Quasi-unital ∞-Categories

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By

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Abstract

The notion of a category has a non-unital analogue, in which one does not require the existence of identity morphisms. Such objects are called **non-unital** categories. Using a natural notion of a functor between non-unital categories one can construct the category $\mathrm{Cat}^{\mathrm{nu}}$ of (small) non-unital categories.

We have a natural functor

$$\mathcal{F} : \mathrm{Cat} \longrightarrow \mathrm{Cat}^{\mathrm{nu}}$$

given by forgetting the identity morphisms. However, this functor does not forget much: given a non-unital category $\mathcal{C}$, one can easily ascertain whether it is in the image of $\mathcal{F}$ by checking if each object admits a neutral endomorphism. If indeed this is the case then $\mathcal{C}$ can be promoted to a (unital) category in a unique way.

Similarly, it is easy to check when a functor $f : \mathcal{F}(\mathcal{C}) \longrightarrow \mathcal{F}(\mathcal{D})$ of non-unital categories comes from a true functor $\mathcal{C} \longrightarrow \mathcal{D}$ of categories. One just needs to verify that f maps neutral morphisms to neutral morphisms. Furthermore, in this case f comes from a unique functor $\mathcal{C} \longrightarrow \mathcal{D}$. In other words, the functor $\mathcal{F}$ is **faithful**. In particular, **no information is lost** by regarding an ordinary category as a non-unital category: everything can be recovered by locating the neutral morphisms.

The goal of this dissertation is to obtain a generalization of the above observations to the realm of **∞-categories**. The theory of ∞-categories, though of fundamental importance in modern homotopy theory, has not yet found a canonical formalism. Instead, there are several **models** for the theory of ∞-categories, all known to be equivalent. These models try to capture the following intuitive notion which arises in classical algebraic topology when one first encounters the category of topological spaces : an ∞-category is a category-like structure $\mathcal{C}$ in which, in addition to objects and morphisms, there is also a good notion of **homotopies** between morphisms, **homotopies between homotopies**, etc. This structure is what enables one to do homotopy theory in $\mathcal{C}$. In particular, like in the category of spaces, in every ∞-cateogry there is a notion of a **homotopy equivalence** (or simply an **equivalence**) between objects, and this notion is central to the study of $\mathcal{C}$.

One possible way to get an ∞-category $\mathcal{C}$ is to consider it as a topological category, i.e. a category enriched in **Kan simplicial sets** (equivalently, one can take categories enriched in topological spaces). The corresponding notions of homotopy between morphisms and homotopy equivalences between objects are then easily derived. This model for ∞-categories is concrete and easy to digest. However, it carries a crippling drawback which manifests itself when one tries to describe the **∞-category of (small) ∞-categories**.

The problem arises when one attempts to describe the category of small topological categories itself as a topological category. If $\mathcal{C}, \mathcal{D}$ are two small topological categories, then clearly there is a natural way to promote $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ into a Kan simplicial set. However, the resulting topological category, which we shall denote by $\mathrm{Cat}_{\mathrm{Top}}$, will not capture the

desired ∞-category of ∞-categories: it would yield a far to strong notion of **equivalence** between topological categories which will not coincide with the desired notion of equivalence, namely that of a **Dwyer-Kan equivalence** (see [DK1]), or DK-equivalences for short.

To remedy this situation one can rely on the theory of **localizations**. This theory enables one to take a a topological category $\mathcal{C}$, equipped with a class of morphisms $\mathcal{W}$ (called **weak equivalences**), and produce a new topological category $\mathcal{C} \longrightarrow \mathcal{W}^{-1}\mathcal{C}$ in which all the arrows of $\mathcal{W}$ become equivalences, and that is universal with respect to this property (see [DK2]).

The desired ∞-category of ∞-categories can then be described formally as the localization $\mathcal{W}^{-1}\operatorname{Cat}_{\mathrm{Top}}$ of the category of topological categories with respect to the class $\mathcal{W}$ of DK-equivalences. The problem with this definition is that the above localization procedure does not always admit an effective description. In particular, given two topological categories, it can be very hard to obtain direct information about the space of morphisms between them in $\mathcal{W}^{-1}\operatorname{Cat}_{\mathrm{Top}}$ (in localization theory this space is sometimes called the **derived mapping space**). This remains true even when one uses very powerful tools to compute the localization $\mathcal{W}^{-1}\operatorname{Cat}_{\mathrm{Top}}$, e.g., the technology of model categories.

In order overcome this problem various alternative (yet equivalent) models were suggested for ∞-categories (see [Ber] for a few such models including proofs of equivalence). The model which will be most relevant to the work in this dissertation is that of **complete Segal spaces**, which are **simplicial spaces** satisfying certain properties. This model was first introduced by Rezk in his beautiful paper [Rez]. The advantage of this model over topological categories is that given two complete Segal spaces the derived mapping space between them is extremely accessible - it is exactly the mapping space between them as simplicial spaces.

The price one pays for the transition from topological categories to complete Segal spaces is that one has to replace the strict categorical structure with an analogous **weak structure**. In paritcular, the associative composition operation is replaced with a "composition up to homtopy", satisfying certain "associativity up to coherent homotopy". Similarly, one replaces the identity morphisms with an appropriate weak analogue, which is encoded in the degeneracy maps of the complete Segal space. This "weakening" procedure is one the deepest underlying themes of modern homotopy theory. It further allows one to include in the realm of ∞-categories candidates whose structure is naturally non-strict, such as the cobordism categories (see §§ 3.3). We refer the reader to [Rez] and [Lur3] for further discussion.

As the weak analogue of units is encoded in all the degeneracy maps together it carries **much more structure** then units in ordinary categories. Our task in this work is to understand what happens when one **forgets this structure**, i.e., forgets the degeneracy maps. Our strategy will be to identify the essentail image of this forgetful functor F_∞ and show that it consists of non-unital ∞-categories which contain certain "units-up-to-homotopy" in an appropriate sense. Such objects will be called **quasi-unital** ∞-categories.

Using model categorical tools we will construct a formal model for the homotopy theory of quasi-unital ∞-categories and show that it is equivalent to the homotopy theory of (unital) ∞-categories, via the forgetful functor. This means that the unital structure of an ∞-category is **canonical** once its existence is enabled by the underlying non-unital structure.

Our main motivation for developing this theory is an application to the proof of the cobordism hypothesis conjectured by Baez and Dolan in 95. In 2009 Lurie published an expository article outlining a proof of this conjecture. Many details in the proof have yet to be written explicitly. The theory of quasi-unital ∞-categories can be used in order to write down a formal proof for the dimension $n = 1$ case of the cobordism hypothesis. This application is described in [Har].

Contents

1 Introduction

The notion of **units**, or **identity morphisms**, is fundamental in category theory. When one proceeds to higher category theory this notion gains considerably more structure - one does not only have for each object X an identity morphism $I_X : X \longrightarrow X$, but also homotopies of the form $I_X \circ f \sim f$, as well as suitable coherence homotopies. Similarly, a functor between ∞-categories carries a bundle of extra information specifying the way it interacts with the unital structure.

In some situations there are no canonical candidates for the I_X's or for these additional homotopies. A typical case where this can happen is when we are trying to describe an ∞-category in which the mapping spaces appear naturally as classifying spaces of ∞-groupoids. A motivating example for us are the **cobordism ∞-categories**.

Suppose that we want to describe the ∞-category whose objects are closed n-manifolds and morphisms are cobordisms between them. Since cobordisms have their own automorphisms (boundary respecting diffeomorphisms) we can't simply take them as a set, but rather as the space classifying the corresponding ∞-groupoid. Gluing of cobordisms induces a weak composition operation on these classifying spaces.

Now given an n-manifold M there will certainly be an equivalence class of cobordisms $M \longrightarrow M$ which are candidates for being the "identity" - all cobordisms which are diffeomorphic to $M \times I$. However it is a bit unnatural to choose any **specific one** of them. Note that even if we choose a specific identity cobordism $M \times I$ we will still have to arbitrarily choose diffeomorphisms of the form $[M \times I] \coprod_M W \cong W$ for each cobordism W out of M as well as various other coherence homotopies.

These choice problems can be overcome in various ways, some more ad-hoc than others, and in the end a unital structure can be obtained (see [Lur3] §2.2 for more details). However, there is great convenience in **not having to specify this structure**. For example, it can be very useful to know that the unital structure (for both ∞-categories and functors) can be uniquely recovered once certain conditions are met. Informally, one can phrase the following natural questions:

1. Given a non-unital associative composition rule - when does a unital structure exist?

2. If a unital structure exists, is it essentially determined by the non-unital structure?

In order to answer such questions one should start by formalizing what exactly a **non-unital** ∞-category is. To allow for a more simple discussion in the introduction let us consider the more rigid case of **topological categories** (i.e., categories enriched in simplicial sets such that each mapping space is Kan). In this case it is clear what the non-unital analogue will be. A **non-unital topological category** $\mathcal{C}$ consists of

1. A class of objects $\mathrm{Ob}(\mathcal{C})$.
2. For every two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a Kan simplicial set of morphisms $\mathrm{Map}_{\mathcal{C}}(X, Y)$ (referred to as the **mapping space** from X to Y).
3. For every three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a strictly associative composition rule
$$\mathrm{Map}_{\mathcal{C}}(X, Y) \times \mathrm{Map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Map}_{\mathcal{C}}(X, Y)$$

Example 1.0.1. Every topological category has a natural underlying non-unital topological category obtained by forgetting the units.

Example 1.0.2. Given a set A one can endow it with a structure of a non-unital topological category by setting all mapping spaces to be **empty**. This construction is left adjoint to the forgetful functor $\mathcal{C} \mapsto \mathrm{Ob}(\mathcal{C})$.

Now let $\mathcal{C}$ be a non-unital topological category. Note that even though there are no identity morphisms, there is still a natural notion of **invertible** morphisms: we can say that $f : X \longrightarrow Y$ is **invertible** if for every Z the maps
$$f_* : \mathrm{Map}_{\mathcal{C}}(Z, X) \longrightarrow \mathrm{Map}_{\mathcal{C}}(Z, Y)$$
and
$$f^* : \mathrm{Map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Map}_{\mathcal{C}}(X, Z)$$
induced by composition with f are **weak equivalences**.

In a similar way one can define when a morphism behaves like an identity morphism. We will say that a morphism $q : X \longrightarrow X$ in $\mathcal{C}$ is a **quasi-unit** if for each $Z \in \mathcal{C}$ the maps
$$q_* : \mathrm{Map}_{\mathcal{C}}(Z, X) \longrightarrow \mathrm{Map}_{\mathcal{C}}(Z, X)$$
and
$$q^* : \mathrm{Map}_{\mathcal{C}}(X, Z) \longrightarrow \mathrm{Map}_{\mathcal{C}}(X, Z)$$
induced by composition with q are **homotopic** to the identity. In particular, quasi-units are always invertible morphisms. If $\mathcal{C}$ admits quasi-units for every object then we will say that $\mathcal{C}$ is a **quasi-unital** topological category. We will say that a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is **unital** if it maps quasi-units to quasi-units.

Remark 1.0.3. It is not hard to show that the existence of a quasi-unit $q : X \longrightarrow X$ is equivalent to the seemingly weaker condition of having an invertible morphism with source X. Furthermore a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is unital if and only if it preserves invertible morphisms. These statements are proven in the beginning of section § 4.

Example 1.0.4. Let X be an infinite simplicial set and let $\mathcal{C}_0$ be the non-unital topological category whose objects are Kan fibrations $p : Y \longrightarrow X$ over X equipped with a section $s : X \longrightarrow Y$, and whose morphisms are compactly supported maps over X (i.e., maps which factor through the prescribed section outside a finite sub simplicial set of X). Then $\mathcal{C}_0$ will not be quasi-unital in general.

We can say that two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ are **equivalent** if there exists an invertible morphism $f : X \longrightarrow Y$. Note the mildly surprising fact that, in general, this definition does not automatically give an equivalence relation. For example, there might be an object which admits no invertible morphism to itself. However, if we assume that $\mathcal{C}$ is quasi-unital then it is fairly easy to show that this relation does become an equivalence relation. Hence when $\mathcal{C}$ is quasi-unital it is rather natural to talk about **equivalence types** of objects.

Once one has a notion of equivalence types one can say when a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ of quasi-unital topological categories is an **equivalence**, namely, when F is surjective on equivalence types and induces a weak equivalence on mapping spaces. Such functors will be called **Dwyer-Kan equivalences**, or DK-equivalences for short.

We will consider DK-equivalences as the natural notion of **weak equivalences** for the category of quasi-unital topological categories and unital functors. Informally speaking, this allows us to consider small quasi-unital topological categories as forming an ∞-category $\mathrm{Cat}^{\mathrm{qu}}_{\mathrm{Top}}$. One can then consider the forgetful functor

$$\mathcal{F} : \mathrm{Cat}_{\mathrm{Top}} \longrightarrow \mathrm{Cat}^{\mathrm{qu}}_{\mathrm{Top}}$$

as a functor of ∞-categories, where $\mathrm{Cat}_{\mathrm{Top}}$ denotes the ∞-category of small topological categories.

The purpose of this dissertation is to show that the forgetful functor $\mathcal{F}$ is an **equivalence of ∞-categories**. As explained in the abstract, the model of topological categories is extremely ineffective for computing derived mapping spaces. As such, it provides little support for proving that a functor such as $\mathcal{F}$ is an equivalence, or even fully-faithful. For this reason, we have chosen to work instead with the model CS of **complete Segal spaces**, first constructed by Rezk in his fundamental paper [Rez]. We shall hence construct a model for the ∞-category of small quasi-unital ∞-categories (and unital functors) in an analogous form of **complete semiSegal spaces**, to be denoted CsS. The forgetful functor $\mathcal{F}$ will then appear as a natural functor

$$\mathcal{F} : \mathrm{CS} \longrightarrow \mathrm{CsS}$$

which we will show to be an **equivalence** of ∞-categories. We can state this main result as follows:

Theorem 1.0.5. *The forgetful functor from ∞-categories to quasi-unital ∞-categories is an equivalence.*

In particular, this gives the following answers to the questions above:

1. Given a non-unital associative composition rule, a unital structure exists if and only if quasi-units exist for every object.

2. When quasi-units exists then the unital structure is essentially determined.

As established in [Rez], the topological category of complete Segal spaces serves as the localization of the topological category of **Segal spaces** by the class of Dwyer-Kan equivalences. Similarly we will show that the topological category of complete semiSegal spaces can be obtained as the localization of a certain category of **quasi-unital semiSegal spaces** by an appropriate analogue of the notion of DK-equivalences. As quasi-unital semiSegal spaces describe quasi-unital ∞-categories in a very natural way, this can be considered as a justification for modeling quasi-unital ∞-categories as complete semiSegal spaces.

It is worthwhile to mention explicitly an interesting particular case. We will say that a non-unital ∞-category $\mathcal{C}$ is a **semi-groupoid** if all the morphisms in $\mathcal{C}$ are invertible. Then we get as a particular case of Theorem 1.0.5 that the homotopy theory of quasi-unital semi-groupoids is equivalent to the homotopy theory of ∞-groupoids, which in turn is equivalent to the (weak) homotopy theory of spaces. This particular case will be proved before the main theorem, and is the main occupation of subsection §§ 4.3.

It is natural to ask whether the theory developed in this dissertation could have been developed relative to some **other model** of ∞-categories. As we explained above the model of topological (or simplicial) categories is not convenient for computing derived mapping spaces. Other popular models for ∞-categories include **Segal categories** and **quasi-categories**. The Segal category model is somewhat close to complete Segal spaces in that the objects in question are also Segal spaces, but the completeness condition is replaced by a discreteness condition on the 0'th space.

It is quite straightforward to realize the notion of quasi-unital ∞-categories in the setting of Segal categories, for example by taking quasi-unital semiSegal spaces with discrete 0'th space. The author expects that the theory developed in this thesis could in fact be reproduced in this language. However, the proof of the main theorem (see §§ 6.2) would have to undergo a complete reformulation as it uses in a crucial way the fact that (marked) complete semiSegal spaces are especially susceptible to the application of the (marked) right Kan extension functor. In some sense this is another incarnation of the high accessibility of derived mapping spaces in the setting of complete Segal (or semiSegal) spaces.

As for the model of quasi-categories, the author feels that it is less suitable for the theory of quasi-unital ∞-categories. As opposed to complete Segal spaces and Segal categories, quasi-categories are simplicial **sets** satisfying certain properties. A natural approach will then be to replace them by semi-simplicial sets satisfying analogous properties. However, the degeneracies play a far more important role in the theory of quasi-categories than in the former two theories. In particular, in order to describe explicitly the mapping space between two vertices in a quasi-category one makes crucial use of degenerate simplices. For example, paths in mapping spaces correspond to degenerate 2-simplices, etc. This is somewhat unfortunate as the theory of quasi-categories has enjoyed a considerable theoretical development, see [Lur2].

1.0.1 Relation to other work

The theory developed here is closely related and much inspired by the theory of quasi-unital algebras developed by Lurie in [Lur1], §6.1.3. There he considers non-unital algebra objects in a general monoidal ∞-category $\mathcal{D}$. Enforcing an existence condition for quasi-units and an appropriate unitality condition for morphisms one obtains the ∞-category of quasi-unital algebra objects in $\mathcal{D}$. It is then proven that

Theorem 1.0.6 ([Lur1])**.** *The forgetful functor from the ∞-category of algebra objects in $\mathcal{D}$ to the ∞-category of quasi-unital algebra objects in $\mathcal{D}$ is an equivalence of ∞-categories.*

Note that if $\mathcal{D}$ is the monoidal ∞-category of spaces (with the Cartesian product) then algebra objects in $\mathcal{D}$ are also known as (non-strict) monoids. We shall therefore refer to non-unital algebra objects in $\mathcal{D}$ as **semi-monoids**. Now note that semi-monoids can be considered as pointed non-unital ∞-categories with one object, and similarly quasi-unital semi-monoids can be considered as pointed quasi-unital ∞-categories with one object. Hence we see that there is a strong link between a result such as 1.0.5 and Theorem 1.0.6. However, it is worth while to point out that even when restricting attention to quasi-unital ∞-categories with one object, Theorem 1.0.5 is not strictly contained in Theorem 1.0.6. This is due to the fact that the mapping space between quasi-unital ∞-categories with one object does not coincide, in general, with the mapping space betweem them as **pointed quasi-unital ∞-categories**, i.e., as quasi-unital semi-monoids.

One can also consider the special case of **semi-groups**, i.e., semi-monoids in which all elements act in a homotopy invertible way (or alternatively, pointed semi-groupoids with one object). In light of Remark 1.0.3 we see that a semi-group is quasi-unital if and only if it is non-empty. Note that unital semi-group objects in spaces are exactly loop spaces. A common corollary of 1.0.6 and 1.0.5 is then the following:

Corollary 1.0.7. *Every non-empty semi-group is homotopy equivalent to the underlying semi-group of a loop space.*

In the context of **strict n-categories** the notion of **quasi-units** has enjoyed a fair amount of interest as well. In [Koc], Kock defines the notion of a **fair n-category**, which in our terms can be called a **strict quasi-unital n-category**. For $n = 2$ and for a variation of the $n = 3$ case Kock and Joyal have shown that a (non-strict) unital structure can be uniquely recovered (see [JK1]). In [JK2] Kock and Joyal further show that every simply connected homotopy 3-type can be modeled by a fair 3-groupoid (see [JK2]). The main difference between their work and the present paper is that we address the (manifestly non-strict) case of quasi-unital ∞-categories (or $(\infty, 1)$-categories, as apposed to (n, n)-categories). Furthermore, our results are framed in terms of a full equivalence between the notions of unital and quasi-unital ∞-categories.

1.0.2 Structure of the essay

The structure of this essay is as follows. In § 2 we introduce the model categorical setting underlying all the constructions of the paper. Specifically, in §§ 2.1 we recall the Reedy model structure on the category of semi-simplicial spaces, and in §§ 2.2 we consider a useful variant obtained by adding a **marking** on the space of 1-simplices.

In § 3 we study the semi-simplicial analogue of Segal spaces, namely that of **semiSegal spaces**. In §§ 3.4 we show that one can study semiSegal spaces in a model categorical setting similar to the way Segal spaces are studied in [Rez].

In § 4 we introduce the notion of **quasi-units** in the semiSegal setting and study the topological category QsS of **quasi-unital** semiSegal spaces and **unital maps** between them. The main interest of this essay is the ∞-category obtained by localizing QsS with respect to DK-equivalences. The localized ∞-category is our model for small **quasi-unital ∞-categories**. In §§ 4.3 we will restrict attention to quasi-unital semiSegal spaces in which all morphisms are invertible and show that in this case the DK-localized ∞-category coincides with the ∞-category of ∞-groupoids. This particular case of the main theorem will provide the first step towards the general proof.

In § 5 we show that QsS can be considered as a **full subcategory** of the category of marked semi-simplicial spaces. This observation is exploited throughout the various subsections of § 5 in order to construct basic tools and methods to work with quasi-unital semiSegal spaces. In particular, in §§ 5.3 we show that QsS has a good notion of **internal mapping objects**. This can be considered as another principal ingredient of the proof of Theorem 1.0.5 as it essentially says that when quasi-units exists they can be chosen coherently over arbitrary families of objects.

In § 6 we introduce the notion of **complete semiSegal spaces** and construct a completion functor from quasi-unital semiSegal spaces to complete semiSegal spaces, analogous to the completion functor constructed in [Rez]. We then show that this completion functor exhibits the category of complete semiSegal spaces as the left localization of QsS with respect to DK-equivalences. Finally, in §§ 6.2 we prove the core part of the main theorem by showing that the ∞-category of complete semiSegal spaces is equivalent to the ∞-category of complete Segal spaces.

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2 Preliminaries

Let Δ denote the simplicial category, i.e., the category whose objects are the finite ordered sets $[n] = \{0, ..., n\}$ and whose morphisms are non-decreasing maps. Let $\mathcal{S} = \mathrm{Set}^{\Delta^{\mathrm{op}}}$ denote the category of **simplicial sets** endowed with the **Kan model structure**, i.e., the weak equivalences are the maps which induce isomorphisms on all homotopy groups and fibrations are Kan fibrations. The category $\mathcal{S}$ admits Cartesian products and internal mapping objects making it into a **monoidal model category**.

We will refer to objects $K \in \mathcal{S}$ as **spaces**. We will say that two maps $f, g : K \longrightarrow L$ in $\mathcal{S}$ are **homotopic** (denoted $f \sim g$) if they induce the same map in the homotopy category associated to the Kan model structure. A **point** in a space K will mean a 0-simplex and a **path** in K will mean a 1-simplex. Note that most spaces which will appear here will be Kan fibrant, so this terminology is somewhat justified.

Many of the categories which we will come across will be enriched in $\mathcal{S}$, so that we have mapping spaces $\mathrm{Map}(X, Y) \in \mathcal{S}$ which carry strictly associative composition rules. We will say that an $\mathcal{S}$-enriched category is **topological** if all the mapping spaces are Kan. A second kind of $\mathcal{S}$-enriched categories which we will come across are $\mathcal{S}$-enriched categories equipped with extra structure. In particular, we recall the following definition:

Definition 2.0.8. Let $\mathcal{C}$ be an $\mathcal{S}$-enriched category.

1. An $\mathcal{S}$**-tensor** structure on $\mathcal{C}$ is a functor $T : \mathcal{S} \times \mathcal{C} \longrightarrow \mathcal{C}$, usually denoted by
$$T(K, X) = K \otimes X$$
together with isomorphisms
$$\mathrm{Map}(K \otimes X, Y) \cong \mathrm{Map}_{\mathcal{S}}(K, \mathrm{Map}(X, Y))$$
which are natural in $K \in \mathcal{S}^{\mathrm{op}}$, $X \in \mathcal{C}^{\mathrm{op}}$ and $Y \in \mathcal{C}$.

2. An $\mathcal{S}$**-power** structure on $\mathcal{C}$ is a functor $P : \mathcal{S}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$, usually denoted by
$$P(K, X) = X^K$$
together with isomorphisms
$$\mathrm{Map}\left(X, Y^K\right) \cong \mathrm{Map}_{\mathcal{S}}(K, \mathrm{Map}(X, Y))$$
which are natural in $K \in \mathcal{S}^{\mathrm{op}}$, $X \in \mathcal{C}^{\mathrm{op}}$ and $Y \in \mathcal{C}$.

3. A **simplicial structure** on $\mathcal{C}$ is just a pair (T, P) of an $\mathcal{S}$-tensor structure and an $\mathcal{S}$-power structure.

As in other places, a simplicial structure on an $\mathcal{S}$-enriched category typically appears in the context of **simplicial model categories** (where it is assumed to be compatible with the model structure in an appropriate sense).

In this section we will set up the model categorical scene in which the rest of the essay takes place. In particular, we will describe two monoidal simplicial model categories, namely the Reedy model category $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ of **semi-simplicial spaces** and an analogous model category $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ of **marked semi-simplicial spaces**. These model categories sit in a sequence of Quillen adjunctions

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \rightleftarrows \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \rightleftarrows \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

Where $\mathcal{S}^{\Delta^{\mathrm{op}}}$ is the Reedy model category of simplicial spaces. This model category was used in [Rez] as the underlying framework in which a model for the homotopy theory of ∞-categories could be set up. In this work we will be using the analogous Reedy model category of semi-simplicial spaces as the underlying framework for studying **non-unital ∞-categories**. The intermediate categories of marked semi-simplicial spaces will be used as a tool in order to bridge the gap between the unital and non-unital case.

2.1 Semi-simplicial Spaces

The purpose of this subsection is to recall the relevant theory and definitions regarding the Reedy model structure on semi-simplicial spaces.

Let $\Delta_s \subseteq \Delta$ denote the subcategory consisting only of **injective** maps. A **semi-simplicial set** is a functor $\Delta_s^{\mathrm{op}} \longrightarrow \mathrm{Set}$. Similarly, a **semi-simplicial spaces** is a functor $\Delta_s^{\mathrm{op}} \longrightarrow \mathcal{S}$.

We will denote by Δ^n the standard n-simplex considered as a **semi-simplicial set** (it is given by the functor $\Delta_s^{\mathrm{op}} \longrightarrow \mathrm{Set}$ represented by $[n]$). If we will want to refer to the standard simplex as a **space** (i.e., an object in $\mathcal{S}$) we will write it as the realization $|\Delta^n| \in \mathcal{S}$ of the corresponding semi-simplicial set. Given a subset $I \subseteq [n]$ we will denote by $\Delta^I \subseteq \Delta^n$ the sub semi-simplicial set corresponding to the sub-simplex spanned by I. We will occasionally abuse notation and consider Δ^n as a semi-simplicial space as well (which is levelwise discrete).

The category of semi-simplicial spaces will be denoted by $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$. This category carries the **Reedy model structure** with respect to the Kan model structure on $\mathcal{S}$ and the obvious Reedy structure on Δ_s. Since Δ_s is a Reedy category in which all non-trivial morphisms are increasing the Reedy model structure coincides with the **injective model structure**. This is a particularly nice situation because we have a concrete description for all three classes of maps, namely:

1. Reedy **weak equivalences** in $\mathcal{S}^{\Delta_s}$ are defined levelwise.
2. Reedy **cofibrations** in $\mathcal{S}^{\Delta_s}$ are defined levelwise.
3. A map $f : X \longrightarrow Y$ is a **Reedy fibration** if the induced maps

$$X_n \longrightarrow M_n(X) \times_{M_n(Y)} Y_n$$

are Kan fibrations, where $M_n(-)$ is the corresponding **matching object** (see [Hir] §15 for more details).

The Reedy model category $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is a **simplicial model category**. The simplicial structure is defined as follows: for a space $K \in \mathcal{S}$ and a semi-simplicial space $X \in \mathcal{S}^{\Delta_s^{\mathrm{op}}}$ we have the semi-simplicial spaces $K \otimes X$ and X^K given by

$$(K \otimes X)_n = K \times X_n$$
$$\left(X^K\right)_k = X_n^K$$

It is then not hard to verify that the Reedy model category is a simplicial model category with respect to this structure. In fact, the simplicial structure extends to a structure of a monoidal model category, as we will see in subsection 2.1.1.

We will sometimes consider a space $K \in \mathcal{S}$ as a semi-simplicial space which is concentrated in degree zero, i.e., as the semi-simplicial space $K \otimes \Delta_0$.

