A NEW CEMENT TO GLUE NON-CONFORMING GRIDS WITH ROBIN INTERFACE CONDITIONS: THE P_1 FINITE ELEMENT CASE

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Abstract. We design and analyze a new non-conforming domain decomposition method, named the NICEM method, based on Schwarz type approaches that allows for the use of Robin interface conditions on non-conforming grids. The method is proven to be well posed. The error analysis is performed in 2D and in 3D for P_1 elements. Numerical results in 2D illustrate the new method.

Key words. Optimized Schwarz domain decomposition; Robin transmission conditions; finite element methods; non-conforming grids; error analysis; NICEM method.

1. Introduction. Our goal in writing this paper is to propose and analyze a non-conforming domain decomposition generalization to P.L. Lions initial idea, [32], in view of an extension of the approach to optimized interface conditions algorithms. This type of algorithm has proven indeed to be an efficient approach to domain decomposition methods in the case of conforming approximations, [12, 25]. This paper presents the basic material related to so called optimized zero$^{\text{th}}$ order method in case of finite element discretizations, see citeGJMN for a short presentation. In the companion paper [1], the case of the finite volume discretization was introduced and analyzed. In the finite element case, our method is based on a new interface cement using Robin conditions, and correspond to an equilibrated mortar approach (i.e. there is no master and slave sides). Thus we name this new method "New Interface Cement Equilibrated Mortar" (NICEM) method.

In Section 2, we present the method at the continuous level and then at the discrete level. Then in Section 3, we give in details the numerical analysis, with the proofs of well-posedness and error estimates both in 2D and 3D for P_1 elements. Given the length of the paper, the numerical analysis for 2D piecewise polynomials of higher order as well as convergence proofs for the Schwarz algorithm used to solve the discrete problem is the subject of another paper. We finally present in Section 4 simulations for two and four subdomains, that fit the theoretical estimates.

We first consider the problem at the continuous level: Find u such that

$$\mathcal{L}(u) = f \text{ in } \Omega \tag{1.1}$$

$$\mathcal{C}(u) = g \text{ on } \partial\Omega \tag{1.2}$$

where $\mathcal{L}$ and $\mathcal{C}$ are partial differential equations. The original Schwarz algorithm is based on a decomposition of the domain Ω into overlapping subdomains and the resolution of Dirichlet boundary value problems in the subdomains. It has been proposed in [32] to use more general boundary conditions for the problems on the

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subdomains in order to use a non-overlapping decomposition of the domain. The convergence factor is also dramatically reduced.

More precisely, let Ω be a $\mathcal{C}^{1,1}$ (or convex polygon in 2D or polyhedron in 3D) domain of $I\!R^d$, $d = 2$ or 3. This assumption is necessary to obtain minimal H^2 regularity that provides the full first order convergence of the P_1 finite element approximation. We could deal with lower regularity on the solution at the price of more technical proofs in non integer Sobolev spaces.

We assume that Ω is decomposed into K non-overlapping subdomains:

$$\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}^k. \tag{1.3}$$

We suppose that the subdomains $\Omega^k,\ 1 \le k \le K$ are either $\mathcal{C}^{1,1}$ or polygons in 2D or polyhedrons in 3D. We assume also that this decomposition is geometrically conforming in the sense that the intersection of the closure of two different subdomains, if not empty, is either a common vertex, a common edge, or a common face in 3D [1]. Let $\mathbf{n}_k$ be the outward normal from Ω^k. Let $(\mathcal{B}_{k,\ell})_{1\le k,\ell \le K, k\ne \ell}$ be the chosen transmission conditions on the interface between subdomains Ω^k and Ω^ℓ (e.g. $\mathcal{B}_{k,\ell} = \frac{\partial}{\partial \mathbf{n}_k} + \alpha_k$). What we shall call here a Schwarz type method for the problem (1.1)-(1.2) is its reformulation: Find $(u_k)_{1\le k\le K}$ such that

$$\begin{aligned} \mathcal{L}(u_k) &= f \text{ in } \Omega^k \\ \mathcal{C}(u_k) &= g \text{ on } \partial\Omega^k \cap \partial\Omega \\ \mathcal{B}_{k,\ell}(u_k) &= \mathcal{B}_{k,\ell}(u_\ell) \text{ on } \partial\Omega^k \cap \partial\Omega^\ell, \end{aligned}$$

leading to the iterative procedure

$$\begin{aligned} \mathcal{L}(u_k^{n+1}) &= f \text{ in } \Omega^k \\ \mathcal{C}(u_k^{n+1}) &= g \text{ on } \partial\Omega^k \cap \partial\Omega \\ \mathcal{B}_{k,\ell}(u_k^{n+1}) &= \mathcal{B}_{k,\ell}(u_\ell^{n}) \text{ on } \partial\Omega^k \cap \partial\Omega^\ell. \end{aligned}$$

Let us focus first on the interface conditions $\mathcal{B}_{k,\ell}$. The convergence factor of associated Schwarz-type domain decomposition methods is very sensitive to the choice of these transmission conditions. The use of exact artificial (also called absorbing) boundary conditions as interface conditions leads to an optimal number of iterations, [22, 34, 21, 20]. Indeed, for a domain decomposed into K strips, the number of iterations is K, see [34]. Let us remark that this result is rather surprising since exact absorbing conditions refer usually to truncation of infinite domains rather than interface conditions in domain decomposition. Nevertheless, this approach has some drawbacks: first, the explicit form of these boundary conditions is known only for constant coefficient operators and simple geometries. Secondly, these boundary conditions are pseudo-differential. The cost per iteration is high since the corresponding discretization matrix is not sparse for the unknowns on the boundaries of the subdomains. For this reason, it is usually preferred to use partial differential approximations to the exact absorbing boundary conditions. This approximation problem is classical in the field of computation on unbounded domains since the seminal paper of Engquist and Majda, [15]. The approximations correspond to “low frequency” approximations of the exact absorbing boundary conditions. In domain decomposition methods, many

[1] This assumption is actually not much restrictive since in the case of a geometrically nonconforming partition, the faces can be decomposed into subfaces to obtain a geometric conformity

authors have used them for wave propagation problems, [13, 14, 31, 5, 38, 29, 8] and in fluid dynamics, [33, 19]. Instead of using "low frequency" in space approximations to the exact absorbing boundary conditions, it has been proposed to design approximations which minimize the convergence factor of the algorithm. Such optimization of the transmission conditions for the performance of the algorithm was done in [25, 26, 27] for a convection-diffusion equation, where coefficients in second order transmission conditions where optimized. These approximations, named OO2 (Optimized Order 2), are quite different from the "low frequency" approximations and reduce dramatically the convergence factor of the method.

When the grids are conforming, the implementation of such interface conditions on the discretized problem is not too difficult. On the other hand, using non-conforming grids is very appealing since their use allows for parallel generation of meshes, for local adaptive meshes and fast and independent solvers. The mortar element method, first introduced in [7], enables the use of non-conforming grids. It is also well suited to the use of the so-called "Dirichlet-Neumann", [19], or "Neumann-Neumann" preconditioned conjugate gradient method applied to the Schur complement matrix, [30, 2, 37]. In the context of finite volume discretizations, it was proposed in [36] to use a mortar type method with arbitrary interface conditions. To our knowledge, such an approach has not been extended to a finite element discretization. Moreover, the approach we present here is different and simpler.

The purpose of this paper is to set the basics, and present the associated analysis in full details of such Robin type boundary conditions. Here we consider only interface conditions of order 0 : $\mathcal{B}_{k,\ell} = \frac{\partial}{\partial \mathbf{n}_k} + \alpha_k$. The approach we propose and study was introduced in [17] and independently implemented in [28] for the Maxwell equations but without numerical analysis. These results are the prerequisite for the goal in designing this non overlapping method: use interface conditions such as OO2 interface conditions (see [25, 27]). The implementation of such optimized order 2 transmission conditions is already available for advection-diffusion problems, [23, 24].

2. Definition of the method. We consider the following problem : Find u such that

$$(Id - \Delta)u = f \quad \text{in } \Omega \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

where Ω is a $\mathcal{C}^{1,1}$ (or convex polygon in 2D or polyhedron in 3D) domain of $I\!R^d$, $d = 2$ or 3, and f is given in $L^2(\Omega)$.
The variational statement of the problem (2.1)-(2.2) consists in writing the problem as follows : Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega (\nabla u \nabla v + uv)\, dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega). \tag{2.3}$$

Making use of the domain decomposition (1.3), the problem (2.3) can be written as follows : Find $u \in H_0^1(\Omega)$ such that

$$\sum_{k=1}^K \int_{\Omega^k} \left(\nabla(u_{|\Omega^k})\ \nabla(v_{|\Omega^k}) + u_{|\Omega^k} v_{|\Omega^k}\right) dx = \sum_{k=1}^K \int_{\Omega^k} f_{|\Omega^k} v_{|\Omega^k} dx, \quad \forall v \in H_0^1(\Omega).$$

Let us introduce the space $H_*^1(\Omega^k)$ defined by

$$H_*^1(\Omega^k) = \{\varphi \in H^1(\Omega^k), \quad \varphi = 0 \text{ over } \partial\Omega \cap \partial\Omega^k\}.$$

It is standard to note that the space $H_0^1(\Omega)$ can then be identified with the subspace of the K-tuple $\underline{v} = (v_1, ..., v_K)$ that are continuous on the interfaces:

$$V = \{\underline{v} = (v_1, ..., v_K) \in \prod_{k=1}^{K} H_*^1(\Omega^k),\ \forall k, \ell, k \neq \ell,\ 1 \leq k, \ell \leq K,\ v_k = v_\ell \text{ over } \partial\Omega^k \cap \partial\Omega^\ell\}.$$

This leads to introduce also the notation of the interfaces of two adjacent subdomains

$$\Gamma^{k,\ell} = \partial\Omega^k \cap \partial\Omega^\ell.$$

In what follows, for the sake of simplicity, the only fact to refer to a pair (k, ℓ) preassumes that $\Gamma^{k,\ell}$ is not empty. The problem (2.3) is then equivalent to the following one : Find $\underline{u} \in V$ such that

$$\sum_{k=1}^{K} \int_{\Omega^k} (\nabla u_k \nabla v_k + u_k v_k)\, dx = \sum_{k=1}^{K} \int_{\Omega^k} f_k v_k dx, \quad \forall \underline{v} \in V.$$

The mortar element method cannot be used easily and efficiently with Robin interface conditions in the framework of Schwarz type methods. In order to glue non-conforming grids with Robin transmission conditions, it turns out to be useful to impose the constraint $v_k = v_\ell$ over $\partial\Omega^k \cap \partial\Omega^\ell$ through a Lagrange multiplier in $H^{-1/2}(\partial\Omega^k)$.

Lemma 1. *For $\underline{v} \in \prod_{k=1}^{K} H_*^1(\Omega^k)$, the constraint $v_k = v_\ell$ across the interface $\Gamma^{k,\ell}$ is equivalent to*

$$\forall \underline{p} \equiv (p_k) \in \prod_{k=1}^{K} H^{-1/2}(\partial\Omega^k) \text{ with } p_k = -p_\ell \text{ over } \Gamma^{k,\ell}, \quad \forall k, \ell,$$
$$\sum_{k=1}^{K} {}_{H^{-1/2}(\partial\Omega^k)} < p_k, v_k >_{H^{1/2}(\partial\Omega^k)} = 0. \tag{2.4}$$

Proof: The proof is similar to the one of proposition III.1.1 in [11] but can't be directly derived from this proposition. Let $\underline{p} \equiv (p_k) \in \prod_{k=1}^{K} H^{-1/2}(\partial\Omega^k)$ with $p_k = -p_\ell$ over $\Gamma^{k,\ell}$, in $(H_{00}^{1/2}(\Gamma^{k,\ell}))'$ sense. Then, there exists over each Ω^k a lifting of the normal trace p_k in $H(\mathrm{div}, \Omega^k)$. The global function $\mathbf{P}$, which restriction to each Ω^k is defined as being equal to the lifting, belongs to $H(\mathrm{div}, \Omega)$ and is such that $(\mathbf{P}.\mathbf{n})_{|\partial\Omega^k} = p_k$. Let now $\underline{v} \in V$. From the previously quoted identification, we know that there exists $\mathbf{v} \in H_0^1(\Omega)$ such that $\mathbf{v}_{|\Omega^k} = v_k$. In addition,

$$\int_\Omega \mathbf{v}\nabla \cdot \mathbf{P} - \int_\Omega \mathbf{P}\nabla\mathbf{v} = 0.$$

On the other hand,

$$\int_\Omega \mathbf{v}\nabla \cdot \mathbf{P} - \int_\Omega \mathbf{P}\nabla\mathbf{v} = \sum_{k=1}^{K} (\int_{\Omega^k} \mathbf{v}\nabla \cdot \mathbf{P} - \int_{\Omega^k} \mathbf{P}\nabla\mathbf{v}) = \sum_{k=1}^{K} \int_{\partial\Omega^k} (\mathbf{P}.\mathbf{n})\underline{v} = \sum_{k=1}^{K} \int_{\partial\Omega^k} p_k v_k,$$

so that (2.4) is satisfied.
Conversely, let $\underline{v} = (v_1, ..., v_K) \in \prod_{k=1}^{K} H_*^1(\Omega^k)$ such that (2.4) is satisfied. Let $x \in \Gamma^{k,\ell}$, and let $\gamma_x \subset \bar{\gamma}_x \subset \Gamma_x \subset \bar{\Gamma}_x \subset \Gamma^{k,\ell}$ be open sets. There exists a function φ in

$\mathcal{D}(\Gamma_x)$ such that $\varphi(y) = 1$ for all y in γ_x. With any $q \in (H^{1/2}_{00}(\Gamma_x))'$, let us associate $\underline{p} \equiv (p_k)$ defined by

$$
\begin{gathered}
{}_{H^{-1/2}(\partial\Omega^k)}< p_k, w_k >_{H^{1/2}(\partial\Omega^k)} = {}_{(H^{1/2}_{00}(\Gamma_x))'}< q, \varphi w_k >_{H^{1/2}_{00}(\Gamma_x)}, \forall w_k \in H^{1/2}(\partial\Omega^k),\\
{}_{H^{-1/2}(\partial\Omega^\ell)}< p_\ell, w_\ell >_{H^{1/2}(\partial\Omega^\ell)} = -\, {}_{(H^{1/2}_{00}(\Gamma_x))'}< q, \varphi w_\ell >_{H^{1/2}_{00}(\Gamma_x)}, \forall w_\ell \in H^{1/2}(\partial\Omega^\ell),\\
\text{and } p_j = 0,\ \forall j \neq k, \ell.
\end{gathered}
$$

By construction, $\underline{p} \in \prod_{k=1}^K H^{-1/2}(\partial\Omega^k)$ and $p_k = -p_\ell$ over $\Gamma^{k,\ell}$. Hence from (2.4),

$$\sum_{k=1}^K {}_{H^{-1/2}(\partial\Omega^k)}< p_k, v_k >_{H^{1/2}(\partial\Omega^k)} = 0.$$

We derive

$${}_{H^{-1/2}(\partial\Omega^k)}< p_k, v_k >_{H^{1/2}(\partial\Omega^k)} = -\, {}_{H^{-1/2}(\partial\Omega^\ell)}< p_\ell, v_\ell >_{H^{1/2}(\partial\Omega^\ell)},$$

thus,

$${}_{(H^{1/2}_{00}(\Gamma_x))'} < q, \varphi v_k >_{H^{1/2}_{00}(\Gamma_x)} = {}_{(H^{1/2}_{00}(\Gamma_x))'} < q, \varphi v_\ell >_{H^{1/2}_{00}(\Gamma_x)},$$

and this is true for any $q \in (H^{1/2}_{00}(\Gamma_x))'$, hence $\varphi v_k = \varphi v_\ell$ over Γ_x, and thus

$$v_k = v_\ell \text{ over } \gamma_x, \quad \forall x \in \Gamma^{k,\ell}.$$

We derive $v_k = v_\ell$ a.e. over $\Gamma^{k,\ell}$, which ends the proof of Lemma 1.

