A posteriori error estimates for space-time domain decomposition method for two-phase flow problem

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OUTLINE

- Motivations and problem setting
  1. Robin domain decomposition for a two-phase flow problem
  2. Estimates and stopping criteria in a two-phase flow problem
  3. Numerical experiments
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Geological disposal of nuclear waste

Deep underground repository
(High-level radioactive waste)

Challenges:
- Different materials → strong heterogeneity, different time scales.
- Large differences in spatial scales.
- Long-term computations.
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Develop stopping criteria to stop the DD iterations as soon as the discretization error has been reached
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2. Robin domain decomposition for a two-phase flow problem
3. Estimates and stopping criteria in a two-phase flow problem
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Domain decomposition in space

- Discretize in time and apply the DD algorithm at each time step:
Domain decomposition in space

Discretize in time and apply the DD algorithm at each time step:

- Solve *stationary* problems in the subdomains, in parallel,
- Exchange information through the interface

Following [Halpern-Nataf-Gander (03), Martin (05)]

Different time steps can be used in each subdomain according to its physical properties.

Following [Halpern-C.J.-Szeftel (12), Hoang-C.J.-Jaffré-Kern-Roberts (13)]
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- \textcolor{red}{\textbf{Same time step}} on the whole domain.
Robin domain decomposition for a two-phase flow problem

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Robin domain decomposition for a two-phase flow problem

Two phase flow equation and DD in time

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- Following [Halpern-C.J.-Szeftel (12), Hoang-C.J.-Jaffré-Kern-Roberts (13)]
Two–phase immiscible flow with discontinuous capillary pressure curves

Following [Enchery-Eymard-Michel 06]

Nonlinear (degenerate) diffusion equation in each subdomain

For $f \in L^2(\Omega \times (0, T))$ and a final time $T > 0$, find $u_i : \Omega_i \times [0, T] \rightarrow [0, 1]$, $i = 1, 2$, such that:

$$
\partial_t u_i - \Delta \varphi_i(u_i) = f, \quad \text{in } \Omega_i \times (0, T),
$$

$$
u_i(\cdot, 0) = u_0, \quad \text{in } \Omega_i,
$$

$$
u_i = g_i, \quad \text{on } \Gamma_i^D \times (0, T).
$$

Kirchhoff transform $\varphi_i$

$$
\varphi_i(u_i) = \int_0^{u_i} \lambda_i(a)\pi_i'(a)da
$$

Capillary pressure

$\pi_i(u_i) : [0, 1] \rightarrow \mathbb{R}$

Global mobility of the gas

$\lambda_i(u_i) : [0, 1] \rightarrow \mathbb{R}$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$
- $u$ scalar unknown gas saturation
- $1 - u$ is the water saturation
- $u_0$ initial gas saturation
- $g$ boundary gas saturation
with the nonlinear interface conditions (physical transmission conditions)

\[
\nabla \varphi_1(u_1) \cdot n_1 = -\nabla \varphi_2(u_2) \cdot n_2, \quad \text{on } \Gamma \times (0, T),
\]
\[
\pi_1(u_1) = \pi_2(u_2), \quad \text{on } \Gamma \times (0, T),
\]
with the nonlinear interface conditions (physical transmission conditions)

\[
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\]
\[
\pi_1 (u_1) = \pi_2 (u_2), \quad \text{on } \Gamma \times (0, T),
\]

\[
\Pi_i (u) := \int_{\pi_i \pi_2 (0)} \min_{j \in \{1, 2\}} (\lambda_j \circ \pi_j - 1_j (u)) \, du.
\]

\[\cdots\] Following [Chavent - Jaffré (86), Enchéry et al. (06), Cances (08), Ern et al (10), Brenner et al. (13)]
with the nonlinear interface conditions (physical transmission conditions)

\[
\nabla \varphi_1(u_1) \cdot n_1 = -\nabla \varphi_2(u_2) \cdot n_2, \quad \text{on } \Gamma \times (0, T),
\]

\[
\bar{\pi}_1(u_1) = \bar{\pi}_2(u_2), \quad \text{on } \Gamma \times (0, T),
\]

\[
\pi_{1(0)} \leq u_1 \leq \pi_{1(1)}, \quad \pi_{2(0)} \leq u_2 \leq \pi_{2(1)}.
\]

\[
\Pi_i(u) := \int_\pi_i \pi_2(0) \min_{j \in \{1, 2\}} (\lambda_j \circ \pi - \pi_{1j}(u)) \, du.
\]

