The Relative Étale Homotopy Type

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Let X be a scheme. A fundamental invariant of X is its étale site $X_{\text{ét}}$. In particular, many essential invariants of X can be described in terms of sheaf cohomology groups

$$H^n(X_{\text{ét}}, F)$$

with respect some étale sheaf. In the realm of algebraic topology cohomology groups appear as homotopy invariants of **spaces**. This leads one to ask whether the abelian invariants of the form $H^n(X_{\text{ét}}, F)$ can be described as cohomology groups of some space. For this to make sense one needs the coefficient sheaf F to have meaning as a coefficient group in which one can take cohomology. This can make sense, for example, if $F = F_A$ is the constant sheaf with value in an abelian group A. In this case one can ask weather there exists a space $|X_{\text{ét}}|$ such that

$$H^n(X_{\text{ét}}, F_A) \cong H^n(|X_{\text{ét}}|, A)$$

Unfortunately the answer to this question in general is no. However, if we agree to except slightly more general objects than spaces then one can obtain a positive answer.

Classical shape theory - associate to each site $\mathcal{C}$ a **pro-object** $\{Y_\alpha\}$ in the **homotopy category of nice spaces** such that

$$H^n(X_{\text{ét}}, F_A) \cong \varinjlim_{\alpha} H^n(Y_\alpha, A) \stackrel{\text{def}}{=} H^n(\{Y_\alpha\}, A)$$

Examples:

1. Classical application - study general spaces (and in particular their Čech cohomology groups) by replacing them with an inverse family of nice spaces.

2. If C is a site with a trivial Grothendieck topology then $|C| = \{N(C)\}$ is an inverse family containing just one space - the nerve of C.

3. When $C = X_{\text{ét}}$ for a scheme X then $|C|$ is known as the étale homotopy type of X, and is denoted by $\text{Ét}(X)$. This notion was first defined and studied by Artin and Mazur in 1969. When X is a variety over $\mathbb{C}$ the étale homotopy type is closely related to the space $X(\mathbb{C})$ of complex points.

Now suppose we are in a relative setting of a scheme X over a base scheme S. A natural problem is to understand the set of sections

$$\overset{\curvearrowleft}{X \longrightarrow S}$$

The basic strategy we wish to consider is to use homotopy theory. In particular, one can hope to use something similar to classical obstruction theory. To understand how this works consider fiber bundle map

$$p : E \longrightarrow B$$

with fiber F (i.e. one can cover B by open sets U such that $p^{-1}(U) \cong U \times F$). Furthermore suppose that B is a nice space, e.g. a CW complex. In this case we can filter B by its skeleton and try to define a section on bigger and bigger the skeletons. This process will result in a series of obstructions which live in the cohomology groups

$$H^{n+1}(B, \pi_n(F))$$

When all obstructions vanish one can (at least in good cases) deduce that a section $s : B \longrightarrow E$ exists. Furthermore when a section exists one has a spectral sequence of pointed sets

$$H^s(B, \pi_t(F)) \Rightarrow \pi_{t-s}(\mathrm{Sec}(p))$$

So that we have a tool to "compute" the space of sections.

How can we apply such homotopical tools to a map of schemes? One approach can be to replace both X and S with their respective étale homotopy types and use homotopy theory to determine whether a section exists on the level of pro-homotopy types. However, this method will often not be sensitive enough. For example consider the map

$$\mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{C}}$$

Since both sides have contractible étale homotopy types one will obtain no information by using homotopy-theoretical obstruction theory. In order to solve this problem we have developed a **relative version** of the étale homotopy type.

By using recent work of Schlank and Barnea one can define for every map

$$f : \mathcal{D} \longrightarrow \mathcal{C}$$

of Grothendieck sites a **relative shape** object. Informally speaking this object is a pro-object in the category of "sheaves of spaces" on $\mathcal{C}$. When $\mathcal{D}$ is the trivial site one obtains a pro-space whose image in the pro-homotopy category is the classical shape of $\mathcal{C}$. Similarly to the absolute case, the relative shape is an object which "represents" the sheaf cohomology groups of $\mathcal{C}$-sheaves which are pulled back from $\mathcal{D}$.

Now given a map $X \longrightarrow S$ one obtains a map of Grothendieck sites

$$S_{\text{ét}} \longrightarrow X_{\text{ét}}$$

We denote by $\acute{\mathrm{E}}\mathrm{t}_{/S}(X)$ the relative shape of X over S. This is a pro-object in the category of "sheaves of spaces" on S. What is a sheaf of spaces on S?

1. When $S = \mathrm{spec}(k)$ for an algebraically closed field then the étale site of S is the trivial site and consequently a sheaf of spaces on S is just a space.
2. When $S = \mathrm{spec}(k)$ for a general field then a sheaf of spaces on S is a space equipped with a continuous action of $\Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.
3. In the general case one can formulate this notion using Quillen's notion of **model categories**.

Now write $\acute{\mathrm{E}}\mathrm{t}_{/S}(X) = \{F_\alpha\}$ where each F_α is a sheaf of spaces on S. Each section

$$X \overset{\curvearrowleft}{\longrightarrow} S$$

gives us a compatible choice of global sections for the F_α's. One can construct an obstruction theory to the existence of such compatible family which leaves in continuous sheaf cohomology groups

$$H^{n+1}_{\mathrm{cont}}(S, \{\pi_n(F_\alpha)\})$$

Furthermore if a section does exist one can construct a spectral sequence

$$H^s_{\mathrm{cont}}(S, \{\pi_t(F_\alpha)\}) \Rightarrow \pi_{t-s}(\mathrm{Sec}(\{F_\alpha\}))$$

References

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