QUILLEN OBSTRUCTION THEORY - HGA, TOULOUSE

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One of Quillen's important insights was that the notion of **cohomology**, as constructed by algebraic topologists by identifying "holes" in spaces, and by homological algebraists by taking projective/injective resolutions, admit a common view point. This common view point is essentially category theoretic in nature: the notion of cohomology which we associate to some object should be defined intrinsically in terms of the category in which we view our object. To do this right one needs a robust framework for higher category theory. One of the first such frameworks was indeed suggested by Quillen: the theory of model categories.

Given a model category $\mathcal{M}$ (satisfying suitable conditions), and an object $X \in \mathcal{M}$, Quillen considers the category $\mathrm{Ab}(\mathcal{M}_{/X})$ of abelian group objects in the slice category $\mathcal{M}_{/X}$. Under suitable conditions one may endow $\mathrm{Ab}(\mathcal{M}_{/X})$ with a model structure in such a way that the free-forgetful adjunction $\mathcal{F} : \mathcal{M}_{/X} \rightleftarrows \mathrm{Ab}(\mathcal{M}_{/X}) : \mathcal{U}$ becomes a Quillen adjunction. Given an abelian group object $M \in \mathrm{Ab}(\mathcal{M}_{/X})$, Quillen defines the n'th cohomology group $\mathrm{H}^n(X; M)$ of X with coefficients in M by the formula $\mathrm{H}^n(X; M) := \pi_0 \mathrm{Map}_{\mathrm{Ab}(\mathcal{M})}(\mathbb{L}\mathcal{F}(\mathrm{Id}_X), M[n])$, where $M[n]$ denotes the n'th suspension of M in $\mathrm{Ab}(\mathcal{M}_{/X})$. The object $\mathbb{L}\mathcal{F}(\mathrm{Id}_X) \in \mathrm{Ab}(\mathcal{M}_{/X})$ later became known as the **cotangent complex** of X, and consequently denoted by $L_X \in \mathrm{Ab}(\mathcal{M}_{/X})$.

For example, if X is a simplicial set and $M = X \times M_0$ with M_0 a discrete abelian group then this definition recovers the usual definition of cohomology with coefficients in M_0. More generally, we recover cohomology with local and simplicial coefficients. On the more algebraic side, Quillen's framework became most well-known for providing a useful cohomology theory for various types of **algebras**. For example, if A is a commutative dg-algebra then the category of abelian group objects over A is equivalent to the category of A-modules, and L_A can be identified with the classical cotangent complex. In this case Quillen cohomology groups can be viewed in homological algebra terms as the derived functors of derivations.

Despite its success, the classical notion of Quillen cohomology can be limited in others contexts. For example, from the formal point of view, the construction of abelian group objects in a model category is problematic, since it is not invariant under Quillen equivalences, and doesn't always produce the most relevant object for the purpose of taking coefficients. One way to overcome these difficulties is to replace abelianization by **stabilization**. In the setting of ∞-categories this approach was developed by Lurie, who referred to it as the **abstract cotangent complex formalism** (see [Lu14, §7.4]). Given a presentable ∞-category $\mathcal{C}$ and an object $X \in \mathcal{C}$, one can, as above, consider the ∞-category of $\mathcal{C}_{/X}$ of objects over X, but now, instead of taking abelian group obejcts, one considers the ∞-category $\mathrm{Sp}(\mathcal{C}_{/X})$ of **Ω-spectrum objects** in $\mathcal{C}_{/X}$. The latter is a stable presentable ∞-category which is related to $\mathcal{C}_{/X}$ via a canonical adjunction $\Sigma^\infty_+ : \mathcal{C}_{/X} \rightleftarrows \mathrm{Sp}(\mathcal{C}_{/X}) : \Omega^\infty$, analogous