The constrained space is then defined as follows

$$
\begin{aligned}
\mathcal{V} = \{(\underline{v}, \underline{q}) \in \left(\prod_{k=1}^K H^1_*(\Omega^k)\right) \times \left(\prod_{k=1}^K H^{-1/2}(\partial\Omega^k)\right),\\
v_k = v_\ell \text{ and } q_k = -q_\ell \text{ over } \Gamma^{k,\ell},\ \forall k, \ell\},
\end{aligned}
\tag{2.5}
$$

and problem (2.3) is equivalent to the following one : Find $(\underline{u}, \underline{p}) \in \mathcal{V}$ such that

$$
\begin{aligned}
\sum_{k=1}^K \int_{\Omega^k} (\nabla u_k \nabla v_k + u_k v_k)\, dx - \sum_{k=1}^K {}_{H^{-1/2}(\partial\Omega^k)} < p_k, v_k >_{H^{1/2}(\partial\Omega^k)}\\
= \sum_{k=1}^K \int_{\Omega^k} f_k v_k dx, \quad \forall \underline{v} \in \prod_{k=1}^K H^1_*(\Omega^k).
\end{aligned}
\tag{2.6}
$$

Being equivalent to the original problem, with $p_k = \frac{\partial u}{\partial \mathbf{n}_k}$ over $\partial\Omega^k$ (recall that f is assumed to be in $L^2(\Omega)$ so that $\frac{\partial u}{\partial \mathbf{n}_k}$ actually belongs to $H^{-1/2}(\partial\Omega^k)$), this problem is naturally well posed.

Let us describe the method in the non-conforming discrete case. Standard mortar methods are based on Galerkin approximation where both the trial spaces and test spaces are defined by imposing a gluing condition on the Dirichlet values on the interface by integral matching through mortar Lagrange multipliers. Here, we wish to match Robin conditions (i.e. the combination of Dirichlet and Neumann condition) we thus need to introduce a new independent entity representing the normal derivative of the trial function on the interface by increasing the set of trial function. This leads in turn to an increase in the set of test functions that appear to be defined with no glue. The method is no longer of Galerkin type but rather of Petrov Galerkin.

In all what follows we restrict the analysis to P_1 finite elements. The more general case is the subject of another paper for sake of brevity.

2.1. Discrete case. We introduce now the discrete spaces. Each Ω^k is provided with its own mesh (classical and locally conforming) $\mathcal{T}_h^k,\ 1 \le k \le K$, such that

$$\overline{\Omega}^k = \cup_{T \in \mathcal{T}_h^k} T.$$

For $T \in \mathcal{T}_h^k$, let h_T be the diameter of T ($h_T = \sup_{x,y \in T} d(x,y)$) and h the discretization parameter

$$h = \max_{1 \le k \le K} h_k, \quad \text{with} \quad h_k = \max_{T \in \mathcal{T}_h^k} h_T.$$

At the price of (even) more technicalities in the analysis, possible large variations in the norms of the solution $u_{|\Omega^k}$ can be compensated by tuning the parameter h_k. This requires in particular that the uniform h is not used but all the analysis is performed with h_k. For the sake of readability we prefer to use h instead of h_k. Let ρ_T be the diameter of the circle (in 2D) or sphere (in 3D) inscribed in T, then $\sigma_T = \frac{h_T}{\rho_T}$ is a measure of the nondegeneracy of T. We suppose that $\mathcal{T}_h^k$ is uniformly regular: there exists σ and τ independent of h such that

$$\forall T \in \mathcal{T}_h^k, \quad \sigma_T \le \sigma \quad \text{and} \quad \tau h \le h_T.$$

We consider that the sets belonging to the meshes are of simplicial type (triangles or tetrahedron), but the analysis made hereafter can be applied as well for quadrangular or hexahedral meshes. Let $\mathcal{P}_1(T)$ denote the space of all polynomials defined over T of total degree less than or equal to 1. The finite elements are of Lagrangian type, of class $\mathcal{C}^0$. We define over each subdomain two conforming spaces Y_h^k and X_h^k by :

$$\begin{aligned} Y_h^k &= \{v_{h,k} \in \mathcal{C}^0(\overline{\Omega}^k), \quad v_{h,k|T} \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}_h^k\}, \\ X_h^k &= \{v_{h,k} \in Y_h^k,\ v_{h,k|\partial\Omega^k \cap \partial\Omega} = 0\}. \end{aligned}$$

The space of traces over each $\Gamma^{k,\ell}$ of elements of Y_h^k is a finite element space denoted by $\mathcal{Y}_h^{k,\ell}$. As we assumed that the domain decomposition is geometrically conforming, then the space $\mathcal{Y}_h^k$ is the product space of the $\mathcal{Y}_h^{k,\ell}$ over each ℓ such that $\Gamma^{k,\ell} \neq \emptyset$. With each such interface we associate a subspace $\tilde{W}_h^{k,\ell}$ of $\mathcal{Y}_h^{k,\ell}$ in the same spirit as in the mortar element method, see [7] in 2D or [4] and [10] in 3D. To be more specific, let us recall the situation in 2D. If the space X_h^k consist of continuous piecewise polynomials of degree ≤ 1, then it is readily noticed that the restriction of X_h^k to $\Gamma^{k,\ell}$ consists in finite element functions adapted to the (possibly curved) side $\Gamma^{k,\ell}$ of piecewise polynomials of degree ≤ 1. This side has two end points that we denote as $x_0^{k,\ell}$ and $x_n^{k,\ell}$ that belong to the set of vertices of the corresponding triangulation of $\Gamma^{k,\ell}$: $x_0^{k,\ell}, x_1^{k,\ell}, ..., x_{n-1}^{k,\ell}, x_n^{k,\ell}$. The space $\tilde{W}_h^{k,\ell}$ is then the subspace of those elements of $\mathcal{Y}_h^{k,\ell}$ that are polynomials of degree 0 over both $[x_0^{k,\ell}, x_1^{k,\ell}]$ and $[x_{n-1}^{k,\ell}, x_n^{k,\ell}]$. As before, the space $\tilde{W}_h^k$ is the product space of the $\tilde{W}_h^{k,\ell}$ over each ℓ such that $\Gamma^{k,\ell} \neq \emptyset$. In 3D, we

used specific notations from [10], given in Section 3.4.
The discrete constrained space is then defined as

$$\mathcal{V}_h = \{(\underline{u}_h, \underline{p}_h) \in \left(\prod_{k=1}^{K} X_h^k\right) \times \left(\prod_{k=1}^{K} \tilde{W}_h^k\right),$$
$$\int_{\Gamma^{k,\ell}} ((p_{h,k} + \alpha u_{h,k}) - (-p_{h,\ell} + \alpha u_{h,\ell}))\psi_{h,k,\ell} = 0,\ \forall \psi_{h,k,\ell} \in \tilde{W}_h^{k,\ell},\quad \forall k,\ell\}. \tag{2.7}$$

Note that, for regular enough function

$$\int_{\Gamma^{k,\ell}} ((p_k + \alpha u_k) - (-p_\ell + \alpha u_\ell))\psi_{k,\ell} = 0,\ \forall \psi_{k,\ell} \in L^2(\Gamma^{k,\ell}),\quad \forall k,\ell,$$

then $p_k = -p_\ell$ and $u_k = u_\ell$, which allows us to make the link between the Robin condition (2.7) and the Dirichlet-Neumann condition in (2.5).

Let $\pi_{k,\ell}$ denote the orthogonal projection operator from $L^2(\Gamma^{k,\ell})$ onto $\tilde{W}_h^{k,\ell}$. Then, for $v \in L^2(\Gamma^{k,\ell})$, $\pi_{k,\ell}(v)$ is the unique element of $\tilde{W}_h^{k,\ell}$ such that

$$\int_{\Gamma^{k,\ell}} (\pi_{k,\ell}(v) - v)\psi = 0, \quad \forall \psi \in \tilde{W}_h^{k,\ell}. \tag{2.8}$$

We remark that the constraint in (2.7) also reads

$$p_k + \alpha\pi_{k,\ell}(u_k) = \pi_{k,\ell}(-p_\ell + \alpha u_\ell) \quad \text{over } \Gamma^{k,\ell}, \quad \forall k,\ell. \tag{2.9}$$

The discrete problem is the following one : Find $(\underline{u}_h, \underline{p}_h) \in \mathcal{V}_h$ such that

$\forall \underline{v}_h = (v_{h,1}, ...v_{h,K}) \in \prod_{k=1}^K X_h^k$,

$$\sum_{k=1}^{K} \int_{\Omega^k} (\nabla u_{h,k} \nabla v_{h,k} + u_{h,k} v_{h,k})\, dx - \sum_{k=1}^{K} \int_{\partial\Omega^k} p_{h,k} v_{h,k} ds = \sum_{k=1}^{K} \int_{\Omega^k} f_k v_{h,k} dx. \tag{2.10}$$

For the numerical analysis, we have to precise the norms that can be used on the Lagrange multipliers $\underline{p}_h$. For any $\underline{p} \in \prod_{k=1}^K L^2(\partial\Omega^k)$, in addition to the natural norm, we can define two better suited norms as follows

$$\|\underline{p}\|_{-\frac{1}{2},*} = \left(\sum_{k=1}^{K} \sum_{\substack{\ell=1 \\ \ell\neq k}}^{K} \|p_k\|^2_{(H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}))'}\right)^{\frac{1}{2}}, \quad \text{and} \quad \|\underline{p}\|_{-\frac{1}{2}} = \left(\sum_{k=1}^{K} \|p_k\|^2_{H^{-\frac{1}{2}}(\partial\Omega^k)}\right)^{\frac{1}{2}},$$

where $\|.\|_{(H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}))'}$ stands for the dual norm of $H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})$.

We also need a stability result for the Lagrange multipliers, and refer to [3] for 2D and to the appendix in 3D (the proof is postponed to the appendix because it needs ingredients that are developed later, in the analysis of the best approximation), in which it is proven that,

LEMMA 2. *There exists a constant c_* such that, for any $p_{h,k,\ell}$ in $\tilde{W}_h^{k,\ell}$, there exists an element $w^{h,k,\ell}$ in X_h^k that vanishes over $\partial\Omega^k \setminus \Gamma^{k,\ell}$ and satisfies*

$$\int_{\Gamma^{k,\ell}} p_{h,k,\ell} w^{h,k,\ell} \geq \|p_{h,k,\ell}\|^2_{(H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}))'} \tag{2.11}$$

with a bounded norm

$$\|w^{h,k,\ell}\|_{H^1(\Omega^k)} \leq c_* \|p_{h,k,\ell}\|_{(H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}))'}. \tag{2.12}$$

We now provide an analysis of the approximation properties of this scheme.

3. Numerical Analysis.

3.1. Well posedness. The first step in this error analysis is to prove the stability of the discrete problem and thus its well posedness. Let us introduce over $(\prod_{k=1}^K H_*^1(\Omega^k) \times \prod_{k=1}^K L^2(\partial\Omega^k)) \times \prod_{k=1}^K H_*^1(\Omega^k)$ the bilinear form

$$\tilde{a}((\underline{u}, \underline{p}), \underline{v})) = \sum_{k=1}^K \int_{\Omega^k} (\nabla u_k \nabla v_k + u_k v_k)\, dx - \sum_{k=1}^K \int_{\partial\Omega^k} p_k v_k ds. \tag{3.1}$$

The space $\prod_{k=1}^K H_*^1(\Omega^k)$ is endowed with the norm

$$\|\underline{v}\|_* = \left(\sum_{k=1}^K \|v_k\|_{H^1(\Omega^k)}^2 \right)^{\frac{1}{2}}.$$

Lemma 3. *There exists $c' > 0$ and a constant $\beta > 0$ such that*

$$\text{for } \alpha h \leq c', \quad \forall (\underline{u}_h, \underline{p}_h) \in \mathcal{V}_h,\ \exists \underline{v}_h \in \prod_{k=1}^K X_h^k,$$
$$\tilde{a}((\underline{u}_h, \underline{p}_h), \underline{v}_h)) \geq \beta (\|\underline{u}_h\|_* + \|\underline{p}_h\|_{-\frac{1}{2},*}) \|\underline{v}_h\|_*. \tag{3.2}$$

Moreover, we have the continuity argument : there exists a constant $c > 0$ such that

$$\forall (\underline{u}_h, \underline{p}_h) \in \mathcal{V}_h,\ \forall \underline{v}_h \in \prod_{k=1}^K X_h^k, \quad \tilde{a}((\underline{u}_h, \underline{p}_h), \underline{v}_h)) \leq c(\|\underline{u}_h\|_* + \|\underline{p}_h\|_{-\frac{1}{2}})(\|\underline{v}_h\|_*). \tag{3.3}$$

Proof of Lemma 3: In (2.11) and (2.12), we have introduced local $H_0^1(\Gamma_{k,\ell})$ functions that can be put together in order to provide an element $\underline{w}_h$ of $\prod_{k=1}^K X_h^k$ that satisfies

$$\sum_{k=1}^K \int_{\partial\Omega^k} p_k w_k ds \geq \|\underline{p}_h\|_{-\frac{1}{2},*}^2. \tag{3.4}$$

Let us now choose a real number γ, $0 < \gamma < \frac{2}{c_*^2}$ (where c_* is introduced in (2.12)) and choose $\underline{v}_h = \underline{u}_h - \gamma \underline{w}_h$ in (3.1) that yields

$$\tilde{a}((\underline{u}_h, \underline{p}_h), \underline{v}_h)) = \sum_{k=1}^K \int_{\Omega^k} (\nabla u_k \nabla (u_k - \gamma w_k) + u_k (u_k - \gamma w_k))\, dx$$
$$- \sum_{k=1}^K \int_{\partial\Omega^k} p_k (u_k - \gamma w_k) ds \tag{3.5}$$

By (2.9), we can write

$$\begin{aligned}
\int_{\Gamma^{k,\ell}} p_k u_k ds &= \frac{1}{4\alpha}\int_{\Gamma^{k,\ell}} ((p_k+\alpha\pi_{k,\ell}(u_k))^2-(p_k-\alpha\pi_{k,\ell}(u_k))^2)ds \\
&= \frac{1}{4\alpha}\int_{\Gamma^{k,\ell}} ((\pi_{k,\ell}(-p_\ell+\alpha u_\ell))^2-(p_k-\alpha\pi_{k,\ell}(u_k))^2)ds \\
&\le \frac{1}{4\alpha}\int_{\Gamma^{k,\ell}} ((p_\ell-\alpha u_\ell)^2-(p_k-\alpha\pi_{k,\ell}(u_k))^2)ds \\
&\le \frac{1}{4\alpha}\int_{\Gamma^{k,\ell}} ((p_\ell-\alpha\pi_{\ell,k}(u_\ell))^2-(p_k-\alpha\pi_{k,\ell}(u_k))^2)ds \\
&\quad + \frac{1}{4\alpha}\int_{\Gamma^{k,\ell}} \alpha^2(\pi_{\ell,k}(u_\ell)-u_\ell)^2 ds
\end{aligned}$$

so that

$$\sum_{k=1}^{K}\int_{\partial\Omega^k} p_k u_k ds \le \frac{\alpha}{4}\sum_{k=1}^{K}\sum_{k<\ell}\int_{\Gamma^{k,\ell}}(u_k-\pi_{k,\ell}(u_k))^2 ds \le c\alpha h\|\underline{u}_h\|_*^2.$$

We refer to [7] in 2D and [4] or [10] equation (5.1) in 3D, where the approximation properties of $\pi_{k,\ell}$ are proven.