... Following [Chavent - Jaffré (86), Enchéry et al. (06), Cances (08), Ern et al (10), Brenner et al. (13)]

where \( \bar{\pi}_1 : u \mapsto \max(\pi_1(u), \pi_2(0)) \) and \( \bar{\pi}_2 : u \mapsto \min(\pi_2(u), \pi_1(1)) \)
with the nonlinear interface conditions (physical transmission conditions)

\[ \nabla \varphi_1(u_1) \cdot n_1 = - \nabla \varphi_2(u_2) \cdot n_2, \quad \text{on } \Gamma \times (0, T), \]
\[ \bar{\pi}_1(u_1) = \bar{\pi}_2(u_2), \quad \text{on } \Gamma \times (0, T), \]

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Robin domain decomposition for a two-phase flow problem

Multidomain problem: Physical form

with the nonlinear interface conditions (physical transmission conditions)

\[
\nabla \varphi_1(u_1) \cdot n_1 = -\nabla \varphi_2(u_2) \cdot n_2, \quad \text{on } \Gamma \times (0, T),
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\[
\overline{\pi}_1(u_1) = \overline{\pi}_2(u_2), \quad \text{on } \Gamma \times (0, T), \quad \Leftrightarrow \quad \Pi_1(u_1) = \Pi_2(u_2)
\]

⋯ Following [Chavent - Jaffré (86), Enchéry et al. (06), Cances (08), Ern et al (10), Brenner et al. (13)]

- \( \overline{\pi}_1 : u \mapsto \max(\pi_1(u), \pi_2(0)) \) and \( \overline{\pi}_2 : u \mapsto \min(\pi_2(u), \pi_1(1)) \)

- \( \Pi_i(u) := \int_{\pi_2(0)}^{\overline{\pi}_i} \min_j (\lambda_j \circ \pi_j^{-1}(u)) \, du \quad \cdots \quad \text{smoother than } \overline{\pi}_i \)
Robin domain decomposition for a two-phase flow problem

Multidomain problem: Physical form

with the nonlinear interface conditions (Robin transmission conditions)

\[
\nabla \varphi_1(u_1) \cdot n_1 + \alpha_{1,2} \Pi_1(u_1) = -\nabla \varphi_2(u_2) \cdot n_2 + \alpha_{1,2} \Pi_2(u_2),
\]

\[
\nabla \varphi_2(u_2) \cdot n_2 + \alpha_{2,1} \Pi_2(u_2)) = -\nabla \varphi_1(u_1) \cdot n_1 + \alpha_{2,1} \Pi_1(u_1),
\]

where \(\alpha_{i,j}\) are free parameters which optimized convergence rates.

\[
\pi_i : u \mapsto \max(\pi_1(u), \pi_2(0)) \quad \text{and} \quad \pi_2 : u \mapsto \min(\pi_2(u), \pi_1(1))
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\[\pi_1 : u \mapsto \max(\pi_1(u), \pi_2(0)) \text{ and } \pi_2 : u \mapsto \min(\pi_2(u), \pi_1(1))\]

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\[\cdots \text{Following [Chavent - Jaffré (86), Enchéry et al. (06), Cances (08), Ern et al (10), Brenner et al. (13)]}\]

\[\cdots \text{Extended to the Ventcell DD method in [Ahmed-S-A.H.-Japhet-Kern-Vohralík (18)]}\]
We now define a **weak solution** to this problem which satisfies:

1. \( u \in H^1(0, T; H^{-1}(\Omega)) \);
2. \( u(\cdot, 0) = u_0 \);
3. \( \varphi_i(u_i) \in L^2(0, T; H^1_{\varphi_i(g_i)}(\Omega_i)) \), where \( u_i := u|_{\Omega_i}, i = 1, 2 \);
   
   \[ \cdots \text{ where } H^1_{\varphi_i(g_i)}(\Omega_i) := \{ v \in H^1(\Omega_i), v = \varphi_i(g_i) \text{ on } \Gamma_i^D \} \]
4. \( \Pi(u, \cdot) \in L^2(0, T; H^1_{\Pi(g, \cdot)}(\Omega)) \);
   
   \[ \cdots \text{ where } H^1_{\Pi(g, \cdot)}(\Omega) := \{ v \in H^1(\Omega), v = \Pi(g, \cdot) \text{ on } \partial \Omega \} \]
5. For all \( \psi \in L^2(0, T; H^1_0(\Omega)) \), the following integral equality holds:

\[
\int_0^T \left\{ \langle \partial_t u, \psi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + \sum_{i=1}^2 (\nabla \varphi_i(u_i), \nabla \psi)_{\Omega_i} - (f, \psi) \right\} dt = 0.
\]
OSWR algorithm