Remark 2.1.1. As mentioned above, there is also a Reedy model structure on the category $\mathcal{S}^{\Delta^{\mathrm{op}}}$ of **simplicial spaces**. According to Theorem 15.8.7 of [Hir] the Reedy model structure on simplicial spaces coincides with the injective model structure, i.e., weak equivalences and cofibrations are defined levelwise and the fibrations are defined using matching objects.

The forgetful functor

$$F : \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

admits a right adjoint

$$\mathrm{RK} : \mathcal{S}^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$$

known as **right Kan extension**. We then see that F preserves cofibrations and trivial cofibrations and so we have a Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathrm{RK}}{\overset{F}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

In subsection § 2.2 we will show that this adjunction factors through an intermediate model category of **marked semi-simplicial spaces**.

Remark 2.1.2. The forgetful functor is also a **right** Quillen functor, i.e., it has a left adjoint, the **left Kan extension** LK, which preserves cofibrations and trivial cofibrations. However, the adjunction in this direction is less relevant for our specific purposes.

2.1.1 Products and Mapping Objects

One of the classical differences between simplicial sets and semi-simplicial sets is that Cartesian products don't behave as well in semi-simplicial sets. For example, topological realization does not commute in general with Cartesian products of semi-simplicial sets. However, there is an alternative symmetric monoidal product $\otimes$ on the category of semi-simplicial sets (and similarly semi-simplicial spaces) for which it **is** true that

$$|X \otimes Y| \cong |X| \times |Y|$$

Let us now describe this monoidal product more explicitly. Since it will be of no additional effort and much additional gain let us go directly to the case of semi-simplicial **spaces**. Let Pos_s be the category of partially ordered sets (posets) and injective order preserving maps between them. One can then consider Δ_s as a full subcategory of Pos_s. Given two posets $A, B \in \mathrm{Pos}_s$ we will denote by $A \times B$ the poset whose underlying set is the Cartesian product of A and B and such that $(a, b) < (a', b')$ if and only if $a < a'$ and $b < b'$. Now given two semi-simplicial spaces X, Y one can consider the functor

$$P_{X,Y} : \Delta_s^{\mathrm{op}} \times \Delta_s^{\mathrm{op}} \longrightarrow \mathcal{S}$$

given by

$$([n], [m]) \mapsto X_n \times Y_m$$

and can take the **left Kan extension** of $P_{X,Y}$ along the functor

$$\Delta_s^{\mathrm{op}} \times \Delta_s^{\mathrm{op}} \longrightarrow \mathrm{Pos}_s^{\mathrm{op}}$$

given by

$$([n], [m]) \mapsto [n] \times [m]$$

This results in a functor

$$\overline{P_{X,Y}} : \mathrm{Pos}_s^{\mathrm{op}} \longrightarrow \mathcal{S}$$

We will denote by $X \otimes Y$ the semi-simplicial space obtained by restricting $\overline{P_{X,Y}}$ to Δ_s.

Remark 2.1.3. The $\otimes$-product of two semi-simplicial spaces can also be described in terms of Cartesian products of simplicial spaces. Let

$$\mathrm{LK} : \mathcal{S}^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$$

denote the left Kan extension functor. If X, Y are two semi-simplicial spaces then it is not hard to show that

$$\mathrm{LK}(X \otimes Y) \cong \mathrm{LK}(X) \times \mathrm{LK}(Y),$$

i.e., that LK becomes a symmetric monoidal functor. In particular $(X \otimes Y)_k$ can be reconstructed as the subspace of **non-degenerate** k-simplices in

$$\mathrm{LK}(X) \times \mathrm{LK}(Y)$$

Remark 2.1.4. One can also obtain a completely explicit description of $X \otimes Y$ as follows. Let $P_k^{n,m}$ denote the set of injective order preserving maps

$$\rho : [k] \longrightarrow [n] \times [m]$$

such that

$$p_{[n]} \circ \rho : [k] \longrightarrow [n]$$

and

$$p_{[m]} \circ \rho : [k] \longrightarrow [m]$$

are **surjective**. Then one can write an explicit formula for the k-simplices of $X \otimes Y$ by

$$(X \otimes Y)_k = \coprod_{n,m \leq k} P_k^{n,m} \times X_n \times Y_m$$

In particular, the set of k-simplices of $\Delta^n \otimes \Delta^m$ can be identified with the set of all injective order preserving maps

$$[k] \longrightarrow [n] \times [m]$$

It is not hard to verify that $\otimes$ is a symmetric monoidal product with unit Δ^0. Furthermore in any of the approaches above one can show that

$$|X \otimes Y| \cong |X| \times |Y|$$

In addition to respecting realizations, the product $\otimes$ also commutes with colimits separately in each variable. This means that $\otimes$ has a corresponding **internal mapping object** which can be described explicitly as follows: if X, Y are two semi-simplicial spaces then the mapping object Y^X is given by

$$(Y^X)_n = \mathrm{Map}(\Delta^n \otimes X, Y)$$

This internal mapping object corresponds to $\otimes$ in the sense that we have a natural isomorphism (the "exponential law")

$$\mathrm{Map}(X \otimes Y, Z) \cong \mathrm{Map}\left(X, Y^Z\right)$$

In other words, the monoidal product $\otimes$ is **closed**.

Remark 2.1.5. The closed monoidal product $\otimes$ extends the simplicial structure of $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ in the sense that we have canonical isomorphisms

$$K \otimes X \cong (K \otimes \Delta_0) \otimes X$$

$$X^K \cong X^{K \otimes \Delta^0}$$

where $K \in \mathcal{S}$ is a space and $X \in \mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is a semi-simplicial space. Note that we have abusively used the same notation for the **simplicial structure** and the **closed monoidal structure**. We hope that this will not result in any confusion.

We claim that the Reedy model structure on $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is **compatible** with the closed monoidal product $\otimes$ in the following sense:

Definition 2.1.6. Let $\mathcal{M}$ be a model structure with a closed monoidal product $\otimes$ such that the unit of $\otimes$ is cofibrant. We say that $\mathcal{M}$ is **compatible** with $\otimes$ if for every pair of cofibrations $f : X' \longrightarrow X, g : Y' \longrightarrow Y$ the induced map

$$h : [X' \otimes Y] \coprod_{X' \otimes Y'} [X \otimes Y'] \longrightarrow X \otimes Y$$

is a cofibration, and is further a trivial cofibration if at least one of f, g is trivial. This condition is commonly referred to as the **pushout-product axiom**. See [Hov] for a slightly more general definition of compatibility (Definition 4.2.6) which does not assume that the unit is cofibrant.

Now since Reedy cofibrations and Reedy weak equivalences are levelwise one can easily verify these conditions using the explicit formula in Remark 2.1.4. In particular, the Reedy model category $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is a monoidal simplicial model category with respect to $\otimes$.

2.2 Marked Semi-simplicial Spaces

The purpose of this subsection is to set up a basic theory of **marked semi-simplicial spaces**. More precisely, we will set up a model category of marked semi-simplicial spaces which will serve as an intermediate step between the Reedy model categories of simplicial and semi-simplicial spaces. In particular, marked semi-simplicial spaces will be used to bridge between unital and non-unital ∞-categories.

We will begin with the basic definitions and then proceed to introduce a model structure on this category which is analogous to the Reedy model structure on semi-simplicial spaces. We will then extend the monoidal structure $\otimes$ to marked semi-simplicial spaces and show that it is compatible with the model structure. We will finish by presenting a factorization of the Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathrm{RK}}{\overset{F}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

through the category of marked semi-simplicial spaces.

Let us open with the main definition:

Definition 2.2.1. A **marked semi-simplicial space** is a pair (X, A) where X is a semi-simplicial space and $A \subseteq X_1$ is a subspace. In order to keep the notation clean we will often denote a marked semi-simplicial space (X, A) simply by X.

Given two marked semi-simplicial spaces $(X, A), (Y, B)$ we denote by

$$\mathrm{Map}^+(X, Y) \subseteq \mathrm{Map}(X, Y)$$

the subspace of maps which send A to B. We will refer to this kind of maps as **marked** maps. We denote by

$$\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

The $\mathcal{S}$-enriched category of marked semi-simplicial spaces and marked maps between them.

Remark 2.2.2. In a similar way one can define **marked semi-simplicial sets** as semi-simplicial sets equipped with a distinguished set of edges. Alternatively one can think of marked semi-simplicial sets as marked semi-simplicial spaces which are levelwise discrete. We will usually not distinguish between these two points of view.

Remark 2.2.3. The category $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ carries a natural **simplicial structure** given by

$$(X, A) \otimes K = (X \otimes K, A \times K)$$

$$(X, A)^K = \left(X^K, A^K\right)$$

Definition 2.2.4. Given a semi-simplicial space X we will denote by $X^\sharp$ the marked semi-simplicial (X, X_1) in which **all** edges are marked. The association $X \mapsto X^\sharp$ is right adjoint to the forgetful functor $(X, A) \mapsto X$.

Definition 2.2.5. Given a semi-simplicial space X we will denote by $X^\flat$ the marked semi-simplicial space $(X, \emptyset)$ in which **no** edges are marked. The association $X \mapsto X^\flat$ is left adjoint to the forgetful functor $(X, A) \mapsto X$.

2.2.1 The Marked Model Structure

The purpose of this subsection is to introduce a model structure on the category $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ which we call the **marked model structure**. We start with the necessary definitions:

Definition 2.2.6. Let (X, A) be a marked semi-simplicial space. We will denote by $\overline{A} \subseteq \pi_0(X_1)$ the **image** of the map

$$\pi_0(A) \longrightarrow \pi_0(X),$$

i.e., the set of connected components of X_1 which meet A. We refer to $\overline{A}$ as the set of **marked connected components** of X_1.

Definition 2.2.7. We will say that a map $f : (X, A) \longrightarrow (Y, B)$ of marked semi-simplicial spaces is a **marked equivalence** if

1. The underling map $f : X \longrightarrow Y$ is a levelwise equivalence.
2. The induced map
$$f_* : \overline{A} \longrightarrow \overline{B}$$
is an isomorphism of sets.

Theorem 2.2.8. *There exists a left proper combinatorial model category structure on $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ such that*

1. *The weak equivalences are the marked equivalences.*
2. *The cofibrations are the maps $f : (X, A) \longrightarrow (Y, B)$ for which the underlying map $X \longrightarrow Y$ is a cofibration (i.e., levelwise injective).*
3. *A map is a fibration if and only if it satisfies the right lifting property with respect to all morphisms which are both cofibrations and weak equivalences.*

Proof. We will use the following general existence theorem which is a slightly weaker version of Proposition A.2.6.13 of [Lur2] (which in turn is based on work of Smith):

Theorem 2.2.9 (Lurie, Smith)**.** *Let $\mathcal{M}$ be a presentable category. Let C, W be two classes of morphisms in $\mathcal{M}$ such that*

1. *C is weakly saturated and is generated (as a weakly saturated class of morphisms) by a* ***set*** *of morphisms C_0.*

2. *W is* ***perfect*** *(see Definition A.2.6.10 of [Lur2]).*

3. *W is stable under pushouts along C, i.e., if*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "g'"] \\ Z \arrow[r] & W \end{tikzcd}$$

is a pushout square such that $f \in C$ and $g \in W$ then $g' \in W$ as well.

4. *If a morphism f in $\mathcal{M}$ has the right lifting property with respect to every morphism in C (or equivalently in C_0) then $f \in W$.*

Then there exists a left proper combinatorial model structure on $\mathcal{M}$ such that the weak equivalences are W and the cofibrations are C.

First we need to verify that $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ is presentable. Note that $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ has all colimits and in particular

$$\operatorname*{colim}_\alpha (X_\alpha, A_\alpha) = \left(\operatorname*{colim}_\alpha X_\alpha, \mathrm{Im}\left(\operatorname*{colim}_\alpha A_\alpha \longrightarrow \operatorname*{colim}_\alpha (X_\alpha)_1\right)\right)$$

We now need to find a subcategory of small objects generating $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$. We will say that a semi-simplicial space X is **finite** if

1. For each k the space X_k contains only finitely many non-degenerate simplices.

2. $X_n = \emptyset$ for large enough n.

It is not hard to verify that each semi-simplicial space is a filtered colimit of finite semi-simplicial spaces (this is part of a general theory - $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is a functor category and finite semi-simplicial spaces are just those which are finite colimits of representables).

We will say that a marked semi-simplicial space (X, A) is **finite** if X is finite. It is then not hard to check that

1. Every marked semi-simplicial space is a filtered colimit of finite marked semi-simplicial spaces.

2. Finite marked semi-simplicial spaces are small in $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$.

3. Finite marked semi-simplicial spaces are closed under finite colimits.

This shows that $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ is presentable.

Now let W be the class of marked equivalences and C the class of marked maps which are levelwise injective. We need to show that the classes (W, C) meet the requirements of Theorem 2.2.9. We start by finding a set of morphisms which generates C as a weakly saturated class.

Let C_0 to be the **set** containing all the morphisms

$$\left[|\partial\Delta^k| \otimes (\Delta^n)^\flat\right] \coprod_{|\partial\Delta^k| \otimes (\partial\Delta^n)^\flat} \left[|\Delta^k| \otimes (\partial\Delta^n)^\flat\right] \hookrightarrow |\Delta^k| \otimes (\Delta^n)^\flat$$

and all the morphisms

$$\left[|\partial\Delta^k| \otimes \left(\Delta^1\right)^\sharp\right] \coprod_{|\partial\Delta^k| \otimes (\Delta^1)^\flat} \left[|\Delta^k| \otimes \left(\Delta^1\right)^\flat\right] \hookrightarrow |\Delta^k| \otimes \left(\Delta^1\right)^\sharp$$

It is not hard to check that C is exactly the weakly saturated class generated from this set (these are standard arguments).

We will now show that (W, C) satisfy the assumptions 2 and 3 of Theorem 2.2.9. Consider the category Set with its trivial model structure (i.e., the weak equivalences are the isomorphisms and all maps are fibrations and cofibrations). We endow $\mathcal{S}^{\Delta_s^{\mathrm{op}}} \times \mathrm{Set}$ with the **product model structure** (i.e., weak equivalences, fibrations and cofibrations are defined coordinate-wise, where on the left we use the Reedy model structure). Let W', C' be the classes of weak equivalences and cofibrations in $\mathcal{S}^{\Delta_s^{\mathrm{op}}} \times \mathrm{Set}$ respectively.

Since both $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ and Set are left proper combinatorial model categories it follows that $\mathcal{S}^{\Delta_s^{\mathrm{op}}} \times \mathrm{Set}$ is a left proper combinatorial model category. This means that W' is stable under pushouts along C' and that W' is perfect (this is part of Smith's theory of combinatorial model categories, cited for example in [Lur2] A.2.6.6).

Now let $F : \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}} \times \mathrm{Set}$ be the functor given by

$$F(X, A) = (X, \overline{A})$$

Then it is clear that F preserves colimits. Since

$$W = F^{-1}(W')$$

and

$$C = F^{-1}(C')$$

we get that W is stable under pushouts along C and that W is perfect (see [Lur2] A.2.6.12). It is then left to check the last assumption of Theorem 2.2.9.

Let $f : (X, A) \longrightarrow (Y, B)$ be a morphism which has the right lifting property with respect to all maps in C_0. Since C_0 contains all maps of the form

$$\left[|\partial\Delta^k| \otimes (\Delta^n)^\flat\right] \coprod_{|\partial\Delta^k| \otimes \partial(\Delta^n)^\flat} \left[|\Delta^k| \otimes (\partial\Delta^n)^\flat\right] \hookrightarrow |\Delta^k| \otimes (\Delta^n)^\flat$$

it follows that f is a levelwise equivalence. It is left to show that f induces an isomorphism

$$\overline{A} \longrightarrow \overline{B}$$

Note that since f is a levelwise equivalence it induces an isomorphism $\pi_0(X_1) \longrightarrow \pi_0(Y_1)$ and so the map $\overline{A} \longrightarrow \overline{B}$ is injective. The fact that it is surjective follows from having the right lifting property with respect to

$$\left(\Delta^1\right)^\flat \hookrightarrow \left(\Delta^1\right)^\sharp$$

which is one of the maps in C_0. This completes the proof of Theorem 2.2.8. □

Definition 2.2.10. We will use the terms **marked fibrations** (**marked-fibrant**) and **marked cofibrations** (**marked-cofibrant**) to denote fibrations (fibrant objects) and cofibrations (cofibrant objects) in the marked model structure.

Remark 2.2.11. The forgetful functor $(X, A) \mapsto X$ from $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ to $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ is both a left and a right Quillen functor. As mentioned above, it has a right adjoint $X \mapsto X^\sharp$ and a left adjoint $X \mapsto X^\flat$. Furthermore it is easy to verify that both the forgetful functor and its left adjoint preserve cofibrations and weak equivalences.

Lemma 2.2.12. *A marked semi-simplicial space* (X, A) *is* ***marked-fibrant*** *if and only if*

1. *X is Reedy fibrant.*
2. *A is a union of connected components of X.*

Proof. Let (X, A) be a marked-fibrant object. From remark 2.2.11 we see that X is Reedy fibrant. Now consider the maps

$$\left[|\Lambda_i^k| \otimes \left(\Delta^1\right)^\sharp\right] \coprod_{|\Lambda_i^k| \otimes (\Delta^1)^\flat} \left[|\Delta^k| \otimes \left(\Delta^1\right)^\flat\right] \hookrightarrow |\Delta^k| \otimes \left(\Delta^1\right)^\sharp$$

for $k \geq 1$ and $0 \leq i \leq k$. By definition we see that these maps are trivial marked cofibrations. Since (X, A) is Reedy fibrant it satisfies the right lifting property with respect to such maps, which in turn means that the inclusion $A \hookrightarrow X_1$ satisfies the right lifting property with respect to the inclusion of spaces $|\Lambda_i^k| \hookrightarrow |\Delta^k|$ for $k \geq 1$. This means that the inclusion $A \hookrightarrow X_1$ is Kan fibration and hence a union of components of X_1.

In the other direction assume that X is Reedy fibrant and $A \subseteq X_1$ is a union of components. Consider an extension problem

$$\begin{tikzcd} (Y, B) \arrow[r, "f"] \arrow[d] & (X, A) \\ (Z, C) & \end{tikzcd}$$

such that $(Y,B) \hookrightarrow (Z,C)$ is a trivial marked cofibration. In this case $Y \hookrightarrow Z$ will be a trivial Reedy cofibration and so there will exist an extension $\overline{f} : Z \longrightarrow X$ in the category of semi-simplicial spaces. We claim that $\overline{f}$ will necessarily send C to A. In fact, let $W \subseteq Z_1$ be a connected component which meets C. Since $(Y,B) \hookrightarrow (Z,C)$ is a marked equivalences it follows that W also meets the image of B. Since A is a union of components of X_1 we get $\overline{f}$ sends all of W to A. This means that $\overline{f}$ sends C to A and we are done.

□

Remark 2.2.13. The same argument as in the proof of Lemma 2.2.12 shows that if $f : (X,M) \longrightarrow (Y,N)$ is a map such that the underlying map $X \longrightarrow Y$ is a Reedy fibration and $M \subseteq X_1$ is a union of components then f is a marked fibration (the converse however is not true in general).

Corollary 2.2.14. *A map $f : X \longrightarrow Y$ between marked-fibrant objects is a marked equivalence if and only if it is a levelwise equivalence which induces a weak equivalence on the corresponding spaces of marked edges.*

2.2.2 Marked Products and Mapping Objects

Let $(X,A),(Y,B)$ be two marked semi-simplicial spaces. Recall the monoidal product $\otimes$ on semi-simplicial spaces defined in §§ 2.1.1. According to Remark 2.1.4 one has

$$(X \otimes Y)_1 = (X_1 \times Y_0) \coprod (X_0 \times Y_1) \coprod (X_1 \times Y_1)$$

We will extend the monoidal product $\otimes$ to marked semi-simplicial spaces by defining $(X,A) \otimes (Y,B)$ to be the marked semi-simplicial space $(X \otimes Y, C)$ where the marking C is given by

$$C = (A \times Y_0) \coprod (X_0 \times B) \coprod (A \times B)$$

The product $\otimes$ on $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ again commutes with colimits separately in each variable and hence has a corresponding **internal mapping object** which is defined as follows:

Definition 2.2.15. Let X,Y be two marked semi-simplicial spaces. The marked **mapping object** from X to Y is the marked semi-simplicial space $\left(Y^X, H\right)$ given by

$$\left(Y^X\right)_n = \mathrm{Map}^+\left(X \times (\Delta^n)^\flat, Y\right)$$

where the marking H is given

$$H = \mathrm{Map}^+\left(X \times \left(\Delta^1\right)^\sharp, Y\right) \subseteq \mathrm{Map}^+\left(X \times \left(\Delta^1\right)^\flat, Y\right) = \left(Y^X\right)_1$$

This internal mapping object corresponds to $\otimes$ in the sense that we have isomorphisms

$$\mathrm{Map}(X \otimes Y, Z) \cong \mathrm{Map}\left(X, Y^Z\right)$$

which are natural in $X, Y \in \left(\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}\right)^{\mathrm{op}}$ and $Z \in \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$.

Lemma 2.2.16. *The marked model structure on $\mathcal{S}_+^{\Delta_s^{\text{op}}}$ is **compatible** with the closed monoidal structure $\otimes$ (see Definition 2.1.6).*

Proof. Since the Reedy model structure on $\mathcal{S}^{\Delta_s^{\text{op}}}$ is compatible with the unmarked version of $\otimes$ we only need to verify the following: if $X' \longrightarrow X$ is a marked cofibration and $Y' \longrightarrow Y$ is a trivial marked cofibration then the map

$$h : [X \otimes Y']_1 \coprod_{(X' \otimes Y')_1} [X' \otimes Y]_1 \longrightarrow (X \otimes Y)_1$$

induces an isomorphism on the set of **marked connected components**. But this is a direct consequence of the fact that the map $Y_1' \longrightarrow Y_1$ is a weak equivalence which induces an isomorphism on the set of marked connected components. $\square$

Corollary 2.2.17. *The marked model structure is compatible with the simplicial structure on $\mathcal{S}_+^{\Delta_s^{\text{op}}}$.*

We finish this subsection with the following definition which we frame for future use:

Definition 2.2.18. Let W be a marked semi-simplicial space with marking $M \subseteq W_1$. We will denote by $\widetilde{W} \subseteq W$ the marked semi-simplicial space such that

$$\widetilde{W}_n = \{\sigma \in W_n | f^*\sigma \in M, \forall f : [1] \longrightarrow [n]\}.$$

In particular, all the edges of $\widetilde{W}$ are marked.

This definition will be mostly applied to mapping objects Y^X. In this case the marked semi-simplicial space $\widetilde{Y^X}$ has the following explicit formula:

$$\left(\widetilde{Y^X}\right)_n = \text{Map}^+ \left(X \otimes (\Delta^n)^\sharp, Y\right).$$

2.2.3 The Marked Right Kan Extension

Recall the Quillen adjunction

$$\mathcal{S}^{\Delta^{\text{op}}} \underset{\text{RK}}{\overset{F}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\text{op}}}$$

between the corresponding Reedy model categories, where F is the forgetful functor and RK is the right Kan extension. The purpose of this section is to construct an analogous Quillen adjunction between $\mathcal{S}^{\Delta^{\text{op}}}$ and the marked model structure on $\mathcal{S}_+^{\Delta^{\text{op}}}$. In order to establish notation it will be useful to recall the definition of the right Kan extension.

Let $\mathcal{C}, \mathcal{D}$ be small categories and $\mathcal{E}$ a category which has all small limits. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{E}$ be two functors. The **right Kan extension** of $\mathcal{G}$ along $\mathcal{F}$ is the functor $\text{RK}(\mathcal{G}) : \mathcal{D} \longrightarrow \mathcal{E}$ given as follows. For each object $d \in \mathcal{D}$

consider the **under category** $\mathcal{D}_{d/}$ whose objects are maps of the form $d \longrightarrow d'$ in $\mathcal{D}$ and whose morphisms are commutative diagrams of the form

$$\begin{tikzcd} & d \arrow[dl] \arrow[dr] & \\ d' \arrow[rr] & & d'' \end{tikzcd}$$

We have a natural forgetful functor $\mathcal{U} : \mathcal{D}_{d/} \longrightarrow \mathcal{D}$ which maps the object $d \longrightarrow d'$ to d'. Using $\mathcal{F}$ and $\mathcal{U}$ we can then form the fiber product category

$$\mathcal{C}_{d/} \stackrel{\text{def}}{=} \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$$

whose objects can be identified with pairs (c, α) where c is an object of $\mathcal{C}$ and $\alpha : d \longrightarrow \mathcal{F}(c)$ is a morphism in $\mathcal{D}$. A morphism in $\mathcal{C}_{d/}$ from (c, α) to (c', α') is then given by a morphism $f : c \longrightarrow c'$ such that the diagram

$$\begin{tikzcd} & d \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ \mathcal{F}(c) \arrow[rr, "\mathcal{F}(f)"] & & \mathcal{F}(c') \end{tikzcd}$$

commutes. One then defines

$$\mathrm{RK}(\mathcal{G})(d) = \lim_{(c,\alpha) \in \mathcal{C}_{d/}} c$$

The construction $\mathcal{G} \mapsto \mathrm{RK}(\mathcal{G})$ yields a functor

$$\mathrm{RK} : \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{E})$$

It is a classical fact that this functor is right adjoint to the pullback functor

$$\mathcal{F}^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

In particular, setting $\mathcal{C} = \Delta_s^{\mathrm{op}}, \mathcal{D} = \Delta^{\mathrm{op}}, \mathcal{E} = \mathcal{S}$ and $\mathcal{F} : \Delta_s^{\mathrm{op}} \hookrightarrow \Delta^{\mathrm{op}}$ the natural inclusion one obtains the adjunction

$$\begin{tikzcd} \mathcal{S}^{\Delta^{\mathrm{op}}} \arrow[r, shift left, "F"] & \mathcal{S}^{\Delta_s^{\mathrm{op}}} \arrow[l, shift left, "\mathrm{RK}"] \end{tikzcd}$$

given above.

Now in order to replace $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ with $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ we first need to find the relevant marked variant of the forgetful functor. Define the **marked forgetful functor**

$$F^+ : \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

as follows: given a simplicial space X we will define $F^+(X)$ to be the marked semi-simplicial space $\left(\overline{X}, D\right)$ where $\overline{X}$ is the underlying semi-simplicial space of X and $D \subseteq \overline{X}_1 = X_1$ is the set of degenerate 1-simplices, i.e., the image of

$$s_0 : X_0 \longrightarrow X_1.$$

The functor F^+ admits a right adjoint

$$\mathrm{RK}^+ : \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$$

which we shall now describe. For this we will need the following definition:

Definition 2.2.19. Let $f : [m] \longrightarrow [n]$ be a map in Δ. We will say that an edge $e \in (\Delta^m)_1$ is ***f*-degenerate** if f maps both its vertices to the same element of $[n]$. We will denote by $(\Delta^m)^f$ the marked semi-simplicial space (Δ^m, A_f) where Δ^m is considered as a semi-simplicial space which is levelwise discrete and $A_f \subseteq (\Delta^m)_1$ is the set of f-degenerate edges.

Now let (X, A) be a marked semi-simplicial space, and write

$$X_m^f = \mathrm{Map}^+\left((\Delta^m)^f, (X, A)\right).$$

Note that we have a natural inclusion

$$X_m^f \subseteq X_m = \mathrm{Map}(\Delta^m, X).$$

The m-simplices in X_m^f will be called ***f*-unital** simplices.

We will now construct the functor

$$\mathrm{RK}^+ : \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}.$$

as follows. For each $[n] \in \Delta$ consider the fiber product category

$$\mathcal{C}_n = \Delta_s^{op} \times_{\Delta^{op}} \Delta_{[n]/}^{op}$$

as above. Then the objects of $\mathcal{C}_n$ can be identified with maps $f : [m] \longrightarrow [n]$ in Δ and a morphism from $f : [m] \longrightarrow [n]$ to $g : [k] \longrightarrow [n]$ in $\mathcal{C}_n$ can then be described as a commutative triangle

$$\begin{tikzcd}
{[k]} \arrow[rr, "h"] \arrow[dr, "g"'] & & {[m]} \arrow[dl, "f"] \\
& {[n]} &
\end{tikzcd}$$

such that h is **injective**. Now let (X, A) be a marked semi-simplicial space and let

$$\mathcal{G}_n : \mathcal{C}_n \longrightarrow \mathcal{S}$$

be the functor which associates to each $f : [m] \longrightarrow [n]$ the space

$$\mathcal{G}_n(f) = X_m^f$$

Note that for each map $[n] \longrightarrow [n']$ in Δ one has a natural functor

$$\mathcal{F}_n : \mathcal{C}_n \longrightarrow \mathcal{C}'_n$$

and a natural transformation

$$\mathcal{F}_n^* \mathcal{G}'_n \longrightarrow \mathcal{G}_n.$$

We can then define $\mathrm{RK}^+(X, A)$ by setting

$$\mathrm{RK}^+(X, A)_n = \lim_{\mathcal{C}_n} \mathcal{G}_n.$$

Remark 2.2.20. The category $\mathcal{C}_n$ carries a Reedy structure which is induced from that of Δ_s. If (X, A) is marked-fibrant then the functor $f \mapsto X_m^f$ will be a Reedy fibrant functor from $\mathcal{C}_n$ to $\mathcal{S}$. This means that in this case the limit above will coincide with the respective **homotopy limit**.