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to the free-forgetful adjunction we had before. Furthermore, this adjuction exhibits $\mathrm{Sp}(\mathcal{C}_{/X})$ as the **stabilization** of $\mathcal{C}_{/X}$, namely, as the universal stable presentable ∞-category admitting such an adjunction. Following [Lu14, §7.4], we will refer to $\mathrm{Sp}(\mathcal{C}_{/X})$ as the **tangent ∞-category** to $\mathcal{C}$ at X, and denote it by $\mathcal{T}_X\mathcal{C} \coloneqq \mathrm{Sp}(\mathcal{C}_{/X})$. Similarly, the coCartesian fibration $\mathcal{T}\mathcal{C} \longrightarrow \mathcal{C}$ classifying the functor $X \mapsto \mathrm{Sp}(\mathcal{C}_{/X})$ is called the **tangnt bundle** of $\mathcal{C}$. Now if $X \in \mathcal{S}$ is a space then $\mathrm{Sp}(\mathcal{S}_{/X})$ is the ∞-category of **parameterized spectra** over X, i.e., families of spectra parameterized by points of X. We will consequently adapt this terminology for any $\mathcal{C}$ and refer to objects of $\mathrm{Sp}(\mathcal{C}_{/X})$ as parameterized spectra over X. Working in this setting one is lead to redefine the cotangent complex of $X \in \mathcal{C}$ simply as its **suspension spectrum** $L_X \coloneqq \Sigma^{\infty}_{+}(\mathrm{Id}_X) \in \mathrm{Sp}(\mathcal{C}_{/X})$. This yields a natural analogue of Quillen cohomology: given a coefficients object $M \in \mathrm{Sp}(\mathcal{C}_{/X})$ one defines the n'th (spectral) Quillen cohomology by the formula $\mathrm{H}^n_{\mathrm{Q}}(X;M) = \pi_0(\mathrm{Map}_{\mathrm{Sp}(\mathcal{C}_{/X})}(L_X, M[n])$. In fact, under the conditions imposed by Quillen one may view classical Quillen cohomology as a particular case of the spectral one, where one restricts attention to coefficients which are Eilenberg-Maclane spectra of strict abelian group objects. We will hence drop the term spectral and simply refer to these invariants as Quillen cohomology.

The Bousfield Kan obstruction theory. When $\mathcal{C} = \mathcal{S}$ is the ∞-category of spaces the above form of Quillen cohomology reproduces generalized cohomology with twisted coefficients: if $X \in \mathcal{S}$ is a space then $\mathrm{Sp}(\mathcal{S}_{/X})$ is the ∞-category of parameterized spectra over X and the Quillen cohomology groups are then given by global sections up to homotopy. The following classical context can be viewed from the perspective of Quillen cohomology: recall that a powerful tool to understand maps $X \longrightarrow Y$ between spaces is to apply the machinery of Bousfield and Kan to the Postnikov tower of Y. This yields an obstruction theory and an associated spectral sequence to compute the homotopy groups $\pi_n \mathrm{Map}(X,Y)$ starting from the cohomology of X with (local) coefficients in the homotopy groups of Y, something that we may think of as a particular case of Quillen cohomology. The main reason why this works is that for $n \geq 1$ the map $P_{n+1}(X) \longrightarrow P_n(X)$ is a **torsor** under the $\mathbb{E}_\infty$-group $K(\pi_{n+1}(X), n+1)$ over $\mathcal{P}_n(X)$.Such torsors can be in fact be defined in a completely abstract setting, where they are known as **small extensions**. In particular, whenever an ∞-category admits a suitable structure of "Postnikov" decompositions into towers of small extensions, one obtain a **Quillen obstruction theory** and associated spectral sequence allowing one to compute homotopy groups of mapping spaces. Such Postnikov structures are actually quite common: under mild conditions, they are inherited from a symmetric monoidal $\mathcal{C}$ to any ∞-category of algebras in $\mathcal{C}$, as well as to the ∞-category of ∞-categories enriched in $\mathcal{C}$.

Deformation theory. A theorem of Lurie (special cases of which were previously established by Basterra-Mandell and Schwede) asserts that if $\mathcal{D}$ is a presentably symmetric monoidal stable ∞-category and $\mathcal{P}$ is some (unital, coherent) ∞-operad then the notion of an Ω-spectrum over a $\mathcal{P}$-algebra A in $\mathcal{D}$ can be identified with the relevant notion of an **A-module**. In other words, we have a natural equivalence of ∞-categories $\mathrm{Sp}(\mathrm{Alg}_{\mathcal{P}}(\mathcal{D}))_{/A}) \simeq \mathrm{Mod}_A(\mathcal{D})$ (see [Lu14, Theorem 7.3.4.13]). Given an A-module M the Quillen cohomology of A with coefficients in M can be understood in terms **derivations**, when the last term is suitable interpreted.