Going back to (3.5), using (3.4) and Lemma 2 yields

$$\begin{aligned}
\tilde{a}((\underline{u}_h,\underline{p}_h),\underline{v}_h) &\ge (1-c\alpha h)\|\underline{u}_h\|_*^2-\gamma\|\underline{u}_h\|_*\|\underline{w}_h\|_*+\gamma\|\underline{p}_h\|_{-\frac{1}{2},*}^2 \\
&\ge (\frac{1}{2}-c\alpha h)\|\underline{u}_h\|_*^2+\gamma\|\underline{p}_h\|_{-\frac{1}{2},*}^2-\frac{\gamma^2}{2}\|\underline{w}_h\|_*^2 \\
&\ge (\frac{1}{2}-c\alpha h)\|\underline{u}_h\|_*^2+(\gamma-\frac{\gamma^2c_*^2}{2})\|\underline{p}_h\|_{-\frac{1}{2},*}^2.
\end{aligned}$$

Due to the choice of γ, we know that, for αh small enough, (3.2) holds. The continuity (3.3) follows from standard arguments (note that the norm on the right-hand side of (3.3) is not the $\|.\|_{-\frac{1}{2},*}$–norm), which ends the proof of Lemma 3.

From this lemma, we have the following result :

Theorem 1. *Let us assume that $\alpha h \le c$, for some constant c small enough. Then, the discrete problem (2.10) has a unique solution $(\underline{u}_h,\underline{p}_h)\in\mathcal{V}_h$, and there exists a constant $c>0$ such that*

$$\|\underline{u}_h\|_*+\|\underline{p}_h\|_{-\frac{1}{2},*}\le c\|f\|_{L^2(\Omega)}.$$

From Lemma 3, we are also in position to state that the discrete solution $(\underline{u}_h,\underline{p}_h)$ satisfies the following optimal error bound

$$\|\underline{u}-\underline{u}_h\|_*+\|\underline{p}-\underline{p}_h\|_{-\frac{1}{2},*}\le c\inf_{(\underline{\tilde{u}}_h,\underline{\tilde{p}}_h)\in\mathcal{V}_h}(\|\underline{u}-\underline{\tilde{u}}_h\|_*+\|\underline{p}-\underline{\tilde{p}}_h\|_{-\frac{1}{2}}) \tag{3.6}$$

and we are naturally led to the analysis of the best approximation of $(\underline{u},\underline{p}=\frac{\partial u}{\partial n})$ solution to (2.6) (or equivalently u solution to (2.1)-(2.2)) by elements in $\mathcal{V}_h$.

3.2. Analysis of the best approximation in 2D. In this part we analyze the best approximation of $(\underline{u},\underline{p})$ by elements in $\mathcal{V}_h$. As the proof is very technical for the analysis of the best approximation, we restrict ourselves in this section to the complete analysis of the 2D. The extension to 3D first order approximation is postponed to a next subsection.

The first step in the analysis is to prove the following lemma

LEMMA 4. *There exist two constants $c_1>0$ and $c_2>0$ independent of h such that for all $\eta_{\ell,k}$ in $\mathcal{Y}_h^{\ell,k}\cap H_0^1(\Gamma^{k,\ell})$, there exists an element $\psi_{\ell,k}$ in $\tilde{W}_h^{\ell,k}$, such that*

$$\int_{\Gamma^{k,\ell}}(\eta_{\ell,k}+\pi_{k,\ell}(\eta_{\ell,k}))\psi_{\ell,k}\geq c_1\|\eta_{\ell,k}\|^2_{L^2(\Gamma^{k,\ell})}, \tag{3.7}$$

$$\|\psi_{\ell,k}\|_{L^2(\Gamma^{k,\ell})}\leq c_2\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})}. \tag{3.8}$$

Note that $\eta_{\ell,k}$ and $\pi_{k,\ell}(\eta_{\ell,k})$ are associated with different grids. Then, we can prove the following interpolation estimates :

THEOREM 2. *For any $u\in H^2(\Omega)\cap H_0^1(\Omega)$, let $u_k=u_{|\Omega^k}$, $1\leq k\leq K$, $\underline{u}=(u_k)_{1\leq k\leq K}$ and let $p_{k,\ell}=\frac{\partial u_k}{\partial \mathbf{n}_k}$ over each $\Gamma^{k,\ell}$. Then there exists an element $\underline{\tilde{u}}_h$ in $\prod_{k=1}^K X_h^k$ and $\underline{\tilde{p}}_h=(\tilde{p}_{k\ell h})$, $\tilde{p}_{k\ell h}\in\tilde{W}_h^{k,\ell}$ such that $(\underline{\tilde{u}}_h,\underline{\tilde{p}}_h)$ satisfy the coupling condition (2.7), and*

$$\|\underline{\tilde{u}}_h-\underline{u}\|_*\leq ch\sum_{k=1}^K\|u_k\|_{H^2(\Omega^k)}+\frac{c}{\alpha}\sum_{k<\ell}\|p_{k,\ell}\|_{H^{\frac12}(\Gamma^{k,\ell})}$$

$$\|\tilde{p}_{k\ell h}-p_{k,\ell}\|_{H^{-\frac12}(\Gamma^{k,\ell})}\leq c\alpha h^2(\|u_k\|_{H^2(\Omega^k)}+\|u_\ell\|_{H^2(\Omega^\ell)})+ch\|p_{k,\ell}\|_{H^{\frac12}(\Gamma^{k,\ell})}$$

where c is a constant independent of h and α.

If we assume more regularity on the normal derivatives on the interfaces, we have

THEOREM 3. *Let $u\in H^2(\Omega)\cap H_0^1(\Omega)$, $u_k=u_{|\Omega^k}$, $1\leq k\leq K$, $\underline{u}=(u_k)_{1\leq k\leq K}$ and $p_{k,\ell}=\frac{\partial u_k}{\partial \mathbf{n}_k}$ is in $H^{\frac32}(\Gamma_{k,\ell})$. Then there exists an element $\underline{\tilde{u}}_h$ in $\prod_{k=1}^K X_h^k$ and $\underline{\tilde{p}}_h=(\tilde{p}_{k\ell h})$, $\tilde{p}_{k\ell h}\in\tilde{W}_h^{k,\ell}$ such that $(\underline{\tilde{u}}_h,\underline{\tilde{p}}_h)$ satisfy the coupling condition (2.7), and*

$$\|\underline{\tilde{u}}_h-\underline{u}\|_*\leq ch\sum_{k=1}^K\|u_k\|_{H^2(\Omega^k)}+\frac{ch}{\alpha}|\log h|\sum_{k<\ell}\|p_{k,\ell}\|_{H^{\frac32}(\Gamma^{k,\ell})}$$

$$\|\tilde{p}_{k\ell h}-p_{k,\ell}\|_{H^{-\frac12}(\Gamma^{k,\ell})}\leq c\alpha h^2(\|u_k\|_{H^2(\Omega^k)}+\|u_\ell\|_{H^2(\Omega^\ell)})+\ ch^2|\log h|\|p_{k,\ell}\|_{H^{\frac32}(\Gamma^{k,\ell})}$$

where c is a constant independent of h and α.

Proof of Lemma 4: We consider $\Gamma^{\ell,k}$ to be on the line $y=0$. Remind that we have denoted as $x_0^{\ell,k},x_1^{\ell,k},...,x_{n-1}^{\ell,k},x_n^{\ell,k}$ the vertices of the triangulation of $\Gamma^{\ell,k}$ that belong to $\Gamma^{\ell,k}$. To any $\eta_{\ell,k}$ in $\mathcal{Y}_h^{\ell,k}\cap H_0^1(\Gamma^{k,\ell})$ we then associate the element $\psi_{\ell,k}$ in $\tilde{W}_h^{\ell,k}$ as follows

$$\psi_{\ell,k}=\begin{cases}\frac{\eta_{\ell,k}(x_1^{\ell,k}-x_0^{\ell,k})}{(x-x_0^{\ell,k})}\text{ over }]x_0^{\ell,k},x_1^{\ell,k}[\\ \eta_{\ell,k}\text{ over }]x_1^{\ell,k},x_{n-1}^{\ell,k}[\\ \frac{\eta_{\ell,k}(x_n^{\ell,k}-x_{n-1}^{\ell,k})}{(x_n^{\ell,k}-x)}\text{ over }]x_{n-1}^{\ell,k},x_n^{\ell,k}[\end{cases}=\begin{cases}\eta^1_{\ell,k}\text{ over }]x_0^{\ell,k},x_1^{\ell,k}[\\ \eta_{\ell,k}\text{ over }]x_1^{\ell,k},x_{n-1}^{\ell,k}[\\ \eta^{n-1}_{\ell,k}\text{ over }]x_{n-1}^{\ell,k},x_n^{\ell,k}[\end{cases}$$

where $\eta^i_{\ell,k} = \eta_{\ell,k}(x_i^{k,\ell})$. By using a mapping onto the reference element $[0,1]$ and by recalling that all norms are equivalent over the space of polynomials of degree 1 we deduce in a classical way that there exists a constant c such that

$$\|\psi_{\ell,k}\|_{L^2(\Gamma^{k,\ell})} \leq c\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})}.$$

Moreover, it is straightforward to derive

$$\int_{\Gamma^{k,\ell}} (\eta_{\ell,k} + \pi_{k,\ell}(\eta_{\ell,k}))\psi_{\ell,k} = \int_{\Gamma^{k,\ell}} \eta_{\ell,k}\psi_{\ell,k} + \int_{\Gamma^{k,\ell}} (\pi_{k,\ell}(\eta_{\ell,k}))^2 + \int_{\Gamma^{k,\ell}} \pi_{k,\ell}(\eta_{\ell,k})(\psi_{\ell,k} - \eta_{\ell,k}).$$

Then, by using the relation

$$\pi_{k,\ell}(\eta_{\ell,k})(\psi_{\ell,k} - \eta_{\ell,k}) \geq -\frac{1}{2}(\pi_{k,\ell}(\eta_{\ell,k}))^2 - \frac{1}{2}(\psi_{\ell,k} - \eta_{\ell,k})^2,$$

we obtain

$$\int_{\Gamma^{k,\ell}} (\eta_{\ell,k} + \pi_{k,\ell}(\eta_{\ell,k}))\psi_{\ell,k} \geq \int_{\Gamma^{k,\ell}} \eta_{\ell,k}\psi_{\ell,k} + \frac{1}{2}\int_{\Gamma^{k,\ell}} (\pi_{k,\ell}(\eta_{\ell,k}))^2 - \frac{1}{2}\int_{\Gamma^{k,\ell}} (\psi_{\ell,k} - \eta_{\ell,k})^2.$$

We realize now that, over the first interval,

$$\int_{]x_0^{\ell,k},x_1^{\ell,k}[} (\eta_{\ell,k}\psi_{\ell,k} - \frac{1}{2}(\psi_{\ell,k} - \eta_{\ell,k})^2) = \int_{]x_0^{\ell,k},x_1^{\ell,k}[} (\frac{(x - x_0^{\ell,k})}{(x_1^{\ell,k} - x_0^{\ell,k})} - \frac{1}{2}\frac{(x - x_1^{\ell,k})^2}{(x_1^{\ell,k} - x_0^{\ell,k})^2})\psi^2_{\ell,k}.$$

We observe, by computing separately each integral, that

$$\int_{]x_0^{\ell,k},x_1^{\ell,k}[} (\frac{(x - x_0^{\ell,k})}{(x_1^{\ell,k} - x_0^{\ell,k})} - \frac{1}{2}\frac{(x - x_1^{\ell,k})^2}{(x_1^{\ell,k} - x_0^{\ell,k})^2}) = \int_{]x_0^{\ell,k},x_1^{\ell,k}[} \frac{(x - x_0^{\ell,k})^2}{(x_1^{\ell,k} - x_0^{\ell,k})^2}.$$

By recalling that $\psi_{\ell,k}$ is constant on $]x_0^{\ell,k}, x_1^{\ell,k}[$, we get that

$$\int_{]x_0^{\ell,k},x_1^{\ell,k}[} (\frac{(x - x_0^{\ell,k})}{(x_1^{\ell,k} - x_0^{\ell,k})} - \frac{1}{2}\frac{(x - x_1^{\ell,k})^2}{(x_1^{\ell,k} - x_0^{\ell,k})^2})\psi^2_{\ell,k} = \int_{]x_0^{\ell,k},x_1^{\ell,k}[} \frac{(x - x_0^{\ell,k})^2}{(x_1^{\ell,k} - x_0^{\ell,k})^2}\psi^2_{\ell,k},$$

and thus

$$\int_{]x_0^{\ell,k},x_1^{\ell,k}[} (\frac{(x - x_0^{\ell,k})}{(x_1^{\ell,k} - x_0^{\ell,k})} - \frac{1}{2}\frac{(x - x_1^{\ell,k})^2}{(x_1^{\ell,k} - x_0^{\ell,k})^2})\psi^2_{\ell,k} = \int_{]x_0^{\ell,k},x_1^{\ell,k}[} \eta^2_{\ell,k}.$$

The same holds true over the interval $]x_{n-1}^{\ell,k}, x_n^{\ell,k}[$. By summing up, we derive that

$$\int_{\Gamma^{k,\ell}} (\eta_{\ell,k} + \pi_{k,\ell}(\eta_{\ell,k}))\psi_{\ell,k} \geq \int_{\Gamma^{k,\ell}} \eta^2_{\ell,k},$$

which ends the proof of Lemma 4.

Proof of Theorem 2: In order to prove this theorem, let us build an element that will belong to the discrete space and will be as close as the expected error to the solution. Let u^1_{kh} be the unique element of X_h^k defined as follows :

- $(u^1_{kh})_{|\partial\Omega^k}$ is the best approximation of u_k over $\partial\Omega^k$ in $\mathcal{Y}_h^{k,\ell}$,
- u^1_{kh} at the inner nodes of the triangulation (in Ω^k) coincide with the interpolate of u_k.