Remark 2.2.21. The functor RK^+ is closely related to the right Kan extension functor. One has natural maps

$$\mathrm{RK}^+(X, A)_n = \lim_{\mathcal{C}_n} X_m^f \longrightarrow \lim_{\mathcal{C}_n} X_m = \mathrm{RK}(X)_n$$

which assemble together to form a natural transformation

$$\mathrm{RK}^+(X, A) \longrightarrow \mathrm{RK}(X).$$

From Lemma 2.2.12 we see that when (X, A) is marked-fibrant the map above identifies $\mathrm{RK}^+(X, A)_n$ with a union of path components of $\mathrm{RK}(X)_n$.

Let us now describe the counit map $F^+ \circ \mathrm{RK}^+ \longrightarrow \mathrm{Id}$. Let X be a marked semi-simplicial space. For each n we have the object $\mathrm{Id} \in \mathcal{C}_n$ corresponding to the identity $\mathrm{Id} : [n] \longrightarrow [n]$. We then get a natural projection map

$$\mathrm{RK}^+(X)_n = \lim_{\mathcal{C}_n} \mathcal{G}_n \longrightarrow \mathcal{G}_n(\mathrm{Id}) = X_n$$

which is functorial in n. Furthermore, unwinding the definition we see that this map sends degenerate edges in $\mathrm{RK}^+(X)_1$ to marked edges in X_1. Hence we obtain a map of marked semi-simplicial spaces

$$\nu_X : F^+\left(\mathrm{RK}^+(X)\right) \longrightarrow X.$$

The unit map is also quite easy to define. Let X be a simplicial space. For each $f : [k] \longrightarrow [n]$ in Δ we have a map $f^* : X_n \longrightarrow X_k$ whose image lies inside $(F^+(X))_k^f \subseteq X_k$. These maps fit together to form a map

$$X_n \longrightarrow \lim_{C_n} (F^+(X))_k^f = \mathrm{RK}^+(F^+(X))_n$$

which is natural in n, hence giving us the unit map

$$X \longrightarrow \mathrm{RK}^+(F^+(X)).$$

It is not hard to verify that these unit and counit maps exhibit RK^+ as a right adjoint of F^+. In particular, if X is a simplicial space and (Y, B) is a marked semi-simplicial space then the composition

$$\mathrm{Map}(X, \mathrm{RK}^+(Y, B)) \longrightarrow \mathrm{Map}(X, \mathrm{RK}(Y)) \xrightarrow{\cong} \mathrm{Map}(F(X), Y)$$

identifies the space $\mathrm{Map}(X, \mathrm{RK}^+(Y,B))$ with the subspace of $\mathrm{Map}(F(X),Y)$ consisting of maps which send degenerate 1-simplices of X_1 to B, i.e., with the space

$$\mathrm{Map}^+\left(F^+(X),(Y,B)\right).$$

We now claim that F^+, RK^+ form a Quillen adjunction. Since any Reedy cofibration in $\mathcal{S}^{\Delta^{\mathrm{op}}}$ is a levelwise injection it follows that F^+ preserves cofibrations. Furthermore, it is not hard to check that F^+ maps levelwise equivalences to marked equivalences, and hence trivial cofibrations to trivial marked cofibrations. Hence we have a Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathrm{RK}^+}{\overset{F^+}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} .$$

Now the forgetful functor

$$F : \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

factors through F^+. This means that the Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathrm{RK}}{\overset{F}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

factors through $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ as the composition

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathrm{RK}^+}{\overset{F^+}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \underset{(\bullet)^\sharp}{\overset{\overline{\bullet}}{\rightleftarrows}} \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

where $\overline{\bullet}$ denotes the forgetful functor $(X,A) \mapsto X$.

3 SemiSegal Spaces

In this section we will describe the formalism underlying the main constructions of this essay, namely that of a **semiSegal space**. SemiSegal spaces are formal models for the notion of **non-unital ∞-categories**. We will begin in §§ 3.1 by reviewing the basic definitions and proceed in §§ 3.2 to explain how the notion of a semiSegal space encodes the information of a non-unital ∞-category.

SemiSegal spaces were used in [Lur3] in order to describe various ∞-categories of manifolds and cobordisms. These types of ∞-categories form a motivating example for the theory developed in this essay: although they carry a unital structure it is simpler and more direct to first obtain a description of their underlying non-unital ∞-categories in terms of semiSegal spaces. We will review this example in §§ 3.3.

Finally, in §§ 3.4 we will revisit the notion of semiSegal spaces in a model categorical framework. More precisely, we will show that there exists a monoidal simplicial model structure on $\mathcal{S}^{\Delta^{\mathrm{op}}}$ whose fibrant objects are exactly the semiSegal spaces. This is completely analogous to the construction of the Segal model structure in [Rez], although some of the proofs require technical modifications to the semi-simplicial setting.

3.1 Basic Definitions

Definition 3.1.1. Let X be a semi-simplicial space. Let $[n],[m] \in \Delta_s$ be two objects and consider the commutative (pushout) diagram

$$\begin{tikzcd}
{[0]} \arrow[r, "0"] \arrow[d, "n"] & {[m]} \arrow[d, "g_{n,m}"] \\
{[n]} \arrow[r, "f_{n,m}"'] & {[n+m]}
\end{tikzcd}$$

where $f_{n,m}(i) = i$ and $g_{n,m}(i) = i + n$. We will say that X satisfies the **Segal condition** if for each $[n],[m]$ as above the induced commutative diagram

$$\begin{tikzcd}
X_{m+n} \arrow[r, "g^*_{n,m}"] \arrow[d, "f^*_{n,m}"'] & X_m \arrow[d, "0^*"] \\
X_n \arrow[r, "n^*"] & X_0
\end{tikzcd}$$

is a **homotopy pullback** diagram. We will say that X is a **semiSegal space** if it is **Reedy fibrant** and satisfies the Segal condition. Note that in that case the above square will induce a homotopy equivalence

$$X_{m+n} \simeq X_m \times_{X_0} X_n.$$

The Segal condition can also be formulated in term of **spines**:

Definition 3.1.2. We will denote by

$$\mathrm{Sp}_n = \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} ... \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subseteq \Delta^n$$

the **spine** of Δ^n, i.e., the sub semi-simplicial set consisting of all the vertices and all the edges of the form $\{i, i+1\}$.

The Segal condition on a **Reedy fibrant** semi-simplicial space X is equivalent to the assertion that the map

$$X_n = \mathrm{Map}(\Delta^n, X) \longrightarrow \mathrm{Map}(\mathrm{Sp}_n, X) = X_1 \times_{X_0} ... \times_{X_0} X_1$$

is a weak equivalence for every $n \geq 2$. This means that the Segal condition can be considered as a **locality** condition: semiSegal spaces are exactly the Reedy fibrant semi-simplicial spaces which are **local** with respect to spine inclusions. We will revisit this point in §§ 3.4.

Example 3.1.3. Let $\mathcal{C}$ be a non-unital small topological category. We can represent $\mathcal{C}$ as a semiSegal space as follows. For each n, let $\mathfrak{C}^{\mathrm{nu}}([n])$ denote the non-unital $\mathcal{S}$-enriched category whose objects are the numbers $0, ..., n$ and whose mapping spaces are

$$\mathrm{Map}_{\mathfrak{C}^{\mathrm{nu}}([n])}(i,j) = \begin{cases} \emptyset & i \geq j \\ I^{(i,j)} & i < j \end{cases}$$

where $(i,j) = \{x \in \{0, ..., n\} | i < x < j\}$. The composition is given by the inclusion

$$I^{(i,j)} \times I^{(j,k)} \cong I^{(i,j)} \times \{0\} \times I^{(j,k)} \subseteq I^{(i,k)}.$$

Note that $\mathfrak{C}^{\mathrm{nu}}([n])$ depends functorially on $[n] \in \Delta_s$. Hence for every non-unital topological category $\mathcal{C}$ we can form a semi-simplicial space $N(\mathcal{C})$ by setting

$$N(\mathcal{C})_n = \mathrm{Fun}(\mathfrak{C}^{\mathrm{nu}}([n]), \mathcal{C}).$$

We endow $N(\mathcal{C})_n$ with a natural topology that comes from the topology of the mapping space of $\mathcal{C}$ (while treating the set of objects of $\mathcal{C}$ as discrete). One can then check that $N(\mathcal{C})$ is a **semiSegal space**.

We think of general semiSegal spaces as relaxed versions of Example 3.1.3. We will explain this point of view further in the next subsection. However, before we do so let us introduce one last piece of terminology which will be very useful for us in studying semiSegal spaces.

Let $\Lambda^n_i \subseteq \Delta^n$ be the sub semi-simplicial set obtained by removing the single n-simplex of Δ^n together with the $(n-1)$-face which is opposite to the i'th vertex. The semi-simplicial set Λ^n_i is called the i'th **horn** of Δ^n. We will refer to an inclusion of the form

$$\Lambda^n_i \subseteq \Delta^n$$

as a **horn inclusion** of dimension n. When $0 < i < n$ we will say that Λ^n_i is an **inner horn** of Δ^n and that the inclusion above is an **inner horn inclusion**.

For various purposes it is useful to replace the spine inclusions $\mathrm{Sp}_n \subseteq \Delta^n$ (see Definition 3.1.2) with inner horn inclusions. The following proposition is the semi-simplicial analogue of a standard fact about Segal spaces:

Proposition 3.1.4. *Let X be a Reedy fibrant semi-simplicial space. Then X is a semiSegal space if and only if the restriction map*

$$\mathrm{Map}(\Delta^n, X) \longrightarrow \mathrm{Map}(\Lambda^n_l, X)$$

is a weak equivalence for every inner horn inclusion $\Lambda^n_l \subseteq \Delta^n$.

Proof. The proposition will follow easily from the following lemma:

Lemma 3.1.5. *Let $n \geq 2$ and $0 < l < n$. Then the semi-simplicial set Λ^n_l can be obtained from Sp_n by successively performing pushouts along inner horn inclusions of dimension $< n$.*

Proof. For $n = 2$ the claim is trivial because $\mathrm{Sp}_2 = \Lambda^2_1$. Now take $n \geq 3$ and assume the claim is true for all m's such that $2 \leq m < n$. This implies that for each $2 \leq m < n$ the full m-simplex Δ^m can be obtained from Sp_m by performing pushouts along inner horn inclusions of dimension $\leq m$.

Let $\Delta^{\{v_l\}} \subseteq \Delta^n$ be the l'th vertex. By expressing Sp_n as a pushout $\mathrm{Sp}_l \coprod_{\Delta^{\{v_l\}}} \mathrm{Sp}_{n-l}$ we see that $\Delta^{\{0,...,l\}} \coprod_{\Delta^{\{v_l\}}} \Delta^{\{l,...,n\}}$ can be obtained from Sp_n by performing pushouts along inner horn inclusions of dimension $< n$.

We will say that a simplex $\Delta^I \subseteq \Delta^n$ is **two-sided** if I contains l, a vertex strictly smaller then l and a vertex strictly larger then l. Set

$$X_1 = \Delta^{\{0,...,l\}} \coprod_{\Delta^{\{v_l\}}} \Delta^{\{l,...,n\}}$$

and for $j = 2, ..., n-1$ define inductively $X_j \subseteq \Delta^n$ to be the union of X_{j-1} and all the two-sided j-simplices. It is then not hard to verify that X_j can be obtained from X_{j-1} by a sequence of pushouts along inner horn inclusions of dimension j and that $X_{n-1} = \Lambda^n_l$.

□

Now from Lemma 3.1.5 one gets that Δ^n can be obtained from Sp_n by a sequence of inner horn inclusions. This proves the "if" direction of Proposition 3.1.4. The "only if" direction follows from Lemma 3.1.5 as well by using a simple inductive argument. □

3.2 Homotopy Theory in a semiSegal Space

We are now ready to explain how a general semiSegal space encodes the information of a **non-unital ∞-category**, i.e., a relaxed version of a non-unital topological category. Let X be a semiSegal space. The objects of the corresponding non-unital ∞-category are the points of X_0. Given two points $x, y \in X_0$ we define the **mapping space** between them by

$$\mathrm{Map}_X(x, y) = \{x\} \times_{X_0} X_1 \times_{X_0} \{y\},$$

i.e., as the fiber of the (Kan) fibration

$$X_1 \xrightarrow{(d_0,d_1)} X_0 \times X_0$$

over the point (x,y). Informally, the space X_2 of triangles induces a "homotopy-composition" operation on these mapping spaces which is homotopy associative in a coherent way. In order to describe this structure in more precise terms one can use the notion of **spans**. We recall the basic definition:

Definition 3.2.1. Let $X, Y \in \mathcal{S}$ be spaces. A **span** from X to Y is a space C equipped with a pair of maps

$$\begin{tikzcd} & C \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\ X & & Y \end{tikzcd}$$

We will say that a span as above is a **fibration span** if the map

$$\varphi \times \psi : C \longrightarrow X \times Y$$

is a fibration.

Remark 3.2.2. Unlike the notion of span, the term **fibration span** is not standard. However, since all the spans we will come across involve only fibrations, and since this simplifies a bit the description of compositions, we have chosen to focus attention on this particular case of spans. Note that every span is equivalent (in a sense given below) to a fibration span and so this does not mean any essential loss of generality.

Note that any map $f : X \longrightarrow Y$ gives a span

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

and so spans can be considered as generalizations of maps. In particular, we consider a span $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ as **going from** X **to** Y, so it is of some importance to keep track of the order.

Definition 3.2.3. Let $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ and $X \xleftarrow{\varphi'} C' \xrightarrow{\psi'} Y$ be two spans. An **equivalence** of spans is a commutative diagram of the form

$$\begin{tikzcd} & C \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X & D \arrow[l] \arrow[r] \arrow[u, "f'"] \arrow[d, "f"'] & Y \\ & C' \arrow[ul, "\varphi'"] \arrow[ur, "\psi'"'] & \end{tikzcd}$$

such that f, f' are weak equivalences. In this case we say that the spans $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ and $X \xleftarrow{\varphi'} C' \xrightarrow{\psi'} Y$ are **equivalent**.

We will say that a span $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ is **map-like** if it is equivalent to a span coming from a map as above. This is equivalent to saying that φ is a **weak equivalence**. As we will see below, all the spans that will appear in the semiSegal formalism are map-like. However, it is still useful to keep the framework general.

Now let $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ and $X \xleftarrow{\rho} D \xrightarrow{\tau} Y$ be two **fibration** spans. One defines their **composition** $X \longleftarrow P \longrightarrow Y$ by forming the commutative diagram

$$\begin{tikzcd}
 & & P \arrow[dl] \arrow[dr] & & \\
 & C \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & & D \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\
X & & Y & & X
\end{tikzcd}$$

where the internal square exhibits P as the fiber product of $C \times_Y D$.

Remark 3.2.4. One can define composition also without the assuming that the spans are fibration spans. However, if one wants the correct notion of composition from the ∞-categorical point of view in the general case one should replace the fiber product above with a suitable **homotopy fiber product**. Since all spans occurring in this text are fibration spans we chose not to address this additional subtlety.

It is not hard to see that this composition generalizes the usual composition of maps. Furthermore this composition rule **respects the notion of equivalence** described above and in particular the composition of two map-like spans is map-like. The latter statement can also be seen directly from the fact that a pullback of a weak equivalence along a fibration is a weak equivalence.

Let us now explain how to use the notion of spans in order to describe the homotopy-composition operation in a semiSegal space. Suppose first that we had a topological category $\mathcal{C}$ and we were given three objects $x, y, z \in \mathcal{C}$ and a morphism $f : x \longrightarrow y$. One would then obtain a composition-by-f maps

$$f_* : \mathrm{Hom}_{\mathcal{C}}(z, x) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(z, y)$$

and

$$f^* : \mathrm{Hom}_{\mathcal{C}}(y, z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x, z).$$

In the semiSegal model we do not have such strict composition. Instead one can describe the composition-by-f maps as **spans**. If $x, y, z \in X_0$ are objects and $f : x \longrightarrow y$ is a morphism (i.e., an element in X_1 such that $d_0(f) = x$ and $d_1(f) = y$) one can consider the space $C^R_{f,z} \subseteq X_2$ given by

$$C^R_{f,z} = \{\sigma \in X_2 \mid \sigma|_{\Delta^{\{1,2\}}} = f, \sigma|_{\Delta^{\{0\}}} = z\} .$$

Then the two restriction maps $\sigma \mapsto \sigma|_{\Delta^{\{0,1\}}}$ and $\sigma \mapsto \sigma|_{\Delta^{\{0,2\}}}$ give us a span (which is a **fibration span** due to X being Reedy fibrant):

$$\begin{tikzcd} & C^R_{f,z} \arrow[dl] \arrow[dr] & \\ \mathrm{Map}_X(z,x) & & \mathrm{Map}_X(z,y) \end{tikzcd}$$

This span describes the operation of composing with f on the right. Similarly we have a fibration span

$$\begin{tikzcd} & C^L_{f,z} \arrow[dl] \arrow[dr] & \\ \mathrm{Map}_X(y,z) & & \mathrm{Map}_X(x,z) \end{tikzcd}$$

describing composition with f on the left.

Now the Segal condition ensures that both $C^R_{f,z}$ and $C^L_{f,z}$ are **map-like** spans, i.e., the left hand side maps are **weak equivalences**. In that sense composition is "almost" well-defined. In particular any map-like span

$$\begin{tikzcd} & C \arrow[dl, "\simeq"'] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

induces a well-defined map

$$\pi_0(X) \longrightarrow \pi_0(Y)$$

This observation enables the following definition:

Definition 3.2.5. Let X be a semiSegal space. We define its **homotopy category** $\mathrm{Ho}(X)$ to be the non-unital category whose objects are X_0 and morphisms set are

$$\mathrm{Hom}_{\mathrm{Ho}(X)}(x,y) \overset{\mathrm{def}}{=} \pi_0(\mathrm{Map}_X(x,y)).$$

The composition is induced by the map-like composition spans described above. More explicitly, the composition of the connected components $[f] \in \pi_0(\mathrm{Map}_X(x,y))$ and $[g] \in \pi_0(\mathrm{Map}_X(y,z))$ is the unique connected component $[h] \in \pi_0(\mathrm{Map}_X(x,z))$ for which there exists a triangle $\sigma \in X_2$ of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z \end{tikzcd}$$

To finish this subsection let us prove a simple lemma which will illustrate the behavior of these composition spans. We will prove that if a morphism h is the "homotopy-composition" of f and g then the span $C^R_{h,z}$ is equivalent to the composition of the fibration spans $C^R_{f,z}$ and $C^R_{g,z}$. This will imply, for instance, the associativity of the composition operation in $\mathrm{Ho}(X)$.

Lemma 3.2.6. *Let $\sigma : \Delta^2 \longrightarrow X$ be a triangle of the form*

$$\begin{tikzcd} & v_2 \arrow[dr, "g"] & \\ v_1 \arrow[ur, "f"] \arrow[rr, "h"] & & v_3 \end{tikzcd}$$

Let $v_0 \in X_0$ be a point and let

$$\mathrm{Map}_X(v_0, v_1) \longleftarrow C^R_{f,v_0} \longrightarrow \mathrm{Map}_X(v_0, v_2),$$

$$\mathrm{Map}_X(v_0, v_2) \longleftarrow C^R_{g,v_0} \longrightarrow \mathrm{Map}_X(v_0, v_3)$$

and

$$\mathrm{Map}_X(v_0, v_1) \longleftarrow C^R_{h,v_0} \longrightarrow \mathrm{Map}_X(v_0, v_3)$$

be the composition spans as above. Then C^R_{h,v_0} is equivalent to the composition of C^L_{f,v_0} and C^L_{g,v_0}. The analogous statement regarding left-composition is true as well.

Proof. We prove the lemma for right-composition (the proof for left-composition is completely analogous). For $i = 1, 2$ define

$$P_i = \{\rho : \Lambda^3_i \longrightarrow X | \rho|_{\Delta^{\{1,2,3\}}} = \sigma, \rho|_{\Delta^{\{0\}}} = v_0\}.$$

The restriction maps to $\Delta^{0,1}, \Delta^{0,3} \subseteq \Lambda^3_i$ induce a fibration span

$$\mathrm{Map}_X(v_0, v_1) \longleftarrow P_i \longrightarrow \mathrm{Map}_X(v_0, v_3).$$

Now note that P_2 can be identified with the composed span

$$\mathrm{Map}_X(v_0, v_1) \longleftarrow C^R_{f,v_0} \times_{\mathrm{Map}_X(v_0,v_2)} C^R_{g,v_0} \longrightarrow \mathrm{Map}_X(v_0, v_3).$$

In a similar manner we observe that P_1 can be identified with the space

$$C^R_{f,v_0} \times_{\mathrm{Map}_X(v_0,v_1)} C^R_{h,v_0}.$$

Since the fibration $C^R_{f,v_0} \longrightarrow \mathrm{Map}_X(v_0, v_1)$ is a weak equivalence we get that the projection map

$$P_1 \longrightarrow C^R_{h,v_0}$$

is a weak equivalence, and in particular induces an equivalence of spans between P_1 and C^R_{h,v_0}. This means that in order to complete the proof it will suffice to show that P_1 and P_2 are equivalent spans.

Consider the space

$$E = \{\rho : \Delta^3 \longrightarrow X | \rho|_{\Delta^{\{1,2,3\}}} = \sigma, \rho|_{\Delta^{\{0\}}} = v_0\}.$$

We have natural restriction maps $E \longrightarrow P_1$ and $E \longrightarrow P_2$ which are weak equivalences by Proposition 3.1.4. Furthermore it is clear that these restriction maps are compatible with the source and target maps of the P_1, P_2 and so induce an equivalence of spans between P_1 and P_2. This finishes the proof of the lemma. $\square$

3.3 The Cobordism Categories

Let us now make a brief digression in order to describe the non-trivial example we have in mind for a semiSegal space (which does not arise naturally from a non-unital topological category). This example occurs when one tries to formally describe the ∞-category of closed n-manifolds and cobordisms between them. For more details about this construction we refer the reader to [Lur3] §2.2.

To begin, note that the ∞-category of n-manifolds and cobordisms contains in it the strict topological groupoid of n-manifolds and **diffeomorphisms**. Hence it will be useful to start by constructing a model for the classifying space of this topological groupoid. Let $V = \mathbb{R}^\infty$ be the infinite dimensional topological vector space obtained as the direct limit (in the category of topological vector spaces)

$$\mathbb{R}^0 \hookrightarrow \mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow ...$$

Then for each closed manifold M, the space $\mathrm{Emb}(M, V)$ of smooth embeddings $M \hookrightarrow V$ (endowed with the C^∞-topology) is contractible (in fact, since M is compact this space is the direct limit of the embedding spaces $\mathrm{Emb}(M, \mathbb{R}^n)$, which become more and more connected as $n \longrightarrow \infty$). The topological group $\mathrm{Diff}(M)$ acts freely on $\mathrm{Emb}(M, \mathbb{R}^n)$ and we denote the quotient space by

$$\mathrm{Sub}(M, V) = \mathrm{Emb}(M, V)/\,\mathrm{Diff}(M).$$

The space $\mathrm{Sub}(M, V)$ can be considered as the space of submanifolds of V which are diffeomorphic to M (without a choice of diffeomorphism). In particular, if we let Man_n be a set of closed n-manifolds which contains each diffeomorphism type exactly once then the space

$$\mathrm{Sub}^n(V) = \coprod_{M \in \mathrm{Man}_n} \mathrm{Sub}(M, V)$$

is the space of all n-submanifolds of V. Since each of the embedding spaces $\mathrm{Emb}(M, V)$ is contractible we can also think of $\mathrm{Sub}^n(V)$ as the **space of all closed n-manifolds**. More precisely, the space $\mathrm{Sub}^n(V)$ is a model for the classifying space of the topological groupoid of n-dimensional closed manifolds and diffeomorphisms between them.

This method can be extended in order to model the cobordism ∞-category of n-manifolds as a semiSegal space Cob^n. The space Cob^n_1 should be the "space of all cobordisms". More formally, this will be the classifying space of tuples (W, M_0, M_1, T) where M_0, M_1 are closed n-manifolds, W is an $(n+1)$-manifold and T is a diffeomorphism from $M_0 \coprod M_1$ to ∂W. Such tuples form a topological groupoid with morphisms being compatible triples of diffeomorphisms. Similarly the space Cob_k^n should be the classifying space of k-composable sequences cobordisms. In order to construct an explicit model for Cob_k^n we will use the embedding technique as above.

Let

$$\tau : [k] \longrightarrow \mathbb{R}$$

be an order preserving map. We will denote the image $\tau(i)$ by τ_i, so that we have an increasing sequence of real numbers

$$\tau_0 < \tau_1 < ... < \tau_k.$$

Let W be a cobordism from M_0 to M_1 (and identify ∂W with $M_1 \coprod M_2$). We will say that an embedding

$$\iota : W \hookrightarrow V \times [\tau_0, \tau_k]$$

is **proper** if

1. ι maps M_0 to $V \times \{\tau_0\}$ and M_1 to $V \times \{\tau_k\}$.
2. The submanifold $\iota(W)$ meets the subspace $V \times \{\tau_i\}$ transversely for each $i = 0, ..., k$.

In particular, if ι is proper then

$$\iota(W) \cap V \times \{\tau_0\} = M_0$$

and

$$\iota(W) \cap V \times \{\tau_n\} = M_1.$$

We will denote by $\mathrm{Emb}_p(W, V, \tau)$ the space of proper embeddings of W in $V \times [\tau_0, \tau_k]$. As in the case of closed manifolds, one can show that the space $\mathrm{Emb}_p(W, V, \tau)$ is a colimit of embedding spaces $\mathrm{Emb}_p(W, \mathbb{R}^n, \tau)$ which become more and more connected. In particular, $\mathrm{Emb}_p(W, V, \tau)$ is **contractible**.

Now let $\mathrm{Diff}(W, M_0, M_1)$ denote the group of diffeomorphisms of W which map M_0 to itself and M_1 to itself. Then $\mathrm{Diff}(W, M_0, M_1)$ acts freely on the space $\mathrm{Emb}_p(W, V, \tau)$ of proper embeddings by reparameterization. We will denote the quotient space by

$$\mathrm{Cob}(W, V, \tau) \stackrel{\mathrm{def}}{=} \mathrm{Emb}_p(W, V, \tau) / \mathrm{Diff}(W, M_0, M_1).$$

Let CMan_n be a set of cobordisms (W, M_0, M_1) between n-manifolds which covers each diffeomorphism type (of cobordism) exactly once. For $k \geq 1$ and $\tau : [k] \longrightarrow \mathbb{R}$ we will denote

$$\mathrm{Cob}^n(V, \tau) \stackrel{\mathrm{def}}{=} \coprod_{\mathrm{CMan}_n} \mathrm{Cob}(W, V, \tau)$$

and for $\tau : [0] \longrightarrow \mathbb{R}$ we will denote

$$\mathrm{Cob}^n(V, \tau) \stackrel{\mathrm{def}}{=} \mathrm{Sub}^n(V \times \{\tau_0\}),$$

where $\mathrm{Sub}^n(V \times \{\tau_0\})$ is the space of all sub n-manifolds of $V \times \{\tau_0\}$, constructed as above.

How should we think of $\mathrm{Cob}^n(V, \tau)$ when the domain of τ is $[k]$ for $k \geq 1$? Note that one can consider $V \times [\tau_0, \tau_1]$ as an (infinite-dimensional) cobordism from $V \times \{\tau_0\}$ to $V \times \{\tau_1\}$. One can then think of $\mathrm{Cob}^n(V, \tau)$ as the space of **subcobordisms** of $V \times [\tau_0, \tau_1]$. Since each $\mathrm{Emb}_p(W, V, \tau)$ is contractible we get that when the domain of τ is $[1]$ then $\mathrm{Cob}^n(V, \tau)$ is a model for the classifying space of cobordisms.