For $k \geq 0$, at step $k$, solve in parallel the space-time Robin subdomain problems $(i = 1, 2)$:

$$\partial_t u_i^k - \Delta \varphi_i(u_i^k) = f_i, \quad \text{in } \Omega_i \times (0, T),$$
$$u_i^k(\cdot, 0) = u_0, \quad \text{in } \Omega_i,$$
$$\varphi_i(u_i^k) = \varphi_i(g_i), \quad \text{on } \Gamma_i^D \times (0, T),$$
$$\nabla \varphi_i(u_i^k) \cdot n_i + \alpha_{i,j} \Pi_i(u_i^k) = \Psi_i^{k-1}, \quad \text{on } \Gamma \times (0, T),$$

with

- $\Psi_i^{k-1} := -\nabla \varphi_j(u_j^{k-1}) \cdot n_j + \alpha_{i,j} \Pi_j(u_j^{k-1}), \quad j = (3 - i), \ k \geq 2,$
- $\Psi_i^0$ is an initial Robin guess on $\Gamma \times (0, T)$.

\cdots \text{well-posedness of Robin problem following [Ahmed-Japhet-Kern, in preparation]}
The discrete solution is found using the cell centered finite volume scheme in space and the backward Euler scheme in time for the subdomain problem. Following [Enchéry-Eymard-Michel (2006)]

\[ u_{h,i}^{k,n} \in P_0(\mathcal{T}_h,i) \times P_0(\mathcal{E}_h^\Gamma) \]: unknown discrete saturation at each time step \( 0 \leq n \leq N \).
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\[ u_{h,i}^{k,n} \in \mathbb{P}_0(T_{h,i}) \times \mathbb{P}_0(E_{h}^\Gamma) : \text{unknown discrete saturation at each time step } 0 \leq n \leq N \]

At each OSWR DD step \( k \geq 1 \) and each time step \( n \geq 1 \), Newton–Raphson iterative linearization procedure is used to linearize the local Robin problem. At each linearization step \( m \geq 1 \), find \( u_{h,i}^{k,n,m} \in \mathbb{P}_0(T_{h,i}) \times \mathbb{P}_0(E_{h}^\Gamma) \)

Define \( u_{h,i}^{k,m}|_{I_n} := u_{h,i}^{k,n,m} \) where \( I_n \) is a subinterval in time.

For a posteriori estimates: \( P_1^\tau \) continuous, piecewise affine in time functions.
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Estimates and stopping criteria in a two-phase flow problem

Strategy

\[ \| u - \tilde{u}_{h_T}^{k,m} \|_\# \leq \eta_{k,m}^{sp} + \eta_{k,m}^{DD} + \eta_{k,m}^{tm} + \eta_{k,m}^{lin} \]

more recent results on coupling DD and a posteriori error estimates

[Becker-Johnson-Rannacher (95), Arioli (04), Arioli-Loghin (07), Patera & Rønquist (01), Meidner-Rannacher-Vihharev (09), Jiránek-Strakoš-Vohralík (10), Ern-Vohralík (13)]

more recent results on Dirichlet & Neumann subdomain problems

\[ H(\text{div}, \Omega) \] flux at each DD iteration

following [Prager-Synge (47), Ladevèze-Pelle (05), Repin (08), Ern-Vohralík (15)]

not applicable to more general (e.g., Robin, Ventcell) transmission conditions

in our contribution: develop a posteriori estimates for DD algorithms where on the interfaces, neither the conformity of the flux nor that of the saturation are preserved for unsteady degenerated non-linear problem

following [Nochetto-Schmidt-Verdi (00), Cancès-Pop-Vohralík (14), Di Pietro-Vohralík-Yousef (15)]

unknown
\[ \| u - u_{h,T}^{k,m} \|_{\#} \leq \text{Fully computable estimators} \]
\[ \| u - \tilde{u}_{h^T}^{k,m} \| \leq \text{Fully computable estimators} \]

\[ \text{Goal: } \| u - \tilde{u}_{h^T}^{k,m} \| \leq \eta_{sp}^{k,m} + \eta_{DD}^{k,m} + \eta_{tm}^{k,m} + \eta_{lin}^{k,m} \]
 Estimates and stopping criteria in a two-phase flow problem

