The Étale Homotopy Type and the Local-Global Principle

Yonatan Harpaz and Tomer Schlank

Department of Mathematics, Hebrew U

tomer.schlank@gmail.com harpazy@gmail.com

November 7, 2012

Outline

Outline

- Obstructions to the local global principle

Outline

- Obstructions to the local global principle
- Fixed points and homotopy fixed points

Outline

- Obstructions to the local global principle
- Fixed points and homotopy fixed points
- Homotopy theoretic obstructions - a sketch

Outline

- Obstructions to the local global principle
- Fixed points and homotopy fixed points
- Homotopy theoretic obstructions - a sketch
- The étale homotopy type

Outline

- Obstructions to the local global principle
- Fixed points and homotopy fixed points
- Homotopy theoretic obstructions - a sketch
- The étale homotopy type
- Results and applications

Obstructions to the local global principle

Given a variety X over a number field K one wants to know:

Obstructions to the local global principle

Given a variety X over a number field K one wants to know:

$$X(K) \stackrel{?}{=} \emptyset$$

If $\mathbb{A}_K$ is the ring of K-adeles we have $X(K) \subset X(\mathbb{A}_K)$ and thus:

Obstructions to the local global principle

Given a variety X over a number field K one wants to know:

$$X(K) \stackrel{?}{=} \emptyset$$

If $\mathbb{A}_K$ is the ring of K-adeles we have $X(K) \subset X(\mathbb{A}_K)$ and thus:

$$X(\mathbb{A}_K) = \emptyset \Rightarrow X(K) = \emptyset$$

Obstructions to the local global principle

Given a variety X over a number field K one wants to know:

$$X(K) \stackrel{?}{=} \emptyset$$

If $\mathbb{A}_K$ is the ring of K-adeles we have $X(K) \subset X(\mathbb{A}_K)$ and thus:

$$X(\mathbb{A}_K) = \emptyset \Rightarrow X(K) = \emptyset$$

Theorem

Hasse-Minkowski If $X \subseteq \mathbb{P}^n$ is hypersurface given by one quadratic equation then

$$X(\mathbb{A}) \neq \emptyset \Rightarrow X(K) \neq \emptyset$$

Obstructions to the local global principle

Given a variety X over a number field K one wants to know:

$$X(K) \stackrel{?}{=} \emptyset$$

If $\mathbb{A}_K$ is the ring of K-adeles we have $X(K) \subset X(\mathbb{A}_K)$ and thus:

$$X(\mathbb{A}_K) = \emptyset \Rightarrow X(K) = \emptyset$$

Theorem

Hasse-Minkowski If $X \subseteq \mathbb{P}^n$ is hypersurface given by one quadratic equation then

$$X(\mathbb{A}) \neq \emptyset \Rightarrow X(K) \neq \emptyset$$

I.e quadrics satisfy the 'local-global principle'.

Obstructions to the local global principle

Not all varieties satisfy the 'local-global principle'. There many known examples where

$$X(\mathbb{A}_K) \neq \emptyset$$

But

$$X(K) = \emptyset$$

Obstructions to the local global principle

Not all varieties satisfy the 'local-global principle'. There many known examples where

$$X(\mathbb{A}_K) \neq \emptyset$$

But

$$X(K) = \emptyset$$

E.g.

$$2y^2 = x^4 - 17$$

[Lind].

Obstructions to the local global principle

Not all varieties satisfy the 'local-global principle'. There many known examples where

$$X(\mathbb{A}_K) \neq \emptyset$$

But

$$X(K) = \emptyset$$

E.g.

$$2y^2 = x^4 - 17$$

[Lind].

Obstructions to the local global principle

In 1971 Manin described a unified way to explain all counter-examples that were known until then.

Obstructions to the local global principle

In 1971 Manin described a unified way to explain all counter-examples that were known until then. There is a natural paring:

$$X(\mathbb{A}) \times \operatorname{Br} X \to \mathbb{Q}/\mathbb{Z}$$

such that

Obstructions to the local global principle

In 1971 Manin described a unified way to explain all counter-examples that were known until then. There is a natural paring:

$$X(\mathbb{A}) \times \operatorname{Br} X \to \mathbb{Q}/\mathbb{Z}$$

such that

$$X(K) \subset X(\mathbb{A})^{\mathrm{Br}} \subset X(\mathbb{A})$$

where $X(\mathbb{A})^{\mathrm{Br}}$ is the left kernel of the pairing. All counter-examples that were known until 1999 can be explained by

$$X(\mathbb{A})^{Br} = \emptyset$$

Obstructions to the local global principle

In 1999 Skorobogatov defined a smaller obstruction set

$$X(K) \subset X(\mathbb{A})^{\acute{e}t,\mathrm{Br}} \subset X(\mathbb{A})^{\mathrm{Br}} \subset X(\mathbb{A})$$

Obstructions to the local global principle

In 1999 Skorobogatov defined a smaller obstruction set

$$X(K) \subset X(\mathbb{A})^{\acute{e}t,\mathrm{Br}} \subset X(\mathbb{A})^{\mathrm{Br}} \subset X(\mathbb{A})$$

And constructed a variety such that

$$X_{Sk}(\mathbb{A})^{\mathrm{Br}} \neq \emptyset$$

but

$$X_{Sk}(\mathbb{A})^{\acute{e}t,\mathrm{Br}} = \emptyset$$

Obstructions to the local global principle

Later [2002] Harari and Skorobogatov defined other obstruction sets related to torsors under algebraic groups over X.