Similarly when the domain of τ is a larger $[k]$ then one can consider the corresponding sequence

$$V \times [\tau_0, \tau_1], V \times [\tau_1, \tau_2], ..., V \times [\tau_{k-1}, \tau_k]$$

as a composable sequence of infinite dimensional cobordisms

$$V \times \{\tau_0\} \longrightarrow V \times \{\tau_1\} \longrightarrow ... \longrightarrow V \times \{\tau_k\}.$$

One can then consider $\mathrm{Cob}^n(V, \tau)$ as the space of sequences of cobordisms contained in the sequence above. This will be a model for the classifying space for composable k-sequences of cobordisms.

We define Cob^n_k to be the space of tuples (x, τ) where $\tau : [k] \longrightarrow \mathbb{R}$ is an order preserving map and x is a point in $\mathrm{Cob}^n(V, \tau)$. This space can be topologized in a natural way because of the nice dependence of $\mathrm{Cob}^n(V, \tau)$ on τ. We claim that the collection of spaces $\{\mathrm{Cob}^n_k\}$ carries a natural structure of a semi-simplicial space: for every $\rho : [k] \longrightarrow [m]$ in Δ_s one obtains a natural map

$$\rho^* : \mathrm{Cob}^n_m \longrightarrow \mathrm{Cob}^n_k$$

by sending a pair (x, τ) to the pair

$$\left(x \cap \left(V \times [\tau_{\rho(0)}, \tau_{\rho(k)}]\right), \tau \circ \rho\right).$$

The fact that Cob^n satisfies the Segal condition follows directly from the construction. It reflects the fact that we can glue cobordisms together, and that the result is well defined up to a contractible space of choices. Note that Cob^n is not Reedy fibrant in general, but this can be fixed by taking a Reedy fibrant replacement

$$\mathrm{Cob}^n \longrightarrow \widehat{\mathrm{Cob}^n}$$

resulting in a semiSegal space $\widehat{\mathrm{Cob}^n}$. This semiSegal space is a model for the underlying non-unital ∞-category of n-manifolds and cobordisms between them.

3.4 The Segal Model Structure

In §§ 3.1 we saw that a Reedy fibrant semi-simplicial space is a semiSegal space if and only if it is **local** with respect to **inner horn inclusions**

$$\Lambda^n_l \hookrightarrow \Delta^n,$$

where we consider Λ^n_l, Δ^n as semi-simplicial spaces which are levelwise discrete. In particular, one can study semiSegal spaces in the framework of model categories by **localizing** the Reedy model category $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ with respect to inner horn inclusions.

The Reedy model category $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ has many nice features (e.g., it is a left proper combinatorial model category) which guaranty that the left Bousfield localization of it with respect to any set of maps exists (see [Hir], Chapter 4). Furthermore, the localized model category will inherit the simplicial structure of $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$. In particular, there exists a simplicial model category Seg whose underlying simplicial category is $\mathcal{S}^{\Delta^{op}}$ such that:

1. A map $f : X \longrightarrow Y$ in Seg is a **cofibration** if and only if it is a Reedy cofibration (i.e., levelwise cofibration).

2. A map $f : X \longrightarrow Y$ in Seg is a **weak equivalence** if and only if for each semiSegal space Z the restriction map

$$\mathrm{Map}(Y, Z) \longrightarrow \mathrm{Map}(X, Z)$$

is a weak equivalence. We will refer to such weak equivalences as **Segal weak equivalences**.

Remark 3.4.1. The fact that Seg is a simplicial model category means in particular that if $K \in \mathcal{S}$ is a space and $X \hookrightarrow Y$ a trivial Segal cofibration (i.e., a trivial cofibration in Seg) then the map

$$K \otimes X \hookrightarrow K \otimes Y$$

is a trivial Segal cofibration. This means that if W is a semiSegal space and $K \in \mathcal{S}$ is a space then W^K is a semiSegal space as well.

Example 3.4.2. All inner horn inclusion and all spine inclusions are trivial Segal cofibrations.

We now claim that the model structure of Seg is **compatible** (see Definition 2.1.6) with the monoidal product $\otimes$. Since the cofibrations in Seg are the same as in the Reedy structure, it is enough to check that if $f : X' \hookrightarrow X$ is a cofibration and $g : Y' \hookrightarrow Y$ is a trivial Segal cofibration then the induced map

$$h : [X \otimes Y'] \coprod_{X' \otimes Y'} [X' \otimes Y] \longrightarrow X \otimes Y$$

is a trivial Segal cofibration. We will start with the following standard reduction argument which shows that it is enough to consider the case of $f : \emptyset \longrightarrow X$ and

$g : \Lambda^n_l \hookrightarrow \Delta^n$. Note that there is nothing specific to semiSegal spaces in this argument - we just use the fact that the localization was done along a set of cofibrations and that every object is cofibrant.

Proposition 3.4.3. *The following statements are equivalent:*

1. *The Segal model structure is compatible with* $\otimes$.
2. *For each semi-simplicial space* X *the maps*
$$X \otimes \Lambda^n_l \hookrightarrow X \otimes \Delta^n$$
are trivial Segal cofibrations.
3. *For each semi-simplicial space* X *and semiSegal space* W *the internal mapping object* W^X *is a semiSegal space.*
4. *For each semi-simplicial space* X *and trivial Segal cofibration* $Y' \hookrightarrow Y$ *the induced map*
$$X \otimes Y' \hookrightarrow X \otimes Y$$
is a trivial Segal cofibration.

Proof. Since every object in Seg is cofibrant (2) is a particular case of (1). Using the exponental law we deduce that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). Hence it is enough to show (4) $\Rightarrow$ (1). Let $X' \longrightarrow X$ be a cofibration and $Y' \longrightarrow Y$ a trivial Segal cofibration. Assuming (4) we get that the map

$$X \otimes Y' \longrightarrow [X \otimes Y'] \coprod_{X' \otimes Y'} [X' \otimes Y]$$

is a pushout along a Segal trivial cofibrations, and hence itself a trivial Segal cofibration. Again from (4) we get that the map

$$X \otimes Y' \longrightarrow X \otimes Y$$

is also a trivial Segal cofibration. Applying the 2-out-of-3 rule we now get that the map

$$h : [X \otimes Y'] \coprod_{X' \otimes Y'} [X' \otimes Y] \longrightarrow X \otimes Y$$

is a Segal weak equivalence and hence a trivial Segal cofibration.

$\square$

Hence, in order to prove the compatibility of $\otimes$ with the Segal model structure it is enough to prove the following:

Proposition 3.4.4. *Let* X *be a semi-simplicial space. Then the maps*

$$\rho : X \otimes \Lambda^n_l \hookrightarrow X \otimes \Delta^n$$

are trivial Segal cofibrations.

Proof. Since X can be obtained as a sequential colimit of inclusions of finite dimensional semi-simplicial spaces (and hence also the homotopy colimit of them) we see that it is enough to prove for finite dimensional X. Proceeding with the reductionist approach we note that any finite-dimensional semi-simplicial space can be built up from $\emptyset$ by successively performing pushouts along maps of the form

$$K \otimes \partial\Delta^m \hookrightarrow K \otimes \Delta^m$$

for $K \in \mathcal{S}$. This means that $X \otimes \Delta^n$ can be built from $X \otimes \Lambda^n_l$ by successively performing pushouts along maps of the form

$$[K \otimes \partial\Delta^m \otimes \Delta^n] \coprod_{K \otimes \partial\Delta^m \otimes \Lambda^n_l} [K \otimes \Delta^m \otimes \Lambda^n_l] \hookrightarrow K \otimes \Delta^m \otimes \Delta^n.$$

In view of Remark 3.4.1 we see that it will be enough to prove that the inclusions

$$[\partial\Delta^m \otimes \Delta^n] \coprod_{\partial\Delta^m \otimes \Lambda^n_l} [\Delta^m \otimes \Lambda^n_l] \hookrightarrow \Delta^m \otimes \Delta^n$$

are trivial Segal cofibrations for every $n, m, l \geq 0$ such that $0 < l < n$. For later applications it will be useful to prove a slightly more general result. The following definition appeared first in Joyal's fundamental paper [Joy]:

Definition 3.4.5. We will say that a horn inclusion

$$\Lambda^n_l \subseteq \Delta^n$$

is a **right** (respectively **left**) horn inclusion if $l \leq 0$ (respectively $l \geq 0$). In particular, inner horn inclusions are those which are both right and left horn inclusions.

We will now prove the following:

Lemma 3.4.6. *Let*

$$\Lambda^n_l \subseteq \Delta^n$$

be an inner (right) horn inclusion. Then for each $m \geq 0$ the semi-simplicial set $\Delta^m \otimes \Delta^n$ can be obtained from the semi-simplicial set

$$X = \partial\Delta^m \otimes \Delta^n \coprod_{\partial\Delta^m \otimes \Lambda^n_l} \Delta^m \otimes \Lambda^n_l$$

by successively performing pushouts along inner (right) horn inclusions of dimension $\geq \max(n, m+1)$.

Proof. According to Remark 2.1.4 the k-simplices of $\Delta^m \otimes \Delta^n$ are in one-to-one span with injective order preserving maps

$$\sigma = (f, g) : [k] \longrightarrow [m] \times [n].$$

We will say that a k-simplex of $\Delta^m \otimes \Delta^n$ is **full** if it is **not** contained in X. If we describe our k-simplex by a map $\sigma = (f, g)$ as above this translates to the condition that f is surjective and that the image of g contains $\{0, ..., n\} \setminus \{l\}$ (so that g is either surjective or misses l). Our purpose is to add all the full simplices to X in a way that involves only pushouts along inner horn inclusions. For this we distinguish between two kinds of k-simplices of $\Delta^m \otimes \Delta^n$:

Definition 3.4.7. Let

$$\sigma = (f, g) : [k] \longrightarrow [m] \times [n]$$

be a **full** k-simplex of $\Delta^m \otimes \Delta^n$. We will say that σ is **special** if $g^{-1}(l) \neq \emptyset$ and

$$f\left(\min g^{-1}(l)\right) = f\left(\max g^{-1}(l-1)\right).$$

Otherwise we will say that σ is **regular**.

Now for $i = 0, ..., m+1$ let X_i denote the union of X and all **special** $(i+n-1)$-simplices of $\Delta^m \otimes \Delta^n$. We now claim the following:

1. $X_0 = X$.
2. For $i = 0, ..., m$ the semi-simplicial set X_{i+1} is obtained from X_i by a sequence of pushouts along inner horn inclusions of dimension $i+n$.
3. $X_{m+1} = \Delta^m \otimes \Delta^n$.

The first claim just follows from the fact that there are no special simplices of dimension less than n. Now X_{i+1} is the union of X_i and all special $(i+n)$-simplices. Hence in order to prove the second claim we will need to find the right **order** in which to add these special $(i+n)$-simplices to X_i. We will do this by sorting them according to the following quantity:

Definition 3.4.8. Let

$$\sigma = (f, g) : [k] \longrightarrow [m] \times [n]$$

be a **full** k-simplex of $\Delta^m \otimes \Delta^n$. We define the **index** of σ to be the quantity

$$\mathrm{ind}(\sigma) = k + 1 - n - |g^{-1}(l)|.$$

Note that for a general full simplex the index is a number between 0 and $k+1-n$. By definition we see that for a **special** k-simplex the index is a number between 0 and $k-n$.

Now fix an $i = 0, ..., m$ and for each $j = 0, ..., i+1$ define $X_{i,j}$ to be the union of X_i and all special $(i+n)$-simplices σ whose index is strictly less than j. We obtain a filtration of the form

$$X_i = X_{i,0} \subseteq X_{i,1} \subseteq ... \subseteq X_{i,i+1} = X_{i+1}$$

We will show that if σ is a special $(i+n)$-simplex of index j then the intersection

$$\sigma \cap X_{i,j}$$

is an inner horn of σ. This means that $X_{i,j+1}$ can be obtained from $X_{i,j}$ by performing pushouts along inner horn inclusions of dimension $m+i$, implying the second claim above. We start by noting that if $\tau = (f, g)$ is a **regular** k-simplex then τ is a face of the special $(k+1)$-simplex

$$\sigma = (f \circ s_{\max g^{-1}(l-1)}, g \circ s_{\min g^{-1}(l)})$$

where $s_r : [k+1] \longrightarrow [k]$ is the degeneracy map hitting r twice. Furthermore we see that $\mathrm{ind}(\sigma) = \mathrm{ind}(\tau)$. This means that $X_{i,j}$ contains in particular all regular $(i+n-1)$-simplices whose index is $< j$. Since taking faces cannot increase the index we see that an $(i+n-1)$-simplex τ is contained in $X_{i,j}$ exactly when τ is not regular of index $\geq j$.

Now let $\sigma = (f, g)$ be a **special** $(i+n)$-simplex of index j and let τ be the $(i+n-1)$-face of σ which is apposed to the v'th vertex for $v = 0, ..., i+n$. Then we see that τ will be regular of index $\geq j$ if and only if $v = \min g^{-1}(l)$, in which case $\mathrm{ind}(\tau) = \mathrm{ind}(\sigma) = j$. Since g is surjective we get that

$$0 < \min g^{-1}(l) \leq i + l \leq i + n$$

and so $X_{i,j} \cap \sigma$ is a right horn of σ which is inner if $l < n$.

It is left to prove the third claim, i.e., that $X_{m+1} = \Delta^m \otimes \Delta^n$. From the considerations above we see that X_{i+1} contains all full k-simplices for $k < n+i$ (as well as all special $(n+i)$-simplices). Since all the full $(m+n)$-simplices are special we get that X_{m+1} contains all full simplices of $\Delta^m \otimes \Delta^n$ of dimension up to $m+n$, yielding the desired result.

$\square$

This completes the proof of Proposition 3.4.4. $\square$

4 Quasi-unital SemiSegal Spaces

In this section we will introduce and study the notion of **quasi-unital** semiSegal spaces. These semiSegal spaces correspond to **quasi-unital** ∞-categories, i.e., non-unital ∞-categories in which each object has an identity-up-to-homotopy in an appropriate sense. Such identities will be called **quasi-units**.

We will begin in §§ 4.1 where we will explain how the notions of **equivalences** and **quasi-units** are encoded in the semiSegal formalism. In §§ 4.2 we will define a topological category QsS consisting of **quasi-unital** semiSegal spaces and **unital** maps between them and study a natural notion of weak equivalences on QsS, namely that of a **Dwyer-Kan equivalence**.

The ∞-category obtained by localizing QsS with respect to DK-equivalences is our proposed model for the ∞-category of small quasi-unital ∞-categories. In §§ 4.3 we will show that when restricted to **quasi-unital semi-groupoids** (i.e., quasi-unital semiSegal spaces in which every morphism is invertible) this corresponding localized ∞-category is equivalent to the ∞-category of ∞-groupoids. This proves the groupoid version of the main theorem 1.0.5, and will be an important step towards the general case.

4.1 Equivalences and Quasi-units

Let X be a semiSegal space. In section 3.1 we saw that the homotopy-composition in X can be described in terms of spans. In particular, if $x, y, z \in X_0$ are points and $f : x \longrightarrow y$ is a morphism in X from x to y (i.e., an edge $f \in X_1$ with vertices x, y) then we have right-composition-by-f map-like span

$$\mathrm{Map}_X(z, x) \xleftarrow{\simeq} C^R_{f,z} \longrightarrow \mathrm{Map}_X(z, y)$$

and a left-composition-by-f map-like span

$$\mathrm{Map}_X(y, z) \xleftarrow{\simeq} C^L_{f,z} \longrightarrow \mathrm{Map}_X(x, z).$$

We want to define properties of f via analogous properties of the spans $C^R_{f,z}, C^L_{f,z}$. In particular we will want to define when a morphism is a **quasi-unit** and when it is **invertible**. For this we will need to first understand how to say this in terms of spans.

Recall that from each space X to itself we have the **identity span** $X \xleftarrow{\mathrm{Id}} X \xrightarrow{\mathrm{Id}} X$. We will say that a span $X \xleftarrow{\varphi} C \xrightarrow{\psi} X$ is **unital** if it is **equivalent** to the identity span (see §§ 3.1 for the definition of equivalence). It is not hard to check that a span as above is unital if and only if both φ, ψ are weak equivalences and are **homotopic** to each other in the Kan model structure.

We will say that a span $X \xleftarrow{\varphi} C \xrightarrow{\psi} Y$ is **invertible** if it admits an **inverse**, i.e., if there exists a span $Y \xleftarrow{\varphi} D \xrightarrow{\psi} X$ such that the compositions

$$X \longleftarrow C \times_Y D \longrightarrow X$$

and

$$Y \longleftarrow D \times_X C \longrightarrow Y$$

are unital.

Remark 4.1.1. Note that if a span

$$X \stackrel{\varphi}{\longleftarrow} C \stackrel{\psi}{\longrightarrow} Y$$

is map-like (i.e., if φ is invertible) then it is equivalent to a span of the form

$$X \stackrel{\mathrm{Id}}{\longleftarrow} D \stackrel{f}{\longrightarrow} Y$$

such that f represents the class $[\psi] \circ [\varphi]^{-1}$ in the Kan homotopy category. In this case the invertibility of $X \stackrel{\varphi}{\longleftarrow} C \stackrel{\psi}{\longrightarrow} Y$ is equivalent to f being a weak equivalence, i.e., to ψ being a weak equivalence.

Now let X be a semiSegal space. Through the point of view of spans we have a natural way to define invertibility and unitality of morphisms:

Definition 4.1.2.

1. Let $x, y \in X_0$ be two objects and $f : x \longrightarrow y$ a morphism in X. We will say that f is **right-invertible** if for every $z \in X_0$ the right composition span

$$\mathrm{Map}_X(z, x) \longleftarrow C^R_{f,z} \longrightarrow \mathrm{Map}_X(z, y)$$

is invertible. Similarly one says that f is **left-invertible** if for every $z \in X_0$ the left composition span

$$\mathrm{Map}_X(y, z) \longleftarrow C^L_{f,z} \longrightarrow \mathrm{Map}_X(x, z)$$

is invertible. We say that f is **invertible** if it is both left invertible and right invertible.

2. Let $x \in X_0$ be an object and $f : x \longrightarrow x$ a morphism in X. We will say that f is a **quasi-unit** if for each $z \in X_0$ the spans

$$\mathrm{Map}_X(x, z) \longleftarrow C^R_{f,z} \longrightarrow \mathrm{Map}_X(x, z)$$

and

$$\mathrm{Map}_X(z, x) \longleftarrow C^L_{f,z} \longrightarrow \mathrm{Map}_X(z, x)$$

are **unital**.

Remark 4.1.3. From Remark 4.1.1 we see that a morphism $f : x \longrightarrow y$ in X is invertible if and only if for each $z \in X_0$ the restriction maps

$$C^R_{f,z} \longrightarrow \mathrm{Map}_X(z, y)$$

and

$$C^L_{f,z} \longrightarrow \mathrm{Map}_X(x, z)$$

are weak equivalences.

Invertible morphisms can be described informally as morphisms such that composition with them induces a weak equivalence on mapping spaces. Note that the notion of invertibility does not presupposed the existence of identity morphisms, i.e., it makes sense in the non-unital setting as well.

We will denote by

$$X_1^{\mathrm{inv}} \subseteq X_1$$

the maximal subspace spanned by the invertible vertices $f \in (X_1)_0$. Using Reedy fibrancy it is not hard to show that if $f, g \in X_1$ are connected by a path in X_1 then f is invertible if and only if g is invertible. Hence X_1^{inv} is just the **union of connected components of** X_1 which meet invertible edges.

We will denote by

$$\mathrm{Map}_X^{\mathrm{inv}}(x, y) = \{x\} \times_{X_0} X_1^{\mathrm{inv}} \times_{X_0} \{y\} \subseteq \mathrm{Map}_X(x, y)$$

the subspace of invertible morphisms from x to y.

Remark 4.1.4. Let X be a semiSegal space and $f : x \longrightarrow y$ be a morphism in X. The right-invertibility of f can be phrased without quantifying over z by saying that the fibration

$$\{\sigma \in X_2 \mid \sigma|_{\Delta^{\{1,2\}}} = f\} \stackrel{(\bullet)|_{\Delta^{\{0,2\}}}}{\longrightarrow} \{g \in X_1 \mid g|_{\Delta^{\{1\}}} = y\}$$

is a weak equivalence. Similarly, the left-invertibility of f can be phrased by saying that the fibration

$$\{\sigma \in X_2 \mid \sigma|_{\Delta^{\{0,1\}}} = f\} \stackrel{(\bullet)|_{\Delta^{\{0,2\}}}}{\longrightarrow} \{g \in X_1 \mid g|_{\Delta^{\{0\}}} = x\}$$

is a weak equivalence. One way to see it is to note that the triviality of a fibration can be detected on the fibers - if all the fibers of the above fibration are weakly contractible then all the fibers of the fibrations

$$C_{f,z}^R \longrightarrow \mathrm{Map}_X(z, y)$$

and

$$C_{f,z}^L \longrightarrow \mathrm{Map}_X(x, z)$$

are weakly contractible, and vice-versa if we quantify over z.

The notion of invertibility can be used in order to obtain a non-unital analogue of the notion of an ∞-groupoid:

Definition 4.1.5. Let X be a semiSegal space. We will say that X is a **semi-groupoid** if $X_1^{\mathrm{inv}} = X_1$.

Remark 4.1.6. In view of Remark 4.1.4 it is not hard to see that a semiSegal space X is a semi-groupoid if and only if the (fibration) maps

$$\mathrm{Map}(\Delta^2, X) \longrightarrow \mathrm{Map}(\Lambda_0^2, X)$$

and

$$\mathrm{Map}(\Delta^2, X) \longrightarrow \mathrm{Map}(\Lambda_2^2, X)$$

are weak equivalences. In particular being a semi-groupoid can be phrased as a locality condition.

The following collection of lemmas describes some basic expected features of invertible morphisms and quasi-units:

Lemma 4.1.7 (two-out-of-three)**.** *Let $\sigma : \Delta^2 \longrightarrow X$ be a triangle with two of the edges being invertible. Then the third edge is invertible as well.*

Proof. Applying Lemma 3.2.6 and Remark 4.1.1 the claim is reduced to the fact that invertible map-like spans satisfy the two-out-of-three property, which is clear. ☐

Lemma 4.1.8. *Let $f : x \longrightarrow y$ be a morphism in X such that there exist triangles of the form*

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "q"] & & x \end{tikzcd}$$

and

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "h"] \arrow[rr, "r"] & & y \end{tikzcd}$$

such that q and r are ***quasi-units****. Then f is a* ***invertible****.*

Proof. Applying Lemma 3.2.6 and Remark 4.1.1 the claim is reduced to showing that if a map-like span F has both a left inverse and a right inverse then it is invertible. Again this is quite immediate from the definition. ☐

Lemma 4.1.9. *Let $\sigma : \Delta^2 \longrightarrow X$ be a triangle of the form*

$$\begin{tikzcd} & y & \\ x \arrow[ur, "f"] \arrow[rr, "q"] & & x \arrow[ul, "f"'] \end{tikzcd}$$

such that f is ***invertible****. Then q is a* ***quasi-unit****.*

Proof. Applying Lemma 3.2.6 and Remark 4.1.1 the claim is reduced to showing that if Q, F are spans such that $Q \circ F \simeq F$ then Q is unital. Again this is easy to verify. ☐

Corollary 4.1.10. *Let X be a semiSegal space and $x \in X_0$ a point. Then x admits a quasi-unit if and only if there exists an invertible morphism with source x.*

Now let X be a semiSegal space. We will denote by $X_1^{\mathrm{qu}} \subseteq X_1$ the space of quasi-units. We have a map $d : X_1^{\mathrm{qu}} \longrightarrow X_0$ given by either d_0 or d_1 (which coincide on X_1^{qu}). If $x \in X_0$ is a point then we will denote by $X_x^{\mathrm{qu}} \subseteq X_1^{\mathrm{qu}}$ the fiber $d^{-1}(x)$, i.e., the space of quasi-units of x. It is an easy exercise to show that X_x^{qu} is a union of connected components of $\mathrm{Map}_X(x, x)$, i.e., that the property of being a quasi-unit is preserved under homotopy.

Lemma 4.1.11. *Let X be a semiSegal space and $x \in X_0$ a point. If X_x^{qu} is not empty then it is connected.*

Proof. Let $q_1, q_2 : x \longrightarrow x$ be two quasi-units. We need to show that q_1, q_2 are in the same connected component of X_x^{qu}. Since X_x^{qu} is a union of components of $\mathrm{Map}_X(x,x)$ it is enough to show that q_1, q_2 are in the same connected component of $\mathrm{Map}_X(x,x)$.

Now from the Segal condition there exists a triangle of the form

$$\begin{tikzcd} & x \arrow[dr, "q_2"] & \\ x \arrow[ur, "q_1"] \arrow[rr, "q_3"] & & x \end{tikzcd}$$

for some $q_3 : x \longrightarrow x$. Now since

$$\mathrm{Map}_X(x,x) \longleftarrow C^R_{q_2,x} \longrightarrow \mathrm{Map}_X(x,x)$$

is a unital span we get that q_1 and q_3 are in the same connected component of $\mathrm{Map}_X(x,x)$. Similarly since

$$\mathrm{Map}_X(x,x) \longleftarrow C^L_{q_1,x} \longrightarrow \mathrm{Map}_X(x,x)$$

is a unital span we get that q_2 and q_3 are in the same connected component of $\mathrm{Map}_X(x,x)$. This means that q_1, q_2 are in the same connected component of $\mathrm{Map}_X(x,x)$ and we are done. $\square$

4.2 Quasi-unital semiSegal spaces

We start with the basic definitions:

Definition 4.2.1. Let X be a semiSegal space. We will say that X is **quasi-unital** if every $x_0 \in X_0$ admits a quasi-unit from x_0 to x_0. We say that a map $f : X \longrightarrow Y$ of quasi-unital semiSegal spaces is **unital** if it maps quasi-units to quasi-units. We will denote by

$$\mathrm{QsS}$$

the topological category of quasi-unital semi-simplicial spaces and unital maps between them.

Remark 4.2.2. Let $\varphi : X \longrightarrow Y$ be a map between quasi-unital semiSegal spaces and let $x \in X_0$ a point. From Lemma 4.1.11 one sees that φ maps quasi-units of x to quasi-units of $\varphi(x)$ if and only if φ maps **at least one** quasi-unit of x to a quasi-unit of $\varphi(x)$.

Remark 4.2.3. If X is quasi-unital then the non-unital category $\mathrm{Ho}(X)$ has units and so can be considered as a (unital) category in a unique way.

Example 4.2.4. Recall the semiSegal space $\widehat{\mathrm{Cob}^n}$ defined in §§ 3.3. It is not hard to show that $\widehat{\mathrm{Cob}^n}$ is quasi-unital. Recall that the space of objects $\widehat{\mathrm{Cob}^n}_0 = \mathrm{Cob}^n_0$ consists of pairs (M, τ_0) where $\tau_0 \in \mathbb{R}$ and M is a submanifold of $V \times \{\tau_0\}$. In particular if M is a submanifold of V and $\tau_0 < \tau_1 \in \mathbb{R}$ are numbers then we can interpret M as a submanifold of both $V \times \{\tau_0\}$ and $V \times \{\tau_1\}$. Furthermore we have a subcobordism

$$M \times [\tau_0, \tau_1] \subseteq V \times [\tau_0, \tau_1]$$

from (M, τ_0) to (M, τ_1). It is not hard to show that this cobordism is an **invertible** morphism in the semiSegal space $\widehat{\mathrm{Cob}}^n$ (in the sense of definition 4.1.2). To see this, note that even though Cob^n is not Reedy fibrant the various face maps in it **are** fibrations. This means that their fibers have the "correct" homotopy types, i.e., the homotopy types they will have after passage to the Reedy fibrant model.

