A new cement to glue non-conforming grids with Robin interface conditions: The Finite Volume Case

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Abstract

Robin interface conditions in domain decomposition methods enable the use of non overlapping subdomains and a speed up in the convergence. Non conforming grids make the grid generation much easier and faster since it is then a parallel task. The goal of this paper is to propose and analyze a new discretization scheme which allows to combine the use of Robin interface conditions with non-matching grids. We consider both a symmetric definite positive operator and the convection-diffusion equation discretized by finite volume schemes. Numerical results are shown.

1 Introduction

The goal of our project is to design a general domain decomposition method that allows to combine the use of optimized interface conditions with non-matching grids. In this paper, we consider the convection-diffusion equation

$$\eta u + div(au) - \nu \Delta(u) = f \text{ in } \Omega \subset \mathbb{R}^d$$

discretized by a finite volume method where η and ν are positive but arbitrarily small and a is a vector field. In a joint paper, the case of a finite element discretization is analyzed.

The original Schwarz algorithm is based on a decomposition of the domain Ω into overlapping subdomains and on solving Dirichlet boundary value problems in each subdomain. It has been proposed in [28], [11] to use more general boundary conditions for the subproblems in order to use a non-overlapping decomposition of the domain. By using optimized interface condition (see [24],

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[8], [25]), the convergence speed is also increased dramatically.

When the grids are matched at the boundaries of the subdomains, the implementation of such interface conditions on the discretized problem is not too difficult. On the other hand, using non-matching grids is very appealing. Indeed, it is possible in a decoupled fashion to build complex meshes which are nevertheless locally structured. The advantages are numerous:

1. the generation of the mesh is then much easier and faster since it can be done mostly in parallel.
2. The structured local mesh enables the use of fast solvers in the subdomains.
3. adaptive procedures are much easier to implement.

The mortar element method, first introduced in [3], enables the use of nonconforming grids. It is also well suited to the use of the so-called "Dirichlet-Neumann" (see [4], [16]) or "Neumann-Neumann" (see [6], [14]) preconditioned conjugate gradient method applied to the Schur complement matrix, see [26]. However, it seems that the original mortar element method cannot be used easily with optimized interface conditions in the framework of Schwarz type methods (also called two-field methods, see [15]).

The goal of our work is to design and study a domain decomposition method which allows for the use of Robin interface conditions ($\frac{\partial}{\partial n}+\alpha$) for the convection-diffusion problem above. We focus here on the error analysis. We do not consider the convergence of the Schwarz method for non-matching grids, although it can be easily derived by using techniques developed here and in [31]. Moreover, usually the additive Schwarz method is replaced by much more efficient Krylov type methods and in addition, the small enough problems can also be solved by a direct method.

To our knowledge, only a few papers address the error analysis of domain decomposition methods with non-matching grids and Robin interface conditions for finite volume or mixed finite element methods. In [2], a non consistent discretization of the fluxes is proposed at interfaces, and an error analysis is performed for the discrete L^2 norm.

In [1], this question has been addressed for the convection-diffusion equation discretized by a mixed finite element method under the assumption that $\exists \gamma_* > 0$ such that $\forall w \in L^2(\Omega)$, $\forall v \in L^2(\Omega)^d$, we have

$$\gamma_* \{ \|w\|^2_{L^2(\Omega)} + \|v\|^2_{(L^2(\Omega))^d} \} \leq \eta \|w\|^2_{L^2(\Omega)} + \frac{1}{\nu} \|v\|^2_{(L^2(\Omega))^d} - \frac{1}{\nu} (a.v, w)_{L^2 \times L^2}.$$

As an example, if a is a positive constant and $d = 1$ the above condition is equivalent to $\frac{a^2}{\nu^2} - 4\frac{\eta}{\nu} < 0$. The convection cannot be dominant. In the present work, we do not need such an assumption but we have geometric assumptions on the grids.