Obstructions to the local global principle

Later [2002] Harari and Skorobogatov defined other obstruction sets related to torsors under algebraic groups over X.We shall consider in this talk the obstructions

$$X(K) \subset X(\mathbb{A})^{fin} \subset X(\mathbb{A})^{fin-Ab} \subset X(\mathbb{A})$$

Obstructions to the local global principle

Later [2002] Harari and Skorobogatov defined other obstruction sets related to torsors under algebraic groups over X.We shall consider in this talk the obstructions

$$X(K) \subset X(\mathbb{A})^{fin} \subset X(\mathbb{A})^{fin-Ab} \subset X(\mathbb{A})$$

Where $X(\mathbb{A})^{fin}$ and $X(\mathbb{A})^{fin-Ab}$ are the obstruction sets related to torsors under finite and finite-abelian groups respectively.

Obstructions to the local global principle

Later [2002] Harari and Skorobogatov defined other obstruction sets related to torsors under algebraic groups over X.We shall consider in this talk the obstructions

$$X(K) \subset X(\mathbb{A})^{fin} \subset X(\mathbb{A})^{fin-Ab} \subset X(\mathbb{A})$$

Where $X(\mathbb{A})^{fin}$ and $X(\mathbb{A})^{fin-Ab}$ are the obstruction sets related to torsors under finite and finite-abelian groups respectively. Harari and Skorobogatov also shoed that $X_{Sk}(\mathbb{A})^{fin} = \emptyset$.

Obstructions to the local global Principle

To conclude there is a diagram of obstruction sets

$$
\begin{array}{ccccccc}
 & & X(\mathbb{A})^{\mathrm{Br}} & \hookrightarrow & X(\mathbb{A})^{fin-ab} & \hookrightarrow & X(\mathbb{A}) \\
 & & \uparrow & & \uparrow & & \\
X(K) & \hookrightarrow & X(\mathbb{A})^{fin,\mathrm{Br}} & \hookrightarrow & X(\mathbb{A})^{fin} & &
\end{array}
$$

Obstructions to the local global Principle

To conclude there is a diagram of obstruction sets

$$\begin{array}{ccccc}
 & & X(\mathbb{A})^{\mathrm{Br}} & \hookrightarrow & X(\mathbb{A})^{fin-ab} & \hookrightarrow & X(\mathbb{A}) \\
 & & \uparrow & & \uparrow & & \\
X(K) & \hookrightarrow & X(\mathbb{A})^{fin,\mathrm{Br}} & \hookrightarrow & X(\mathbb{A})^{fin} & &
\end{array}$$

We shall present a reinterpretation of this diagram in terms of homotopy theoretic properties of X.

Obstructions Related to "homotopical realization"

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a geomtircal/topological object with a continues action by Γ_K and study the fixed points.

Obstructions Related to "homotopical realization"

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a geomtircal/topological object with a continues action by Γ_K and study the fixed points.For that assume we have such a "topological Realization". I.e assume that we are given functor

$$F : Var/K \to Top^{\Gamma_K}$$

. Giving for a variety over K a topological space with a Galois action We are going to think of $F(X)$ as some kind of topological realization of X. We shall assume further that $F(Speck)$ is contractible. Note that Since F is a functor we get a map

$$X(K) \to Map_{\Gamma}(F(Speck), F(X))$$

. What is

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$ Problems:

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$ Problems:

- there is no continuous action of Γ_K on $X(\mathbb{C})$.

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$ Problems:

- there is no continuous action of Γ_K on $X(\mathbb{C})$.
- $X(\mathbb{C})$ actually depend on a non-unique embedding $K \to \mathbb{C}$

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$ Problems:

- there is no continuous action of Γ_K on $X(\mathbb{C})$.
- $X(\mathbb{C})$ actually depend on a non-unique embedding $K \to \mathbb{C}$
- It is quite hard in general to analyze fixed points of a action on topological spaces

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$ Problems:

- there is no continuous action of Γ_K on $X(\mathbb{C})$.
- $X(\mathbb{C})$ actually depend on a non-unique embedding $K \to \mathbb{C}$
- It is quite hard in general to analyze fixed points of a action on topological spaces

Solutions:

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$ Problems:

- there is no continuous action of Γ_K on $X(\mathbb{C})$.
- $X(\mathbb{C})$ actually depend on a non-unique embedding $K \to \mathbb{C}$
- It is quite hard in general to analyze fixed points of a action on topological spaces

Solutions:

- Use Artin and Mazur's étale homotopy type instead of $X(\mathbb{C})$ (we explain what this is later).