**Strategy**

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- Results on a posteriori error estimates valid during the iteration of an algebraic solver
  [Becker-Johnson-Rannacher (95), Arioli (04), Arioli-Loghin (07), Patera & Rønquist (01), Meidner-Rannacher-Vihharev (09), Jiránek-Strakoš-Vohralík (10), Ern-Vohralík (13)]
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More recent results on coupling DD and a posteriori error estimates [V.Rey-C.Rey-Gosselet (14)]
Dirichlet & Neumann subdomain problems \( \Rightarrow H(\text{div}, \Omega) \) flux at each DD iteration
Following [Prager-Synge (47), Ladevèze-Pelle (05), Repin (08), Ern-Vohralík (15)]

\[ \text{not applicable to more general (e.g. Robin, Ventcell) transmission conditions} \]
Estimates and stopping criteria in a two-phase flow problem

**Strategy**

\[ \| u - \tilde{u}_{h,T}^{k,m} \|_{\text{unknown}} \leq \text{Fully computable estimators} \]

**Goal:**

\[ \| u - \tilde{u}_{h,T}^{k,m} \|_{\text{unknown}} \leq \eta_{\text{isp}}^{k,m} + \eta_{\text{DD}}^{k,m} + \eta_{\text{tm}}^{k,m} + \eta_{\text{lin}}^{k,m} \]

Results on a posteriori error estimates valid during the iteration of an algebraic solver

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In our contribution: develop a posteriori estimates for DD algorithms where on the interfaces, neither the conformity of the flux nor that of the saturation are preserved for unsteady degenerated non linear problem

Following [Nochetto-Schmidt-Verdi (00), Cancès-Pop-Vohralík (14), Di Pietro-Vohralík-Yousef (15)]
Unsteady diffusion equation

\[ u = -S \nabla p, \quad \text{in} \quad \Omega \times (0, T) \]

\[ \phi \frac{\partial p}{\partial t} + \nabla \cdot u = f, \quad \text{in} \quad \Omega \times (0, T) \]

\[ p = g_D \quad \text{on} \quad \Gamma_D \cap \partial \Omega \times (0, T) \]

\[ -u \cdot n = g_N \quad \text{on} \quad \Gamma_N \cap \partial \Omega \times (0, T) \]

\[ p(\cdot, 0) = p_0 \quad \text{in} \quad \Omega \]


In this contribution: we take up the path initiated in the two papers above
Estimates and stopping criteria in a two-phase flow problem

Strategy

\[ \| u - \tilde{u}^{k,m}_{h,T} \| \leq \]  

**Fully computable estimators**

depend on \( H(\text{div}, \Omega) \) flux and a saturation which have good properties
Estimates and stopping criteria in a two-phase flow problem

\[ \| u - \tilde{u}_{h,\tau}^k,m \|_{\#} \leq \text{unknown} \]

Fully computable estimators depend on \( H(\text{div}, \Omega) \) flux and a saturation which have good properties

\[ \text{FV method gives } u_{h,i}^{k,n,m} \notin H^1(\Omega_i), \ i = 1, 2 \implies \left\{ \begin{array}{l} \varphi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \\ \Pi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \implies \Pi(u_{h}^{k,n,m}) \notin H^1(\Omega) \end{array} \right. \]
Strategy

\[ \| u - \tilde{u}_{h}^{k,m} \|_{\#} \leq \text{unknown} \]

**Fully computable estimators**

depend on \( H(\text{div}, \Omega) \) flux and a saturation which have good properties

- FV method gives \( u_{h,i}^{k,n,m} \notin H^1(\Omega_i), i = 1, 2 \)  \[ \begin{cases} 
\varphi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \\
\Pi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \end{cases} \]  \[ \Rightarrow \]  \( \Pi(u_{h}^{k,n,m}) \notin H^1(\Omega) \)

- Robin DD method gives \( u_{h}^{k,n,m} \notin H(\text{div}, \Omega) \) and \( \Pi(u_{h}^{k,n,m}) \) jumps across \( \Gamma \)
Estimates and stopping criteria in a two-phase flow problem