We first introduce our new matching scheme in the symmetric definite positive case ($a = 0$) and then extend it to the convective case ($a \neq 0$) studied in § 7. More precisely, in § 2 the domain decomposition method is defined at the continuous level, in § 3 the finite volume scheme is defined along with the treatment of the interface conditions, in § 4 the non-conforming domain decomposition method is proved to be well-posed under no extra assumption compared with the original finite volume scheme (cf. [23]), and the error analysis is performed in § 5. At the expense of some extra assumptions on the geometry of the interface cells and if $\alpha = O(1)$, we have the same error estimate as for the original finite volume. In § 6, we explain how it is possible to relax these assumptions and replace them by weaker geometric assumptions. In § 7, the convective term is taken into account and no extra assumption is made compared with the symmetric positive definite case. In § 8, numerical results are given.

2 Domain Decomposition at the continuous level

Let Ω be a bounded domain in $\mathbb{R}^d$ for $d \geq 2$ and $\eta > 0$. We consider the following problem:

find u such that

$$\begin{aligned}(-\Delta + \eta)(u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega.\end{aligned} \tag{1}$$

The domain Ω is decomposed into N non-overlapping subdomains, $\bar{\Omega} = \cup_{1\leq i\leq N}\bar{\Omega}_i$. Given $\alpha > 0$, the above problem is reformulated as a domain decomposition problem:

Find $(u_i)_{1\leq i\leq N}$ such that

$$(-\Delta + \eta)(u_i) = f \text{ in } \Omega_i \tag{2}$$

$$u_i = 0 \text{ on } \partial\Omega \cap \partial\Omega_i \tag{3}$$

$$\frac{\partial u_i}{\partial n_i} + \alpha u_i = -\frac{\partial u_j}{\partial n_j} + \alpha u_j \text{ on } \partial\Omega_j \cap \partial\Omega_i \tag{4}$$

A simple iterative method for solving the above domain decomposition method is:

$$\begin{aligned}(-\Delta + \eta)(u_i^{n+1}) &= f \text{ in } \Omega_i, \\ u_i^{n+1} &= 0 \text{ on } \partial\Omega \cap \partial\Omega_i, \\ \frac{\partial u_i^{n+1}}{\partial n_i} + \alpha u_i^{n+1} &= -\frac{\partial u_j^n}{\partial n_j} + \alpha u_j^n \text{ on } \partial\Omega_j \cap \partial\Omega_i.\end{aligned} \tag{5}$$

The well-posedness and convergence of the above problems and algorithm have been studied in [28]. It is also possible to use (5) as a preconditioner for Krylov type methods, see for example [10], [30] and [25]. We are interested in the discretization of (2) by a finite volume scheme with non matching grids on the interface between the subdomains.

3 Finite volume discretization

The scheme is taken from [23]. We choose this scheme as an example but the interface matching conditions that we propose would work with other schemes, see § 6. Associated with each domain Ω_i, let $\mathcal{T}_i$ be a set of closed polygonal subsets of Ω_i such that $\bar{\Omega}_i = \cup_{K\in\mathcal{T}_i} K$ and $\mathcal{E}_{\Omega_i}$ be the set of faces associated with $\mathcal{T}_i$, i.e. a set of closed subsets of dimension d such that for any $(K, K') \in \mathcal{T}_i^2$ with $K \neq K'$, one has either

- $K \cap K' \in \mathcal{E}_{\Omega_i}$. In this case, $\partial K \cap \partial K'$ is denoted by $[K, K']$.
- $dim(K \cap K') < d - 1$
- $K \cap K' = \emptyset$

We also assume that no face intersects both $\partial\Omega_i \backslash \partial\Omega$ and $\partial\Omega_i \cap \partial\Omega$.

Remark 3.1 *As a result, $\partial\Omega_i \cap \partial\Omega$ can be written as an union of (whole) faces and the same holds for $\partial\Omega_i \backslash \partial\Omega$.*