Now using Remark 4.1.4 one can express the invertibility of a morphism using a map between two fibers of face maps. For example, the left-invertibility of $M \times [\tau_0, \tau_1]$ is the claim that for each $\tau_2 > \tau_1$ the map

$$\{\sigma \in \mathrm{Cob}^n(V, \{\tau_0, \tau_1, \tau_2\}) \mid \sigma \cap (V \times [\tau_0, \tau_1]) = M \times [\tau_0, \tau_1]\} \longrightarrow$$

$$\{g \in \mathrm{Cob}^n (V, \{\tau_0, \tau_2\}) \mid g \cap (V \times \{\tau_0\}) = M\}$$

is a weak equivalence, which can be proved using standard techniques. This means that every object in $\widehat{\mathrm{Cob}}^n$ has an invertible morphism out of it. By Corollary 4.1.10 we see that $\widehat{\mathrm{Cob}}^n$ is **quasi-unital**.

4.2.1 Dwyer-Kan Equivalences

We will be interested in studying the category QsS after localization by a certain class of weak equivalences, called **Dwyer-Kan equivalences**. This is a direct adaptation of the notion of DK-equivalence of ∞-categories to the quasi-unital setting. We propose to model the ∞-category of (small) quasi-unital ∞-categories as the localization of QsS by DK-equivalences. Alternatively, one can think of this object as a relative topological category $(\mathrm{QsS}, \mathrm{DK})$ where DK denotes the class of DK-equivalences. We will give $(\mathrm{QsS}, \mathrm{DK})$ an equivalent model in § 6 in the form of **complete semiSegal spaces**.

We start with a slightly more general notion of fully-faithful maps:

Definition 4.2.5. Let $f : X \longrightarrow Y$ be map of semiSegal spaces. We will say that f is **fully-faithful** if for all $x, y \in X_0$ the induced map

$$\mathrm{Map}_X(x, y) \longrightarrow \mathrm{Map}_Y(f_0(x), f_0(y))$$

is a weak equivalence.

The notion of Dwyer-Kan equivalences will be obtained from the notion of fully-faithful maps by requiring the appropriate analogue of "essential surjectivity". For this let us introduce some terminology.

Definition 4.2.6. Let $x, y \in X_0$ be two points. We say that x and y are **equivalent** (denoted $x \simeq y$) if there exists an invertible morphism $f :\in X_1^{\mathrm{inv}}$ from x to y.

Lemma 4.2.7. *Let X be a quasi-unital semiSegal space. Then $\simeq$ is an equivalence relation. We will refer to the corresponding set of equivalence classes as the set of **equivalence-types** of X.*

Proof. Having quasi-units implies reflexivity. Lemma 4.1.7 then gives transitivity and for symmetry one uses the fact the if $f : x \longrightarrow y$ is an invertible morphism and $q : x \longrightarrow x$ is a quasi-unit then there exists a triangle $\sigma : \Delta^2 \longrightarrow X$ of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "q"] & & x \end{tikzcd}$$

where g is invertible by Lemma 4.1.7. $\square$

Lemma 4.2.8. *Let X be a quasi-unital semiSegal space and $x, y \in X_0$ points in the same path-component of X_0. Then $x \simeq y$.*

Proof. Let $q : x \longrightarrow x$ be a quasi-unit. Consider the fibration

$$X_1^{\mathrm{inv}} \longrightarrow X_0 \times X_0.$$

Restricting this fibration to $\{x\} \times X_0$ we get a fibration

$$F_x \longrightarrow \{x\} \times X_0$$

where

$$F_x = \{f \in X_1^{\mathrm{inv}} | d_0(f) = x\}.$$

Now let $\gamma : I \longrightarrow X_0$ be a path from x to y. Considering γ as a path in $\{x\} \times X_0$ from (x, x) to (x, y) we can lift it to a path $\widetilde{\gamma} : I \longrightarrow F_x$ starting at $q \in X^{\mathrm{inv}}$ and ending at some $f : X^{\mathrm{inv}}$ satisfying $d_0(f) = x$ and $d_1(f) = y$. Hence $x \simeq y$ and we are done. $\square$

Definition 4.2.9. Let $f : X \longrightarrow Y$ be a map between **quasi-unital** semiSegal spaces. We will say that f is a **Dwyer-Kan equivalence** (DK for short) if it is fully faithful and induces a surjective map on the set of equivalence-types.

Remark 4.2.10. A DK-equivalence $f : X \longrightarrow Y$ is automatically a unital map.

Remark 4.2.11. One can replace the surjectivity condition on equivalence types by f inducing an equivalence of homotopy categories $\mathrm{Ho}(X) \longrightarrow \mathrm{Ho}(Y)$ (see Remark 4.2.3).

Lemma 4.2.12. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

be a sequence of maps of quasi-unital semiSegal spaces. If both $g \circ f$ and $h \circ g$ are DK-equivalences then all of f, g, h are DK-equivalences.

Proof. Follows from the analogous statement for mapping spaces and homotopy categories (see Remark 4.2.11). $\square$

4.3 Quasi-unital Semi-groupoids

Let

$$\mathrm{GsS} \subseteq \mathrm{QsS}$$

be the full subcategory spanned by semi-groupoids. Then the localization of GsS by DK-equivalences can serve as a model for the ∞-category of quasi-unital semi-groupoids.

In this subsection we will study this localized category via the geometric realization functor $X \mapsto |X|$. Note that the realization functor

$$|\bullet| : \mathcal{S}^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}$$

has a right adjoint

$$\Pi : \mathcal{S} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

given by

$$\Pi(Z)_n = \mathrm{Map}_{\mathcal{S}}(|\Delta^n|, Z)$$

for $Z \in \mathcal{S}$. When Z is a Kan complex, the semi-simplicial space $\Pi(Z)$ is Reedy fibrant, and one can easily verify that it satisfies the Segal condition. In fact, in this case $\Pi(Z)$ will be a **semi-groupoid**. We will refer to $\Pi(Z)$ as the **fundamental semi-groupoid** of Z. Note that $\Pi(Z)$ naturally extends to a **simplicial space**, yielding an honest ∞-groupoid (known as the **fundamental ∞-groupoid** of Z).

Let $\mathrm{K} \subseteq \mathcal{S}$ be the full subcategory spanned by Kan complexes. For a space $Z \in \mathcal{S}$ let us denote by $\widehat{Z} \in \mathrm{K}$ the (functorial) Kan replacement of Z. The functor $\widehat{\bullet}$ is then homotopy-left adjoint to the full inclusion $\mathrm{K} \subseteq \mathcal{S}$. Furthermore, it exhibits K as the left localization of $\mathcal{S}$ with respect to weak equivalences.

From the above considerations we see that the adjoint pair

$$|\bullet| : \mathcal{S}^{\Delta_s^{\mathrm{op}}} \rightleftarrows \mathcal{S} : \Pi$$

induces a **homotopy-adjoint** pair

$$\widehat{|\bullet|} : \mathrm{GsS} \rightleftarrows \mathrm{K} : \Pi|_{\mathrm{K}}.$$

Since the category Δ_s is weakly contractible we get that the functor $\Pi_{|\mathrm{K}}$ is actually **fully-faithful**, i.e., induces an equivalence between K and the full subcategory of GsS spanned by its image. Hence we can consider $\widehat{|\bullet|}$ as a **left localization** functor. The class of morphisms by which it localizes are the morphisms which it sends to equivalences.

Remark 4.3.1. It is worthwhile to obtain a simple characterization of the essential image of the functor $\Pi : \mathrm{K} \longrightarrow \mathrm{GsS}$ (which can be called **homotopy-constant** objects). Note that for any semi-groupoid of the form $\Pi(Z)$ the two maps

$$d_0, d_1 : \Pi(Z)_1 \longrightarrow \Pi(Z)_0$$

are weak equivalences. We claim that this condition is also sufficient: if X is a semi-groupoid such that

$$d_0, d_1 : X_1 \longrightarrow X_0$$

are weak equivalences then the Segal condition implies that for every $f : [k] \longrightarrow [n]$ in Δ_s the map

$$f^* : X_n \longrightarrow X_k$$

is a weak equivalence. Since the category Δ_s is weakly contractible this will imply that the map

$$X_0 \longrightarrow |X| \simeq \operatorname{hocolim}_{\Delta_s} X$$

is a weak equivalence and hence that the unit map

$$X \longrightarrow \Pi\left(\widehat{|X|}\right)$$

is a levelwise equivalence.

In this section we will prove that this class of equivalences are exactly the DK-equivalences (Theorem 4.3.10). This means that $\widehat{|\bullet|}$ serves as a **left localization** functor with respect to DK-equivalences between quasi-unital semi-groupoids. We can frame this theorem as follows:

Theorem 4.3.2. *The ∞-category of quasi-unital semi-groupoids (i.e., the localization of* GsS *by DK-equivalences) is equivalent to* K*. The equivalence is given by sending a semi-groupoid X to the Kan replacement of its realization $\widehat{|X|}$.*

Note that this is exactly what happens in the case of **unital** ∞-groupoids (where the realization functor is sometimes referred to as **classifying space**). In particular, quasi-unital and unital ∞-groupoids have the same homotopy theory, i.e., the homotopy theory of Kan complexes. Hence we get the main conclusion of this subsection:

Theorem 4.3.3. *The forgetful functor induces an equivalence between the ∞-category of ∞-groupoids and the ∞-category of quasi-unital semi-groupoids.*

Remark 4.3.4. In the unital case one has a slightly nicer situation because the realization of a groupoid (i.e., a Segal space in which all morphisms are invertible) is automatically a Kan complex. This is not true for general semi-groupoids. However, this part is not essential in any way to the general theory.

Now before we get to Theorem 4.3.2 we need some basic terminology. We will say that X is **connected** if for each $x, y \in X_0$ one has $\operatorname{Map}_X(x, y) \neq \emptyset$. Every connected semi-groupoid is quasi-unital (see Corollary 4.1.10) and every quasi-unital semi-groupoid is a levelwise disjoint union of connected semi-groupoids (see Lemma 4.2.7). We will refer to these as the **connected components** of X. Since geometric realization commutes with coproducts we see that $\widehat{|X|}$ will be a disjoint union of the realizations of the connected components of X.

The core argument for proving Theorem 4.3.2 is a generalization of Segal's theorem (see [Seg] and [Pup]) to semi-simplicial spaces X for which X_0 is not necessarily contractible (the proof is a slight modification of the proof for this special case):

Theorem 4.3.5. *Let X be a connected semi-groupoid and let $x_0 \in X_0$ be a point. We will consider x_0 as a point in $\widehat{|X|}$ via the natural inclusion $X_0 \hookrightarrow \widehat{|X|}$. Then $\widehat{|X|}$ is a connected space and the natural map*

$$\mathrm{Map}_X(x_0, x_0) \longrightarrow \Omega\left(\widehat{|X|}, x_0\right)$$

is a weak equivalence.

Proof. We will rely on the main result of [Pup] which can be stated as follows:

Theorem 4.3.6. *Let X, Y be two semi-simplicial spaces and let $\varphi : X \longrightarrow Y$ be a map such that for each $f : [k] \longrightarrow [n]$ in Δ_s the square*

$$\begin{tikzcd} X_n \arrow[r, "f^*"] \arrow[d, "\varphi_n"] & X_k \arrow[d, "\varphi_k"] \\ Y_n \arrow[r, "f^*"] & Y_k \end{tikzcd}$$

is a homotopy pullback square. Then the square

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & {|X|} \arrow[d] \\ Y_0 \arrow[r] & {|Y|} \end{tikzcd}$$

is a homotopy pullback square as well.

Now let $x \in X_0$ be a point. Define the path semi-groupoid $P(X, x)$ as follows:

$$P(X, x)_n = \{\sigma \in X_{n+1} | d_0(\sigma) = x\} \subseteq X_{n+1}.$$

It is not hard to see that $P(X, x)$ is also a semi-groupoid. We have a natural map

$$p : P(X, x) \longrightarrow X$$

which maps $\sigma \in P(X, x)_n$ to $\sigma|_{\Delta^{\{1,...,n+1\}}} \in X_n$. Note that for each n the map

$$p_n : P(X, x)_n \longrightarrow X_n$$

is a fibration whose fiber over $\sigma \in X_n$ is homotopy equivalent to the mapping space

$$\mathrm{Map}_X(x, \sigma|_{\Delta^{\{0\}}}).$$

Furthermore, it is not hard to see that for each $f : [k] \longrightarrow [n]$ in Δ_s the square

$$\begin{tikzcd} P(X,x)_n \arrow[r, "f^*"] \arrow[d, "p_n"] & P(X,x)_k \arrow[d, "p_k"] \\ X_n \arrow[r, "f^*"] & X_k \end{tikzcd}$$

is a homotopy pullback square. Hence by Puppe's Theorem above the square

$$\begin{tikzcd} P(X,x)_0 \arrow[r] \arrow[d, "p_0"] & {|P(X,x)|} \arrow[d] \\ X_0 \arrow[r] & {|X|} \end{tikzcd}$$

is also a homotopy pullback square. Since the map $p_0 : P(X,x)_0 \longrightarrow X_0$ is a fibration we get that for each $x \in X_0$ the square above induces a weak equivalence from $p_0^{-1}(x) = \mathrm{Map}_X(x_0, x)$ to the homotopy fiber of the map

$$|P(X,x)| \longrightarrow |X|$$

over the image of x. Hence we get a homotopy fibration sequence

$$\mathrm{Map}_X(x,x) \longrightarrow |P(X,x)| \longrightarrow |X|.$$

This homotopy fibration sequence admits a map to the "geometric" path-fibration sequence

$$\begin{tikzcd} \mathrm{Map}_X(x,x) \arrow[r] \arrow[d] & {|P(X,x)|} \arrow[r] \arrow[d] & {|X|} \arrow[d] \\ \Omega\left(\widehat{|X|}, x\right) \arrow[r] & P\left(\widehat{|X|}, x\right) \arrow[r] & \widehat{|X|} \end{tikzcd}$$

where $P\left(\widehat{|X|}, x\right)$ is the (contractible) space of paths $\gamma : I \longrightarrow \widehat{|X|}$ such that $\gamma(0) = x$ and the projection map $P\left(\widehat{|X|}, x\right) \longrightarrow \widehat{|X|}$ is the evaluation $\gamma \mapsto \gamma(1)$.

Since X is a connected semi-groupoid we get that the space $|X|$ is connected. Hence in order to show that the natural map

$$\mathrm{Map}_X(x,x) \longrightarrow \Omega\left(\widehat{|X|}, x\right)$$

is a weak equivalence it is enough to show that the space $|P(X,x)|$ is contractible.

Lemma 4.3.7. *The space $|P(X,x)|$ is contractible.*

Proof. The face maps $d_1, ..., d_{n+1} : P(X, x)_n \longrightarrow P(X, x)_0$ induce maps

$$P(X, x)_n \longrightarrow (P(X, x)_0)^{n+1}$$

which fit together to form a map of semi-simplicial spaces

$$P(X, x) \longrightarrow \mathrm{cosk}^s_0(P(X, x)_0)$$

where $\mathrm{cosk}^s_0 : \mathcal{S} \longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}_s}$ is the right adjoint to the functor $X_\bullet \mapsto X_0$. Unwinding the definition of $P(X, x)$ and using the fact that X is a semi-groupoid we see that this map is actually a levelwise weak equivalence. Note that the realization of a semi-simplicial space coincides with its homotopy colimit and so is preserved by levelwise equivalences. Hence it is enough to show that

$$|\mathrm{cosk}^s_0(P(X, x)_0)|$$

is contractible. But this is true because every semi-simplicial space of the form $\mathrm{cosk}^s_0(Z)$ for $Z \neq \emptyset$ admits a canonical semi-simplicial null-homotopy $\Delta^1 \otimes \mathrm{cosk}^s_0(Z) \longrightarrow \mathrm{cosk}^s_0(Z)$. $\square$

This finishes the proof of Theorem 4.3.5. $\square$

Corollary 4.3.8. *Let X be a semi-groupoid. Then the counit map*

$$X \longrightarrow \Pi\left(\widehat{|X|}\right)$$

is a DK-equivalence.

Proof. By Theorem 4.3.5 the counit map is fully-faithful. Since the map $X_0 \longrightarrow \widehat{|X|}$ is surjective on connected components we see that the map f is in fact a DK-equivalence. $\square$

Corollary 4.3.9. *Let X be a quasi-unital semi-groupoid. Then X is DK-equivalent to the underlying semi-groupoid of an ∞-groupoid.*

We are now ready to prove the main result of this subsection:

Theorem 4.3.10. *Let $f : X \longrightarrow Y$ be a map between quasi-unital semi-groupoids. Then f is a DK-equivalence if and only if the induced map*

$$\widehat{|X|} \longrightarrow \widehat{|Y|}$$

is a weak equivalence.

Proof. First note that the connected components of X as a semi-groupoid are in bijection with the connected components of $\widehat{|X|}$ as a space. Furthermore, any DK-equivalence induces an isomorphism on the set of connected components. Hence it is enough to prove the lemma for the case where both X and Y are connected and non-empty.

In this case every map is surjective on equivalence types, so we get that a map is a DK-equivalence if and only if it induces an equivalence on mapping spaces. Furthermore, since X and Y are non-empty semi-groupoids it is enough to choose just one point $x \in X_0$ and check whether f induces a weak equivalence

$$f_* : \mathrm{Map}_X(x, x) \longrightarrow \mathrm{Map}_Y(y, y),$$

where $y = f_0(x)$.

Let $P(X, x), P(Y, y)$ be the path semi-groupoids as in the proof of Theorem 4.3.5. Then we have a commutative diagram of spaces

$$\begin{tikzcd} \mathrm{Map}_X(x,x) \arrow[r] \arrow[d] & {|P(X,x)|} \arrow[r] \arrow[d] & {|X|} \arrow[d] \\ \mathrm{Map}_Y(y,y) \arrow[r] & {|P(Y,y)|} \arrow[r] & {|Y|} \end{tikzcd}$$

Now the middle vertical map is a weak equivalence because both spaces are contractible. Since both $|X|, |Y|$ are connected we get that the map

$$|X| \longrightarrow |Y|$$

is a weak equivalence if and only if the map

$$\mathrm{Map}_X(x, x) \longrightarrow \mathrm{Map}_Y(y, y)$$

is a weak equivalence. Now the map $|X| \longrightarrow |Y|$ is a weak equivalence if, and only if, the map $\widehat{|X|} \longrightarrow \widehat{|Y|}$ is a weak equivalence, and hence the required result follows.

□

We finish this subsection with an application which we record for future use. Recall that in general geometric realization does not commute with **Cartesian products** of semi-simplicial spaces (i.e., levelwise products). The following corollary shows that in the specific case of semi-groupoids, geometric realization **does** commute with Cartesian products:

Corollary 4.3.11. *Let X, Y be two quasi-unital semi-groupoids. Then the natural map*

$$|X \times Y| \longrightarrow |X| \times |Y|$$

is a weak equivalence.

Proof. First note that if X, Y are semi-groupoids then $X \times Y$ is a semi-groupoid as well. Furthermore we can assume without loss of generality that both X and Y are connected and non-empty, in which case $X \times Y$ is connected and non-empty as well. Let $x \in X_0$ and $y \in Y_0$ be base points. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{X\times Y}((x,y),(x,y)) \arrow[r, "\cong"] \arrow[d] & \mathrm{Map}_X(x,x)\times \mathrm{Map}_Y(y,y) \arrow[d] \\
\Omega_{(x,y)}\left(\widehat{|X\times Y|}\right) \arrow[r] & \Omega_x\left(\widehat{|X|}\right)\times \Omega_y\left(\widehat{|Y|}\right)
\end{tikzcd}$$

in which the upper horizontal map is an isomorphism and the two vertical maps are weak equivalences by Theorem 4.3.5. This means that the lower horizontal map is a weak equivalence.

Composing the horizontal map with the natural isomorphism

$$\Omega_x\left(\widehat{|X|}\right)\times \Omega_y\left(\widehat{|Y|}\right) \cong \Omega_{(x,y)}\left(\widehat{|X|}\times\widehat{|Y|}\right)$$

and using the fact that $\widehat{|X|}$ and $\widehat{|Y|}$ are connected and non-empty we get that the map

$$\widehat{|X\times Y|} \longrightarrow \widehat{|X|}\times\widehat{|Y|}$$

is a weak equivalence. This implies that the map

$$|X\times Y| \longrightarrow |X|\times|Y|$$

is a weak equivalence as desired. $\square$

5 Quasi-unital semiSegal Spaces in the Marked Setting

In this section we will study quasi-unital semiSegal spaces in the setting of **marked semi-simplicial spaces**. We begin with the following basic observation:

Proposition 5.0.12. *Let $\varphi : X \longrightarrow Y$ be a map between quasi-unital semiSegal spaces. The following are equivalent:*

1. *φ is unital.*
2. *φ sends invertible edges to invertible edges.*

Proof. First assume that φ sends invertible edges to invertible edges and let $x \in X_0$ a point. Since X is quasi-unital there exists a quasi-unit $q : x \longrightarrow x$. Since every quasi-unit is invertible there exists a triangle of the form

$$\begin{tikzcd} & x & \\ x \arrow[ur, "q"] \arrow[rr, "q'"] & & x \arrow[ul, "q"'] \end{tikzcd}$$

By Lemma 4.1.9 we get that q' is a quasi-unit. The map φ then sends this triangle to a triangle of the form

$$\begin{tikzcd} & \varphi(x) & \\ \varphi(x) \arrow[ur, "\varphi(q)"] \arrow[rr, "\varphi(q')"] & & \varphi(x) \arrow[ul, "\varphi(q)"'] \end{tikzcd}$$

By Lemma 4.1.9 we get that $\varphi(q')$ is a quasi-unit. Hence by Remark 4.2.2 we get that φ maps quasi-units of x to quasi-units of $\varphi(x)$.

Now assume that φ is unital and let $f : x \longrightarrow y$ be an invertible edge. Since X is quasi-unital there exist quasi-unit $q : x \longrightarrow x$ and $r : y \longrightarrow y$. Since f is invertible there exist triangles of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "q"] & & x \end{tikzcd}$$

and

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "h"] \arrow[rr, "r"] & & y \end{tikzcd}$$

Applying φ to these triangles and using the fact that $\varphi(q), \varphi(r)$ are quasi-units we get from Lemma 4.1.8 that $\varphi(f)$ is invertible. This finishes the proof of Proposition 5.0.12.

$\square$

Remark 5.0.13. By a similar argument one can show that the two equivalent conditions of Proposition 5.0.12 are equivalent to the seemingly weaker condition of φ sending quasi-units to invertible edges.

Now from Lemma 5.0.12 we get that we can think of unital maps alternatively as maps which **preserve equivalences**. This observation can be used in order to describe unital maps in terms of **marked maps**. For this we will need the following definition:

Definition 5.0.14. Let X be a quasi-unital semiSegal space. We will denote by

$$X^{\natural} = \left(X, X_1^{\mathrm{inv}}\right)$$

the marked semi-simplicial space having X as its underlying semi-simplicial space such that the marked edges are exactly the equivalences.

Now let $f : X \longrightarrow Y$ be a unital map between two quasi-unital semiSegal spaces. Then we get that f induces a **marked map**

$$f^{\natural} : X^{\natural}_{\bullet} \longrightarrow Y^{\natural}_{\bullet}.$$

Furthermore, the entire space of unital maps from X to Y can be identified with the space of marked maps

$$\mathrm{Map}^{+}\left(X^{\natural}, Y^{\natural}\right).$$

This means that the association $X \mapsto X^{\natural}$ identifies the topological category QsS with a **full subcategory**

$$\mathrm{QsS} \subseteq \mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}.$$

Definition 5.0.15. Let W be a marked semi-simplicial space. We will say that W is **quasi-unital** if it belongs to the essential image of QsS, i.e., if there exists a quasi-unital semiSegal space X such that

$$W \cong X^{\natural}.$$

In other words, a marked semiSegal space is quasi-unital if the underlying semiSegal space is quasi-unital and in addition all invertible edges are marked.

Definition 5.0.16. We will say that a map $f : W \longrightarrow Z$ of quasi-unital marked semiSegal spaces is a **DK-equivalence** if the corresponding map of quasi-unital semiSegal spaces is a DK-equivalence.

The purpose of this section is to translate the embedding $\mathrm{QsS} \subseteq \mathcal{S}_{+}^{\Delta_s^{\mathrm{op}}}$ into useful tools for manipulating quasi-unital semiSegal spaces. Our first order of business is to construct a good notion of **marked semiSegal spaces**. This will be achieved in §§ 5.1 where we will **localize** the marked model structure on $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ in an analogous way to the Segal localization of the Reedy model structure on $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$. The fibrant objects in this model structure will be called **marked semiSegal spaces**. We will then show that this localization is **compatible** with the marked monoidal product $\otimes$. This will give us a good notion of internal mapping objects between marked semiSegal spaces.

In §§ 5.2 we will study the notion of fully-faithful maps in the context of marked semiSegal spaces. We will apply the results of this subsection in §§ 5.3 where we will show that the mapping object between two quasi-unital marked semiSegal spaces is again quasi-unital. This will show that QsS has a good notion of internal mapping objects.

In §§ 5.4 we will study a dual notion to DK-equivalences in the context of marked semi-simplicial spaces, namely that of **DK-anodyne** maps. The main result shows that a certain interesting family of maps are DK-anodyne. Finally in §§ 5.5 we will employ the various tools developed so far in order to define and study a notion of **categorical equivalence** between quasi-unital semiSegal spaces. The results of these last two subsections will be essential in § 6 in order to construct the localization of QsS with respect to DK-equivalences.

5.1 Marked semiSegal Spaces

Recall the **marked model structure** on $\mathcal{S}^{\Delta_+^{\mathrm{op}}}$ described in § 2.2. In this section we will localize this model structure in order to study semiSegal spaces in a marked setting. We start with the basic definitions:

Definition 5.1.1. Let $(W, M) \in \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ be a marked-fibrant object. We will say that (W, M) is a **marked semiSegal space** if the following conditions are satisfied:

1. W is a semiSegal space.
2. Every marked edge of (W, M) is invertible, i.e., $M \subseteq W_1^{\mathrm{inv}}$.
3. M is closed under 2-out-of-3, i.e., if there exists a triangle $\sigma \in W_2$ with two marked edges then the third is marked as well.

Example 5.1.2. Let W be a quasi-unital semiSegal space. Then $W^\natural$ is a marked semiSegal space. In particular we can consider QsS as a full subcategory of the (topological) subcategory spanned by the marked semiSegal spaces.

Remark 5.1.3. In light of Remark 4.1.10 we see that a marked semiSegal space W is quasi-unital if and only if every object $w_0 \in W_0$ has a **marked edge** out of it and all the invertible edges are marked.

Definition 5.1.4. We will say that W is a **marked semi-groupoid** if W is a marked semiSegal space in which all edges are marked. Note that in this case the underlying semiSegal space will be a semi-groupoid.

As for the non-marked case, the property of W being a marked semiSegal space can be described in terms of locality with respect to a certain set of maps. In order to describe this conveniently we will need a bit of terminology.

We will use the phrase **marked horn inclusion** to describe an inclusion of marked semi-simplicial sets of the form

$$(\Lambda^n_i, A) \subseteq (\Delta^n, B)$$

such that $A = B \cap (\Lambda^n_i)_1$. We will be interested in the following kind of marked horn inclusions:

Definition 5.1.5. We will say that a marked horn inclusion

$$(\Lambda^n_i, A) \subseteq (\Delta^n, B)$$

is **admissible** if $B = A$ and in addition **one** of the following three conditions is satisfied:

1. $0 < i < n$ and $A = \emptyset$.
2. $i = 0$ and $A = \left\{\Delta^{\{0,1\}}\right\}$.
3. $i = n$ and $A = \left\{\Delta^{\{n-1,n\}}\right\}$.

The role of admissible marked horn inclusions in the theory of marked semiSegal spaces is explained by the following proposition:

Proposition 5.1.6. *Let $(W, M) \in \mathcal{S}_+^{\Delta_s^{op}}$ be a marked-fibrant object. Then (W, M) satisfies properties $(1), (2)$ of Definition 5.1.1 if and only if (W, M) is **local** with respect to all admissible marked horn inclusions. More explicitly, if for all admissible marked horn inclusions*

$$(\Lambda^n_i, A) \subseteq (\Delta^n, A)$$

the restriction map

$$\mathrm{Map}^+((\Delta^n, A), (W, M)) \longrightarrow \mathrm{Map}^+((\Lambda^n_i, A), (W, M))$$

is a weak equivalence.