Strategy

\[ \| u - \tilde{u}_{h,\tau}^k \|_{\#} \leq \] unknown

- Fully computable estimators depend on \( \mathcal{H}(\text{div}, \Omega) \) flux and a saturation which have good properties

- FV method gives \( u_h^k, n, m \notin H^1(\Omega_i), i = 1, 2 \implies \begin{cases} \varphi_i(u_h^k, n, m) \notin H^1(\Omega_i) \\ \Pi_i(u_h^k, n, m) \notin H^1(\Omega_i) \implies \Pi(u_h^k, n, m) \notin H^1(\Omega) \end{cases} \)

- Robin DD method gives \( u_h^k, n, m \notin H(\text{div}, \Omega) \) and \( \Pi(u_h^k, n, m) \) jumps across \( \Gamma \)

Strategy:

- Follow [Nochetto-Schmidt-Verdi (00), Cancès-Pop-Vohralík (14), Di Pietro-Vohralík-Yousef (15), S-A.H., C. Japhet, M. Kern, and M. Vohralík (18)]
- Extension to Robin DD for nonlinear problem in this work
Estimates and stopping criteria in a two-phase flow problem

\[ \| u - \tilde{u}_{h,T}^{k,m} \|_{\#} \leq \text{unknown} \]

**Fully computable estimators**

depend on \( H(\text{div}, \Omega) \) flux and a saturation which have good properties

- FV method gives \( u_{h,i}^{k,n,m} \notin H^1(\Omega_i), \ i = 1, 2 \Rightarrow \begin{cases} \varphi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \\ \Pi_i(u_{h,i}^{k,n,m}) \notin H^1(\Omega_i) \Rightarrow \Pi(u_{h}^{k,n,m}) \notin H^1(\Omega) \end{cases} \)

- Robin DD method gives \( u_h^{k,n,m} \notin H(\text{div}, \Omega) \) and \( \Pi(u_h^{k,n,m}) \) jumps across \( \Gamma \)

**Strategy:**

- Follow [Nochetto-Schmidt-Verdi (00), Cancès-Pop-Vohralík (14), Di Pietro-Vohralík-Yousef (15), S-A.H., C. Japhet, M. Kern, and M. Vohralík (18)]
- Extension to Robin DD for nonlinear problem in this work

**Postprocessing:** \( \tilde{u}_{h,T}^{k,m} \) (\( u_{h,T}^{k,m} \) is piecewise constant and not suitable for the energy norm)

where \( \tilde{u}_{h,T}^{k,m} := \varphi_i^{-1}(\tilde{\varphi}_{h,T,i}^{k,m}) \) with \( \tilde{\varphi}_{h,T,i}^{k,m} \in P_1(\mathcal{P}_2(T_h,i)) \)

\( \tilde{u}_{h,T}^{k,m} \) used for theoretical analysis and \( \tilde{\varphi}_{h,T,i}^{k,m} \) used in practice for the estimators
Estimates and stopping criteria in a two-phase flow problem

\[ \| u - \tilde{u}_{h,\tau}^{k,m} \| \leq \]  

\textbf{Fully computable estimators}

depend on \( H(\text{div}, \Omega) \) flux and a saturation which have good properties

\[ \times \]  

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\[ \times \]  

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\[ \bullet \]  

\textbf{Saturation and flux reconstructions:}

- Reconstruction saturation \( s_{h,i}^{k,n,m} := \varphi_i^{-1} (\varphi_{h,i}^{k,n,m}) \)

where \( \varphi_{h,\tau,i}^{k,m} \in P_1(T_h,i) \cap H^1(\Omega_i) \)-conforming in each subdoamin
modified to ensure the continuity across the interface: \( \Pi_1(s_{h,1}^{k,n,m}) = \Pi_2(s_{h,2}^{k,n,m}) \)

\( \sigma_{h,\tau}^{k,m} : H(\text{div}, \Omega) \)-conforming and local conservative in each element, piecewise constant in time
Strategy

Potential reconstructions (2 subdomains)

\( u_h^{k,n,m} \) (from DD solver)

\( \varphi_h^{k,n,m} \): postprocessing

\[ \tilde{u}_{h,T}^{k,m} := \varphi_i^{-1}(\varphi_{h,T,i}^{k,m}) \]

\( \varphi_h^{k,n,m} \in H^1(\Omega_i) \)

\[ s_{h,i}^{k,n,m} := \varphi_i^{-1}(\varphi_h^{k,n,m}) \]
Following [Di Pietro-Vohralík-Yousef (14), Cancès-Pop-Vohralík (14)]