Then, it satisfies

$$\|u_{kh}^1 - u_k\|_{L^2(\partial\Omega^k)} \le ch^{\frac{3}{2}}\|u_k\|_{H^2(\Omega^k)}, \tag{3.9}$$

from which we deduce that

$$\|u_{kh}^1 - u_k\|_{L^2(\Omega^k)} + h\|u_{kh}^1 - u_k\|_{H^1(\Omega^k)} \le ch^2\|u_k\|_{H^2(\Omega^k)}, \tag{3.10}$$

and, from Aubin-Nitsche estimate

$$\|u_{kh}^1 - u_k\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le ch^2\|u_k\|_{H^2(\Omega^k)}. \tag{3.11}$$

We define then separately the best approximation $p_{k\ell h}^1$ of $p_{k,\ell} = \frac{\partial u_k}{\partial \mathbf{n}_k}$ over each $\Gamma^{k,\ell}$ in $\tilde{W}_h^{k,\ell}$ in the L^2 norm. These elements satisfy for the error estimate

$$\|p_{k\ell h}^1 - p_{k,\ell}\|_{L^2(\Gamma^{k,\ell})} \le ch^{\frac{1}{2}}\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})} \tag{3.12}$$

$$\|p_{k\ell h}^1 - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le ch\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}. \tag{3.13}$$

But there is very few chance that $(\underline{u}_h^1, \underline{p}_h^1)$ satisfy the coupling condition (2.7). This element of $\left(\prod_{k=1}^K X_h^k\right) \times \left(\prod_{k=1}^K \tilde{W}_h^k\right)$ misses (2.7) of elements $\epsilon_{k,\ell}$ and $\eta_{\ell,k}$ such that

$$\int_{\Gamma^{k,\ell}} (p_{k\ell h}^1 + \epsilon_{k,\ell} + \alpha u_{kh}^1)\psi_{k,\ell} = \int_{\Gamma^{k,\ell}} (-p_{\ell kh}^1 + \alpha\eta_{\ell,k} + \alpha u_{\ell h}^1)\psi_{k,\ell},\ \forall \psi_{k,\ell} \in \tilde{W}_h^{k,\ell} \tag{3.14}$$

$$\int_{\Gamma^{k,\ell}} (p_{\ell kh}^1 + \alpha\eta_{\ell,k} + \alpha u_{\ell h}^1)\psi_{\ell,k} = \int_{\Gamma^{k,\ell}} (-p_{k\ell h}^1 - \epsilon_{k,\ell} + \alpha u_{kh}^1)\psi_{\ell,k},\ \forall \psi_{\ell,k} \in \tilde{W}_h^{\ell,k}. \tag{3.15}$$

In order to correct that, without polluting (3.9)-(3.13), for each couple (k,ℓ) we choose one side, say the smaller indexed one, hereafter we shall also assume that each couple (k,ℓ) is ordered by $k < \ell$. Associated to that choice, we define $\epsilon_{k,\ell} \in \tilde{W}_h^{k,\ell}$, $\eta_{\ell,k} \in \mathcal{Y}_h^{\ell,k} \cap H_0^1(\Gamma^{k,\ell})$, such that $(\underline{\tilde{u}}_h, \underline{\tilde{p}}_h)$ satisfy (2.7) where we define

$$\tilde{u}_{\ell h} = u_{\ell h}^1 + \sum_{k<\ell} \mathcal{R}_{\ell,k}(\eta_{\ell,k}), \quad \tilde{p}_{k\ell h} = p_{k\ell h}^1 + \epsilon_{k,\ell} \quad (\text{for } k < \ell), \tag{3.16}$$

where $\mathcal{R}_{\ell,k}$ is a discrete lifting operator (see [38, 6]) that to any element of $\mathcal{Y}_h^{\ell,k} \cap H_0^1(\Gamma^{k,\ell})$ associates a finite element function over Ω^ℓ that vanishes over $\partial\Omega^\ell \setminus \Gamma^{k,\ell}$ and satisfies

$$\begin{aligned} \forall w \in \mathcal{Y}_h^{\ell,k} \cap H_0^1(\Gamma^{k,\ell}),\ (\mathcal{R}_{\ell,k}(w))_{|\Gamma_{k,\ell}} = w \\ \|\mathcal{R}_{\ell,k}(w)\|_{H^1(\Omega^\ell)} \le c\|w\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} \end{aligned} \tag{3.17}$$

where c is h-independent.

The set of equations (3.14)-(3.15) for $\epsilon_{k,\ell}$ and $\eta_{\ell,k}$ results in a square system of linear algebraic equations that can be written as follows

$$\int_{\Gamma^{k,\ell}} (\epsilon_{k,\ell} - \alpha\eta_{\ell,k})\psi_{k,\ell} = \int_{\Gamma^{k,\ell}} e_1\psi_{k,\ell},\ \forall \psi_{k,\ell} \in \tilde{W}_h^{k,\ell} \tag{3.18}$$

$$\int_{\Gamma^{k,\ell}} (\epsilon_{k,\ell} + \alpha\eta_{\ell,k})\psi_{\ell,k} = \int_{\Gamma^{k,\ell}} e_2\psi_{\ell,k},\ \forall \psi_{\ell,k} \in \tilde{W}_h^{\ell,k} \tag{3.19}$$

with

$$e_1 = -p^1_{k\ell h} - p^1_{\ell k h} + \alpha(u^1_{\ell h} - u^1_{kh}), \tag{3.20}$$
$$e_2 = -p^1_{k\ell h} - p^1_{\ell k h} + \alpha(u^1_{kh} - u^1_{\ell h}). \tag{3.21}$$

Proposition 1. *The linear system (3.18)-(3.19) is well posed.*
Proof: With the notations above, (3.18) yields

$$\epsilon_{k,\ell} = \pi_{k,\ell}(\alpha\eta_{\ell,k} + e_1) \tag{3.22}$$

and (3.19) yields

$$\alpha\eta_{\ell,k} = \pi_{\ell,k}(-\epsilon_{k,\ell} + e_2). \tag{3.23}$$

As (3.18)-(3.19) is a square linear system, it suffices to prove uniqueness for e_1 and e_2 null. From (3.22)-(3.23), we get

$$0 = \eta_{\ell,k} + \pi_{\ell,k}\pi_{k,\ell}(\eta_{\ell,k})$$

so that for all $\psi_{\ell,k}$ in $\tilde{W}^{k,\ell}_h$,

$$0 = \int_{\Gamma^{k,\ell}} (\eta_{\ell,k} + \pi_{k,\ell}(\eta_{\ell,k}))\psi_{\ell,k}.$$

By Lemma 4, this proves that $\eta_{\ell,k}$ is zero, thus by (3.22), $\epsilon_{k,\ell}$ is zero.

Let us resume the proof of Theorem 2: By (3.22) and (3.23) we have

$$\int_{\Gamma^{k,\ell}} (\eta_{\ell,k} + \pi_{k,\ell}(\eta_{\ell,k}))\psi_{\ell,k} = \frac{1}{\alpha}\int_{\Gamma^{k,\ell}} (e_2 - \pi_{k,\ell}(e_1))\psi_{\ell,k}, \ \forall \psi_{\ell,k} \in \tilde{W}^{\ell,k}_h. \tag{3.24}$$

To estimate $\|\tilde{p}_{k\ell h} - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}$ and $\|\tilde{u}_{\ell h} - u_\ell\|_{H^1(\Omega^\ell)}$, we first estimate $\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})}$: from (3.7) and (3.24) we get

$$c_1\|\eta_{\ell,k}\|^2_{L^2(\Gamma^{k,\ell})} \le \frac{1}{\alpha}\|e_2 - \pi_{k,\ell}(e_1)\|_{L^2(\Gamma^{k,\ell})}\|\psi_{\ell,k}\|_{L^2(\Gamma^{k,\ell})} \tag{3.25}$$

and using (3.8) in (3.25)

$$\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})} \le \frac{c_2}{\alpha c_1}\|e_2 - \pi_{k,\ell}(e_1)\|_{L^2(\Gamma^{k,\ell})}$$

hence

$$\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})} \le \frac{c_2}{\alpha c_1}(\|e_2\|_{L^2(\Gamma^{k,\ell})} + \|e_1\|_{L^2(\Gamma^{k,\ell})}). \tag{3.26}$$

Now, from (3.20) and (3.21), for $i = 1, 2$

$$\|e_i\|_{L^2(\Gamma^{k,\ell})} \le \|p^1_{k\ell h} + p^1_{\ell k h}\|_{L^2(\Gamma^{k,\ell})} + \alpha\|u^1_{\ell h} - u^1_{kh}\|_{L^2(\Gamma^{k,\ell})}$$

and recalling that $p_{k,\ell} = \frac{\partial u_k}{\partial \mathbf{n}_k} = -\frac{\partial u_\ell}{\partial \mathbf{n}_\ell} = -p_{\ell,k}$ over each $\Gamma^{k,\ell}$

$$\|p^1_{k\ell h} + p^1_{\ell k h}\|_{L^2(\Gamma^{k,\ell})} \le \|p^1_{k\ell h} - p_{k,\ell}\|_{L^2(\Gamma^{k,\ell})} + \|p^1_{\ell k h} - p_{\ell,k}\|_{L^2(\Gamma^{k,\ell})}$$
$$\|u^1_{\ell h} - u^1_{kh}\|_{L^2(\Gamma^{k,\ell})} \le \|u^1_{kh} - u_k\|_{L^2(\Gamma^{k,\ell})} + \|u^1_{\ell h} - u_\ell\|_{L^2(\Gamma^{k,\ell})}$$

so that, using (3.9) and (3.12), we derive for $i = 1, 2$

$$\|e_i\|_{L^2(\Gamma^{k,\ell})} \le c\alpha h^{\frac{3}{2}}(\|u_k\|_{H^2(\Omega^k)} + \|u_\ell\|_{H^2(\Omega^\ell)}) + ch^{\frac{1}{2}}\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})} \tag{3.27}$$

and (3.26) yields

$$\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})} \le ch^{\frac{3}{2}}(\|u_k\|_{H^2(\Omega^k)} + \|u_\ell\|_{H^2(\Omega^\ell)}) + \frac{ch^{\frac{1}{2}}}{\alpha}\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}. \tag{3.28}$$

We can now evaluate $\|\tilde{p}_{k\ell h} - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}$, using (3.16) :

$$\|\tilde{p}_{k\ell h} - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le \|\epsilon_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} + \|p^1_{k\ell h} - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}. \tag{3.29}$$

The term $\|p^1_{k\ell h} - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}$ is estimated in (3.13), so let us focus on the term $\|\epsilon_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}$. From (3.22) we have,

$$\|\epsilon_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le \alpha\|\eta_{\ell,k}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} + \|e_1\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} + \|(Id - \pi_{k,\ell})(\alpha\eta_{\ell,k} + e_1)\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}. \tag{3.30}$$

To evaluate $\|e_1\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}$ we proceed as for $\|e_1\|_{L^2(\Gamma^{k,\ell})}$ and from (3.11) and (3.13) we have, for $i = 1, 2$:

$$\|e_i\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le c\alpha h^2(\|u_k\|_{H^2(\Omega^k)} + \|u_\ell\|_{H^2(\Omega^\ell)}) + ch\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}. \tag{3.31}$$

The third term in the right-hand side of (3.30) satisfies

$$\|(Id - \pi_{k,\ell})(\alpha\eta_{\ell,k} + e_1)\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le c\sqrt{h}\|\alpha\eta_{\ell,k} + e_1\|_{L^2(\Gamma^{k,\ell})}.$$

Then, using (3.28) and (3.27) yields

$$\|(Id - \pi_{k,\ell})(\alpha\eta_{\ell,k} + e_1)\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le c\alpha h^2(\|u_k\|_{H^2(\Omega^k)} + \|u_\ell\|_{H^2(\Omega^\ell)}) + ch\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}. \tag{3.32}$$

In order to estimate the term $\|\eta_{\ell,k}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}$ in (3.30), we use (3.24):

$$2\int_{\Gamma^{k,\ell}} \eta_{\ell,k}\psi_{\ell,k} = \int_{\Gamma^{k,\ell}} (\eta_{\ell,k} - \pi_{k,\ell}(\eta_{\ell,k}))\psi_{\ell,k} + \frac{1}{\alpha}\int_{\Gamma^{k,\ell}} (e_2 - \pi_{k,\ell}(e_1))\psi_{\ell,k}.$$

Using the symmetry of the operator $\pi_{k,\ell}$ we deduce

$$2\int_{\Gamma^{k,\ell}} \eta_{\ell,k}\psi_{\ell,k} = \int_{\Gamma^{k,\ell}} (\psi_{\ell,k} - \pi_{k,\ell}(\psi_{\ell,k}))\eta_{\ell,k} + \frac{1}{\alpha}\int_{\Gamma^{k,\ell}} (e_2 - \pi_{k,\ell}(e_1))\psi_{\ell,k}.$$

Then, we have

$$|\int_{\Gamma^{k,\ell}} \eta_{\ell,k}\psi_{\ell,k}| \le c\sqrt{h}\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})}\|\psi_{\ell,k}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})} + \frac{1}{\alpha}\|e_2 - \pi_{k,\ell}(e_1)\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}\|\psi_{\ell,k}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}$$

and thus, we obtain

$$\|\eta_{\ell,k}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le c\sqrt{h}\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})} + \frac{c}{\alpha}\|e_2 - \pi_{k,\ell}(e_1)\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}. \tag{3.33}$$

Then, using (3.28) and the fact that

$$\|e_2 - \pi_{k,\ell}(e_1)\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le \|e_2\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} + \|e_1\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} + \|e_1 - \pi_{k,\ell}(e_1)\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})}$$
$$\le \|e_2\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} + \|e_1\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} + c\sqrt{h}\|e_1\|_{L^2(\Gamma^{k,\ell})} \quad (3.34)$$

with (3.27) and (3.31) yields

$$\|\eta_{\ell,k}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le ch^2(\|u_k\|_{H^2(\Omega^k)} + \|u_\ell\|_{H^2(\Omega^\ell)}) + \frac{ch}{\alpha}\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}.$$

Using the previous inequality in (3.30), (3.29) yields

$$\|\tilde{p}_{k\ell h} - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le c\alpha h^2(\|u_k\|_{H^2(\Omega^k)} + \|u_\ell\|_{H^2(\Omega^\ell)}) + ch\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}. \quad (3.35)$$

Let us now estimate $\|\tilde{u}_{\ell h} - u_\ell\|_{H^1(\Omega^\ell)}$:

$$\|\tilde{u}_{\ell h} - u_\ell\|_{H^1(\Omega^\ell)} \le \|u^1_{\ell h} - u_\ell\|_{H^1(\Omega^\ell)} + \sum_{k<\ell}\|\mathcal{R}_{\ell,k}(\eta_{\ell,k})\|_{H^1(\Omega^\ell)} \quad (3.36)$$

and from (3.17)

$$\|\mathcal{R}_{\ell,k}(\eta_{\ell,k})\|_{H^1(\Omega^\ell)} \le c\|\eta_{\ell,k}\|_{H^{\frac{1}{2}}_{00}(\Gamma^{k,\ell})}$$

then, with an inverse inequality

$$\|\mathcal{R}_{\ell,k}(\eta_{\ell,k})\|_{H^1(\Omega^\ell)} \le ch^{-\frac{1}{2}}\|\eta_{\ell,k}\|_{L^2(\Gamma^{k,\ell})}.$$

Hence, from (3.28) we have

$$\|\mathcal{R}_{\ell,k}(\eta_{\ell,k})\|_{H^1(\Omega^\ell)} \le ch(\|u_k\|_{H^2(\Omega^k)} + \|u_\ell\|_{H^2(\Omega^\ell)}) + \frac{c}{\alpha}\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})} \quad (3.37)$$

and (3.36) yields

$$\|\tilde{u}_{\ell h} - u_\ell\|_{H^1(\Omega^\ell)} \le ch\|u_\ell\|_{H^2(\Omega^\ell)} + ch\sum_{k<\ell}\|u_k\|_{H^2(\Omega^k)} + \frac{c}{\alpha}\sum_{k<\ell}\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}, \quad (3.38)$$

which ends the proof of Theorem 2.