This type of Quillen cohomology often arises in the context of **deformation theory**. Indeed, a “metatheorem” in deformation theory states that whenever an object X is “deformable” then the tangent complex T_X of the associated deformation theory can be obtained from the cotangent complex L_X of X by taking the mapping spectrum $\mathrm{Map}_{\mathcal{T}_X\mathcal{C}}(L_X, M_X)$ to a suitable coefficient object $M_X \in \mathcal{T}_X\mathcal{C}$. This happens, essentially, because first order deformations are also small extensions, and so we may consider this metatheorem as identifying deformation theory as a form of a **Quillen obstruction theory**. In many situations one would like to deform an object which is itself categorical, for example, a dg-category, maybe equipped with some kind of a monoidal structure. To compute the tangent complexes of these deformation problems one needs to know how to compute Quillen cohomology of (possibly structured) enriched categories.

Quillen cohomology of enriched categories. Motivated by the points above, the goal of a current work in progress with Matan Prasma and Joost Nuiten is to understand the tangent categories and Quillen cohomology of enriched categories. This includes, on the one hand, structures such as (∞, n)-categories, and on the other hand, more algebraic type of enrichment, such as dg-categories. The first step towards all of these questions is to understand the notion of a parametertized spectrum over an algebra object in an ∞-category $\mathcal{C}$ which is not necessarily stable. Our first result is hence an extension of Lurie's comparison to the non-stable case:

Theorem 1 ([HNP16a])**.** *Let $\mathcal{C}$ be a closed symmetric monoidal, differentiable presentable ∞-category and let $\mathcal{O}^{\otimes} = \mathrm{N}^{\otimes}(\mathcal{P})$ be the operadic nerve a fibrant simplicial operad. Then for any $\mathcal{O}$-algebra A in $\mathcal{C}$ the forgetful functor induces an equivalence of ∞-categories*

$$\mathcal{T}_A \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{T}_A \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}).$$

Theorem 1 can be understood as computing the stablization of the complicated object $(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))_{/A}$ in terms of the stabilziation of the simpler object $(\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))_{/A}$. In some situations, it is better to have a formulation which presents $\mathcal{T}_A \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ in terms which only involve the tangent categories of $\mathcal{C}$ itself. One way to do so is to employ the language of $\mathcal{O}$-monoidal ∞-categories. If $A \in \mathcal{C}$ is an $\mathcal{O}$-algebra object then $\mathcal{C}_{/A}$ acquires a natural structure of an $\mathcal{O}$-monoidal ∞-category such that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}_{/A}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_{/A}$. Furthermore, for every $\mathcal{O}$-algebra object $f : B \longrightarrow A$ in $\mathcal{C}_{/A}$ we have $\mathrm{Mod}_f^{\mathcal{O}}(\mathcal{C}_{/A}) \simeq (\mathrm{Mod}_B^{\mathcal{O}}(\mathcal{C}))_{/A}$, where we consider A as a B-module via f. We may hence identify $\mathrm{Sp}((\mathrm{Mod}_A^{\mathcal{O}}(\mathcal{C}))_{/A})$ with the stabilization of the module category $\mathrm{Mod}_{\mathrm{Id}_A}^{\mathcal{O}}(\mathcal{C}_{/A})$. Using the machinery of **Day convolutions**, as developed in [Lu14, §2.2.6], one can also construct an induced $\mathcal{O}$-monoidal structure on $\mathcal{T}_A\mathcal{C} = \mathrm{Sp}(\mathcal{C}_{/A})$, such that $\Sigma^{\infty}_{+} : \mathcal{C}_{/A} \longrightarrow \mathrm{Sp}(\mathcal{C}_{/A})$ is naturally an $\mathcal{O}$-monoidal functor. One can then show that the functor Σ^{∞}_{+} induces an equivalence

$$\mathrm{Sp}(\mathrm{Mod}_f^{\mathcal{O}}(\mathcal{C}_{/A})) \xrightarrow{\simeq} \mathrm{Mod}_{\Sigma^{\infty}_{+}(f)}(\mathrm{Sp}(\mathcal{C}_{/A})).$$