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$ Problems:

- there is no continuous action of Γ_K on $X(\mathbb{C})$.
- $X(\mathbb{C})$ actually depend on a non-unique embedding $K \to \mathbb{C}$
- It is quite hard in general to analyze fixed points of a action on topological spaces

Solutions:

- Use Artin and Mazur's étale homotopy type instead of $X(\mathbb{C})$ (we explain what this is later).
- Use homotopy fixed points instead of fixed points (we explain what this is in the next slide).

Fixed points and homotopy fixed points

The set $X(K)$ can be viewed ad the set of fixed points under the Galois action of the set $X(\bar{K})$. We want to think about $X(\bar{K})$ as a topological object with a continues action by Γ_K and study the fixed points.First attempt - one can consider the topology of the space $X(\mathbb{C})$ Problems:

- there is no continuous action of Γ_K on $X(\mathbb{C})$.
- $X(\mathbb{C})$ actually depend on a non-unique embedding $K \to \mathbb{C}$
- It is quite hard in general to analyze fixed points of a action on topological spaces

Solutions:

- Use Artin and Mazur's étale homotopy type instead of $X(\mathbb{C})$ (we explain what this is later).
- Use homotopy fixed points instead of fixed points (we explain what this is in the next slide).

Fixed points and homotopy fixed points

Fixed points are not preserve under maps which are both equivariant and homotopy equivalence.

Fixed points and homotopy fixed points

Fixed points are not preserve under maps which are both equivariant and homotopy equivalence. Example :

$$f : \mathbb{R} \to \{*\}$$

$\mathbb{Z}$ acts on $\mathbb{R}$ by translation.

Fixed points and homotopy fixed points

Fixed points are not preserve under maps which are both equivariant and homotopy equivalence. Example :

$$f : \mathbb{R} \to \{*\}$$

$\mathbb{Z}$ acts on $\mathbb{R}$ by translation.

Definition

Let G be a group acting on a topological space X, we define the *homotopy fixed points on* G to be the space

$$X^{hG} = Map_G(EG, X)$$

where EG is a constructible space with a trivial G action.

Fixed points and homotopy fixed points

Fixed points are not preserve under maps which are both equivariant and homotopy equivalence. Example :

$$f : \mathbb{R} \to \{*\}$$

$\mathbb{Z}$ acts on $\mathbb{R}$ by translation.

Definition

Let G be a group acting on a topological space X, we define the *homotopy fixed points on* G to be the space

$$X^{hG} = Map_G(EG, X)$$

where EG is a constructible space with a trivial G action.

Note that every fixed point gives rise to a homotopy fixed point.

Fixed points and homotopy fixed points

Homotopy fixed points:

Fixed points and homotopy fixed points

Homotopy fixed points:

- Preserved under equivariant homotopy equivalence.

Fixed points and homotopy fixed points

Homotopy fixed points:

- Preserved under equivariant homotopy equivalence.
- There is a sequence of obstructions for $X^{hG} \neq \emptyset$ in $H^{n+1}(G, \pi_n(X))$

Fixed points and homotopy fixed points

Homotopy fixed points:

- Preserved under equivariant homotopy equivalence.
- There is a sequence of obstructions for $X^{hG} \neq \emptyset$ in $H^{n+1}(G, \pi_n(X))$
- The first obstruction is an element in the non-abelian cohomology set $H^2(G, \pi_1(x))$. Vanishing of this obstruction is equivalent to the splitting of a certain short exact

  $$1 \to \pi(X) \to \pi(X_{hG}) \to G \to 1$$

  to split.
- There is a spectral sequence

  $$H^p(G, \pi_q(X)) \Rightarrow \pi_{q-p}(X^{hG})$$

Fixed points and homotopy fixed points

Homotopy fixed points:

Fixed points and homotopy fixed points

Homotopy fixed points:

- Preserved under equivariant homotopy equivalence.

Fixed points and homotopy fixed points

Homotopy fixed points:

- Preserved under equivariant homotopy equivalence.
- There is a sequence of obstructions for $X^{hG} \neq \emptyset$ in $H^{n+1}(G, \pi_n(X))$

Fixed points and homotopy fixed points

Homotopy fixed points:

- Preserved under equivariant homotopy equivalence.
- There is a sequence of obstructions for $X^{hG} \neq \emptyset$ in $H^{n+1}(G, \pi_n(X))$
- The first obstruction is an element in the non-abelian cohomology set $H^2(G, \pi_1(x))$. Vanishing of this obstruction is equivalent to the splitting of a certain short exact

$$1 \to \pi(X) \to \pi(X_{hG}) \to G \to 1$$

  to split.
- There is a spectral sequence

$$H^p(G, \pi_q(X)) \Rightarrow \pi_{q-p}(X^{hG})$$