Proof. First assume that (W, M) is local with respect to all admissible marked horn inclusions. This includes in particular the case of **inner** horn inclusions

$$(\Lambda^n_l)^\flat \subseteq (\Delta^n)^\flat$$

which implies that W is a semiSegal space. Next since (W, M) is local with respect to

$$\left(\Lambda^2_0, \left\{\Delta^{\{0,1\}}\right\}\right) \hookrightarrow \left(\Delta^2, \left\{\Delta^{\{0,1\}}\right\}\right)$$

and

$$\left(\Lambda^2_2, \left\{\Delta^{\{1,2\}}\right\}\right) \hookrightarrow \left(\Delta^2, \left\{\Delta^{\{1,2\}}\right\}\right)$$

we get from Remark 4.1.3 that $M \subseteq W_1^{\mathrm{inv}}$.

Now assume that (W, M) is a marked semiSegal space and let

$$f : (\Lambda^n_i, A) \subseteq (\Delta^n, A)$$

be an admissible marked horn inclusion. If $0 < i < n$ or if $n = 2$ then (W, M) is local with respect to f by the same considerations as above. Hence we can assume that $n > 2$ and $i \in \{0, n\}$. We will prove the case $i = n$ and leave the analogous $i = 0$ case to the reader.

Consider the **spine** $\mathrm{Sp}_n \subseteq \Lambda^n_n$. Note that $A = \left\{\Delta^{\{n-1,n\}}\right\}$ is contained in $(\mathrm{Sp}_n)_1$ and that the restriction map

$$\mathrm{Map}^+((\Delta^n, A), (W, M)) \longrightarrow \mathrm{Map}^+((\mathrm{Sp}_n, A), (W, M)) \cong W_1 \times_{W_0} ... \times_{W_0} W_1 \times_{W_0} M$$

is a weak equivalence. The desired claim will then follow by induction from the following Lemma:

Lemma 5.1.7. *Let $n \geq 3$. Then the marked semi-simplicial set (Λ^n_n, A) can be obtained from (Sp_n, A) by successively performing pushouts along admissible marked horn inclusions of dimension $< n$.*

Proof. Let $I = \{0, ..., n\} \setminus \{n-1\}$ and consider the sub marked semi-simplicial set

$$X_1 = \left(\Delta^I\right)^\flat \coprod_{\Delta^{\{n\}}} \left(\Delta^{\{n-1,n\}}\right)^\sharp \subseteq (\Delta^n, A)\,.$$

Since $|I| \geq 3$ we get from Lemma 3.1.5 that X_1 can be obtained from (Sp_n, A) by a sequence of pushouts along admissible **inner** horn inclusions.

We will say that a simplex σ in Δ^n is **good** if it contains the edge $\Delta^{\{n-1,n\}} \subseteq \Delta^n$. Now for $j = 2, ..., n-1$ define X_j to be the union of X_{j-1} and all good j-simplices. One then easily verifies that X_j can be obtained from X_{j-1} be a sequence of pushouts along admissible marked horn inclusions of the form

$$\left(\Lambda^j_j, \left\{\Delta^{\{j-1,j\}}\right\}\right) \subseteq \left(\Delta^j, \left\{\Delta^{\{j-1,j\}}\right\}\right)$$

and that $X_{n-1} = (\Lambda^n_n, A)$. $\square$

This finishes the proof of Proposition 5.1.6. $\square$

Now let $W \in \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ be a marked-fibrant object. In light of Lemma 5.1.6 we see that W will be a semiSegal space if and only if W is local with respect to the set $\mathbf{S}$ defined as follows:

Definition 5.1.8. Let $\mathbf{S}$ be the set which contains:

1. All admissible marked horn inclusions.

2. All the maps of the form

$$\left(\Delta^2, A\right) \hookrightarrow \left(\Delta^2\right)^\sharp$$

   where $A \subseteq \left(\Delta^2\right)_1$ a set of size 2.

Since the marked model structure is combinatorial and left proper the left Bousfield localization of $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ with respect to $\mathbf{S}$ exists. In particular, there exists a simplicial (combinatorial, left proper) model category Seg_+ whose underlying simplicial category is $\mathcal{S}^{\Delta_s^{\mathrm{op}}}$ such that

1. Cofibrations in Seg_+ are the cofibrations of the marked model structure (i.e., levelwise injective maps).
2. Weak equivalences in Seg_+ are maps $f : X \longrightarrow Y$ such that for every marked semiSegal space W the induced map

$$\mathrm{Map}^+(Y, W) \longrightarrow \mathrm{Map}(X, W)$$

   is a weak equivalence.

Definition 5.1.9. We will denote by **MS-equivalences**, **MS-fibrations** and **MS-cofibrations** the weak equivalences, fibrations and cofibrations in Seg_+ respectively (to avoid confusion compare to the terminology in Definition 2.2.10).

Remark 5.1.10. Let X, Y be semi-simplicial spaces and $f : X \longrightarrow Y$ a trivial Segal cofibration. Then by definition we get that

$$f^\flat : X^\flat \longrightarrow Y^\flat$$

will be a trivial MS-cofibration. In particular we see that the adjunction

$$\mathrm{Seg} \underset{\overline{\bullet}}{\overset{(\bullet)^\flat}{\rightleftarrows}} \mathrm{Seg}_+$$

is a Quillen adjunction

The following kind of trivial MS-cofibrations will be useful to note:

Definition 5.1.11. Let X be a marked semi-simplicial space and $B \subseteq C \subseteq X_1$ two subspaces. We will say that the map

$$(X, B) \hookrightarrow (X, C)$$

is a **triangle remarking** if (X, C) can be obtained from (X, B) by a sequence of pushouts along maps of the form

$$K \otimes \left(\Delta^2, A\right) \hookrightarrow K \otimes \left(\Delta^2\right)^\sharp$$

for $K \in \mathcal{S}$ and $|A| = 2$. Note that any triangle remarking is a trivial MS-cofibration.

This notion is exemplified in the following lemma:

Lemma 5.1.12. *For every $i = 0, ..., n$ the map*

$$(\Lambda_i^n)^\sharp \longrightarrow (\Delta^n)^\sharp$$

is a trivial MS-cofibration.

Proof. Let $M \subseteq (\Delta^n)_1$ be the set of edges that are contained in Λ^n_i. Then (Δ^n, M) is obtained from $(\Lambda^n_i)^\sharp$ by performing a pushout along an admissible marked horn inclusion. The desired result now follows from the fact that the map

$$(\Delta^n, M) \hookrightarrow (\Delta^n)^\sharp$$

is a triangle remarking. $\square$

Corollary 5.1.13. *If W is a marked semiSegal space then $\widetilde{W}$ (see Definition 2.2.18) is a marked semiSegal space as well.*

Proof. First of all it is clear that $\widetilde{W}$ is marked-fibrant (see Lemma 2.2.12). From Lemma 5.1.12 it follows that $\widetilde{W}$ is local with respect to all admissible marked horn inclusions and so by Proposition 5.1.6 $\widetilde{W}$ satisfies properties (1) and (2) of Definition 5.1.1. Since clearly the marked edges in $\widetilde{W}$ are closed under 2-out-of-3 we get that $\widetilde{W}$ is a marked semiSegal space. $\square$

Remark 5.1.14. Since all the edges in $\widetilde{W}$ are marked we see that $\widetilde{W}$ is a **marked semi-groupoid**. From Remark 5.1.3 we get that if W is quasi-unital then $\widetilde{W}$ will be quasi-unital as well. Furthermore in this case the inclusion $\widetilde{W} \subseteq W$ will identify $\widetilde{W}$ with the **maximal semi-groupoid** of W.

Now recall that $\mathcal{S}_+^{\Delta_s^{op}}$ is a monoidal model category with respect to the marked monoidal product $\otimes$ (see §§ 2.2.2). We would like to show that this monoidality survives the localization:

Theorem 5.1.15. *The marked Segal model structure is compatible with the marked monoidal product $\otimes$.*

Proof. Arguing as in Proposition 3.4.3 we see that it is enough to show that if X is a marked semi-simplicial space and $f : Y \longrightarrow Z$ is a map in $\mathbf{S}$ then the induced map

$$X \otimes Y \longrightarrow X \otimes Z$$

is a trivial SM-cofibration. Since X can be obtained as a sequential colimit of inclusions of finite dimensional marked semi-simplicial spaces it is enough to prove for finite dimensional X. Now any finite-dimensional semi-simplicial space can be built up from $\emptyset$ by successively performing pushouts along maps of the form

$$K \otimes (\partial\Delta^m)^\flat \hookrightarrow K \otimes (\Delta^m)^\flat$$

and

$$K \otimes \left(\Delta^1\right)^\flat \hookrightarrow K \otimes \left(\Delta^1\right)^\sharp$$

for $K \in \mathcal{S}$. This means that for each injective map $f : Y \longrightarrow Z$ the induced map

$$X \otimes Y \longrightarrow X \otimes Z$$

can be obtained as a composition of pushouts along maps of the form

$$\left[K \otimes (\partial\Delta^m)^\flat \otimes Z\right] \coprod_{K\otimes(\partial\Delta^m)^\flat\otimes Y} \left[K \otimes (\Delta^m)^\flat \otimes Y\right] \hookrightarrow K \otimes (\Delta^m)^\flat \otimes Z$$

and

$$\left[K \otimes \left(\Delta^1\right)^\flat \otimes Z\right] \coprod_{K\otimes(\Delta^1)^\flat\otimes Y} \left[K \otimes \left(\Delta^1\right)^\sharp \otimes Y\right] \hookrightarrow K \otimes \left(\Delta^1\right)^\sharp \otimes Z.$$

Since Seg_+ is simplicial it will be enough to prove that for every $f : Y \longrightarrow Z$ in **S** the inclusions

$$\left[(\partial\Delta^m)^\flat \otimes Z\right] \coprod_{(\partial\Delta^m)^\flat\otimes Y} \left[(\Delta^m)^\flat \otimes Y\right] \hookrightarrow (\Delta^m)^\flat \otimes Z$$

and

$$\left[\left(\Delta^1\right)^\flat \otimes Z\right] \coprod_{(\Delta^1)^\flat\otimes Y} \left[\left(\Delta^1\right)^\sharp \otimes Y\right] \hookrightarrow \left(\Delta^1\right)^\sharp \otimes Z$$

are trivial MS-cofibrations.

We begin by observing that for a pair of inclusions of the form

$$f : (X, A) \hookrightarrow (Y, A)$$

$$g : (Z, B) \hookrightarrow (Z, C)$$

such that $f_0 : X_0 \longrightarrow Y_0$ is surjective the induced map

$$[(Z, C) \otimes (X, A)] \coprod_{(Z,B)\otimes(X,A)} [(Z, B) \otimes (Y, A)] \longrightarrow (Z, C) \otimes (Y, A)$$

is an **isomorphism** of marked semi-simplicial spaces and in particular a trivial MS-cofibration. Hence we just need to prove that the following cases are trivial MS-cofibrations:

1. The maps of the form

   $$\left[\left(\Delta^1\right)^\sharp \otimes \left(\Delta^2, A\right)\right] \coprod_{(\Delta^1)^\flat\otimes(\Delta^2,A)} \left[\left(\Delta^1\right)^\flat \otimes \left(\Delta^2\right)^\sharp\right] \longrightarrow \left(\Delta^1\right)^\sharp \otimes \left(\Delta^2\right)^\sharp = \left(\Delta^1 \otimes \Delta^2\right)^\sharp,$$

   where $|A| = 2$.

2. The maps of the form

   $$(\partial\Delta^m)^\flat \otimes (\Delta^n, A) \coprod_{(\partial\Delta^m)^\flat\otimes\left(\Lambda^n_l, A\right)} (\Delta^m)^\flat \otimes (\Lambda^n_l, A) \hookrightarrow (\Delta^m)^\flat \otimes (\Delta^n, A),$$

   where $(\Lambda^n_l, A) \hookrightarrow (\Delta^n, A)$ is an admissible marked horn inclusion.

As for case (1) note that this map induces an isomorphism on the underlying semi-simplicial sets. Furthermore, the marking on the left hand side contains all edges except exactly **one** edge $e \in \left(\Delta^1 \otimes \Delta^2\right)_1$.

Note that each triangle in $\Delta^1 \otimes \Delta^2$ has three distinct edges. Furthermore every edge in $\Delta^1 \otimes \Delta^2$ lies on some triangle. Hence one can find a triangle which lies on e such that its other two edges are not e. This means that there exists a pushout diagram of marked semi-simplicial sets of the form

$$\begin{tikzcd}
(\Delta^2, A) \arrow[r] \arrow[d] & (\Delta^2)^\sharp \arrow[d] \\
\left[(\Delta^1)^\sharp \otimes (\Delta^2, A)\right] \coprod_{(\Delta^1)^\flat \otimes (\Delta^2, A)} \left[(\Delta^1)^\flat \otimes (\Delta^2)^\sharp\right] \arrow[r] & (\Delta^1 \otimes \Delta^2)^\sharp
\end{tikzcd}$$

Since the upper horizontal row is a trivial MS-cofibration we get that the lower horizontal map is an MS-cofibration as well.

We shall now prove case (2) which is basically a marked generalization of Lemma 3.4.6:

Lemma 5.1.16. *Let $(\Lambda^n_l, A) \hookrightarrow (\Delta^n, A)$ be an admissible marked horn inclusion. Then the marked semi-simplicial set $(\Delta^m)^\flat \otimes (\Delta^n, A)$ can be obtained from the marked semi-simplicial set*

$$X = (\partial\Delta^m)^\flat \otimes (\Delta^n, A) \coprod_{(\partial\Delta^m)^\flat \otimes (\Lambda^n_l, A)} (\Delta^m)^\flat \otimes (\Lambda^n_l, A)$$

by successively performing pushouts along admissible marked horn inclusions. In particular, the inclusion

$$X \subseteq (\Delta^m)^\flat \otimes (\Delta^n, A)$$

is a trivial MS-cofibration.

Proof. If $m = 0$ then the claim is immediate, so we can assume $m > 0$. In this case the marking of $(\Delta^m)^\flat \otimes (\Delta^n, A)$ is the same as the marking of X, so that we don't need to worry about adding marked edges in the course of performing the desired pushouts. Now if $0 < l < n$ then by Lemma 3.4.6 we get that $(\Delta^m)^\flat \otimes (\Delta^n, A)$ can be obtained from X by performing pushouts along admissible inner horn inclusions of the form

$$\left(\Lambda^k_l\right)^\flat \subseteq \left(\Delta^k\right)^\flat .$$

Hence we can assume that $l = 0, n$. Observing the symmetry between the $l = 0$ and $l = n$ we see that it will be enough to prove for the case $l = n$.

In this case we get from Lemma 3.4.6 that $\Delta^m \otimes \Delta^n$ can be obtained from the underlying semi-simplicial set of X by a sequence of **right** horn inclusions (see Definition 3.4.5).

Whenever the the right horn inclusion is inner one can make it into an admissible one by applying the functor $(\bullet)^\flat$. What is left is to verify is that whenever we used a non-inner horn inclusion

$$\Lambda^k_k \subseteq \Delta^k$$

then the $\{k, k-1\}$-edge of Λ^k_k was mapped to a **marked** edge in $(\Delta^m)^\flat \otimes (\Delta^n, A)$. For this we will need to recall the **proof** of Lemma 3.4.6.

Recall the k-simplices of $\Delta^m \otimes \Delta^n$ are in bijection with injective order preserving maps

$$\sigma = (f, g) : [k] \longrightarrow [m] \times [n].$$

We say that a k-simplex of $\Delta^m \otimes \Delta^n$ is **full** if it is **not** contained in X. Given a full simplex σ we say that σ is **special** if $g^{-1}(l) \neq \emptyset$ and

$$f\left(\min g^{-1}(l)\right) = f\left(\max g^{-1}(l-1)\right).$$

In particular if $\sigma = (f, g)$ is special then f, g are **surjective**.

Now in the proof of Lemma 3.4.6 we showed that one can add the special simplices of $\Delta^m \otimes \Delta^n$ to X in a specific order such that when we come to add the special k-simplex σ we have already added all its faces except the face opposite the vertex $\min g^{-1}(l)$. Since g is surjective we get that

$$0 < \min g^{-1}(l) \leq k$$

and so this results in a pushouts along a right horn inclusion. The only case where this right horn inclusion is not inner is when $g^{-1}(l) = k$. By the definition of special we then have

$$f(k) = f(k-1)$$

and so the $\{k-1, k\}$-edge of σ is mapped to a **marked** edge in $(\Delta^m)^\flat \otimes (\Delta^n, A)$. This means that indeed the addition of σ can be done by a pushout along an **admissible** horn inclusion. $\square$

This finishes the proof of Theorem 5.1.15. $\square$

Corollary 5.1.17. *Let W be a marked semiSegal space and X a marked semi-simplicial space. Then W^X is a marked semiSegal space and $\widetilde{W^X}$ is a marked semi-groupoid.*

5.2 Fully-Faithful Maps

The purpose of this section is to study the notion of **fully-faithful maps** in the setting of marked semiSegal spaces. The main result is a characterization of fully-faithful marked fibrations in terms of a certain right lifting property (Proposition 5.2.8) and various corollaries of this characterization which will be used in the following subsections.

We begin with the basic definition:

Definition 5.2.1. Let $f : (W, M) \longrightarrow (Z, N)$ be a map of marked semiSegal spaces. We will say that f is **fully-faithful** if the induced map $W \longrightarrow Z$ is fully-faithful (see Definition 4.2.5) and in addition $M = f^{-1}(N)$. For reasons of completeness we will always consider a map of the form $\emptyset \longrightarrow Z$ as fully-faithful (where $\emptyset$ is the levelwise empty marked semiSegal space).

Remark 5.2.2. Let $f : (W, M) \longrightarrow (Z, N)$ be a map of marked semiSegal spaces. The condition that f is fully-faithful is equivalent to the condition that the diagrams

$$\begin{tikzcd} W_1 \arrow[r] \arrow[d] & Z_1 \arrow[d] \\ W_0 \times W_0 \arrow[r] & Z_0 \times Z_0 \end{tikzcd}$$

and

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ W_0 \times W_0 \arrow[r] & Z_0 \times Z_0 \end{tikzcd}$$

are both homotopy pullback diagrams (recall that in marked semiSegal spaces the space of marked edges is always a union of components of the space of edges). Since the maps $Z_1 \longrightarrow Z_0 \times Z_0$ and $N \longrightarrow Z_0 \times Z_0$ are Kan fibrations this condition is equivalent to

$$W_1 \longrightarrow Z_1 \times_{Z_0 \times Z_0} (W_0 \times W_0)$$

and

$$M \longrightarrow N \times_{Z_0 \times Z_0} (W_0 \times W_0)$$

being weak equivalences.

Example 5.2.3. Consider the functor $\mathrm{tr}_0 : \mathcal{S}_+^{\Delta_s^{\mathrm{op}}} \longrightarrow \mathcal{S}$ given by $\mathrm{tr}_0(W) = W_0$. This functor admits a right adjoint

$$\mathrm{cosk}_0^+ : \mathcal{S} \longrightarrow \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

given by

$$\left(\mathrm{cosk}_0^+(X)\right)_n = X^{n+1}$$

and such that **all** the edges in $\left(\mathrm{cosk}_0^+(X)\right)_1$ are marked. Then one sees that the mapping space between any two objects in $\mathrm{cosk}_0^+(X)$ is **contractible**. In particular, if $f : X \longrightarrow Y$ is any map of spaces then the induced map

$$\mathrm{cosk}_0^+(X) \longrightarrow \mathrm{cosk}_0^+(Y)$$

is fully-faithful.

Example 5.2.3 seems to be quite a degenerate case. However, in some sense it is the **universal case**. More precisely, any fully-faithful map is obtained (up to a marked equivalence) by pulling back a map of this form. First note that pulling back a fully-faithful map along a marked fibration (see Definition 2.2.10) always results in a fully-faithful map:

Lemma 5.2.4. *Let*

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "f'"] & W \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a pullback diagram such that f is fully-faithful and g is a marked fibration. Then f' is fully-faithful.

Proof. First we need to show that the induced map

$$X_1 \longrightarrow (X_0 \times X_0) \times_{(Y_0 \times Y_0)} Y_1$$

is a weak equivalence. Since X is given by the fiber product of W and Y over Z we can write this map as

$$W_1 \times_{Z_1} Y_1 \longrightarrow (W_0 \times W_0) \times_{(Z_0 \times Z_0)} (Y_0 \times Y_0) \times_{(Y_0 \times Y_0)} Y_1 = (W_0 \times W_0) \times_{(Z_0 \times Z_0)} Y_1 =$$

$$(W_0 \times W_0) \times_{(Z_0 \times Z_0)} Z_1 \times_{Z_1} Y_1.$$

Since the map $Z \longrightarrow W$ is fully-faithful the map

$$W_1 \longrightarrow (W_0 \times W_0) \times_{(Z_0 \times Z_0)} Z_1$$

is a weak equivalence. Since the map $Y \longrightarrow Z$ is marked fibration we get that the map $Y_1 \longrightarrow Z_1$ is Kan fibration which means that the map

$$W_1 \times_{Z_1} Y_1 \longrightarrow (W_0 \times W_0) \times_{Z_0 \times Z_0} Z_1 \times_{Z_1} Y_1$$

is a weak equivalence as desired. The condition on the marking follows from the same consideration by replacing X_1, Y_1, W_1 and Z_1 by the respective spaces of marked edges.

$\square$

Definition 5.2.5. Let Y be a marked semiSegal space and $X_0 \in \mathcal{S}$ a space equipped with a map $f_0 : X_0 \longrightarrow Y_0$.

We will denote by $P(f_0, Y)$ the pullback in the square

$$\begin{tikzcd} P(f_0, Y) \arrow[r] \arrow[d] & \operatorname{cosk}_0^+(X_0) \arrow[d] \\ Y \arrow[r] & \operatorname{cosk}_0^+(Y_0) \end{tikzcd}$$

We will denote by

$$p_{f_0} : P(f_0, Y) \longrightarrow Y$$

the canonical projection. In view of Example 5.2.3 and Lemma 5.2.4 we get that p_{f_0} is **fully-faithful**. Note that if f_0 is a Kan fibration then p_{f_0} is a marked fibration.

We now claim that every fully-faithful map is essentially of the form $P(f_0, Y) \longrightarrow Y$ for some Y and f_0. Let $f : X \longrightarrow Y$ be a map of marked semiSegal spaces and consider $f_0 : X_0 \longrightarrow Y_0$. Then f factors naturally as

$$X \xrightarrow{f'} P(f_0, Y) \xrightarrow{p_{f_0}} Y.$$

Furthermore, if f is a marked fibration then both f' and p_{f_0} are marked fibrations.

We now have the following simple observation:

Lemma 5.2.6. *Let $f : X \longrightarrow Y$ be a map of marked semiSegal spaces. Then f is fully-faithful if and only if the map*

$$f' : X \longrightarrow P(f_0, Y)$$

as above is a marked equivalence.

Proof. By definition we see that f is fully-faithful if and only if the map

$$f'_1 : X_1 \longrightarrow P(f_0, Y)_1$$

is a weak equivalence which induces a weak equivalence on the corresponding marked subspaces. Since the map

$$f'_0 : X_0 \longrightarrow P(f_0, Y)_0$$

is an isomorphism and both $X, P(f_0, Y)$ are marked semiSegal spaces we see that f'_1 is a weak equivalence if and only if f' is a levelwise equivalence. The additional condition that f'_1 induces an equivalence on the marked subspaces is then equivalent to f being a marked equivalence (see Corollary 2.2.14). $\square$

Corollary 5.2.7. *A map $f : X \longrightarrow Y$ of marked semiSegal spaces is fully-faithful if and only if the induced square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{cosk}_0^+(X) \arrow[d] \\ Y \arrow[r] & \operatorname{cosk}_0^+(Y) \end{tikzcd}$$

is a homotopy pullback square in the marked model structure.

We can now deduce the main result of this subsection:

Proposition 5.2.8. *Let $f : W \longrightarrow Z$ be a marked fibration between marked semiSegal spaces. Then f is fully-faithful if and only if f satisfies the right lifting property with respect to marked cofibrations $g : X \longrightarrow Y$ such that $g_0 : X_0 \hookrightarrow Y_0$ is a weak equivalence.*

Proof. First assume that f is fully faithful. Since $f : W \longrightarrow Z$ is a marked fibration we get that the map

$$f' : W \longrightarrow P(f_0, Z)$$

is a trivial marked fibration. Hence it will suffice to show that the map $p_{f_0} : P(f_0, Z) \longrightarrow Z$ satisfies the right lifting property with respect to g. But p_{f_0} was pulled back from the map $\mathrm{cosk}_0^+(W_0) \longrightarrow \mathrm{cosk}_0^+(Z_0)$ and so it will suffice to prove that the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"] & \mathrm{cosk}_0^+(W_0) \arrow[d] \\ Y \arrow[r] & \mathrm{cosk}_0^+(Z_0) \end{tikzcd}$$

admits a lift. By adjunction this is equivalent to the square

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "g_0"] & W_0 \arrow[d] \\ Y_0 \arrow[r] & Z_0 \end{tikzcd}$$

admitting a lift. Since f was a marked fibration we get that $f_0 : W_0 \longrightarrow Z_0$ is a Kan fibration. Since g_0 is a trivial Kan cofibration the result follows.

Now assume that $f : (W, M) \longrightarrow (Z, N)$ satisfies the right lifting property with respect to all marked cofibrations $g : X \longrightarrow Y$ such that g_0 is a weak equivalence. Then for each $m \geq 0$ we get that f satisfies the right lifting property with respect to maps of the form

$$\left[(\Delta^1, A) \otimes |\partial\Delta^m|\right] \coprod_{\partial\Delta^1 \otimes |\partial\Delta^m|} \left[\partial\Delta^1 \otimes |\Delta^m|\right] \hookrightarrow (\Delta^1, A) \otimes |\Delta^m|.$$

Substituting in $A = \emptyset$ and $A = \left(\Delta^1\right)_1$ we get that the maps

$$W_1 \longrightarrow Z_1 \times_{Z_0 \times Z_0} (W_0 \times W_0)$$

and

$$M \longrightarrow N \times_{Z_0 \times Z_0} (W_0 \times W_0)$$

are trivial Kan fibration. $\square$

Corollary 5.2.9. *Let $f : W \longrightarrow Z$ be a fully-faithful marked fibration. Let X be a marked semi-simplicial space and $g : X \longrightarrow Z$ a map. Then every lift $\widetilde{g_0} : X_0 \longrightarrow W_0$ of g_0 extends to a lift $\widetilde{g} : X \longrightarrow W$ of g.*

We finish this subsection with an application we frame for future use:

Lemma 5.2.10. *Let $f : X \longrightarrow W$ be a map of marked semi-simplicial spaces such that*

1. *X is quasi-unital.*
2. *W is marked-fibrant.*
3. *The square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{cosk}_0^+(X_0) \arrow[d] \\ W \arrow[r] & \mathrm{cosk}_0^+(W_0) \end{tikzcd}$$

   is a homotopy pullback square.

4. *The map $f_0 : X_0 \longrightarrow W_0$ is surjective on connected components.*

Then W is quasi-unital and f is a DK-equivalence.

Proof. We start by showing that W is a marked semiSegal space. Let $f : Y \longrightarrow Z$ be a map in $\mathbf{S}$. We need to show that the map

$$\mathrm{Map}^+(Z, W) \longrightarrow \mathrm{Map}^+(Y, W)$$

is a weak equivalence. Note that in all cases the 0'th level map

$$f_0 : Y_0 \longrightarrow Z_0$$

is an isomorphism. Hence from (3) we get that the square

$$\begin{tikzcd} \mathrm{Map}^+(Z, X) \arrow[r] \arrow[d] & \mathrm{Map}^+(Z, W) \arrow[d] \\ \mathrm{Map}^+(Y, X) \arrow[r] & \mathrm{Map}^+(Y, W) \end{tikzcd}$$

is a homotopy pullback square. Furthermore both vertical maps are Kan fibrations. We wish to show that the right vertical fibration is trivial. Since X is a marked semiSegal space the left vertical Kan fibration is trivial. Now since Kan fibrations are trivial if and only if all their fibers are contractible it will be enough to show that the lower horizontal map is surjective on connected components. For this it will be enough to show that the map

$$\mathrm{Map}_{\mathcal{S}}(Y_0, X_0) = \mathrm{Map}^+\left(Y, \mathrm{cosk}_0^+(X_0)\right) \longrightarrow \mathrm{Map}^+\left(Y, \mathrm{cosk}_0^+(W_0)\right) = \mathrm{Map}_{\mathcal{S}}(Y_0, W_0)$$

is surjective on connected components. But this just follows from the fact that Y_0 is discrete and the map

$$X_0 \longrightarrow W_0$$

is surjective on components. This shows that W is a marked semiSegal space. Now from (3) we then get that f is fully faithful. From (4) we get that for every object $w_0 \in W_0$ there is a marked edge with source w_0. From Remark 4.1.10 we then see that W is quasi-unital. Using again (4) we get that f is a DK-equivalence.