One may then rephrase Theorem 1 as follows:

Corollary 2. *Let $\mathcal{C}$ and $\mathcal{O}$ be as in Theorem 1. Then for any $\mathcal{O}$-algebra A in $\mathcal{C}$ the forgetful functor induces an equivalence of ∞-categories*

$$\mathcal{T}_A \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Mod}_{L_{\overline{A}}}(\mathcal{T}_A\mathcal{C})$$

where $L_{\overline{A}}$ denotes the cotangent complex of the ***underlying object*** *$\overline{A} \in \mathcal{C}$ of A.*

Let us say a few words about the method of proof of Theorem 1. While the statement of Theorem 1 is phrased for ∞-categories, it is actually proven in the context of **model categories**, where the ∞-categorical statement is obtained by choosing a model categorical lift to each of the entities. This is the reason for the assumption that the ∞-operad in question is the nerve of a simplicial operad - an assumption that is expected to hold for all ∞-operad. The proof on the level of model categories allows one to deal, in effect, with more general cases, since the model categorical tools which can be applied to simplicial operad work equally well for operads enriched in essentially anything else. We are hence able to obtain the above results for say, dg-operads, without extra cost. This extra generality is also used in the proof to reduce to the case where A is the **initial algebra** by replacing the operad $\mathcal{P}$ with the enveloping algebra $\mathcal{P}_A$. Now when A is initial one usually refers to algebras equipped with a map to A as **augmented algebras**. On the other hand, modules over the initial algebras can be identified with $\mathcal{P}_{\leq 1}$-algebras, where $\mathcal{P}_{\leq 1}$ is the operad obtained from $\mathcal{P}$ by removing all operations in arity ≥ 2. The statement of Theorem 3 then reduces to proving that restriction of structure induces an equivalences

$$\mathrm{Sp}(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{P}}) \xrightarrow{\simeq} \mathrm{Sp}(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{P}_{\leq 1}})$$

between the stabilization of augmented $\mathcal{P}$-algebras and the stabilization of augmented $\mathcal{P}_{\leq 1}$-algebras. We note that the restriction of structure functor admits a left adjoint, the free algebra functor, which, under suitable hypothesis, becomes part of a Quillen adjunction $\mathrm{Sp}(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{P}_{\leq 1}}) \overset{\perp}{\rightleftarrows} \mathrm{Sp}(\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{P}})$, in which the right adjoint detects equivalences. The proof then proceed by consideration the natural filtration

$$\mathcal{P}_{\leq 1} \hookrightarrow \mathcal{P}_{\leq 2} \longrightarrow ...\mathcal{P}_{\leq n} \longrightarrow ...$$

of $\mathcal{P}$, together with the induced filtration on the unit map of the free-forgetful adjunction. Finally, one can show that each step in this filtration is a stable equivalence by showing that the gaps are given by diagonals of multi-reduced functors, and are hence stably trivial.

The statement of Theorem 1 can be applied to compute the tangent categories and Quillen cohomology of **enriched categories**. The main idea is that $\mathbf{S}$-enriched categories with a fixed set of objects can be identified with the category of **algebras** over a suitable operad. Theorem 1 can then be used to deduce the following result:

Theorem 3 ([HNP16b]). *Let $\mathbf{S}$ be a sufficiently nice symmetric monoidal model category and let $\mathcal{T}\mathbf{S} \longrightarrow \mathbf{S}$ be the tangent bundle of $\mathbf{S}$. Let $\mathrm{Cat}_{\mathbf{S}}$ be the model category of $\mathbf{S}$-enriched categories and let $\mathcal{C}$ be a fibrant $\mathbf{S}$-enriched category. Then the tangent model category $\mathcal{T}_{\mathcal{C}}\mathrm{Cat}_{\mathbf{S}}$ is naturally Quillen equivalent to the model category $\mathrm{Fun}^{\mathbf{S}}_{/\mathbf{S}}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathcal{T}\mathbf{S})$ consisting of the $\mathbf{S}$-enriched lifts*