Proof of Theorem 3: The proof is the same that for Theorem 2, except that the relation (3.12) is changed using the following lemma

LEMMA 5. *The best L^2 approximation $p^1_{k\ell h}$ of $p_{k,\ell}$ satisfy the error estimate*

$$\|p^1_{k\ell h} - p_{k,\ell}\|_{L^2(\Gamma^{k,\ell})} \le ch^{\frac{3}{2}}\ |\log h|\|p_{k,\ell}\|_{H^{\frac{3}{2}}(\Gamma^{k,\ell})}. \quad (3.39)$$

Therefore, (3.13) is changed in

$$\|p^1_{k\ell h} - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le ch^2\ |\log h|\|p_{k,\ell}\|_{H^{\frac{3}{2}}(\Gamma^{k,\ell})}$$

and (3.35) is changed in

$$\|\tilde{p}_{k\ell h} - p_{k,\ell}\|_{H^{-\frac{1}{2}}(\Gamma^{k,\ell})} \le c\alpha h^2(\|u_k\|_{H^2(\Omega^k)} + \|u_\ell\|_{H^2(\Omega^\ell)}) + ch^2\ |\log h|\|p_{k,\ell}\|_{H^{\frac{3}{2}}(\Gamma^{k,\ell})}$$

and (3.38) is changed in

$$\|\tilde{u}_{\ell h} - u_\ell\|_{H^1(\Omega^\ell)} \le ch\|u_\ell\|_{H^2(\Omega^\ell)} + ch\sum_{k<\ell}\|u_k\|_{H^2(\Omega^k)} + \frac{ch}{\alpha}\ |\log h|\sum_{k<\ell}\|p_{k,\ell}\|_{H^{\frac{3}{2}}(\Gamma^{k,\ell})}.$$

Proof of Lemma 5: Let $\bar{p}_{k\ell h}$ be the unique element of $\tilde{W}^{k,\ell}_h$ defined as follows :

- $(\bar{p}_{k\ell h})_{|[x_1^{\ell,k},x_{n-1}^{\ell,k}]}$ coincide with the interpolate of degree 1 of $p_{k,\ell}$.
- $(\bar{p}_{k\ell h})_{|[x_0^{\ell,k},x_1^{\ell,k}]}$ and $(\bar{p}_{k\ell h})_{|[x_{n-1}^{\ell,k},x_n^{\ell,k}]}$ coincide with the interpolate of degree 0 of $p_{k,\ell}$.

Then, we have

$$\|p^1_{k\ell h}-p_{k,\ell}\|^2_{L^2(\Gamma^{k,\ell})}\leq\|\bar{p}_{k\ell h}-p_{k,\ell}\|^2_{L^2(\Gamma^{k,\ell})}.$$

Using Deny-Lions theorem we have

$$\|p^1_{k\ell h}-p_{k,\ell}\|^2_{L^2((\Gamma^{k,\ell})}\leq\|\bar{p}_{k\ell h}-p_{k,\ell}\|^2_{L^2(]x_0^{\ell,k},x_1^{\ell,k}[)}+ch^3\|p_{k,\ell}\|^2_{H^{\frac{3}{2}}([x_1^{\ell,k},x_{n-1}^{\ell,k}])}+\|\bar{p}_{k\ell h}-p_{k,\ell}\|^2_{L^2(]x_{n-1}^{\ell,k},x_n^{\ell,k}[)}.$$

In order to analyse the two extreme contributions, we use Deny-Lions theorem

$$\|\bar{p}_{k\ell h}-p_{k,\ell}\|^2_{L^2(]x_0^{\ell,k},x_1^{\ell,k}[)}\leq ch^{3-\frac{2}{p}}\|\frac{dp_{k,\ell}}{dx}\|^2_{L^p(]x_0^{\ell,k},x_1^{\ell,k}[)},$$

and thus

$$\|\bar{p}_{k\ell h}-p_{k,\ell}\|^2_{L^2(]x_0^{\ell,k},x_1^{\ell,k}[)}\leq ch^{3-\frac{2}{p}}\|\frac{dp_{k,\ell}}{dx}\|^2_{L^p(\Gamma^{k,\ell})},$$

Then, we use the estimate

$$\|\frac{dp_{k,\ell}}{dx}\|_{L^p(\Gamma^{k,\ell})}\leq cp\|\frac{dp_{k,\ell}}{dx}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})},$$

where c is a constant. Thus we have

$$\|\bar{p}_{k\ell h}-p_{k,\ell}\|^2_{L^2(]x_0^{\ell,k},x_1^{\ell,k}[)}\leq cp^2h^{3-\frac{2}{p}}\|\frac{dp_{k,\ell}}{dx}\|^2_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}.$$

Now we take $p=-\log h$ and thus we obtain

$$\|\bar{p}_{k\ell h}-p_{k,\ell}\|^2_{L^2(]x_0^{\ell,k},x_1^{\ell,k}[)}\leq c(h^{\frac{3}{2}}log(h))^2\|\frac{dp_{k,\ell}}{dx}\|^2_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}.$$

In a same way we have

$$\|\bar{p}_{k\ell h}-p_{k,\ell}\|^2_{L^2(]x_{n-1}^{\ell,k},x_n^{\ell,k}[)}\leq c(h^{\frac{3}{2}}log(h))^2\|\frac{dp_{k,\ell}}{dx}\|^2_{H^{\frac{1}{2}}(\Gamma^{k,\ell})},$$

and thus we obtain

$$\|p^1_{k\ell h}-p_{k,\ell}\|_{L^2(\Gamma^{k,\ell})}\leq ch^{\frac{3}{2}}|\log h|\|p_{k,\ell}\|_{H^{\frac{3}{2}}(\Gamma^{k,\ell})},$$

which ends the proof of Lemma 5.

3.3. Error Estimates. Thanks to (3.6), we have the following error estimates:

THEOREM 4. *Assume that the solution u of (2.1)-(2.2) is in $H^2(\Omega)\cap H^1_0(\Omega)$, and $u_k=u_{|\Omega^k}\in H^2(\Omega^k)$, and let $p_{k,\ell}=\frac{\partial u_k}{\partial \mathbf{n}_k}$ over each $\Gamma^{k,\ell}$. Then, there exists a constant c independent of h and α such that*

$$\|\underline{u}_h-\underline{u}\|_*+\|\underline{p}_h-\underline{p}\|_{-\frac{1}{2},*}\leq c(\alpha h^2+h)\sum_{k=1}^K\|\underline{u}\|_{H^2(\Omega^k)}+\ c(\frac{1}{\alpha}+h)\sum_{k=1}^K\sum_\ell\|p_{k,\ell}\|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})}.$$

THEOREM 5. *Assume that the solution u of (2.1)-(2.2) is in $H^2(\Omega) \cap H^1_0(\Omega)$, $u_k = u_{|\Omega^k} \in H^2(\Omega^k)$, and $p_{k,\ell} = \frac{\partial u_k}{\partial \mathbf{n}_k}$ is in $H^{\frac{3}{2}}(\Gamma_{k,\ell})$. Then there exists a constant c independent of h and α such that*

$$\|\underline{u}_h - \underline{u}\|_* + \|\underline{p}_h - \underline{p}\|_{-\frac{1}{2},*} \leq c(\alpha h^2 + h) \sum_{k=1}^{K} \|\underline{u}\|_{H^2(\Omega^k)} + \ c(\frac{h}{\alpha} + h^2)|\log h| \sum_{k=1}^{K} \sum_{\ell} \|p_{k,\ell}\|_{H^{\frac{3}{2}}(\Gamma^{k,\ell})}$$

REMARK 1. *Note that in most practical situations the normal traces p_ℓ are more regular than what can be expected from the basic trace result that states*

$$\|p_\ell\|_{H^{\frac{1}{2}}(\partial\Omega^\ell)} \leq c\|u_\ell\|_{H^2(\Omega^\ell)}, \tag{3.40}$$

this can be due for instance to the fact that we have local regularity for u in the neighborhood of the interfaces. In such generic cases, Theorem 5 should be used. Indeed provided that the solution $\underline{u}$ of (2.1)-(2.2) is in $\prod_{k=1}^{K} H^2_*(\Omega^k)$ and $p_{k,\ell} = \frac{\partial u_k}{\partial \mathbf{n}_k}$ is in $H^{\frac{3}{2}}(\Gamma_{k,\ell})$ if we choose α as a constant independent of h then

$$\|\underline{u}_h - \underline{u}\|_* = O(h|\log(h)|),$$

that is quasi optimal. In all these genereric cases, any choice of α in the large range $(\frac{C_1}{\log(h)}, \frac{C_2}{h})$ with any positive constants C_1 and C_2, yields an optimal error bound

$$\|\underline{u}_h - \underline{u}\|_* = O(h).$$

The above result on the convergence of the discrete method is interesting as it lets a lot of flexibility to choose α properly for other purpose. Indeed the matching (2.7) is in practice obtained through an iterative algorithm (see (4.1)-(4.2) in the Section 4), the convergence of which depends on α (not the convergence with h). In this respect let us remind that in [16] the optimal choice $\alpha = \frac{C}{\sqrt{h}}$ is proposed for the convergence of the iterative algorithm. This is the subject of a future paper.

Note that the value of $\alpha = \frac{c}{h}$ in the expression $p_{k,\ell} + \alpha u_k$ is actually consistent at the discrete level with the natural norm of the traces of u and the traces of the normal derivative of u on $\partial\Omega_k$.

We want to emphasize however that in some rare and pathological cases where (3.40) is the best that can be stated on the regularity of $\underline{p}$, Theorem 4 is the only one that can be used in order to get an error estimate:

$$\|\underline{u}_h - \underline{u}\|_* \leq c(\frac{1}{\alpha} + h) \sum_{i=1}^{K} \|\underline{u}\|_{H^2(\Omega^k)}.$$

Under such an hypothesis: the solution $\underline{u}$ of (2.1)-(2.2) is in $\prod_{k=1}^{K} H^2_*(\Omega^k)$ and $p_{k,\ell} = \frac{\partial u_k}{\partial \mathbf{n}_k}$ is only in $H^{\frac{1}{2}}(\Gamma_{k,\ell})$ then a choice where α is a constant independent of h yields,

$$\|\underline{u}_h - \underline{u}\|_* \leq c \sum_{i=1}^{K} \|\underline{u}\|_{H^2(\Omega^k)}$$

which does not provide any convergence. In order to get an optimal convergence rate, we have to choose a parameter α that satisfies : $\alpha = \frac{c}{h}$ and then

$$\|\underline{u}_h - \underline{u}\|_* \leq ch \sum_{i=1}^{K} \|\underline{u}\|_{H^2(\Omega^k)}.$$

3.4. Analysis of the best approximation in 3D. In this section, we prove Theorem 2 and Theorem 3 for a P_1-discretization in 3D. The main parts of the proofs of these theorems in section 3.2 are dimension-independent. Only Lemma 4 and Lemma 5 are dimension-dependent, so we prove these lemma for a P_1-discretization in 3D. We shall use the construction proposed in [10]. In order to make the reading easy, we shall recall the notations of the above mentioned paper. The analysis is done on one subdomain Ω^k that will be fixed in what follows. A typical interface between this subdomain and a generic subdomain Ω_ℓ will be denoted by Γ. We denote by $\mathcal{T}$ the restriction to Γ of the triangulation $\mathcal{T}_h^k$. Let $S(\mathcal{T})$ denote the space of piecewise linear functions with respect to $\mathcal{T}$ which are continuous on Γ and vanish on its boundary. The space of the Lagrange multipliers on Γ, defined below, will be denoted by $M(\mathcal{T})$. In 2D, the requirement dim $M(\mathcal{T})$ =dim $S(\mathcal{T})$ can be satisfied by lowering the degree of the finite elements on the intervals next to the end points of the interface. In 3D, it is slightly more complex (see [4]). Thus, we shall use the construction proposed in [10] with the following hypothesis

H.1 All the vertices of the boundary of Γ are connected to zero, one, or two vertices in the interior of Γ.

Making hypothesis H.1, there are four kinds of triangles (see Figure 3.1):
1. Inner triangles i.e they don't touch the boundary of Γ.
2. Triangles labeled 1 which have only one vertex on the boundary
3. Triangles labeled 2 which have two vertices on the boundary
4. Triangles labeled 3 which have three vertices on the boundary

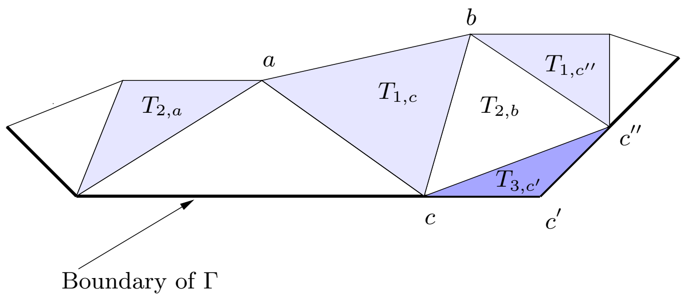

FIG. 3.1. *Two different situations of 2D triangulation of the interface Γ, next to it's boundary (near cross points): in light grey (triangle labeled 1) a vertex c is connected to two vertices in the interior of Γ, in dark grey (triangle labeled 3) a vertex c' is connected to two vertices on the boundary of Γ*

Let $\mathcal{V}$, $\mathcal{V}_0$, $\partial\mathcal{V}$ denote respectively the set of all the vertices of $\mathcal{T}$, the vertices in the interior of Γ, and the vertices on the boundary of Γ. The finite element basis functions will be denoted by $\Phi_a,\ a \in \mathcal{V}$. Thus,

$$S(\mathcal{T}) = \text{ span } \{\Phi_a : a \in \mathcal{V}_0\}.$$

For $a \in \mathcal{V}$, let σ_a denote the support of Φ_a,

$$\sigma_a := \bigcup \{T \in \mathcal{T} : a \in T\},$$

and let $\mathcal{N}_a$ be the set of neighboring vertices in $\mathcal{V}_0$ of a:

$$\mathcal{N}_a := \{b \in \mathcal{V}_0 : b \in \sigma_a\}.$$

Thus,

$$\mathcal{N} := \bigcup_{a \in \partial\mathcal{V}} \mathcal{N}_a$$

is the set of those interior vertices which have a neighbor on the boundary of Γ. If some triangle $T \in \mathcal{T}$ has all its vertices on the boundary of Γ, then there exists one (corner) vertex which has no neighbor in $\mathcal{V}_0$. Let $\mathcal{T}_c$ be the set of triangles $T \in \mathcal{T}$ which have all their vertices on the boundary of Γ. For $T \in \mathcal{T}_c$, we denote by c_T the only vertex of T that has no interior neighbor (such a vertex is unique as soon as the triangulation is fine enough). Let $\mathcal{N}_c$ denote the vertices a_T of $\mathcal{N}$ which belong to a triangle adjacent to a triangle $T \in \mathcal{T}_c$. Now, we define the space $M(\mathcal{T})$ [1] by

$$M(\mathcal{T}) := \text{ span } \{\hat{\Phi}_a,\ a \in \mathcal{V}_0\},$$

where the basis functions $\hat{\Phi}_a$ are defined as follows :

$$\hat{\Phi}_a := \begin{cases} \Phi_a, & a \in \mathcal{V}_0 \setminus \mathcal{N} \\ \Phi_a + \displaystyle\sum_{b \in \partial\mathcal{V} \cap \sigma_a} A_{b,a}\Phi_b & a \in \mathcal{N} \setminus \mathcal{N}_c \\ \Phi_{a_T} + \displaystyle\sum_{b \in \partial\mathcal{V} \cap \sigma_{a_T}} A_{b,a_T}\Phi_b + \Phi_{c_T} & a = a_T \in \mathcal{N}_c \end{cases}$$

the weights $A_{b,a}$ being defined as in (3.41) :

(i) for all boundary nodes $c \in \partial\mathcal{V}$ connected to two interior nodes a and b, if $T_{2,a}$ (resp. $T_{2,b}$) denote the adjacent triangle to abc having a (resp. b) as a vertex and its two others vertices on $\partial\mathcal{V}$, then the weights are defined such that (see [10])

$$A_{c,a} + A_{c,b} = 1 \text{ and } |T_{2,b}|A_{c,a} = |T_{2,a}|A_{c,b}, \tag{3.41}$$

(ii) for all boundary nodes $c \in \partial\mathcal{V}$ connected to only one interior node a, then the weight is defined by

$$A_{c,a} = 1. \tag{3.42}$$

(note that this case — not covered in [10] — actually corresponds to the previous case where the boundary nodes $c \in \partial\mathcal{V}$ is connected to two coincident interior nodes a and $b = a$.

