The Étale Homotopy Type and the Local-Global Principle

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- Obstructions to the local global principle
- Fixed points and homotopy fixed points

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- Homotopy theoretic obstructions a sketch

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- The étale homotopy type
- Results and applications

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I.e quadrics satisfy the 'local-global principle'.

Not all varieties satisfy the 'local-global principle'. There many known examples where

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such that

$$X(K) \subset X(\mathbb{A})^{\mathsf{Br}} \subset X(\mathbb{A})$$

where $X(\mathbb{A})^{B^{r}}$ is the left kernel of the pairing. All counter-examples that were known until 1999 can be explained by

$$X(\mathbb{A})^{Br} = \emptyset$$

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And constructed a variety such that

 $X_{Sk}(\mathbb{A})^{\mathsf{Br}} \neq \emptyset$

but

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$$X(K) \subset X(\mathbb{A})^{\mathit{fin}} \subset X(\mathbb{A})^{\mathit{fin}-\mathit{Ab}} \subset X(\mathbb{A})$$

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Where $X(\mathbb{A})^{fin}$ and $X(\mathbb{A})^{fin-Ab}$ are the obstruction sets related to torsors under finite and finite-abelian groups respectively.

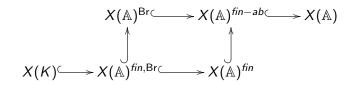
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Where $X(\mathbb{A})^{fin}$ and $X(\mathbb{A})^{fin-Ab}$ are the obstruction sets related to torsors under finite and finite-abelian groups respectively. Harari and Skorobogatov also shoed that $X_{Sk}(\mathbb{A})^{fin} = \emptyset$.

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To conclude there is a diagram of obstruction sets

We shall present a reinterpretation of this diagram in terms of homotopy theoretic properties of X.

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The set X(K) can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a geomtircal/topological object with a continues action by Γ_K and study the fixed points.

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$$F: Var/K \to Top^{\Gamma_K}$$

. Giving for a variety over K a topological space with a Galois action We are going to think of F(X) as some kind of topological realization of X. We shall assume further that F(Speck) is contractible. Note that Since F is a functor we get a map

$$X(K)
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