(1)
$$\begin{array}{ccc} & & \mathcal{T}\mathbf{S} \\ & \nearrow & \downarrow \pi \\ \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} & \xrightarrow[\mathrm{Map}_{\mathcal{C}}]{} & \mathbf{S} \end{array}$$

Furthermore, under this equivalence, the cotangent complex of $\mathcal{C}$ corresponds to the desuspension of the composite functor $\Sigma^{\infty}_{\int} \circ \mathrm{Map}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathbf{S} \longrightarrow \mathcal{T}\mathbf{S}$.

The way to deduce Theorem 3 from Theorem 1 is to show that the stabilization of $(\mathrm{Cat}_{\mathbf{S}})_{/\mathcal{C}}$ is equivalent to the stabilization of $(\mathrm{Cat}_{\mathbf{S}}^{\mathrm{Ob}(\mathcal{C})})_{/\mathcal{C}}$, where $(\mathrm{Cat}_{\mathbf{S}}^{\mathrm{Ob}(\mathcal{C})})$ is the category of $\mathbf{S}$-enriched categories with a fixed object set $\mathrm{Ob}(\mathcal{C})$. On the other hand, $(\mathrm{Cat}_{\mathbf{S}}^{\mathrm{Ob}(\mathcal{C})})$ is now a category of algebras over a suitable (colored) operad $\mathcal{P}_{\mathrm{Ob}(\mathcal{C})}$, and the corresponding category of operadic $\mathcal{C}$-modules is equivalent to the category of enriched functors $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathbf{S}$, where the module corresponding to $\mathcal{C}$ itself is the mapping space functor $\mathrm{Map}_{\mathcal{C}}$. One can then deduce from Theorem 1 that the functor

$$\mathcal{T}_{\mathcal{C}} \mathrm{Cat}_{\mathbf{S}} \simeq \mathcal{T}_{\mathcal{C}} \mathrm{Cat}_{\mathbf{S}}^{\mathrm{Ob}(\mathcal{C})} \xrightarrow{\simeq} \mathcal{T}_{\mathrm{Map}_{\mathcal{C}}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathbf{S})$$

which is induced by the forgetful functor $(\mathrm{Cat}_{\mathbf{S}}^{\mathrm{Ob}(\mathcal{C})})_{\mathcal{C}//\mathcal{C}} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathbf{S})$ is an equivalence. On the other hand, parameterized spectrum objects in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathbf{S})$ over $\mathrm{Map}_{\mathcal{C}}$ can be identified with lifts as in (1). The identification of the cotangent complex then requires identifying $L_{\mathcal{C}} \in \mathcal{T}_{\mathcal{C}} \mathrm{Cat}_{\mathbf{S}}$ on the LHS with $L_{\mathrm{Map}_{\mathcal{C}}}[-1] \in \mathcal{T}_{\mathcal{C}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}, \mathbf{S})$ on the RHS. To show this, the first step is to use the fact that the tensor product of enriched categories preserves homotopy colimits in each variable separately in order to reduce to the case where $\mathcal{C} = *$ is the enriched category with one object whose endomorphism object is $1_{\mathbf{S}}$. In this case the above identification gives $\mathcal{T}_* \mathrm{Cat}_{\mathbf{S}} \simeq \mathcal{T}_{1_{\mathbf{S}}} \mathbf{S}$ and one just need to check that under this equivalence $L_* \in \mathcal{T}_* \mathrm{Cat}_{\mathbf{S}}$ corresponds to $L_{1_{\mathbf{S}}}[-1] \in \mathcal{T}_{1_{\mathbf{S}}} \mathbf{S}$. The proof then proceeds by observing that the image of $1_{\mathbf{S}}$ in $\mathrm{Cat}_{\mathbf{S}}^*$ under the left adjoint to the forgetful functor $\mathrm{Cat}_{\mathbf{S}}^* \longrightarrow \mathrm{Fun}(*, \mathbf{S}) = \mathbf{S}$ sends $1_{\mathbf{S}}$ to the category with one object whose endomorphisms are the free associative algebra object generated from $1_{\mathbf{S}}$. This needs to be compared to the (unpointed) suspension of $*$ in $\mathrm{Cat}_{\mathbf{S}}$, which in turn can be identified with the enriched category with one object whose endomorphism object is the free **group-like** associative algebra generated from $1_{\mathbf{S}}$. While these two one-object categories are not the same, one can show that they become equivalent after a single suspension, and hence have equivalent suspension spectra.