To any $u \in S(\mathcal{T})$, $u = \sum_{a \in \mathcal{V}_0} u(a)\Phi_a$, we associate $v \in M(\mathcal{T})$ where $v = \sum_{a \in \mathcal{V}_0} u(a)\hat{\Phi}_a$. More explicitly, that means that to any $u \in S(\mathcal{T})$, we associate an element $v \in M(\mathcal{T})$ as follows (see Figure 3.1):

(i) v is a piecewise linear finite element on $\mathcal{T}$

(ii) for all interior nodes a, $v(a) := u(a)$

(iii) for all boundary nodes c, by assumption we have two situations:

[1] $M(\mathcal{T})$ is the notation introduced in [10], that we use here for the sake of clarity. Corresponding to our previous notation, $M(\mathcal{T}) \equiv \tilde{W}_h^{k,\ell}$

- c is connected to two interior nodes denoted by a and b.
  Then, $v(c) := Au(a) + Bu(b)$ where

$$A + B = 1 \text{ and } |T_{2,b}|A = |T_{2,a}|B \tag{3.43}$$

- c is not connected to any interior point. We consider the triangle adjacent to the triangle to which c belongs to. This triangle has one interior node denoted by b. Then, we define $v(c) := u(b)$.

We shall need the following technical assumption:

H.2 *For any triangle $T_{3,c'}$ having all three vertices on the boundary of $\mathcal{T}$ (see Figure 3.1), we consider the two triangles $T_{1,c}$ and $T_{1,c''}$ surrounding $T_{3,c'}$. We assume that there exists $\frac{1}{2} \leq C \leq \frac{2}{3}$ such that*

$$\frac{7}{96}\min(|T_{1,c}|, |T_{1,c''}|) > \frac{C}{2}|T_{3,c'}|.$$

We now prove Lemma 4 in 3D :

LEMMA 6. *We assume hypothesis H.2 and that $\mathcal{T}$ is uniformly regular. There exist two constants $c_1 > 0$ and $c_2 > 0$ independent of h such that for all u in $S(\mathcal{T})$, there exists an element v in $M(\mathcal{T})$, such that*

$$\int_\Gamma (u + \pi(u))v \geq c_1\|u\|^2_{L^2(\Gamma)}, \tag{3.44}$$

$$\|v\|_{L^2(\Gamma)} \leq c_2\|u\|_{L^2(\Gamma)}, \tag{3.45}$$

where π denote the orthogonal projection operator from $L^2(\Gamma)$ onto $M(\mathcal{T})$.

Let $u \in S(\mathcal{T})$, and the associate $v \in M(\mathcal{T})$ where $v = \sum_{a\in\mathcal{V}_0} u(a)\hat{\Phi}_a$. In order to prove (3.44), we prove the following lemma:

LEMMA 7. *We assume hypothesis H.2 and that $\mathcal{T}$ is uniformly regular. Then, there exists $\frac{1}{2} \leq C \leq \frac{2}{3}$ and $c > 0$ such that, for $u \in S(\mathcal{T})$ and $v \in M(\mathcal{T})$ constructed from u as explained above ((i)-(iii)), we have*

$$\int_\Gamma (uv - \frac{C}{2}(u - v)^2) \geq c\int_\Gamma u^2. \tag{3.46}$$

Proof of Lemma 7: Let us introduce the notation

$$Q_\Gamma := \int_\Gamma (uv - \frac{C}{2}(u - v)^2).$$

We have

$$Q_\Gamma = \frac{1}{4}\int_\Gamma (u + v)^2 - (1 + 2C)(u - v)^2.$$

In order to estimate Q_Γ, we remark that

$$Q_\Gamma = \sum_{T\in\mathcal{T}} Q_T$$

where

$$Q_T = \frac{1}{4}\int_T (u + v)^2 - (1 + 2C)(u - v)^2.$$

We consider the four kinds of triangles introduced above (after hypothesis H.1).

Inner triangles. On an inner triangle T, $u = v$ so that for all $C > 0$, we have

$$Q_T \geq c \int_T u^2$$

for $c \leq 1$.

Triangles having only one vertex on the boundary. Let $T_{1,c}$ be such a triangle (see Figure 3.1). First notice that we have (remember $u(c) = 0$)

$$\int_{T_{1,c}} u^2 = \frac{|T_{1,c}|}{12}\Big(u(a)^2+u(b)^2+(u(a)+u(b))^2\Big) = \frac{|T_{1,c}|}{12}\Big(2u(a)^2+2u(b)^2+2u(a)u(b)\Big)$$

see for example [9] (II.8.4). As for $Q_{T_{1,c}}$, we have

$$Q_{T_{1,c}} = \frac{|T_{1,c}|}{48}\Big((2u(a))^2 + (2u(b))^2 + (Au(a) + Bu(b))^2$$
$$+(2u(a) + 2u(b) + Au(a) + Bu(b))^2 - 2(1 + 2C)(Au(a) + Bu(b))^2\Big). \quad (3.47)$$

Thus,

$$Q_{T_{1,c}} = \frac{|T_{1,c}|}{48}\Big(8u(a)^2 + 8u(b)^2 + 8u(a)u(b) + 4(u(a) + u(b))(Au(a) + Bu(b))$$
$$-4C(Au(a) + Bu(b))^2\Big). \quad (3.48)$$

If we take $C = 1$ in (3.48) and use $A + B = 1$, we get:

$$Q_{T_{1,c}} = \frac{|T_{1,c}|}{48}\Big(4u(a)^2 + 4u(b)^2 + 4u(a)u(b) + 4AB(u(a) - u(b))^2 + 4(u(a) + u(b))^2\Big)$$
$$\geq \frac{1}{2}\int_{T_{1,c}} u^2.$$

Hence, for all $0 < C \leq 1$, we have:

$$Q_{T_{1,c}} \geq \frac{1}{2}\int_{T_{1,c}} u^2.$$

Therefore,

$$Q_{T_{1,c}} \geq c\int_{T_{1,c}} u^2$$

for $0 < C \leq 1$ and $0 < c \leq \frac{1}{2}$. We shall also use in the sequel the estimate:

$$Q_{T_{1,c}} \geq \frac{|T_{1,c}|}{48}\Big(8u(a)^2 + 8u(b)^2 + 12u(a)u(b)\Big)$$
$$\geq |T_{1,c}|\frac{7}{96}u(b)^2. \quad (3.49)$$

Triangles having two vertices on the boundary. Let $T_{2,a}$ be such a triangle (see Figure 3.1). As we will sum over all the triangles of type $T_{2,a}$, we introduce the following notations: $T_{2,i-1} = T_{2,a}$, $T_{2,i} = T_{2,b}$, $u_i = u(a) = v(a)$, $u_{i+1} = u(b) = v(b)$ and $v_i = v(c)$, see Figure 3.2.

We consider now a triangle $T_{2,i}$ having two vertices on the boundary of the face Γ. Let $\mathcal{N}_2 = \{i,\ T_{2,i}$ has two vertices on the boundary of $\Gamma\}$. First notice that we have

$$\int_{T_{2,i}} u^2 = \frac{|T_{2,i}|}{12}\left(2u_{i+1}^2\right).$$

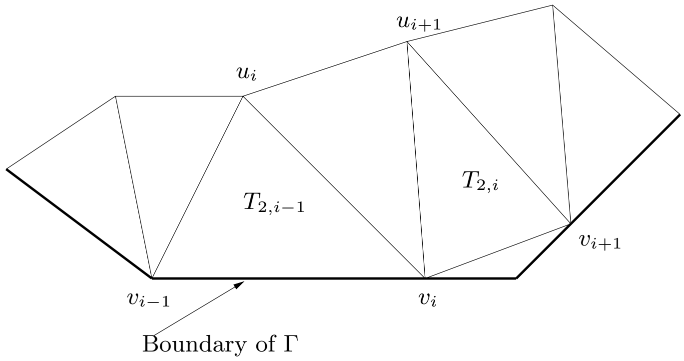

FIG. 3.2. *Notations for the triangles having two vertices on the boundary of* Γ *(triangle of type 2)*

And we have

$$\begin{aligned} Q_{T_{2,i}} &= \frac{|T_{2,i}|}{48}\Big(4u_{i+1}^2 + v_i^2 + v_{i+1}^2 + (2u_{i+1} + v_i + v_{i+1})^2 \\ &\qquad -(1+2C)(v_i^2 + v_{i+1}^2 + (v_i + v_{i+1})^2)\Big) \\ &= \frac{|T_{2,i}|}{48}\Big(8u_{i+1}^2 - 4Cv_i^2 - 4Cv_{i+1}^2 - 4Cv_iv_{i+1} + 4u_{i+1}(v_i + v_{i+1})\Big). \end{aligned}$$

Then,

$$Q_{T_{2,i}} \geq \frac{|T_{2,i}|}{48}\Big(8u_{i+1}^2 - 6Cv_i^2 - 6Cv_{i+1}^2 + 4u_{i+1}(v_i + v_{i+1})\Big).$$

Defining $E_i := u_{i+1}v_i$ and $F_i := u_{i+1}v_{i+1}$ (cf. [10] page 11), we have:

$$Q_{T_{2,i}} \geq \int_{T_{2,i}} u^2 + \frac{|T_{2,i}|}{48}\Big(-6Cv_i^2 + 4E_i - 6Cv_{i+1}^2 + 4F_i\Big).$$

Now we sum these terms over all the triangles having two vertices on the boundary of Γ.

$$\begin{aligned} &\sum_{i\in\mathcal{N}_2} Q_{T_{2,i}} \geq \int_{\cup_{i\in\mathcal{N}_2} T_{2,i}} u^2 + \sum_{i\in\mathcal{N}_2} \frac{|T_{2,i}|}{48}\Big(-6Cv_i^2 + 4E_i - 6Cv_{i+1}^2 + 4F_i\Big) \\ &\geq \int_{\cup_{i\in\mathcal{N}_2} T_{2,i}} u^2 + \frac{1}{48}\sum_{i\in\mathcal{N}_2} (|T_{2,i}|(-6Cv_i^2 + 4E_i) + |T_{2,i-1}|(-6Cv_i^2 + 4F_{i-1})). \end{aligned} \tag{3.50}$$

The condition (3.43) leads to the inequality

$$|T_{2,i}|E_i + |T_{2,i-1}|F_{i-1} = (|T_{2,i}| + |T_{2,i-1}|)v_i^2$$

(see equation after (3.19) in [10]), so that we get:

$$|T_{2,i}|(-6Cv_i^2 + 4E_i) + |T_{2,i-1}|(-6Cv_i^2 + 4F_{i-1}) = (|T_{2,i}| + |T_{2,i-1}|)(4 - 6C)v_i^2.$$

This term is positive for $C \leq 2/3$. Hence for $0 < C \leq 2/3$, inequality (3.50) becomes:

$$\sum_{i\in\mathcal{N}_2} Q_{T_{2,i}} \geq \int_{\cup_{i\in\mathcal{N}_2} T_{2,i}} u^2.$$

Therefore, for $0 < C \leq 2/3$ and $0 < c \leq 1$,

$$\sum_{i\in\mathcal{N}_2} Q_{T_{2,i}} \geq c \int_{\cup_{i\in\mathcal{N}_2} T_{2,i}} u^2.$$

Triangles having all three vertices on the boundary. Let $T_{3,c'}$ be such a triangle (see Figure 3.1). We have to control:

$$Q_{T_{3,c'}} = -\frac{C}{2}|T_{3,c'}||u(b)|^2$$

by the integrals over the two triangles $T_{1,c}$ and $T_{1,c''}$ surrounding $T_{3,c'}$. This can be achieved using the assumption H.2 and using that from (3.49), we have

$$Q_{T_{1,c}\cup T_{1,c''}} \geq \min(|T_{1,c}|, |T_{1,c''}|)u(b)^2\frac{7}{48}.$$

In conclusion, we have that (3.46) holds with $c = 1/4$ for a constant C, $\frac{1}{2} \leq C \leq \frac{2}{3}$.

Proof of Lemma 6: Using the uniform regularity of $\mathcal{T}$, it is easy to check (3.45). Using the definition of π, as in (2.8), it is straightforward to derive

$$\int_\Gamma (u + \pi(u))v = \int_\Gamma uv + \int_\Gamma (\pi(u))^2 + \int_\Gamma \pi(u)(v - u).$$

Then, using the relation

$$\pi(u)(v - u) \geq -(\pi(u))^2 - \frac{1}{4}(v - u)^2$$

leads to

$$\int_\Gamma (u + \pi(u))v \geq \int_\Gamma uv - \frac{1}{4}\int_\Gamma (v - u)^2.$$

Thus, for $C \geq \frac{1}{2}$, we have

$$\int_\Gamma (u + \pi(u))v \geq \int_\Gamma uv - \frac{C}{2}\int_\Gamma (v - u)^2.$$

Then, using (3.46), we obtain (3.44) which ends the proof of Lemma 6.

Proof of Lemma 5 in 3D: Let $\bar{p}_{k\ell h}$ be the unique element of $M(\mathcal{T})$ defined as follows :

1. $\bar{p}_{k\ell h}$ is a piecewise linear finite element on $\mathcal{T}$
2. for all interior nodes a, $\bar{p}_{k\ell h}(a) := p_{k,\ell}(a)$
3. for all boundary nodes c, by assumption we have two situations:
   - c is connected to two interior nodes denoted by a and b.
     Then, $\bar{p}_{k\ell h}(c) := Ap_{k,\ell}(a) + Bp_{k,\ell}(b)$ where
     $$A + B = 1 \text{ and } |T_{2,b}|A = |T_{2,a}|B$$
     where $T_{2,a}$ (resp. $T_{2,b}$) denote the adjacent triangle to abc having a (resp. b) as a vertex and its two others vertices on $\partial\mathcal{V}$.
   - c is not connected to any interior point. We consider the triangle adjacent to the triangle to which c belongs to. This triangle has one interior node denoted by b. Then, we define $\bar{p}_{k\ell h}(b) := p_{k,\ell}(b)$.

Like for the proof in 2D, we introduce the best approximation $p^1_{k\ell h}$ of $p_{k,\ell} = \frac{\partial u_k}{\partial \mathbf{n}_k}$ over Γ in $M(\mathcal{T})$. Then, we have

$$\|p^1_{k\ell h} - p_{k,\ell}\|^2_{L^2(\Gamma)} \le \|\bar{p}_{k\ell h} - p_{k,\ell}\|^2_{L^2(\Gamma)}.$$

The right-hand side in the previous inequality can be written in the form

$$\|\bar{p}_{k\ell h} - p_{k,\ell}\|^2_{L^2(\Gamma)} = \sum_{T\in\mathcal{T}} R_T \tag{3.51}$$

where

$$R_T = \int_T (\bar{p}_{k\ell h} - p_{k,\ell})^2 dx.$$

We consider again the four kinds of triangles introduced above (after hypothesis H.1).