$\square$

5.3 Quasi-unital Mapping Objects

The purpose of this section is to show that the full subcategory $\mathrm{QsS} \subseteq \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ is closed under taking mapping objects. More precisely, we will show that if W, Z are **quasi-unital** marked semiSegal spaces (Definition 5.0.15) then the marked mapping object W^Z is quasi-unital as well.

We begin with the following auxiliary proposition:

Proposition 5.3.1. *Let W be a quasi-unital marked semiSegal space and let $\mathrm{Id} \in \left(\widetilde{W^W}\right)_0$ denote the object corresponding to the identity. Then there exists a **quasi-unit** $H : \mathrm{Id} \longrightarrow \mathrm{Id}$ in the marked semi-groupoid $\widetilde{W^W}$.*

Proof. Since $\widetilde{W^W}$ is a marked semi-groupoid we get from Corollary 4.1.10 that it will be enough to find an edge

$$h \in \left(\widetilde{W^W}\right)_1$$

such that $d_0 = Id \in \left(\widetilde{W^W}\right)_0$. From the definition of $\widetilde{W^W}$ we see that this edge corresponds to a map

$$h : W \otimes \left(\Delta^1\right)^\sharp \longrightarrow W$$

such that $h|_{W \otimes \Delta^{\{0\}}} = Id$. Consider the restriction map

$$p : W^{\left(\Delta^1\right)^\sharp} \longrightarrow W$$

induced by the inclusions $\Delta^{\{0\}} \hookrightarrow \left(\Delta^1\right)^\sharp$. Using the exponential law we see that the existence of h as above corresponds to a map

$$g : W \longrightarrow W^{\left(\Delta^1\right)^\sharp}$$

such that $p \circ g = Id$. In other words, we want to show that the map p admits a section. In light of Corollary 5.2.9 we see that it will be enough to prove the following:

1. The map p is fully-faithful.
2. The 0-level map
$$p_0 : W_0^{\left(\Delta^1\right)^\sharp} = W_1^{\mathrm{inv}} \longrightarrow W_0$$
admits a section.

The first assertion is contained in the following lemma:

Lemma 5.3.2. *Let W be a marked semiSegal space. Let*

$$p^i : W^{\left(\Delta^1\right)^\sharp} \longrightarrow W$$

be the restriction along the inclusion $\Delta^{\{i\}} \hookrightarrow \left(\Delta^1\right)^\sharp$. Then p^i is fully-faithful.

Proof. In light of Remark 5.2.2 and using the fact that the functor $(\bullet)^W$ takes pushout squares to pullback squares we see that the statement of the lemma is equivalent to the claim that the maps

$$\left[\left(\Delta^1\right)^\sharp \times \left(\partial\Delta^1\right)^\flat\right] \coprod_{\Delta^{\{i\}}\times(\partial\Delta^1)^\flat} \left[\Delta^{\{i\}} \times \left(\Delta^1\right)^\flat\right] \longrightarrow \left(\Delta^1\right)^\sharp \otimes \left(\Delta^1\right)^\flat$$

and

$$\left[\left(\Delta^1\right)^\sharp \times \left(\partial\Delta^1\right)^\sharp\right] \coprod_{\Delta^{\{i\}}\times(\partial\Delta^1)^\sharp} \left[\Delta^{\{i\}} \times \left(\Delta^1\right)^\sharp\right] \longrightarrow \left(\Delta^1\right)^\sharp \otimes \left(\Delta^1\right)^\sharp$$

are trivial MS-cofibrations. Now for the first map the right hand side can be obtained from the left hand side by performing two pushouts along admissible horn inclusions of dimension 2. In the second map one needs to perform in addition a triangle remarking (see Definition 5.1.11). $\square$

It is now left to prove the existence of a section on the 0'th level. Let

$$W_1^{\mathrm{aut}} = \left\{f \in W_1^{\mathrm{inv}} | d_0(f) = d_1(f)\right\} \subseteq W_1^{\mathrm{inv}}$$

be the subspace of self equivalences. We have a commutative diagram

$$\begin{tikzcd} W_1^{\mathrm{aut}} \arrow[rr] \arrow[dr, "d"] & & W_1^{\mathrm{inv}} \arrow[dl, "d_0"'] \\ & W_0 & \end{tikzcd}$$

where d is the map induced by d_0 (or d_1). Hence it will be enough to prove that d admits a section. Note that both W_1^{aut} and W_1^{inv} only concern invertible maps. In particular for this purpose we could have replaced W with $\widetilde{W}$. Hence the statement will follow from the following statement regarding semi-groupoids:

Lemma 5.3.3. *Let X be a quasi-unital semi-groupoid. Then the map*

$$d : X_1^{\mathrm{aut}} \longrightarrow X_0$$

as above admits a section.

Proof. Consider the Kan replacement $\widehat{|X|}$ of the realization of X. Let $\widehat{|X|}^{S^1}$ be the space of continuous paths

$$\gamma : S^1 \longrightarrow \widehat{|X|}$$

and let $p : \widehat{|X|}^{S^1} \longrightarrow \widehat{|X|}$ be the map $p(\gamma) = \gamma(1)$. Consider the commutative diagram

$$\begin{tikzcd} X_1^{\mathrm{aut}} \arrow[r] \arrow[d, "d_0"] & \widehat{|X|}^{S^1} \arrow[d, "p"] \\ X_0 \arrow[r] & \widehat{|X|} \end{tikzcd}$$

By Theorem 4.3.5 we get that for each $x \in X_0$ the above square induces a weak equivalence from the fiber on the left hand side

$$d^{-1}(x) = \mathrm{Map}_X(x,x)$$

to the fiber on the right hand side

$$p^{-1}(x) = \Omega\left(\widehat{|X|}, x\right).$$

Since the vertical maps are fibrations (the map d is obtained by pulling the fibration $d_0 \times d_1 : X_1 \longrightarrow X_0 \times X_0$ along the diagonal) this square is a homotopy pullback square. Now since the map

$$\widehat{|X|}^{S^1} \longrightarrow \widehat{|X|}$$

admits a section (given by choosing for each x the constant map at x) we get that the fibration

$$d : X_1^{\mathrm{aut}} \longrightarrow X_0$$

admits a section as well. $\square$

This finishes the proof of Proposition 5.3.1. $\square$

Corollary 5.3.4. *Let W be a quasi-unital semiSegal space and X a marked semi-simplicial space. Then W^X is quasi-unital.*

Proof. Since the object $\mathrm{Id} \in (\widetilde{W^W})_0$ admits a quasi-unit it follows that there is a map of semiSegal spaces

$$u : W \longrightarrow W^{(\Delta^1)^\sharp}$$

which is a section to the restriction map

$$W^{(\Delta^1)^\sharp} \longrightarrow W^{(\Delta^0)} = W.$$

Then the operation $f \mapsto u \circ f$ determines a map of semiSegal spaces

$$W^X \longrightarrow \left(W^{(\Delta^1)^\sharp}\right)^X = \left(W^X\right)^{(\Delta^1)^\sharp}$$

which is a section of the restriction map

$$\left(W^X\right)^{(\Delta^1)^\sharp} \longrightarrow \left(W^X\right)^{\Delta^{\{0\}}} = W^X.$$

This shows that every object in W^X has a marked edge out of it and so the underlying semiSegal space of W^X is quasi-unital. In fact, one sees that for each object $f \in \left(W^X\right)_0$ there exists a quasi-unit $H_f \in (W^X)_1$ from f to f which is **marked**.

It is left to show that all the invertible edges of W^X are marked. Let $x_0 \in X_0$ be a point. The evaluation map

$$\mathrm{ev}_{x_0} : W^X \longrightarrow W$$

sends marked edges to marked edges. From the above considerations we see that for each object $f \in \left(W^X\right)_0$ there is at least one quasi-unit which is sent to a marked (and hence invertible) edge of W. This means that ev_{x_0} sends all quasi-unit to invertible maps in W. By remark 5.0.13 we get that ev_{x_0} sends invertible edges to invertible edges. Since W is quasi-unital all its invertible edges are marked, so ev_{x_0} sends all invertible edges to marked edges. Since this is true for every $x_0 \in X_0$ it follows from the definition of the marking on W^X that each invertible edge in W^X is marked.

□

Corollary 5.3.5. *Let W be a quasi-unital semiSegal space and X a marked semi-simplicial space. Then $\widetilde{W^X}$ is a quasi-unital marked semi-groupoid which is the maximal semi-groupoid contained in W^X.*

Proof. Follows directly from Remark 5.1.14. □

5.4 DK-anodyne maps

In this subsection we will use results from the previous subsections in order to introduce and study an auxiliary notion of **DK-anodyne** maps. This will be used in § 6 in order to study the localization of QsS with respect to DK-equivalences.

We begin with the basic definitions:

Definition 5.4.1. Let $f : W \longrightarrow Z$ be a marked fibration of marked semiSegal spaces. We will say that f is **relatively marked** if it satisfies the right lifting property with respect to the maps $\Delta^{\{0\}} \hookrightarrow \left(\Delta^1\right)^\sharp$ and $\Delta^{\{1\}} \hookrightarrow \left(\Delta^1\right)^\sharp$.

Definition 5.4.2. We will say that a marked cofibration $g : X \longrightarrow y$ of marked semi-simplicial spaces is **DK-anodyne** if it satisfies the left lifting property with respect to all relatively marked maps.

Example 5.4.3. Any trivial MS-cofibration is DK-anodyne.

Example 5.4.4. The maps $\Delta^{\{0\}} \hookrightarrow \left(\Delta^1\right)^\sharp$ and $\Delta^{\{1\}} \hookrightarrow \left(\Delta^1\right)^\sharp$ are DK-anodyne.

The main motivation for studying DK-anodyne maps is the following:

Lemma 5.4.5. *Let $g : X \longrightarrow Y$ be a DK-anodyne map and let W be a quasi-unital marked semiSegal space. Then the induced map*

$$g^* : W^Y \longrightarrow W^X$$

is a DK-equivalence.

Proof. First since g is DK-anodyne we get that the map

$$g_0^* : \left(W^Y\right)_0 \longrightarrow \left(W^X\right)_0$$

is surjective. Hence it will be enough to show that g^* is fully-faithful. Let $i = 0, 1$ and consider the inclusion $\Delta^{\{i\}} \subseteq \left(\Delta^1\right)^\sharp$. From Lemma 5.3.2 we get that the marked fibration

$$p^i : W^{\left(\Delta^1\right)^\sharp} \longrightarrow W^{\Delta^{\{i\}}} = W$$

is fully-faithful. Unwinding the definitions this implies that any diagram of the form

$$\begin{tikzcd} \Delta^{\{i\}} \arrow[r] \arrow[d] & W^{\left(\Delta^1\right)^\flat} \arrow[d] \\ \left(\Delta^1\right)^\sharp \arrow[r] & W^{\left(\partial\Delta^1\right)^\flat} \end{tikzcd}$$

has a **Kan contractible** space of lifts. In other words if we denote

$$T_m = \left[\left(\Delta^1\right)^\flat \otimes |\partial\Delta^m|\right] \coprod_{\left(\partial\Delta^1\right)^\flat \otimes |\partial\Delta^m|} \left[\left(\partial\Delta^1\right)^\flat \otimes |\Delta^m|\right]$$

then for each $m \geq 0$ the map

$$W^{\left(\Delta^1\right)^\flat \otimes |\Delta^m|} \longrightarrow W^{T_m}$$

is relatively marked, and hence satisfies the right lifting property with respect to g. This in turn implies that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"] & W^{\left(\Delta^1\right)^\flat} \arrow[d] \\ Y \arrow[r] & W^{\left(\partial\Delta^1\right)^\flat} \end{tikzcd}$$

has a Kan contractible space of lifts which means that

$$g^* : W^Y \longrightarrow W^Z$$

is fully faithful.

□

Remark 5.4.6. The property of being DK-anodyne is invariant under pushouts, i.e., if we have a pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

of marked semi-simplicial spaces such that the upper horizontal map is DK-anodyne then the lower horizontal one is DK-anodyne as well.

Our main interest is to show that the following types of maps are DK-anodyne. Let $f:[m]\longrightarrow[n]$ be a surjective map in Δ and let $h:[n]\longrightarrow[m]$ be a section of f. Let $M\subseteq(\mathrm{Sp}_n)_1$ be a marking on the n-spine and let $\widetilde{M}\subseteq(\Delta^n)_1$ be the marking generated from it, i.e., the smallest set containing M which is closed under 2-out-of-3. We first observe the following:

Lemma 5.4.7. *The inclusion*

$$\iota:(\mathrm{Sp}_n,M)\hookrightarrow\left(\Delta^n,\widetilde{M}\right)$$

is a trivial MS-cofibration.

Proof. Factorize ι as

$$(\mathrm{Sp}_n,M)\overset{\iota'}{\hookrightarrow}(\Delta^n,M)\overset{\iota''}{\hookrightarrow}\left(\Delta^n,\widetilde{M}\right).$$

Then ι' is a pushout along the trivial MS-cofibration $\mathrm{Sp}_n^\flat\hookrightarrow(\Delta^n)^\flat$ and ι'' is a triangle remarking (Definition 5.1.11). □

Now let $M_f\subseteq(\mathrm{Sp}^m)_1$ be the set of all pairs $\{i,i+1\}$ such that either $f(i)=f(i+1)$ or $\Delta^{\{f(i),f(i+1)\}}$ is in M and let $\widetilde{M}_f\subseteq(\Delta^m)_1$ be the marking generated from it. Our purpose is to prove that the map

$$h:\left(\Delta^m,\widetilde{M}\right)\longrightarrow\left(\Delta^n,\widetilde{M}_f\right)$$

is DK-anodyne. Note that when M contains all edges one obtains a map of the form

$$(\Delta^n)^\sharp\hookrightarrow(\Delta^m)^\sharp$$

and when M is empty one obtains the inclusion

$$(\Delta^n)^\flat\hookrightarrow(\Delta^m)^f$$

which was considered in §§§ 2.2.3.

Proposition 5.4.8. *In the notation above, the map*

$$h:\left(\Delta^m,\widetilde{M}\right)\longrightarrow\left(\Delta^n,\widetilde{M}_f\right)$$

is DK-anodyne.

Proof. For each $i=0,...,n$ let $S_i\subseteq\Delta^m$ be the 1-dimensional sub semi-simplicial set containing all the vertices and all the edges of the form $\Delta^{\{j,j+1\}}$ such that

$$f(j)=f(j+1)=i.$$

Then clearly the inclusion

$$\Delta^{\{h(i)\}}\subseteq S_i$$

is DK-anodyne. Let $S \subseteq \Delta^m$ be the (disjoint) union of all the S_i's.

Let h_1, h_2 be two sections of f. We will define the sub marked semi-simplicial set $T(h_1, h_2) \subseteq \left(\Delta^m, \widetilde{M}_f\right)$ to be the (not necessarily disjoint) union of $S^\sharp \subseteq \left(\Delta^m, \widetilde{M}_f\right)$ in and all edges of the form $\Delta^{\{h_1(i), h_2(i+1)\}}$ for $i = 0, ..., n$, i.e.,

$$T(h_1, h_2) = \left(S \bigcup_i \Delta^{\{h_1(i), h_2(i+1)\}}, N\right) \subseteq \left(\Delta^m, \widetilde{M}_f\right),$$

where N is the marking induced from $\widetilde{M}_f$, i.e., N contains all the edges of S and the edges $\Delta^{\{h_1(i), h_2(i+1)\}}$ for which $\Delta^{\{i,i+1\}}$ is in M. Note that when $h_1 = h_2 = h$ we have

$$T(h, h) = S^\sharp \coprod_{\Delta^0 \times \{0,...,m\}} (h(\mathrm{Sp}_n), h(M)).$$

Let P be the pushout in the square

$$\begin{tikzcd} (\mathrm{Sp}_n, M) \arrow[r] \arrow[d] & T(h,h) \arrow[d] \\ \left(\Delta^n, \widetilde{M}\right) \arrow[r] & P \end{tikzcd}$$

Note that P embeds naturaly in $\left(\Delta^m, \widetilde{M}_f\right)$.

Since the inclusion $\Delta^0 \times \{0, ..., m\} \subseteq S^\sharp$ is DK-anodyne we get that the top horizontal row is DK-anodyne, and so the bottom horizontal row is DK-anodyne. Since the left vertical map is a trivial MS-cofibration we get that the right vertical map is a trivial MS-cofibration.

In order to finish the proof it will be enough to show that the map

$$T(h, h) \hookrightarrow \left(\Delta^m, \widetilde{M}_f\right)$$

is a trivial MS-cofibration. This will imply that the map

$$P \hookrightarrow \left(\Delta^m, \widetilde{M}_f\right)$$

is a trivial MS-cofibration and so that the composition

$$\left(\Delta^n, \widetilde{M}\right) \hookrightarrow P \hookrightarrow \left(\Delta^m, \widetilde{M}_f\right)$$

is DK-anodyne.

Now in order to prove that the map

$$T(h, h) \longrightarrow \left(\Delta^m, \widetilde{M}_f\right)$$

is a trivial MS-cofibration it will be easier to allow for the sections h_1, h_2 to vary. More precisely, we will prove the following:

Lemma 5.4.9. *For every two sections h_1, h_2 of f the inclusion*

$$T(h_1, h_2) \subseteq \left(\Delta^m, \widetilde{M}_f\right)$$

is a trivial MS-cofibration.

Proof. We begin by arguing that that it is enough to prove the lemma for just one pair of sections h_1, h_2. We say that two pairs $(h_1, h_2), (h_1', h_2')$ are **neighbours** if

$$\sum_{i=0}^{n} |h_1(i) - h_1'(i)| + |h_2(i) - h_2'(i)| = 1.$$

It is not hard to see that the resulting neighbouring graph is connected, i.e., that we can get from any pair (h_1, h_2) to any other pair (h_1', h_2') by a sequence of pairs such that each consecutive couple of pairs are neighbours. Hence it is enough to show that property of

$$T(h_1, h_2) \subseteq \left(\Delta^m, \widetilde{M}_f\right)$$

being a trivial MS-cofibration respects the neighbourhood relation. To see why this is true observe that if (h_1, h_2) and (h_1, h_2') are neighbours then one can add to $T(h_1, h_2)$ a single triangle $\sigma \subseteq \Delta^m$ such that

$$R = T(h_1, h_2) \cup \sigma$$

contains $T(h_1', h_2')$ and such that R can be obtained from either $T(h_1, h_2), T(h_1', h_2')$ by performing a pushout along a 2-dimensional admissible marked horn inclusion and possibly a remarking. Hence the claim for either $T(h_1, h_2)$ or $T(h_1', h_2')$ is equivalent to

$$R \subseteq \left(\Delta^m, \widetilde{M}_f\right)$$

being a trivial MS-cofibration.

Now that we know that it is enough to prove for a single choice of (h_1, h_2) let us choose the pair $h_{\max}(i) = \max(f^{-1}(i))$ and $h_{\min}(i) = \min(f^{-1}(i))$. Then we see that

$$T(h_{\max}, h_{\min}) = (\mathrm{Sp}_m, M_f)$$

and the map

$$(\mathrm{Sp}_m, M_f) \hookrightarrow \left(\Delta^m, \widetilde{M}_f\right)$$

is a trivial MS-cofibration from Lemma 5.4.7. □

This finishes the proof of Proposition 5.4.8. □

5.5 Categorical Equivalences

In this section we will use the quasi-unital mapping objects constructed in the previous section in order to define a quasi-unital analogue of the notion of **categorical equivalences** (see [Rez]). The main results of this section is Theorem 5.5.5 and Corollary 5.5.6 which will be used in § 6 in order to localize QsS with respect to DK-equivalences.

We start with the basic definition:

Definition 5.5.1. Let $f : X \longrightarrow Y$ be a map of quasi-unital marked semiSegal spaces. We will say that f is a **categorical equivalence** if there exists a map $g : Y \longrightarrow X$ such that $f \circ g$ is equivalent to the identity in the marked semi-groupoid $\widetilde{Y^Y}$ and $g \circ f$ is equivalent to the identity in the marked semi-groupoid $\widetilde{X^X}$.

Example 5.5.2. Every levelwise equivalence $f : X \longrightarrow Y$ is a categorical equivalence: since X, Y are marked-fibrant f admits a homotopy inverse $g : Y \longrightarrow X$ so that the compositions $f \circ g$ and $g \circ f$ are in the same connected component of the identity in $\widetilde{Y^Y}$ and $\widetilde{X^X}$ respectively. Since these marked semi-groupoids are quasi-unital the result follows from Lemma 4.2.8.

Next we study some basic properties of categorical equivalences:

Proposition 5.5.3. *Let $f : X \longrightarrow Y$ be a categorical equivalence between quasi-unital marked semiSegal spaces. Then f is a DK-equivalence.*

Proof. Let $g : Y \longrightarrow X$ be a homotopy inverse. By Lemma 4.2.12 it is enough to show that both $f \circ g$ and $g \circ f$ are DK-equivalences. Let $h : X \longrightarrow X^{(\Delta^1)^\sharp}$ be a homotopy from the identity to $g \circ f$.

Now let

$$p^i : X^{(\Delta^1)^\sharp} \longrightarrow X^{\Delta^0} = X$$

be the restriction map. Since $p^0 \circ h = \mathrm{Id}$ is a DK-equivalence and p^0 is a DK-equivalence (since it is a restriction along a DK-anodyne map) we get from Lemma 4.2.12 that h is a DK-equivalence. Since p^1 is a DK-equivalence as well we get that $p^1 \circ h = g \circ f$ is a DK-equivalence. A similar argument works for $f \circ g$. $\square$

Proposition 5.5.4. *Let Z be a marked semiSegal space and $f : X \longrightarrow Y$ a categorical equivalence of quasi-unital marked semiSegal spaces. Then the induced maps*

$$f^* : Z^Y \longrightarrow Z^X$$

is a categorical equivalence.

Proof. By definition there exists a map $g : Y \longrightarrow X$ and homotopies

$$H : \left(\Delta^1\right)^\sharp \longrightarrow X^X$$

$$G : \left(\Delta^1\right)^\sharp \longrightarrow Y^Y$$

from the compositions $g \circ f$ and $f \circ g$ to the respective identities. Now we have that the composition

$$\left(\Delta^1\right)^\sharp \otimes Z^X \longrightarrow X^X \otimes Z^X \longrightarrow Z^X$$

gives a homotopy from the identity to $f^* \circ g^*$ and the composition

$$\left(\Delta^1\right)^\sharp \otimes Z^Y \longrightarrow Y^Y \otimes Z^Y \longrightarrow Z^Y$$

gives a homotopy from the identity to $g^* \circ f^*$. □

We now come to the main result of this subsection:

Theorem 5.5.5. *Let $f : X \longrightarrow Y$ be a DK-equivalence between quasi-unital marked semiSegal spaces. Assume further that the map $f_0 : X_0 \longrightarrow Y_0$ admits a section. Then f is a categorical equivalence.*

Proof. Since we can factorize $f = q \circ p$ where p is a levelwise weak equivalence (and hence in particular a categorical equivalence in view of Example 5.5.2) and q is a fully-faithful marked fibration we can assume without loss of generality that f is a marked fibration.

Now from our assumptions we get that $f_0 : X_0 \longrightarrow Y_0$ admits a section $g_0 : Y_0 \longrightarrow X_0$. Since f is a fully-faithful marked fibration we get from Corollary 5.2.9 that we can extend g_0 to a section

$$g : Y \longrightarrow X$$

of f. We claim that g is a homotopy inverse of f. On one direction the composition $f \circ g$ is the identity. We need to show that $g \circ f$ is equivalent to the identity in the marked semi-groupoid $\widetilde{X^X}$.

Since the object $\mathrm{Id} \in (\widetilde{Y^Y})_0$ admits a quasi-unit it follows that there is a map of marked semiSegal spaces

$$u : Y \otimes \left(\Delta^1\right)^\sharp \longrightarrow Y$$

such that

$$u|_{Y \otimes \Delta^{\{0\}}} = u|_{Y \otimes \Delta^{\{1\}}} = Id.$$

Composing with $(f \otimes Id)$ we get a map

$$h = u \circ (f \otimes Id) : X \otimes \left(\Delta^1\right)^\sharp \longrightarrow Y$$

which gives an equivalence from f to itself in $\widetilde{Y^X}$.

Now we want to find a lift

$$\begin{tikzcd} & X \arrow[d, "f"] \\ X \times \left(\Delta^1\right)^\sharp \arrow[ur, dashed, "\widetilde{h}"] \arrow[r, "h"] & Y \end{tikzcd}$$

so that $\widetilde{h}$ will give an equivalence from the identity to $g \circ f$ in $\widetilde{X^X}$. Using again Corollary 5.2.9 it is enough to lift h on the 0'th level. Now

$$\left(X \times (\Delta^1)^\sharp\right)_0 = X_0 \coprod X_0$$

and h_0 maps both copies of X_0 to Y_0 via the map f_0. Hence we can lift h_0 to a map

$$\widetilde{h}_0 : X_0 \coprod X_0 \longrightarrow X_0$$

by sending the first copy of X_0 via the identity and the second via $g_0 \circ f_0$ (using the fact that $f_0 \circ g_0 \circ f_0 = f_0$). Then $\widetilde{h}_0$ extends to the desired $\widetilde{h}$ and we are done. □

In particular, Theorem 5.5.5 gives us the following strengthening of Lemma 5.4.5:

Corollary 5.5.6. *Let $f : X \hookrightarrow Y$ be a DK-anodyne map and let W be a quasi-unital marked semiSegal space. Then the map*

$$f^* : W^Y \longrightarrow W^X$$

is a categorical equivalence.

Proof. From Lemma 5.4.5 we know that f^* is a DK-equivalence. Hence Theorem 5.5.5 tells us that it will be enough to show that the 0-level map

$$\left(W^Y\right)_0 \longrightarrow \left(W^X\right)_0$$

admits a section. Let $K = \left(W^X\right)_0 = \mathrm{Map}^+(X, W)$. Then we have a canonical map

$$f_K : X \longrightarrow W^K.$$

Since W^K can be identified with the mapping object $W^{K \otimes \Delta^0}$ we get that W^K is quasi-unital. Hence f_K admits an extension

$$\begin{tikzcd} X \arrow[r, "f_K"] \arrow[d] & W^K \\ Y \arrow[ur, dashed] & \end{tikzcd}$$

Applying the exponential law we get a lift in the diagram

$$\begin{tikzcd} & \left(W^Y\right)_0 \arrow[d] \\ K \arrow[ur, dashed] \arrow[r, equal] & \left(W^X\right)_0 \end{tikzcd}$$

as desired. □

6 Complete SemiSegal Spaces

In this section we will introduce the ∞-category of complete semiSegal spaces and show that it serves as a model for the localization of QsS by DK-equivalences. The notion of completeness, the construction of the completion functor and many of the related proofs are the semi-simplicial variations of their respective analogues in [Rez].

Definition 6.0.7. Let X be a semiSegal space. We will say that X is **complete** if the restricted maps $d_0 : X_1^{\mathrm{inv}} \longrightarrow X_0$ and $d_1 : X_1^{\mathrm{inv}} \longrightarrow X_0$ are both homotopy equivalences.

Remark 6.0.8. According to Remark 4.3.1 we get that X is complete if and only if the maximal ∞-groupoid of X is homotopy-constant. In particular, in this case the maps

$$d_0, d_1 : X_1^{\mathrm{inv}} \longrightarrow X_0$$

will automatically be **homotopic to each other** in the Kan model structure.

An important observation is that any complete semiSegal space is quasi-unital: since the map $X_1^{\mathrm{inv}} \longrightarrow X_0$ is a trivial fibration every object $x \in X_0$ admits an invertible morphism of the form $f : x \longrightarrow y$ for some y. Form Corollary 4.1.10 one then gets that x admits a quasi-unit.

Let $\mathrm{CsS} \subseteq \mathrm{QsS}$ denote the full topological subcategory spanned by complete semiSegal spaces. We will show in the following sections that the topological category CsS serves as a model for the localization of QsS by DK-equivalences. Formally speaking (see Definition 5.2.7.2 and Proposition 5.2.7.12 of [Lur2]) this means that there exists a functor

$$\widehat{\bullet} : \mathrm{QsS} \longrightarrow \mathrm{CsS}$$

such that:

1. $\widehat{\bullet}$ is homotopy left adjoint to the inclusion $\mathrm{CsS} \subseteq \mathrm{QsS}$.
2. A map in QsS is a DK-equivalence if and only if its image under $\widehat{\bullet}$ is a homotopy equivalence.