**Extension to Robin DD**

\[
Q_{t,i} := L^2(0, t; L^2(\Omega_i)), \quad X_t := L^2(0, t; H^1_0(\Omega)), \quad X'_t := L^2(0, t; H^{-1}(\Omega)).
\]

\[
\| u - \tilde{u}^{k,m}_h \|_\#^2 := \sum_{i=1}^2 \| \varphi_i(u_i) - \varphi_i(\tilde{u}^{k,m}_h) \|^2_{Q_{t,i}} + \frac{L\varphi}{2} \| u - \tilde{u}^{k,m}_h \|_X^2 + \frac{L\varphi}{2} \| (u - \tilde{u}^{k,m}_h)(\cdot, t) \|^2_{H^{-1}(\Omega)}.
\]

\[
\| u - \tilde{u}^{k,m}_h \|_\#^2 := \| u - \tilde{u}^{k,m}_h \|_\#^2 + 2 \sum_{i=1}^2 \int_0^T \left( \| \varphi_i(u_i) - \varphi_i(\tilde{u}^{k,m}_h) \|^2_{Q_{t,i}} + \int_0^t \| \varphi_i(u_i) - \varphi_i(\tilde{u}^{k,m}_h) \|^2_{Q_{s,i}} e^{t-s} ds \right) dt.
\]

where \( L\varphi \) is the maximal Lipschitz constant of the functions \( \varphi_i \)

**Theorem**

If \( \bar{\varphi} \in L^2(0, T; H^1_0(\Omega)) \), where \( \bar{\varphi}|_{\Omega_i} := \varphi_i(u_i) - \varphi_i(s^{k,m}_{h\tau,i}), \quad i = 1, 2 \), then

\[
\| u - \tilde{u}^{k,m}_h \|_\# \leq \sqrt{\frac{L\varphi}{2} \sqrt{2e^T - 1}} \eta_{IC}^{k,m} + \eta_{sp}^{k,m} + \eta_{tm}^{k,m} + \eta_{dd}^{k,m} + \eta_{lin}^{k,m}
\]

which depend on \( \sigma^{k,m}_{h\tau}, \tilde{\varphi}^{k,m}_{h\tau,i}, \tilde{\varphi}^{k,m}_{h\tau,i} \)
0 - Postprocessing function $\tilde{\varphi}_{h,i}^{k,n,m}$ of $\varphi_i(u_{h,\tau,i}^{k,m})$

$\tilde{\varphi}_{h,i}^{k,n,m} \in P_2(T_h,i)$ at each iteration $k$, at each time step $n$, $n = 0, \ldots, N$, and at each linearization step $m$, is constructed as:

$$
-\nabla \tilde{\varphi}_{h,i}^{k,n,m} |_K = u_{h,i}^{k,n,m} |_K, \quad \forall K \in T_h,i,
$$

$$
\frac{(\varphi^{-1}(\tilde{\varphi}_{h,i}^{k,n,m}), 1)_K}{|K|} = u_K^{k,n,m} |_K, \quad \forall K \in T_h,i.
$$

$\tilde{\varphi}_{h,i}^{k,n,m} \notin H^1(\Omega_i)$
0 - Postprocessing function $\tilde{\varphi}_{h,i}^{k,n,m}$ of $\varphi_i(u_{hT,i}^{k,m})$

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$$\frac{(\varphi^{-1}(\tilde{\varphi}_{h,i}^{k,n,m}), 1)_K}{|K|} = u_{h,i}^{k,n,m} |_K, \quad \forall K \in T_{h,i}.$$

- $\tilde{\varphi}_{h,i}^{k,n,m} \notin H^1(\Omega_i)$

1 - Piecewise continuous polynomial $\hat{\varphi}_{h,i}^{k,n,m}$ in each subdomain

$$\hat{\varphi}_{h,i}^{k,n,m}(\mathbf{x}) := \mathcal{I}_{av}(\tilde{\varphi}_{h,i}^{k,n,m})(\mathbf{x}) = \frac{1}{|T_{\mathbf{x}}|} \sum_{K \in T_{\mathbf{x}}} \tilde{\varphi}_{h,i}^{k,n,m} |_K(\mathbf{x}) \in P_2(T_{h,i}) \cap H^1(\Omega_i)$$

$$\hat{\varphi}_{h,i}^{k,n,m}(\mathbf{x}) := \varphi_i(g_i(\mathbf{x}))$$ on $\Gamma^D_i$. 
2 - Reconstruction saturation

