z_1^2 + \omega^2 x_1^2 z_0 z_1 : x_0^2 z_0^2 - \omega^2 x_0 x_1 z_1^2 - \omega x_1^2 z_0 z_1 :$$
$$-x_1^2 z_1^2 + \omega x_0 x_1 z_0^2 + \omega^2 x_0^2 z_0 z_1 : x_1^2 z_1^2 - \omega x_0^2 x_1 z_0 - \omega^2 x_0^2 z_0 z_1)$$

This gives us a map $\varphi : U = Y \setminus S \longrightarrow X$. You will show in the exercise that this map is actually an isomorphism onto its image, which is the dense open set $V \subseteq X$ obtained by removing the corresponding 5 lines. Hence X and Y have an isomorphic dense open subsets so they are birational.

Since Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\overline{\mathbb{Q}}$ we get that X is a rational surface. In fact, if $\widetilde{Y}$ is the blow up of Y along S then we can extend the map φ to $\widetilde{Y}$ which actually results in an isomorphism $\widetilde{Y} \cong X$. You will prove this nice geometric fact in the exercise.

Now what about rational points? Note that up until know we have been working over $\overline{\mathbb{Q}}$, or more precisely over $\mathbb{Q}(\omega)$. Now X is defined over $\mathbb{Q}$, and this structure can be encoded by the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ on $X(\overline{\mathbb{Q}})$. $V \subseteq X$ is defined by removing a set of 5 lines which is actually invariant (as a set) under the Galois action (even though not all the lines are defined over $\mathbb{Q}$). Hence we have a well defined Galois action on $V(\overline{\mathbb{Q}})$.

Now we have an isomorphism φ over $\overline{\mathbb{Q}}$ between U and V so it induces a bijection $U(\overline{\mathbb{Q}}) \cong V(\overline{\mathbb{Q}})$. Hence there exists a unique Galois action on $U(\overline{\mathbb{Q}})$ such that φ becomes equivariant. This action defines on U a structure of a variety over $\mathbb{Q}$ and φ will respect this structure, i.e. it will become an isomorphism over $\mathbb{Q}$.

In order to see what this action looks like, note that:

1. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ satisfies $\sigma(\omega) = \omega$ then

$$\sigma(\varphi((x_0 : x_1),(z_0 : z_1))) = \varphi((\sigma(x_0) : \sigma(x_1)),(\sigma(z_0) : \sigma(z_1)))$$

so we define $\sigma((x_0 : x_1),(z_0 : z_1)) = ((\sigma(x_0) : \sigma(x_1)),(\sigma(z_0) : \sigma(z_1)))$.

2. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ satisfies $\sigma(\omega) = \omega^2$ then

$$\sigma(\varphi((x_0 : x_1),(z_0 : z_1))) = \varphi((\sigma(z_0) : \sigma(z_1)),(\sigma(x_0) : \sigma(x_1)))$$

so we define $\sigma((x_0 : x_1),(z_0 : z_1)) = ((\sigma(z_0) : \sigma(z_1)),(\sigma(x_0) : \sigma(x_1)))$.

Hence we see that $((x_0 : x_1), (z_0 : z_1))$ is Galois invariant if and only if $(x_0 : x_1), (z_0 : z_1) \in \mathbb{P}^1(\mathbb{Q}(\omega))$ and

$$(z_0 : z_1) = (\sigma(x_0) : \sigma(x_1))$$

There is exactly one such point which does not lie on the open set $x_1 \neq 0, z_1 \neq 0$, and that the point

$$p = ((1 : 0), (1 : 0))$$

for which we get

$$\varphi(p) = (-1, 1, 0, 0)$$

The rest of the rational points can be written as

$$((t + (\omega - \omega^2)s : 1), (t + (\omega^2 - \omega)s : 1))$$

for $t, s \in \mathbb{Q}$. Substituting in we get

$$\varphi(t, s) = \varphi((t + (\omega - \omega^2)s : 1), (t + (\omega^2 - \omega)s : 1)) =$$

$$(-(t^2 + 3s^2)^2 - (t + 3s) :$$

$$(t^2 + 3s^2)^2 + (t - 3s) :$$

$$-(t^2 + 3s^2)(t - 3s) - 1 :$$

$$(t^2 + 3s^2)(t + 3s) + 1)$$

The only point where this is ill defined is $(t, s) = (-1, 0)$ (the other four ill definition points of φ are not rational). Hence we get a parametrization of the rational points on the cubic by a pair of rational points. This parametrization doesn't cover the points in X which are not on the image of φ. But these points lie on the 5 lines of which only one, $l^2_{0,0}$ is Galois invariant and contains a rational point. Hence we know that the rational points on the cubic are:

1. Points on $l^2_{0,0}$ which are of the form $(a, a, -a, -a)$ for some $a \in \mathbb{Q}$.
2. The point $(-1 : 1 : 0 : 0)$.
3. Points of the form $\varphi(t, s)$ for $t, s \in \mathbb{Q}$ such that $(t, s) \neq (-1, 0)$.

Examples:

$$\varphi(0, -1) = (-6 : 12 : -10 : -8) = (-3 : 6 : -5 : -4)$$

$$\varphi(-3/2, 1/2) = (-9 : 6 : 8 : 1)$$

$$\varphi(-1/2, -3/2) = (-44 : 53 : -29 : -34)$$