Example 4. When $\mathbf{S}$ is the category chain complexes over a field one obtains the notion of a **dg-categories**. The conclusion is then that parameterized spectrum objects over a dg-category $\mathcal{C}$ coincides with the notion of a **$\mathcal{C}$-bimodule**. Furthermore, the Quillen cohomology groups with coefficients of in a $\mathcal{C}$-bimodule $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathbf{S}$ coincide with the corresponding **Hoschshild cohomology**, up to a shift.

Example 5. When $\mathbf{S}$ is the category of simplicial sets one obtains the notion of **simplicial categories**, which is a model for the theory of ∞-categories. In this case, one can make an additional formal maneuver, and translate Theorem 3 into an identification of $\mathcal{T}_{\mathcal{C}} \mathrm{Cat}_{\infty}$ with the ∞-category of functors $\mathrm{Tw}(\mathcal{C}) \longrightarrow \mathrm{Sp}$ from the **twisted arrow category** of $\mathcal{C}$ to spectra. Under this equivalence the cotangent complex of $\mathcal{C}$ corresponds to the constant functor with value $\mathbb{S}[-1]$ and we may identify Quillen cohomology with **functor cohomology** on $\mathrm{Tw}(\mathcal{C})$, up to a shift.

Example 6. When $\mathbf{S}$ is the category of **marked simplicial sets** one obtains the notion of **marked-simplicial categories**, which is a model for the theory of $(\infty, 2)$-categories. In this case, one can apply Theorem 3 twice and use a series of formal maneuvers in order to identify $\mathcal{T}_{\mathbb{C}} \mathrm{Cat}_{(\infty,2)}$ with the ∞-category of functors $\mathrm{Tw}_2(\mathcal{C}) \longrightarrow \mathrm{Sp}$, where $\mathrm{Tw}_2(\mathbb{C})$ is now a certain analogue of the twisted arrow category, which we call the **twisted 2-cell category**. Under this equivalence the cotangent complex of $\mathcal{C}$ corresponds to the constant functor with value $\mathbb{S}[-2]$.

We note that the cotangent complex of $\mathcal{C}$ as an object of $\mathrm{Cat}_{\mathbf{S}}$ is **not** the same as the cotangent complex of $\mathcal{C}$ as an object of $\mathrm{Cat}_{\mathbf{S}}^{\mathrm{Ob}(\mathcal{C})}$. For example, for $\mathrm{Ob}(\mathcal{C}) = *$ the cotangent complex of a one-object enriched category is not the same as the cotangent complex of the corresponding algebra in $\mathbf{S}$. However, the two objects are not that far from each other. We note that $\mathrm{Fun}(\mathbb{B}A^{\mathrm{op}} \otimes \mathbb{B}A, \mathbf{S}) \cong \mathrm{BiMod}_A(\mathbf{S})$ can simply be identified with the category of A-bimodules. Let us denote the underlying A-bimodule of A by ${}_AA_A$ and the underlying left (resp. right) A-module of A by ${}_AA$ (resp. A_A). We already know that both $\mathcal{T}_A \mathrm{Alg}(\mathbf{S})$ and $\mathcal{T}_{\mathbb{B}A} \mathrm{Cat}_{\mathbf{S}}$ can be identified with $\mathrm{Sp}(\mathrm{BiMod}_A(\mathbf{S})_{A//A})$. One can then show that L_A and $L_{\mathbb{B}A}$ are related inside $\mathrm{Sp}(\mathrm{BiMod}_A(\mathbf{S})_{A//A})$ by a natural homotopy cofiber sequence

$$L_A \longrightarrow \mathbb{L}\Sigma^\infty_+({}_AA \otimes A_A) \longrightarrow \mathbb{L}\Sigma^\infty_+({}_AA_A) \simeq L_{\mathbb{B}A}[1]. \tag{2}$$