Reconstruction saturation in each subdomain: \( s_h^{k,n,m}|_{\Omega_i} := \varphi_i^{-1}(\hat{\varphi}_h^{k,n,m}) \)

According to the weak solution \( u \), we require that

- \( s_h^{k,n,m}|_{\Omega_i} \in H^1(0, T; H^{-1}(\Omega)) \)
- \( \varphi_i(s_h^{k,n,m}) \in L^2(0, T; H^1_{\varphi_i(g_i)}(\Omega_i)) \)

\[ ... \varphi_i(s_h^{k,n,m}|_{\Omega_i}) := \varphi_i(\varphi_i^{-1}(\hat{\varphi}_h^{k,n,m})) = \hat{\varphi}_h^{k,n,m} \in H^1_{\varphi_i(g_i)}(\Omega_i) \]

- \( \Pi_1(s_h^{k,n,m}|_{\Omega_1}) = \Pi_2(s_h^{k,n,m}|_{\Omega_2}) \) on \( \Gamma \)

\[ ... \text{where } \Pi_i, 1 \leq i \leq 2, \text{ is chosen as follows:} \]

\[ \Pi_i(s_h^{k,n,m}|_{\Omega_i}(x_\Gamma)) = \frac{\Pi_i(\varphi_i^{-1}(\hat{\varphi}_h^{k,n,m}(x_\Gamma)))) + \Pi_j(\varphi_j^{-1}(\hat{\varphi}_h^{k,n,m}(x_\Gamma))))}{2}. \]

- \( \frac{1}{|K|}(s_h^{k,n,m}, 1)_K = u_K^{k,n,m}, \quad \forall K \in \mathcal{T}_h \)

\[ ... \text{using suitable constants } \alpha_K^{k,n,m} \text{ and the } b_K \text{ the bubble function on } K. \]
3 - Equilibrated flux reconstruction $\sigma_{h,T}^{k,m}$

$$
\sigma_{h,T}^{k,m} \in P_0^0(\mathbf{H}(\text{div}, \Omega)),
$$

$$
\left( f^n - \frac{u_{K}^{k,n,m} - u_{K}^{k,n-1}}{\tau^n} - \nabla \cdot \sigma_{h}^{k,n,m}, 1 \right)_K = 0, \quad \forall K \in T_h.
$$

- Average of the fluxes on the interface

where $B_1$ and $B_2$ are the two bands surrounding the interface $\Gamma$ in 3D
Estimates and stopping criteria in a two-phase flow problem
Reconsuction techniques

3 - Equilibrated flux reconstruction $\sigma_{h_T}^{k,m}$

$$\sigma_{h_T}^{k,m} \in P_0^0(\mathbf{H}(\text{div}, \Omega)),$$

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- Average of the fluxes on the interface
- Misfit of mass balance in each subdomain

where $B_1$ and $B_2$ are the two bands surrounding the interface $\Gamma$ in 3D
Estimates and stopping criteria in a two-phase flow problem

Reconstruction techniques

3 - Equilibrated flux reconstruction $\sigma_{h_\tau}^{k,m}$

$$\sigma_{h_\tau}^{k,m} \in P^0_{\tau}(H(\text{div}, \Omega)),$$

$$\left( f^n - \frac{u_K^{k,n,m} - u_K^{k,n-1}}{\tau^n} - \nabla \cdot \sigma_{h_\tau}^{k,n,m}, 1 \right)_K = 0, \quad \forall K \in T_h.$$

- Average of the fluxes on the interface
- Misfit of mass balance in each subdomain
- Distribute the misfit by coarse grid problem

where $B_1$ and $B_2$ are the two bands surrounding the interface $\Gamma$ in 3D
3 - Equilibrated flux reconstruction $\sigma_{h,T}^{k,m}$

$$\sigma_{h,T}^{k,m} \in P_0^0(H(\text{div}, \Omega)),$$

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- Average of the fluxes on the interface
- Misfit of mass balance in each subdomain
- Distribute the misfit by coarse grid problem
- Add the corrections to the averages

where $B_1$ and $B_2$ are the two bands surrounding the interface $\Gamma$ in 3D.
3 - Equilibrated flux reconstruction $\sigma^{k,m}_{h\tau}$

$$\sigma^{k,m}_{h\tau} \in P^{0}_\tau(H(\text{div}, \Omega)),$$

$$\left( f^n - \frac{u_{k,n,m}^k - u_{k,n-1}^k}{\tau^n} - \nabla \cdot \sigma^{k,n,m}_{h}, 1 \right)_{K} = 0, \quad \forall K \in T_h.$$