The functor $\widehat{\bullet}$ will be called the **completion** functor. In order to define and study this functor it will be useful to work in the setting of **marked** semi-simplicial spaces.

Definition 6.0.9. Let W be a marked semiSegal space. We will say that W is **complete** if there exists a complete semiSegal space X such that $W \cong X^{\natural}$.

Remark 6.0.10. Via the functor $W \mapsto W^{\sharp}$ we can identify CsS with the full subcategory of $\mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$ spanned by complete marked semiSegal spaces.

Remark 6.0.11. If W is a complete marked semiSegal space then so is $\widetilde{W}$.

Before we proceed to construct the completion functor let us start by proving two basic results concerning complete marked semiSegal spaces. First we show that the subcategory of complete marked semiSegal spaces admits internal mapping objects. In fact, we show something slightly stronger:

Proposition 6.0.12. *Let X be a marked semi-simplicial space and W a complete marked semiSegal space. Then W^X (and hence also $\widetilde{W^X}$) are complete. In particular, $\widetilde{W^X}$ is a homotopy-constant marked semi-simplicial space.*

Proof. For $i = 0, 1$ consider the restriction map

$$p^i : W^{(\Delta^1)^\sharp} \longrightarrow W^{(\Delta^0)} = W.$$

Since W is complete we get by definition that the maps

$$p_0^i : \left(W^{(\Delta^1)^\sharp}\right)_0 \longrightarrow W_0$$

are weak equivalences. By Proposition 5.3.2 we get that p^i is also fully-faithful and hence a marked equivalence. Using the exponential law this implies that the restriction map

$$\mathrm{Map}^+\left(X \otimes \left(\Delta^1\right)^\sharp, W\right) \longrightarrow \mathrm{Map}^+\left(X \otimes \Delta^{\{i\}}, W\right)$$

is a weak equivalence. This means that the maps

$$d_0, d_1 : \left(W^X\right)_1^{\mathrm{inv}} \longrightarrow \left(W^X\right)_0$$

are weak equivalences and so W^X is complete. □

Next we show the notion of DK-equivalence in CsS coincides with homotopy equivalence:

Proposition 6.0.13. *Let $f : X \longrightarrow Y$ be a DK-equivalence between complete marked semiSegal spaces. Then f has a homotopy inverse.*

Proof. Since X, Y are Reedy fibrant it is enough to show that f is a levelwise equivalence. Now since f is a DK-equivalence it will be enough to show that the map

$$f_0 : X_0 \longrightarrow Y_0$$

is a weak equivalence. Since X, Y are quasi-unital it will be enough to show that the map

$$\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$$

of maximal sub-groupoids is a levelwise equivalence. But since X, Y are complete these semi-groupoids are homotopy-constant and so it will be enough to show that the map

$$\left|\widetilde{f}\right| : \left|\widetilde{X}\right| \longrightarrow \left|\widetilde{Y}\right|$$

is a weak equivalence. The result now follows from Theorem 4.3.10 once one observe that if f is a DK-equivalence then $\widetilde{f}$ is a DK-equivalence as well. □

The remainder of this section is organized as follows. In §§ 6.1 we will construct the completion functor

$$\widehat{\bullet} : \mathrm{QsS} \longrightarrow \mathrm{CsS}$$

and show that it satisfies properties (1) and (2) above. In §§ 6.2 we will show that the topological category CsS is equivalent to the category of **complete Segal spaces**.

6.1 Completion

In this section we will construct the completion functor

$$\widehat{\bullet} : \mathrm{QsS} \longrightarrow \mathrm{CsS}$$

and show that it serves as a left localization functor with respect to DK-equivalences. More precisely, we show that $\widehat{\bullet}$ is homotopy left adjoint to the full inclusion $\mathrm{CsS} \hookrightarrow \mathrm{QsS}$ and that a map in QsS is a DK-equivalence if and only if its image under $\widehat{\bullet}$ is a homotopy equivalence.

Let X be a quasi-unital marked semiSegal space. Consider the bi-semi-simplicial spaces $X_{\bullet,\bullet}, Y_{\bullet,\bullet} : \Delta_s^{\mathrm{op}} \times \Delta_s^{\mathrm{op}} \longrightarrow \mathcal{S}$ given by

$$X_{n,m} = \mathrm{Map}^+\left((\Delta^n)^\flat \otimes (\Delta^m)^\sharp, X\right)$$

and

$$Y_{n,m} = \mathrm{Map}((\Delta^n)^\sharp \otimes (\Delta^m)^\sharp, X).$$

Remark 6.1.1. By definition we have

$$X_{\bullet,m} \cong X^{(\Delta^m)^\sharp},\ Y_{\bullet,m} \cong \widetilde{X^{(\Delta^m)^\sharp}},$$

$$X_{n,\bullet} \cong \widetilde{X^{(\Delta^n)^\flat}},\ Y_{n,\bullet} \cong \widetilde{X^{(\Delta^n)^\sharp}}.$$

In particular, $X_{\bullet,m}$ is a marked semiSegal space and $X_{n,\bullet}, Y_{n,\bullet}, Y_{\bullet,m}$ are marked semi-groupoids.

We define the marked semi-simplicial space $\left(\widetilde{X}, M\right)$ by setting

$$\widetilde{X}_n = |X_{n,\bullet}|$$

and

$$M = |Y_{1,\bullet}| \subseteq |X_{1,\bullet}|.$$

We then define the **completion** $\widehat{X}$ of X to be the **marked fibrant replacement** of $\left(\widetilde{X}, M\right)$.

Theorem 6.1.2. *Let X be a quasi-unital marked semiSegal space. Then*

1. *$\widehat{X}$ is a complete marked semiSegal space.*

2. The natural map $X \longrightarrow \widehat{X}$ is a DK-equivalence.

Proof. Let $n \geq 0$ be an integer. From Proposition 5.4.8 we conclude that for each map $f : [k] \longrightarrow [m]$ in Δ_s the map

$$f^* : X_{\bullet,m} \longrightarrow X_{\bullet,k}$$

is a DK-equivalence. From Corollary 5.2.7 we then get that the induced square

$$\begin{tikzcd} X_{n,m} \arrow[r] \arrow[d, "f_n^*"] & (X_{0,m})^{n+1} \arrow[d, "(f_0^*)^{n+1}"] \\ X_{n,k} \arrow[r] & (X_{0,k})^{n+1} \end{tikzcd}$$

is a homotopy pullback square. From Corollary 4.3.11 the natural map

$$\left|X_{0,\bullet}^{n+1}\right| \longrightarrow |X_{0,\bullet}|^{n+1}$$

is a weak equivalence and so Puppe's Theorem (see Theorem 4.3.6 above) implies that the square

$$\begin{tikzcd} X_{n,0} \arrow[r] \arrow[d] & (X_{0,0})^{n+1} \arrow[d] \\ |X_{n,\bullet}| \arrow[r] & |X_{0,\bullet}|^{n+1} \end{tikzcd}$$

is a homotopy pullback square. Observing that the marked subspace of X_1 coincides with the preimage of $|Y_{1,\bullet}| \subseteq |X_{1,\bullet}|$ we see that the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{cosk}_0^+ (X_0) \arrow[d] \\ \widetilde{X} \arrow[r] & \mathrm{cosk}_0^+ \left(\widetilde{X}_0\right) \end{tikzcd}$$

is a homotopy pullback square of marked semi-simplicial space. This means that the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{cosk}_0^+ (X_0) \arrow[d] \\ \widehat{X} \arrow[r] & \mathrm{cosk}_0^+ \left(\widehat{X}_0\right) \end{tikzcd}$$

is again a homotopy pullback square of marked semi-simplicial spaces. Since $\widehat{X}$ is marked fibrant we get from Lemma 5.2.10 that $\widehat{X}$ is quasi-unital and the map $X \longrightarrow \widehat{X}$ is a DK-equivalence.

It is left to show that $\widehat{X}$ is complete. Since all the invertible edges in $\widehat{X}$ are marked it will be enough to show that the maps d_0, d_1 induce weak equivalences from the marked subspace of $\widehat{X}_1$ to X_0.

Now note that in the marked semi-simplicial space $\widetilde{X}$ the marked subspace $|Y_{1,\bullet}| \subseteq |X_{1,\bullet}| = \widetilde{X}_1$ is a union of components. Since the same is true for $\widehat{X}$ we get that the marked equivalence $\widetilde{X} \longrightarrow \widehat{X}$ induces a weak equivalence on the marked subspaces. Hence it will be enough to show that the maps

$$|d_0|, |d_1| : |Y_{1,\bullet}| \longrightarrow |Y_{0,\bullet}| = \widetilde{X}_0$$

are weak equivalences. But this follows from Theorem 4.3.10 since the maps $d_0, d_1 : Y_{1,\bullet} \longrightarrow Y_{0,\bullet}$ are DK-equivalences of marked semi-groupoids.

□

We will now show that $\widehat{\bullet}$ is homotopy left adjoint to the inclusion functor $\mathrm{CsS} \hookrightarrow \mathrm{QsS}$.

Theorem 6.1.3. *Let X be a quasi-unital marked semiSegal space and Z a complete marked semiSegal space. Then the natural map*

$$\mathrm{Map}^+\left(\widehat{X}, Z\right) \longrightarrow \mathrm{Map}(X, Z)$$

is a weak equivalence.

Proof. The marked semi-simplicial space $\widehat{X}$ is the homotopy colimit of the Δ_s-diagram $[m] \mapsto X_{\bullet,m}$. This means that

$$\mathrm{Map}^+\left(\widehat{X}, W\right) \simeq \mathrm{holim}_{\Delta_s} \mathrm{Map}^+\left(X_{\bullet,m}, W\right).$$

We wish to show that this homotopy limit is actually taken over a **homotopy constant** diagram. From Proposition 5.4.8 we know that any map of the form

$$f : \left(\Delta^k\right)^\sharp \hookrightarrow \left(\Delta^m\right)^\sharp$$

is DK-anodyne. By Corollary 5.5.6 we then get that the map

$$f^* : X^{(\Delta^m)^\sharp} = X_{\bullet,m} \longrightarrow X_{\bullet,k} = X^{\left(\Delta^k\right)^\sharp}$$

is a categorical equivalence.

Now Proposition 5.5.4 tells us that each of the induced maps

$$f^* : W^{X_{\bullet,m}} \longrightarrow W^{X_{\bullet,k}}$$

is a categorical equivalence. But by Proposition 6.0.12 the above mapping objects are complete and so by Proposition 6.0.13 the map

$$\mathrm{Map}^+\left(X_{\bullet,m}, W\right) = \left(W^{X_{\bullet,m}}\right)_0 \longrightarrow \left(W^{X_{\bullet,k}}\right)_0 = \mathrm{Map}^+\left(X_{\bullet,k}, W\right)$$

is a weak equivalence. This implies that the restriction map

$$\mathrm{Map}^+\left(\widehat{X}, W\right) \simeq \mathrm{holim}_{\Delta_s} \mathrm{Map}^+\left(X_{\bullet,m}, W\right) \longrightarrow \mathrm{Map}^+\left(X_{\bullet,0}, W\right)$$

is a weak equivalence as required. □

Next we show that the class of maps which are localized by $\widehat{\bullet}$ are exactly the DK-equivalences.

Proposition 6.1.4. *Let $f : X \longrightarrow Y$ be a map of quasi-unital marked semiSegal spaces. Then f is a DK-equivalence if and only if the induced map*

$$f_* : \widehat{X} \longrightarrow \widehat{Y}$$

is an equivalence in CsS.

Proof. From the second part of Theorem 6.1.2 we get that f is a DK-equivalence if and only if

$$f_* : \widehat{X} \longrightarrow \widehat{Y}$$

is a DK-equivalence. Hence the result follows from Proposition 6.0.13.

□

6.2 Proof of the Main Theorem

In this section we will finally address the main goal of this essay by showing that the topological category of complete marked semiSegal spaces is equivalent to the topological category of **complete Segal spaces**, which is one of the standard models for the ∞-category of ∞-categories. We will denote the topological category of complete Segal spaces by CS.

Recall the Quillen adjunction

$$\mathcal{S}^{\Delta^{\mathrm{op}}} \underset{\mathrm{RK}^+}{\overset{F^+}{\rightleftarrows}} \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

described in §§§ 2.2.3 and let $X \in \mathcal{S}^{\Delta^{\mathrm{op}}}$ be a **Segal space**. Then $F^+(X)$ will **almost** be a marked semiSegal space: the underlying semi-simplicial space of $F^+(X)$ will indeed by a semiSegal space, but the marking of F^+ will not necessarily satisfy the requirement of the marked Segal condition. In fact, $F^+(X)$ might not even be fibrant as a marked semi-simplicial space. However, this situation can be rectified in a canonical way.

Define a functor

$$F^\natural : \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{S}_+^{\Delta_s^{\mathrm{op}}}$$

as follows: for each simplicial space X consider

$$F^\natural(X) = (F(X), A),$$

where $F(X)$ is the underlying semi-simplicial space of X and $A \subseteq F(X)_1 = X_1$ is the union of connected components which meet the image of $s_0 : X_0 \longrightarrow X_1$. We have a natural transformation

$$F^+(X) \longrightarrow F^\natural(X)$$

which is a marked equivalence. Furthermore when X is Reedy fibrant we will get that $F^\natural(X)$ is fibrant and so the map above can be considered as a canonical fibrant replacement of $F^+(X)$. Note that the functor RK^+ sends marked-fibrant objects to Reedy fibrant objects. In particular we have the following observation

Lemma 6.2.1. *The functors $F^\natural$ and RK^+ induce an adjunction between the full topological subcategory spanned by the Reedy fibrant simplicial spaces and the full topological subcategory spanned by the marked-fibrant objects.*

Proof. This basically boils down to the fact that if (X, A) is marked-fibrant then A is a union of connected components of X_1 (see Lemma 2.2.12). $\square$

The situation is even a little bit better:

Lemma 6.2.2.
1. *If X is a Segal space then $F^\natural(X)$ is a marked semiSegal space.*
2. *If in addition X is complete then $F^\natural(X)$ is complete as well.*

Proof. Let us start with the first claim. If X is a Segal space then the underlying semi-simplicial space of $F^\natural(X)$ (which is just $F(X)$) is a semiSegal space. Now the marked edges of $F^\natural(X)$ are in the same connected component as degenerate edges, and hence invertible. Furthermore the condition of being in the same connected component as a degenerate map is closed under 2-out-of-3. Hence $F^\natural(X)$ is a marked semiSegal space.

As for the second claim, we note that the restricted face maps $d_0^{\mathrm{inv}}, d_1^{\mathrm{inv}} : X_1^{\mathrm{inv}} \longrightarrow X_0$ are both left-inverses to the map $s_0^{\mathrm{inv}} : X_0 \longrightarrow X_1^{\mathrm{inv}}$ induced by the degeneracy s_0. If X is complete then s_0^{inv} is a weak equivalence which means that both $d_0^{\mathrm{inv}}, d_1^{\mathrm{inv}}$ are weak equivalences. To show that $F^\natural(X)$ is complete it is left to check that all invertible edges are marked - but this is again immediate from the completeness of X. $\square$

Corollary 6.2.3. *If X is a complete Segal space then*

$$F^\natural(X) = (F(X))^\natural.$$

From the above considerations we obtain a functor

$$F^\natural : \mathrm{CS} \longrightarrow \mathrm{CsS}\,.$$

We can consider $F^\natural$ as the forgetful functor which takes an ∞-category and forgets the **identity maps**.

The rest of this section is devoted to proving the following incarnation of our main theorem:

Theorem 6.2.4. *The functor*

$$F^\natural : \mathrm{CS} \longrightarrow \mathrm{CsS}$$

is an equivalence of ∞-categories.

Proof. Let X be a complete marked semiSegal space and consider the counit map

$$F^\natural\left(\mathrm{RK}^+(X)\right) \longrightarrow X.$$

The main ingredient of the proof of Theorem 6.2.4 is given by the following theorem:

Theorem 6.2.5. *Let X be a complete marked semiSegal space. Then the counit map*

$$\nu_X : F^\natural\left(\mathrm{RK}^+(X)\right) \longrightarrow X$$

is a marked equivalence.

Proof. We will start with a lemma which will help us compute $\mathrm{RK}^+(X)$ more easily by replacing the indexing category $\mathcal{C}_n$ with a simpler subcategory.

Lemma 6.2.6. *Let $n \geq 0$ and consider the full subcategory $\mathcal{C}_n^0 \subseteq \mathcal{C}_n$ spanned by objects of the form $f : [m] \longrightarrow [n]$ such that f is **surjective**. Then the inclusion $\mathcal{C}_n^0 \hookrightarrow \mathcal{C}_n$ is cofinal.*

Proof. We need to show that for every object $X \in \mathcal{C}_n$ the category

$$\mathcal{C}_n^0 \times_{\mathcal{C}_n} \mathcal{C}_{n/X}$$

is weakly contractible. Let X be the object corresponding to a morphism $g : [k] \longrightarrow [n]$. The objects of the category $\mathcal{C}_n^0 \times_{\mathcal{C}_n} \mathcal{C}_{n/X}$ can be identified with commutative diagrams of the form

$$\begin{tikzcd} {[k]} \arrow[rr, "h"] \arrow[dr, "g"'] & & {[m]} \arrow[dl, "f"] \\ & {[n]} & \end{tikzcd}$$

such that f is surjective and h is injective (and g remains fixed). A morphism $\mathcal{C}_n^0 \times_{\mathcal{C}_n} \mathcal{C}_{n/X}$ between two diagrams as above as a morphism of diagrams in the opposite direction which is the identity on $[k]$ and $[n]$. A careful examination shows that the category $\mathcal{C}_n^0 \times_{\mathcal{C}_n} \mathcal{C}_{n/X}$ is then in fact isomorphic to the product

$$\mathcal{C}_n^0 \times_{\mathcal{C}_n} \mathcal{C}_{n/X} \cong \prod_{i=0}^{n} \mathcal{E}_i$$

such that

$$\mathcal{E}_i = \begin{cases} \Delta_s^{op} & g^{-1}(i) = \emptyset \\ {\Delta_s^{op}}_{/g^{-1}(i)} & g^{-1}(i) \neq \emptyset \end{cases}$$

When $g^{-1}(i) \neq \emptyset$ then $\mathcal{E}_i$ has a terminal object and so is weakly contractible. When $g^{-1}(i) = \emptyset$ then $\mathcal{E}_i = \Delta_s^{op}$ is contractible as well. In fact, the category Δ_s^{op} is sifted: the inclusion

$$\Delta_s^{\mathrm{op}} \hookrightarrow \Delta_s^{\mathrm{op}} \times \Delta_s^{\mathrm{op}}$$

is cofinal.

This finishes the proof of the lemma.

□

In view of Remark 2.2.20 this means in particular that

$$\mathrm{RK}^{+}(X)_n \simeq \mathrm{holim}_{\mathcal{C}_n^0} \mathcal{G}_n.$$

We now observe the following:

Lemma 6.2.7. *The category $\mathcal{C}_n^0$ is weakly contractible.*

Proof. The category $\mathcal{C}_n^0$ is isomorphic to $(\Delta_s^{op})^n$ (the isomorphism is given by sending a surjective map $f : [m] \longrightarrow [n]$ to the vector of linearly ordered sets $(f^{-1}(0), ..., f^{-1}(n))$ considered as an object of $(\Delta_s^{op})^n$). This means that $\mathcal{C}_n^0$ is weakly contractible. $\square$

Now the claim that

$$\mathrm{RK}^{+}(X)_n \longrightarrow X_n$$

is a homotopy equivalence will follow from the following proposition:

Proposition 6.2.8. *Let X be a complete marked semiSegal space. Suppose we are given a diagram*

$$\begin{tikzcd} {[k]} \arrow[rr, "h"] \arrow[dr, "g"'] & & {[m]} \arrow[dl, "f"] \\ & {[n]} & \end{tikzcd}$$

such that both f, g are surjective and h is injective. Then h induces an equivalence

$$h^* : X_m^f \xrightarrow{\simeq} X_k^g.$$

Proof. Since g is a surjective map between simplices it admits a section $s : [n] \longrightarrow [k]$. One then obtain a sequence

$$X_m^f \xrightarrow{h^*} X_k^g \xrightarrow{s^*} X_n^{\mathrm{Id}}.$$

From the 2-out-of-3 rule we see that it will be enough to prove the lemma for $k = n$ and $g = \mathrm{Id}$. Note that in this case $X_n^{\mathrm{Id}} = X_n$ and we can consider h as a section of f.

According to Proposition 5.4.8 we get that the map

$$X^{(\Delta^m, A_f)} \longrightarrow X^{(\Delta^n)^\flat}$$

is a DK-equivalence. By Propositions 6.0.12 and 6.0.13 this map is a levelwise equivalence. Evaluating at level 0 we get the desired result. $\square$

This finishes the proof of Theorem 6.2.5 $\square$

Corollary 6.2.9. *Let X be a complete marked semiSegal space. Then* $\mathrm{RK}^{+}(X)$ *is a complete Segal space.*

Proof. First since X is fibrant as a marked semi-simplicial space we get that $\mathrm{RK}^+(X)$ is Reedy fibrant. Since

$$F^\natural\left(\mathrm{RK}^+(X)\right) \simeq X$$

we get that $\mathrm{RK}^+(X)$ satisfies the Segal condition and hence is a Segal space. Furthermore, we get that

$$\mathrm{RK}^+(X)_1^{\mathrm{inv}} \simeq X_1^{\mathrm{inv}}$$

and in particular the map

$$d_0 : \mathrm{RK}^+(X)_1^{\mathrm{inv}} \longrightarrow \mathrm{RK}^+(X)_0$$

is a weak equivalence. Since s_0 is a section of d_0 we get that s_0 is a weak equivalence and hence $\mathrm{RK}^+(X)$ is a complete Segal space. □

Now let

$$G : \mathrm{CsS} \longrightarrow \mathrm{CS}$$

be the functor induced by the restriction of RK^+. Then by Lemma 6.2.1 we see that G is the right adjoint of the forgetful functor

$$F^\natural : \mathrm{CS} \longrightarrow \mathrm{CsS}\,.$$

Now let Y be a complete **Segal space** and consider the unit of the adjunction

$$u_Y : Y \longrightarrow G\left(F^\natural(Y)\right).$$

Since the composition

$$F^\natural(Y) \overset{F^\natural(u_Y)}{\longrightarrow} F^\natural\left(G\left(F^\natural(Y)\right)\right) \overset{\nu_{F^\natural(Y)}}{\longrightarrow} F^\natural(Y)$$

is the identity and since the latter map is an levelwise equivalence by Theorem 6.2.5 we get that $F^\natural(u_Y)$ is a levelwise equivalence. This means that u_Y is a levelwise equivalence. Combining this with Theorem 6.2.5 we see that the adjunction $\left(F^\natural, G\right)$ is in fact an equivalence of ∞-categories.

This finishes the proof of Theorem 6.2.4. □

We finish this section with a nice application of Theorem 6.2.4. Let

$$F : \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{S}^{\Delta_s^{\mathrm{op}}}$$

be the forgetful functor. Given a quasi-unital semiSegal space $X \in \mathcal{S}^{\Delta_s^{\mathrm{op}}}$ one can ask whether there exists a **Segal space** $Y \in \mathcal{S}^{\Delta^{\mathrm{op}}}$ such that $F(Y)$ is **levelwise equivalent** to X. Note that if such a Y exists then it is unique up to a levelwise equivalence. The following proposition gives a positive answer to this question:

Proposition 6.2.10. *Let X be a quasi-unital semiSegal space. Then* $\mathrm{RK}^{+}\left(X^{\natural}\right)$ *is a Segal space and the natural map*

$$c : F\left(\mathrm{RK}^{+}(X)\right) \longrightarrow X$$

is a levelwise equivalence. In other words, ***every quasi-unital semiSegal space can be promoted to a Segal space in an essentially unique way****.*

Proof. Let W be a complete semiSegal space such that $W^{\natural} \cong \widehat{X^{\natural}}$ so that we have a DK-equivalence $f : X \longrightarrow W$. Then

$$\begin{tikzcd} X^{\natural} \arrow[r] \arrow[d] & \mathrm{cosk}_0^{+}(X_0) \arrow[d] \\ W^{\natural} \arrow[r] & \mathrm{cosk}_0^{+}(W_0) \end{tikzcd}$$

is a homotopy pullback square in which the lower horizontal map is a marked fibration. Since the functor RK^{+} is a right Quillen functor it preserves pullback squares, marked fibrations and marked equivalences between marked-fibrant objects. This means that the square

$$\begin{tikzcd} \mathrm{RK}^{+}\left(X^{\natural}\right) \arrow[r] \arrow[d] & \mathrm{RK}^{+}\left(\mathrm{cosk}_0^{+}(X_0)\right) \arrow[d] \\ \mathrm{RK}^{+}\left(W^{\natural}\right) \arrow[r] & \mathrm{RK}^{+}\left(\mathrm{cosk}_0^{+}(W_0)\right) \end{tikzcd}$$

is a homotopy pullback square in the Reedy model structure on simplicial spaces. From the uniqueness of right adjoints we get that $\mathrm{RK}^{+} \circ \mathrm{cosk}_0^{+} = \mathrm{cosk}_0$ is the right adjoint to the functor $(\bullet)_0 : \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{S}$, i.e., the familiar coskeleton functor. Hence we obtain a homotopy pullback square

$$\begin{tikzcd} \mathrm{RK}^{+}\left(X^{\natural}\right) \arrow[r] \arrow[d] & \mathrm{cosk}_0(X_0) \arrow[d] \\ \mathrm{RK}^{+}\left(W^{\natural}\right) \arrow[r] & \mathrm{cosk}_0(W_0) \end{tikzcd}$$

From Corollary 6.2.9 we know that $\mathrm{RK}^{+}\left(W^{\natural}\right)$ is a complete Segal space, and it is easy to show that $\mathrm{cosk}_0(X_0)$ and $\mathrm{cosk}_0(W_0)$ are Segal spaces. Now since $\mathrm{RK}^{+}\left(X^{\natural}\right)$ is Reedy fibrant and the map

$$\mathrm{RK}^{+}\left(X^{\natural}\right) \longrightarrow \mathrm{RK}^{+}\left(W^{\natural}\right) \times_{\mathrm{cosk}_0(W_0)} \mathrm{cosk}_0(X_0)$$

is a levelwise equivalence it follows that $\mathrm{RK}^{+}\left(X^{\natural}\right)$ is a Segal space. Furthermore since $\mathrm{RK}^{+}\left(W^{\natural}\right)_0 \simeq W_0$ by Theorem 6.2.5 we can deduce that the map

$$c_0 : F\left(\mathrm{RK}^{+}\left(X^{\natural}\right)\right)_0 \longrightarrow X_0$$

is a weak equivalence. Hence the desired result will be established once we show that c is fully-faithful.

Since the forgetful functor F preserves pullback squares, Reedy fibrations and levelwise equivalences we get that the square

$$\begin{tikzcd} F\left(\mathrm{RK}^{+}\left(X^{\natural}\right)\right) \arrow[r] \arrow[d] & F\left(\operatorname{cosk}_0(X_0)\right) \arrow[d] \\ F\left(\mathrm{RK}^{+}\left(W^{\natural}\right)\right) \arrow[r] & F\left(\operatorname{cosk}_0(W_0)\right) \end{tikzcd}$$

is a homotopy pullback square in the Reedy model structure on semi-simplicial spaces. This implies that the map

$$F\left(\mathrm{RK}^{+}\left(X^{\natural}\right)\right) \longrightarrow F\left(\mathrm{RK}^{+}\left(W^{\natural}\right)\right)$$

is fully-faithful. Now consider the diagram

$$\begin{tikzcd} F\left(\mathrm{RK}^{+}\left(X^{\natural}\right)\right) \arrow[r] \arrow[d, "c"] & F\left(\mathrm{RK}^{+}\left(W^{\natural}\right)\right) \arrow[d] \\ X \arrow[r] & W \end{tikzcd}$$

Since the horizontal maps are fully-faithful and the right vertical map is a levelwise equivalence by virtue of Theorem 6.2.5 we get that c is fully-faithful as required. $\square$

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