Inner triangles. On an inner triangle T, $\bar{p}_{k\ell h} = \sum_{a\in\mathcal{V}\cap T} p_{k,\ell}(a)\Phi_a$ is the P_1 finite element interpolation of $p_{k,\ell}$ and we use Deny-Lions theorem :

$$R_T \le ch^3 \|p_{k,\ell}\|^2_{H^{\frac{3}{2}}(T)}. \tag{3.52}$$

Triangles having only one vertex on the boundary. Let $T_{1,c}$ be a triangle with only one vertex on the boundary (see Figure 3.1). Let c be the vertex of $T_{1,c}$ on $\partial\mathcal{V}$, and a and b the two vertices of $T_{1,c}$ which are interior nodes. Then, for $p_{k,\ell} \in P_0(T_{1,c})$ we have $\bar{p}_{k\ell h} = p_{k,\ell}$. For a triangle (or a finite union of triangles) $\sigma \subset \mathcal{T}$, we need to introduce the space $L^{2,p}(\sigma)$ of functions that are L^2 in the tangential direction to $\partial\Gamma$ and L^p in the normal direction to $\partial\Gamma$, where $\partial\Gamma$ is the boundary of Γ. Then, using Deny-Lions theorem, we have

$$R_{T_{1,c}} \le ch^{3-\frac{2}{p}} \|\nabla p_{k,\ell}\|_{L^{2,p}(T_{1,c})}. \tag{3.53}$$

Triangles having two vertices on the boundary. Let $T_{2,b}$ be a triangle with two vertices on the boundary of the face Γ (see Figure 3.1), $T_{1,c}$ and $T_{1,c''}$ the two triangles surrounding $T_{2,b}$. We consider $\bar{p}_{k\ell h}$ on the polygon $\sigma_2 := T_{2,b} \cup T_{1,c} \cup T_{1,c''}$. Then, for $p_{k,\ell} \in P_0(\sigma_2)$ we have $\bar{p}_{k\ell h} = p_{k,\ell}$. Using a piecewise affine transformation and Deny-Lions theorem, we have

$$R_{T_{2,b}} \le \int_{\sigma_{a_2}} (\bar{p}_{k\ell h} - p_{k,\ell})^2 dx \le ch^{3-\frac{2}{p}} \|\nabla p_{k,\ell}\|_{L^{2,p}(\sigma_{a_2})}. \tag{3.54}$$

Triangles having all three vertices on the boundary. Let $T_{3,c'}$ be such a triangle, and let $T_{2,b}$ be the triangle adjacent to $T_{3,c'}$ as on Figure 3.1. Let $T_{1,c}$ and $T_{1,c''}$ be the two triangles surrounding $T_{2,b}$. We consider $\bar{p}_{k\ell h}$ on the polygon $\sigma_3 := T_{3,c'} \cup T_{2,b} \cup T_{1,c} \cup T_{1,c''}$. Then, for $p_{k,\ell} \in P_0(\sigma_3)$, we have $\bar{p}_{k\ell h} = p_{k,\ell}$. Using a piecewise affine transformation and Deny-Lions theorem, we obtain

$$R_{T_{3,c'}} \leq \int_{\sigma_3} (\bar{p}_{k\ell h} - p_{k,\ell})^2 dx c h^{3-\frac{2}{p}} \|\nabla p_{k,\ell}\|_{L^{2,p}(\sigma_{a_3})}. \tag{3.55}$$

We proceed like for the proof of Lemma 5 in 2D and sum up the contribution (3.52) with those derived from (3.53), (3.54) and (3.55). We obtain

$$\|\bar{p}_{k\ell h} - p_{k,\ell}\|^2_{L^2(\Gamma)} \leq ch^3 \|p_{k,\ell}\|^2_{H^{\frac{3}{2}}(\Gamma)} + ch^{3-\frac{2}{p}} p^2 \|\nabla p_{k,\ell}\|^2_{H^{\frac{1}{2}}(\Gamma)}.$$

Then, taking $p = -log(h)$, we get

$$\|\bar{p}_{k\ell h} - p_{k,\ell}\|^2_{L^2(\Gamma)} \leq c(h^3 + h^3(log(h))^2) \|p_{k,\ell}\|^2_{H^{\frac{3}{2}}(\Gamma)},$$

which ends the proof of Lemma 5 in 3D.

4. Numerical results. We introduce the discrete algorithm : let $(u^n_{h,k}, p^n_{h,k}) \in X^k_h \times \tilde{W}^k_h$ be a discrete approximation of (u,p) in Ω^k at step n. Then, $(u^{n+1}_{h,k}, p^{n+1}_{h,k})$ is the solution in $X^k_h \times \tilde{W}^k_h$ of

$$\int_{\Omega^k} \left(\nabla u^{n+1}_{h,k} \nabla v_{h,k} + u^{n+1}_{h,k} v_{h,k}\right) dx - \int_{\partial\Omega^k} p^{n+1}_{h,k} v_{h,k} ds = \int_{\Omega^k} f_k v_{h,k} dx, \ \forall v_{h,k} \in X^k_h, \tag{4.1}$$

$$\int_{\Gamma^{k,\ell}} (p^{n+1}_{h,k} + \alpha u^{n+1}_{h,k}) \psi_{h,k,\ell} = \int_{\Gamma^{k,\ell}} (-p^n_{h,\ell} + \alpha u^n_{h,\ell}) \psi_{h,k,\ell}, \quad \forall \psi_{h,k,\ell} \in \tilde{W}^{k,\ell}_h. \tag{4.2}$$

The convergence analysis of this iterative scheme is the subject of another paper.

We consider the problem

$$(Id - \Delta) u(x,y) = x^3(y^2 - 2) - 6xy^2 + (1 + x^2 + y^2)\sin(xy), \quad (x,y) \in \Omega,$$
$$u(x,y) = x^3 y^2 + \sin(xy), \quad (x,y) \in \partial\Omega,$$

with exact solution $u(x,y) = x^3y^2 + sin(xy)$. In section 4.3 we consider the domain $\Omega = (-1,1) \times (0, 2\pi)$, otherwise the domain is the unit square $\Omega = (0,1) \times (0,1)$.

We decompose Ω into non-overlapping subdomains with meshes generated in an independent manner. The computed solution is the solution at convergence of the discrete algorithm (4.1)-(4.2), with a stopping criterion on the jumps of interface conditions that must be smaller than 10^{-8}.

Remark 2. *In the implementation of the method, the main difficulty lies in computing projections between non matching grids. In [17] we present an efficient algorithm in two dimensions to perform the required projections between arbitrary grids, in the same spirit as in [18] for finite volume discretization with projections on piecewise constant functions.*

4.1. Choice of the Robin parameter α. In our simulations the Robin parameter is either an arbitrary constant or is obtained by minimizing the convergence factor (depending on the mesh size in that case). In the conforming two subdomains case, with constant mesh size h and an interface of length L, the optimal theoretical value of α which minimizes the convergence factor at the continuous level is (see cite-Gander06):

$$\alpha_{opt} = [((\frac{\pi}{L})^2 + 1)((\frac{\pi}{h})^2 + 1)]^{\frac{1}{4}}.$$

In the non-conforming case, the mesh size is different for each side of the interface. We consider the following values : $\alpha_{min} = [((\frac{\pi}{L})^2 + 1)((\frac{\pi}{h_{min}})^2 + 1)]^{\frac{1}{4}}$, $\alpha_{mean} = [((\frac{\pi}{L})^2 + 1)((\frac{\pi}{h_{mean}})^2 + 1)]^{\frac{1}{4}}$, $\alpha_{max} = [((\frac{\pi}{L})^2 + 1)((\frac{\pi}{h_{max}})^2 + 1)]^{\frac{1}{4}}$, where h_{min}, h_{mean} and h_{max} stands respectively for the smallest, meanest or highest step size on the interface.

4.2. H^1 error between the continuous and discrete solutions. In this part, we compare the relative H^1 error in the non-conforming case to the error obtained on a uniform conforming grid.

Definition of the relative H^1 error : Let K be the number of subdomains. Let $u_i = u_{|\Omega^i}$, $1 \leq i \leq K$ (where u is the continuous solution), and let $(\underline{u}_h)_i = (\underline{u}_h)_{|\Omega^i}$ where $\underline{u}_h$ is the solution of the discrete problem (2.10). Now, let $E_{ex} = \|u\|_*$ and let $E_i = \|(\underline{u}_h)_i - u_i\|_{H^1(\Omega^i)}$, $1 \leq i \leq K$. Let $E = (\sum_{i=1}^K E_i^2)^{1/2}$. The relative H^1 error is then E/E_{ex}.
We consider four initial meshes : the two uniform conforming meshes (mesh 1 and 4) of Figure 4.1, and the two non-conforming meshes (mesh 2 and 3) of Figure 4.2. In the non-conforming case, the unit square is decomposed into four non-overlapping subdomains numbered as in Figure 4.2 on the left.

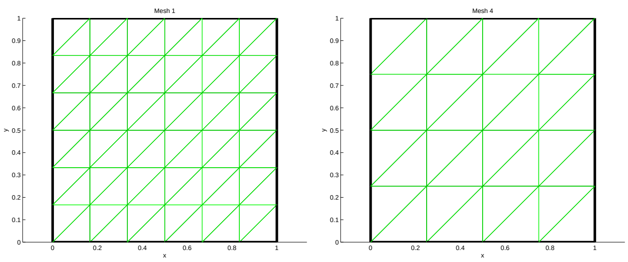

FIG. 4.1. *Uniform conforming meshes : mesh 1 (on the left), and mesh 4 (on the right)*

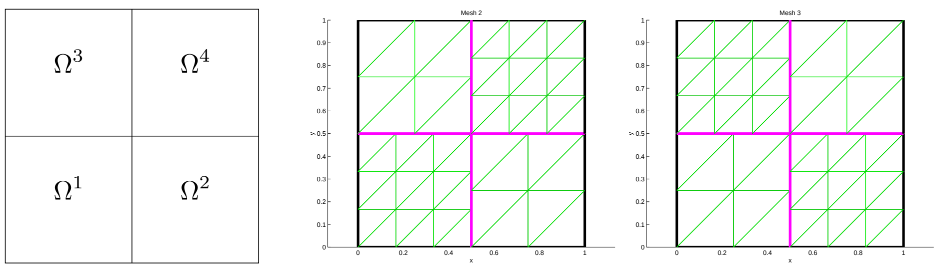

FIG. 4.2. *Domain decomposition (on the left), and non-conforming meshes: mesh 2 (on the middle), and mesh 3 (on the right)*

Figure 4.3 shows the relative H^1 error versus the number of refinement for these four meshes, and the mesh size h versus the number of refinement, in logarithmic scale. At each refinement, the mesh size is divided by two. The results of Figure 4.3 show that the relative H^1 error tends to zero at the same rate than the mesh size, and this fits with the theoretical error estimates of Theorem 5. On the other hand, we observe that the two curves corresponding to the non-conforming meshes (mesh 2 and mesh 3) are between the curves of the conforming meshes (mesh 1 and mesh 4).

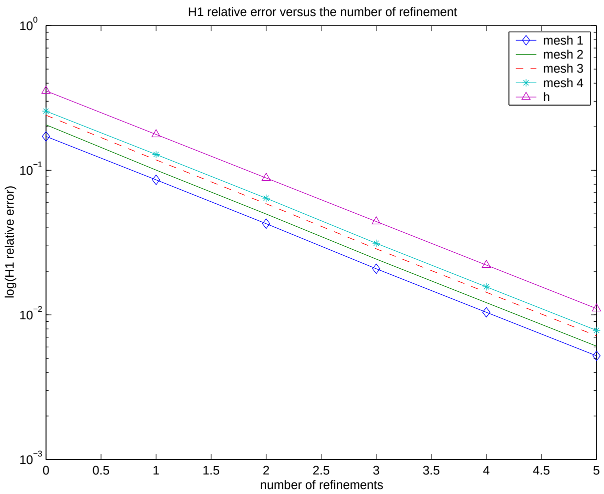

FIG. 4.3. *Relative H^1 error versus the number of refinements for the initial meshes : mesh 1, (diamond line), mesh 2 (solid line), mesh 3 (dashed line), and mesh 4 (star line). The triangle line is the mesh size h versus the number of refinements, in logarithmic scale*

The relative H^1 error for mesh 2 is smaller than the one corresponding to mesh 3, and this is because mesh 2 is more refined than mesh 3 in subdomain Ω^4, where the solution steeply varies. More precisely, let us compare for mesh 2, the relative H^1 error in the domain $\Omega^1 \cup \Omega^2 \cup \Omega^3$ to the relative H^1 error in the subdomain Ω^4 (which is the subdomain where the solution steeply varies). This comparison is done in Table 4.1.

Refinement	$(E_1^2 + E_2^2 + E_3^2)^{1/2}/E_{ex}$	E_4/E_{ex}	E/E_{ex}
0	1.45e-01	1.46e-01	2.06e-01
1	7.17e-02	7.02e-02	1.004e-01
2	3.59e-02	3.49e-02	5.01e-02
3	1.79e-02	1.73e-02	2.49e-02
4	8.73e-03	8.46e-03	1.21e-02

TABLE 4.1
Comparison, in the case of **mesh 2**, *for different refinements (column one), of the relative H^1 error in the domain composed by subdomains Ω^1, Ω^2 and Ω^3 (column 2) to the relative H^1 error in the subdomain Ω^4 (column 3). The fourth column is the relative H^1 error in the whole domain.*

We observe that, as expected, the relative H^1 error in the domain composed by subdomains Ω^1, Ω^2 and Ω^3 (second column of Table 4.1) is balanced with the relative

H^1 error in the subdomain Ω^4 (third column of Table 4.1). Indeed, the mesh 2 is more refined in the subdomain Ω^4 where the solution steeply varies.

Refinement	$(E_1^2 + E_2^2 + E_3^2)^{1/2}/E_{ex}$	E_4/E_{ex}	E/E_{ex}
0	1.26e-01	2.04e-01	2.40e-01
1	5.57e-02	1.04e-01	1.18e-01
2	2.74e-02	5.22e-02	5.90e-02
3	1.36e-02	2.59e-02	2.93e-02
4	6.64e-03	1.26e-02	1.43e-02

TABLE 4.2

Comparison, in the case of **mesh 3***, for different refinements (column one), of the H^1 relative error in the domain composed by subdomains Ω^1, Ω^2 and Ω^3 (column 2) to the H^1 relative error in the subdomain Ω^4 (column 3). The fourth column is the H^1 relative error in the whole domain.*

Let us now do the same comparison in the case of mesh 3. This mesh is coarser in the subdomain Ω^4 where the solution steeply varies. In Table 4.2, we observe that as expected, the H^1 relative error in the domain composed by subdomains Ω^1, Ω^2 and Ω^3 (second column of Table 4.2) is smaller (almost half) than the H^1 relative error in the subdomain Ω^4 (third column of Table 4.2). That one is close to the H^1 relative error in the whole domain (fourth column of Table 4.2).