- Average of the fluxes on the interface
- Misfit of mass balance in each subdomain
- Distribute the misfit by coarse grid problem
- Add the corrections to the averages
- Solve local Neumann problem in the bands

where $B_1$ and $B_2$ are the two bands surrounding the interface $\Gamma$ in 3D
OUTLINE

1. Motivations and problem setting
2. Robin domain decomposition for a two-phase flow problem
3. Estimates and stopping criteria in a two-phase flow problem
4. Numerical experiments
Numerical experiment with two rock types

Let $\Omega = [0, 1]^3$, $\Omega = \Omega_1 \cap \Omega_2$, where $\Gamma = \{x = 1/2\}$. We consider the capillary pressure functions and the global mobilities given respectively by

$$
\pi_1(u) = 5u^2, \quad \pi_2(u) = 5u^2 + 1, \quad \lambda_i(u) = u(1 - u), \ i \in \{1, 2\}.
$$

- Homogeneous Neumann boundary conditions are fixed on the remaining part of $\partial \Omega$
- $f = 0$ in $\Omega$ and $u_0 = 0$

Here, the gas cannot enter the subdomain $\Omega_2$ if $\pi_1(u_1)$ is lower than the entry pressure $\pi_1(u_1^*)$, with $u_1^* = \frac{1}{\sqrt{5}} \approx 0.44$.

- Robin transmission conditions $\alpha = \alpha_{1,2} = \alpha_{2,1}$.
- The implementation is based on the Matlab Reservoir Simulation Toolbox (MRST)
Stopping criterion

**DD:**
- Classical stopping criterion: Residual $\leq 10^{-6}$
- Adaptive stopping criterion:
  \[
  \eta_{dd}^{k,m} \leq 0.1 \max \left\{ \eta_{sp}^{k,m}, \eta_{tm}^{k,m} \right\}.
  \]
Numerical experiments

### Stopping criterion

**DD:**
- Classical stopping criterion: Residual $\leq 10^{-6}$
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  $$\eta_{dd}^{k,m} \leq 0.1 \max \left\{ \eta_{sp}^{k,m}, \eta_{tm}^{k,m} \right\}.$$

Newton at final iteration of OSWR, $t = 6.6$:
- Classical stopping criterion: Residual $\leq 10^{-8}$
- $\eta_{lin,i}^{k,n,m} \leq 0.1 \max \left\{ \eta_{sp,i}^{k,n,m}, \eta_{tm,i}^{k,n,m}, \eta_{dd,i}^{k,n,m} \right\}$, $i = 1, 2$
Saturation $u(t)$ for $t = 2.9$

Estimated error for $t = 2.9$
Saturation $u(t)$ for $t = 6.6$

Estimated error for $t = 6.6$
Saturation $u(t)$ for $t = 13$

Estimated error for $t = 13$
Saturation $u(t)$ for $t = 15$

Estimated error for $t = 15$
Numerical experiments

Saturation $u(t)$ for $t = 15$

Estimated error for $t = 15$

Capillary pressure $\pi(u(t), \cdot)$ for $t = 6.6$

Estimated DD error for $t = 6.6$
Numerical experiments

Saturation $u(t)$ for $t = 15$

Estimated error for $t = 15$

Capillary pressure $\pi(u(t), \cdot)$ for $t = 15$

Estimated DD error for $t = 15$
Conclusions

- The quality of the result is ensured by controlling the error between the approximate solution and the exact solution at each iteration of the DD algorithm.
- Different components of the error have been distinguished.
- An efficient stopping criterion for the DD iterations has been established.
- Many of the DD and linearization iterations usually performed can be saved.

Future work

- Assess how much computing time can be saved
- Develop an a posteriori coarse-grid corrector
- Extend to advection-diffusion
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S-A.H.-Japhet-Kern-Vohralík, accepted, 2018 (steady case)
S-A.H.-Japhet-Vohralík, accepted, 2018 (unsteady case - heterogenous)

Thank you for your attention!