Remark 7. When $\mathbf{S}$ is **stable**, the cofiber sequence 2 can be written as

$$L_A \longrightarrow A^{\mathrm{op}} \otimes A \longrightarrow A \tag{3}$$

where we view all objects as A-bimodules. This is the $n = 1$ case of the cofiber sequence appearing in [Lu14, Theorem 7.3.5.1] and in [Fra13, Theorem 1.1]. When tensored with the A-bimodule A one obtained a long exact sequence relating the Quillen cohomology and **Hochschild cohomology** of A.

Example 8. When $\mathbf{S} = \mathrm{Ch}_k$ is the category of chain complexes over a field k, A is a discrete algebra and M is a discrete A-bimodule, the cofiber sequence (3) identifies the Quillen cohomology groups $\mathrm{H}^n_Q(A, M)$ for $n \geq 1$ with the Hochschild cohomology group $\mathrm{HH}^{n+1}(A, M)$. For $n = 0$ we obtain instead a surjective map $f_0 : \mathrm{H}^0_Q(A, M) \longrightarrow \mathrm{HH}^1(A, M)$. Unwinding the definitions we see that $\mathrm{H}^0_Q(A, M)$ is the group of derivations $A \longrightarrow M$, $\mathrm{HH}^1(A, M)$ is the group of derivations modulo the inner derivations, and f_0 is the natural map between these two types of data.

Example 9. One may try to combine Corollary 2 and Theorem 3 in order to compute Quillen cohomology of **monoidal dg-categories**. Given a monoidal dg-category $\mathcal{C}$ (with underlying dg-category $\overline{\mathcal{C}}$), the dg-category $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}$ inhertis a monoidal structure, and the induced monoidal structure on $\mathcal{T}_{\overline{\mathcal{C}}} \mathrm{Cat}_{\mathrm{dg}} \simeq \mathrm{Fun}(\overline{\mathcal{C}}^{\mathrm{op}} \otimes \overline{\mathcal{C}}, \mathrm{Ch}_k)$ is given by the associated **shifted Day convolution**. By Theorem 3 the cotangent complex $L_{\overline{C}}$ of $\overline{\mathcal{C}}$ is given by the shifted mapping object functor $L_{\overline{C}} \simeq \mathrm{Map}_{\mathcal{C}}[-1]$, which carries a natural algebra structure with respect the shifted Day convolution. Conjugating everything by a shift, we may use Corollary 2 to identify parameterized spectrum objects over the monoidal dg-category $\mathcal{C}$ with bimodules $\mathcal{F} : \overline{\mathcal{C}}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathrm{Ch}_k$ equipped an action of $\mathrm{Map}_{\mathcal{C}}$. Informally speaking, such an action is given by maps of the form

$$\mathrm{Map}_{\mathcal{C}}(x, y) \otimes \mathcal{F}(a, b) \otimes \mathrm{Map}_{\mathcal{C}}(x', y') \longrightarrow \mathcal{F}(xax', yby')$$

for every $x, y, a, b, x', y' \in \mathcal{C}$, subject to natural compatibility conditions. We note, however, that the cotangent complex of $\mathcal{C}$ as a monoidal dg-category is not $\mathrm{Map}_{\mathcal{C}}$. Instead, the underlying bimodule of $\mathcal{L}_{\mathcal{C}}$ sits in a short exact sequence of the form (2) involving $\mathrm{Map}_{\mathcal{C}}$ and the left Kan extension of $\mathrm{Map}_{\mathcal{C}} \otimes \mathrm{Map}_{\mathcal{C}} : (\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \otimes (\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \longrightarrow \mathrm{Ch}_k$ along the map $(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \otimes (\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}) \longrightarrow \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C}$ induced by the monoidal structure.

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