4.3. Error estimates for a solution with minimal regularity. In this section we propose an example where the assumptions of Theorem 4 hold, but not the one of Theorem 5, to illustrate the optimality of Theorem 4 in the minimal regular case. The first difficulty is to construct such a solution. We propose, for $J \geq 1$ a given integer, the solution u_J defined on $\Omega = (-1,1) \times (0, 2\pi)$ by

$$u_J(x,y) = \begin{cases} \phi_J(2x,y) - \phi_J(x,y), & x \geq 0 \\ -(\phi_J(-2x,y) - \phi_J(-x,y)), & x \leq 0 \end{cases},$$

with

$$\phi_J(x,y) = \sum_{j=1}^{J} \sin(jy) \frac{\sinh(j(x-1))}{j^2 \cosh(j)}.$$

The interface Γ is located at $x = 0$. For J sufficiently high, tidious computations show that there exists $c > 0$ such that

$$\|\frac{\partial u_J}{\partial_x}\|^2_{H^{\frac{1}{2}}(\Gamma)} \leq c\,log(J), \quad \|\frac{\partial u_J}{\partial_x}\|^2_{H^{\frac{3}{2}}(\Gamma)} \leq c\,J^2, \quad \|u_J\|^2_{H^2(\Omega)} \leq c\,log(J).$$

Therefore, for J sufficiently high, from Theorem 4 and Theorem 5, we have

$$\|\underline{u}_{J,h} - \underline{u}_J\|_* \leq ch\sqrt{\log(J)} + c\min(\frac{1}{\alpha}\sqrt{\log(J)}, \frac{h}{\alpha}|\log(h)|J). \tag{4.3}$$

Thus, considering the case

$$J = \frac{c_1}{h^{\frac{3}{2}}}, \quad \alpha = \frac{c_2}{h^\theta}, \tag{4.4}$$

with $\theta > 0$ given, the assumptions of Theorem 4 hold uniformly in J. This is not the case for the assumptions of Theorem 5, and from (4.3), there exists a constant c independent of J and h such that

$$\|\underline{u}_{J,h} - \underline{u}_J\|_* \leq ch\sqrt{-log(h)} + cmin(h^\theta\sqrt{-log(h)}, h^{\theta-\frac{1}{2}}|log(h)|). \tag{4.5}$$

Remark 3. *For numerical simulations, c_1 and c_2 must be tuned carefully. First, the frequencies are restricted to $J \le \frac{\pi}{h_I}$ where h_I is the mesh size over the interface. From the definition of J in (4.4), the condition $h_I \ge (\frac{c_1}{\pi})^2$ ensures that $J \le \frac{\pi}{h_I}$. Then, on one hand c_2 must be choosen not too small so that, in the right-hand side of (4.3), the second term does not become too small than the first term. On the other hand c_2 must be small enough to observe the error estimates.*

In order to illustrate the error estimate (4.5), we consider the non-conforming meshes represented on Figure 4.4. Then the meshes are refined four times, by cutting each triangle into four smaller ones (e.g. the mesh size is divided by 2 at each refinement). To compute the H^1 error, we consider a finest grid obtained from the initial one with the mesh size divided five times by a factor 2 (with 744401 vertices in domain 1 and 1090065 vertices in domain 2). The non-conforming solutions are interpolated on the finest grid to compute the error. We take $c_1 = 0.08$ and $c_2 = \frac{1}{1.2\,10^5}$. We start with $J = 1$ on the initial mesh. Then the values of J at each refinement are $2, 7, 22$ and 63. The computations of the H^1 norms are done on a grid obtained from the finest one with the mesh size divided by a factor 2. The non-conforming converged solution, at each refinement, is such that the residual is smaller than 10^{-7}.

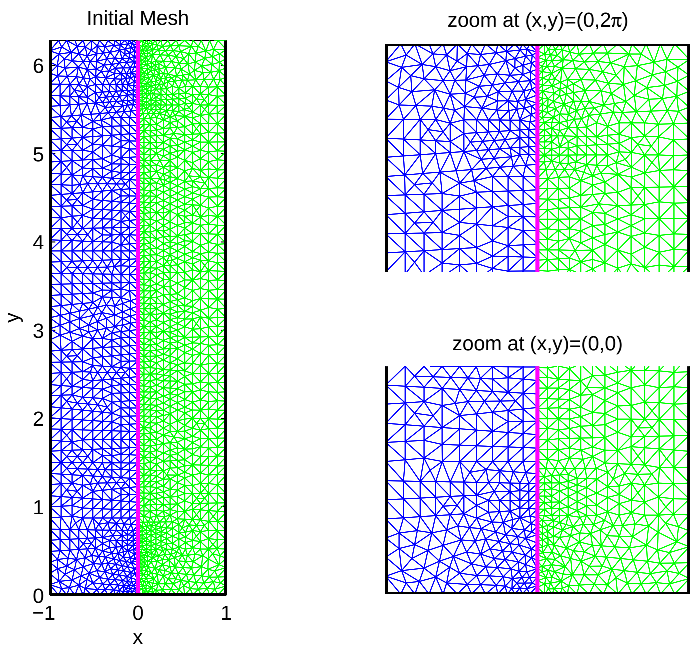

Fig. 4.4. *Initial non-conforming meshes: global meshes (left) with a zoom at corners (right)*

Figure 4.5 (left) shows the relative H^1 error versus the mesh size. We observe that the error tends to zero at the same rate than h^θ, for $\theta = \frac{1}{2}$ (star curve). This result fits with (4.5) and thus illustrates the optimality of the theoretical error estimates of Theorem 4. Figure 4.5 (right) illustrates the dependance of the error versus the Robin parameter α defined by (4.4). We represent on the interface the difference of the exact solution and the computed solution in absolute value, after three refinements (i.e. $h = 0.0233$), for $\theta = \frac{1}{4}$ and for $\theta = \frac{1}{2}$. We observe that decreasing θ increases the error as expected.

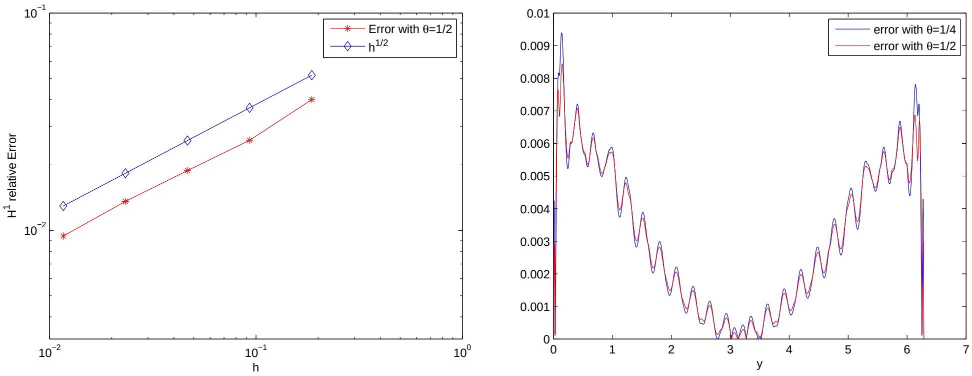

FIG. 4.5. *Relative H^1 error versus the mesh size, in logarithmic scales, for $\theta = \frac{1}{2}$ (left), and error on the interface for $\theta = \frac{1}{4}$ and for $\theta = \frac{1}{2}$ (right)*

4.4. Convergence : Choice of the Robin parameter. Let us now study the convergence speed to reach the discrete solution, for different values of the Robin parameter α, which is taken constant on the interfaces. The unit square is decomposed into four non-overlapping subdomains with non-conforming meshes (with 189, 81, 45 and 153 nodes respectively) generated as shown in Figure 4.6. The Schwarz algorithm can be interpreted as a Jacobi algorithm applied to an interface problem [34]. In order to accelerate the convergence, we can replace the Jacobi algorithm by a Gmres [35] algorithm.

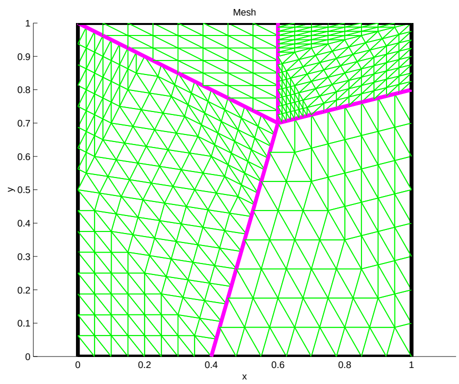

FIG. 4.6. *Domain decomposition in 4 subdomains with non-conforming grids*

On Figure 4.7 we represent the relative H^1 error between the discrete Schwarz (left part) and Gmres (right part) converged solution and the iterate solution, for different values of the Robin parameter α. We observe that the optimal numerical value of the Robin parameter is close to α_{mean} and near α_{min} and α_{max} defined in Section 4.1. Moreover the convergence is accelerated by a factor 2 for Gmres, compared to

Schwarz algorithm, and the Gmres algorithm is less sensitive to the choice of the Robin parameter.

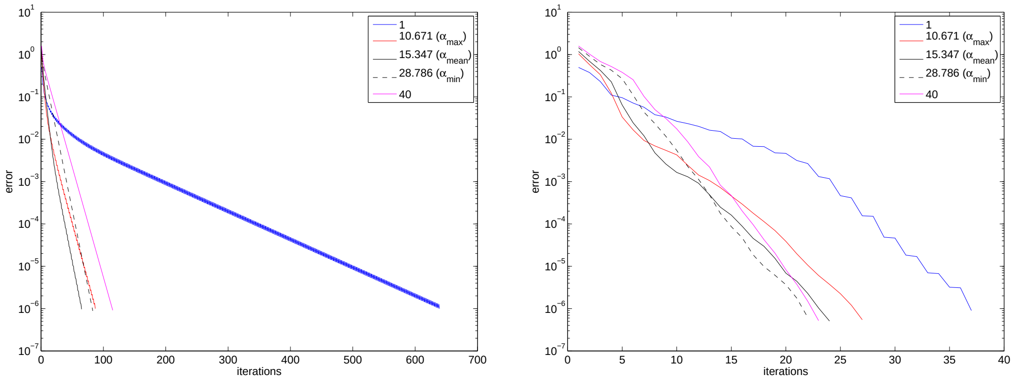

FIG. 4.7. *Relative H^1 error between the discrete Schwarz (left part) and Gmres (right part) converged solution and the iterate solution, for different values of the Robin parameter α*

4.5. Conclusions of the numerical results. The numerical results on the relative H^1 error between the continuous and discrete solutions correspond to the theoretical error estimates of Theorems 4 and 5. As seems natural, we also observe that, for a fixed number of mesh points, the relative H^1 error between the continuous and discrete solutions is smaller for a mesh refined in the region of the domain where the solution steeply varies, than for a mesh which is coarser in that region. Note finally that, in term of convergence speed to reach the discrete solution, the Robin parameter α must depend of the mesh size, and our simulations show that $\alpha = \alpha_{mean}$ is close to the optimal numerical value.

Appendix A. Inf-sup condition. The purpose of this appendix is to show that for Lemma 2, the proof of [3] can be extended to the 3D situation. Indeed the main ingredients required for the extensions have been proven in [10]. Let us first recall a standard stability result in higher norms of the L^2 projection operator $\bar{\pi}_{k,\ell}$ from $L^2(\Gamma^{k,\ell})$ onto $\mathcal{Y}_h^{k,\ell} \cap H_0^1(\Gamma^{k,\ell})$ orthogonal to $\tilde{W}_h^{k,\ell}$.

LEMMA 8. *Making the hypothesis that the triangulation $\mathcal{T}_h^k$ is uniformly regular, there exists a constant $c > 0$ such that*

$$\forall v \in H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}),\ \|\bar{\pi}_{k,\ell} v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} \leq c\|v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})}.$$

Proof of Lemma 8: From (3.46) we deduce a uniform inf-sup condition between $\mathcal{Y}_h^{k,\ell} \cap H_0^1(\Gamma^{k,\ell})$ and $\tilde{W}_h^{k,\ell}$ in $L^2(\Gamma^{k,\ell})$. It results that the projection operator $\bar{\pi}_{k,\ell}$ is stable in $L^2(\Gamma^{k,\ell})$ and thus there exists a constant $c_1 > 0$ such that

$$\forall v \in H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}),\ \|v - \bar{\pi}_{k,\ell} v\|_{L^2(\Gamma^{k,\ell})} \leq c_1 h^{\frac{1}{2}} \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})}.$$

Let $\tilde{\pi}_{k,\ell}$ denote the orthogonal projection operator from $H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})$ onto $\mathcal{Y}_h^{k,\ell} \cap H_0^1(\Gamma^{k,\ell})$ for $H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})$ inner product. Then, for all v in $H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})$,

$$\|\bar{\pi}_{k,\ell} v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} \leq \|\tilde{\pi}_{k,\ell} v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} + \|\bar{\pi}_{k,\ell} v - \tilde{\pi}_{k,\ell} v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})}.$$

Then, with an inverse inequality, there exists a constant $c_2 > 0$ such that

$$\|\bar{\pi}_{k,\ell} v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} \leq \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} + c_2 h^{-\frac{1}{2}} \|\bar{\pi}_{k,\ell} v - \tilde{\pi}_{k,\ell} v\|_{L^2(\Gamma^{k,\ell})}.$$

Thus,

$$\|\bar{\pi}_{k,\ell} v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} \leq \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} + c_2 h^{-\frac{1}{2}} c' h^{\frac{1}{2}} \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})},$$

and then, with $c = 1 + c'c_2$,we have

$$\|\bar{\pi}_{k,\ell} v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} \leq c \|v\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})}, \forall v \in H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}),$$

which ends the proof of Lemma 8.

Proof of Lemma 2: From the definition of the $(H_{00}^{1/2}(\Gamma^{k,\ell}))'$ norm, for any $p_{h,k,\ell}$ in $\tilde{W}_h^{k,\ell}$, there exists an element $w^{k,\ell}$ in $H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})$ such that

$$\int_{\Gamma^{k,\ell}} p_{h,k,\ell} w^{k,\ell} =_{(H_{00}^{1/2}(\Gamma^{k,\ell}))'} < p_{h,k,\ell}, w^{k,\ell} >_{H_{00}^{1/2}(\Gamma^{k,\ell})} = \|p_{h,k,\ell}\|_{(H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}))'} \|w^{k,\ell}\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})},$$

and $w^{k,\ell}$ can be chosen such that

$$\|w^{k,\ell}\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} = \|p_{h,k,\ell}\|_{(H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}))'}.$$

We apply now the projection operator on $w^{k,\ell}$ from Lemma 8. We derive that $\bar{\pi}_{k,\ell}(w^{k,\ell}) = w_h^{k,\ell} \in \mathcal{Y}_h^{k,\ell} \cap H_0^1(\Gamma^{k,\ell})$ and

$$\|w_h^{k,\ell}\|_{H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})} \leq c \|p_{h,k,\ell}\|_{(H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}))'},$$

and

$$\int_{\Gamma^{k,\ell}} p_{h,k,\ell} w_h^{k,\ell} = \int_{\Gamma^{k,\ell}} p_{h,k,\ell} w^{k,\ell} = \|p_{h,k,\ell}\|^2_{(H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell}))'}.$$

It remains to lift $w_h^{k,\ell}$ over Ω^k, this is done by prolongating $w_h^{k,\ell}$ by zero over $\partial\Omega^k \setminus \Gamma^{k,\ell}$ and lifting this element of $H^{\frac{1}{2}}(\partial\Omega^k)$ over Ω^k as proposed in [6], which ends the proof of Lemma 2.

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