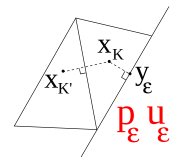
We shall use the following notations

- Let ϵ_i be a face of $\mathcal{E}_{\Omega_i}$ located on the boundary of Ω_i, $K(\epsilon_i)$ denotes the control cell $K \in \mathcal{T}_i$ such that $\epsilon_i \in K$.
- $\mathcal{E}_{iD}$ is the set of faces such that $\partial\Omega \cap \partial\Omega_i = \cup_{\epsilon\in\mathcal{E}_{iD}} \epsilon$. Let us recall that a Dirichlet boundary condition is imposed on $\partial\Omega \cap \partial\Omega_i$.
- $\mathcal{E}_i$ is the set of faces such that $\partial\Omega_i \backslash \partial\Omega = \cup_{\epsilon\in\mathcal{E}_i} \epsilon$. Let us recall that a Robin interface condition is imposed on $\partial\Omega_i \backslash \partial\Omega$.
- $\mathcal{E}(K)$ denotes the set of the faces of $K \in \mathcal{T}_i$.
- $\mathcal{E}_{iD}(K) = \mathcal{E}(K) \cap \mathcal{E}_{iD}$ is the set of the faces of $K \in \mathcal{T}_i$ which are on $\partial\Omega \cap \partial\Omega_i$.
- $\mathcal{E}_i(K) = \mathcal{E}(K) \cap \mathcal{E}_i$ is the set of the faces of $K \in \mathcal{T}_i$ which are on $\partial\Omega_i \backslash \partial\Omega$.
- $\mathcal{N}_i(K)$ is the set of the control cells adjacent to K: $\mathcal{N}_i(K) = \{K' \in \mathcal{T}_i : K \cap K' \in \mathcal{E}_{\Omega_i}\}$

We make the following

Assumption 3.2 *We assume that there exist points $(y_\epsilon)_{\epsilon\in\mathcal{E}_{\Omega_i}}$ on the faces ($y_\epsilon \in \epsilon$) and points $(x_K)_{K\in\mathcal{T}_i}$ inside the control cells such that*

- *For any adjacent control cells, K and K', the straight line $[x_K, x_{K'}]$ is perpendicular to the face $[K, K']$ and $[x_K, x_{K'}] \cap [K, K'] = \{y_{[K,K']}\}$.*
- *For any face $\epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}$, the straight line $[x_{K(\epsilon)}, y_\epsilon]$ is perpendicular to ϵ.*



Geometric assumption

It is then possible to write a finite volume scheme for the equation (2). We shall use the primary unknowns $(u_K)_{K\in\mathcal{T}}$ which aim at being approximations to $u(x_K)$. The scheme is obtained by integrating (2) over each control volume K:

$$\int_K \eta u - \int_{\partial K} \frac{\partial u}{\partial n} = \int_K f.$$

This relation is discretized by

$$\eta \; meas(K)\, u_K - \sum_{K'\in\mathcal{N}_i(K)} p_{K,K'} meas([K,K']) - \sum_{\epsilon\in\mathcal{E}_{iD}(K)} p_\epsilon meas(\epsilon) - \sum_{\epsilon\in\mathcal{E}_i(K)} p_\epsilon meas(\epsilon) = F_K \qquad (6)$$

where $meas([K,K'])$ is the measure of $[K,K']$, F_K is an approximation to $\int_K f$ such that

$$S_K = \int_K f - F_K = meas(K)O(diam(K)) \qquad (7)$$

and p_ϵ is a discretization (defined below) of the normal derivative $\partial u/\partial n_K$ on the face $[K,K']$.
For an face $[K,K']$ common to two control volumes K and K',

$$p_{K,K'} = \frac{u_{K'} - u_K}{d(x_{K'}, x_K)} \qquad (8)$$

We have the useful property that $p_{K,K'} = -p_{K',K}$. For an face ϵ on the boundary $\partial\Omega$, the homogeneous Dirichlet boundary condition (3) is taken into account by

$$p_\epsilon = \frac{0 - u_K}{d(y_\epsilon, x_K)} \qquad (9)$$

When there is no domain decomposition, this scheme has been analyzed in [23] in the more general case of discontinuous coefficients and it is proved to be of order one for a discrete H^1-norm.
In the next subsection, a scheme is written for the discretization of the interface condition (4).

3.1 Discretization of the interface conditions

On each interface face $\epsilon \in \mathcal{E}_i$ of a control volume $K = K(\epsilon)$, we introduce p_ϵ and u_ϵ related by the relation

$$p_\epsilon = \frac{u_\epsilon - u_K}{d(y_\epsilon, x_K)} \tag{10}$$

Then, the interface condition (4) on an interface face $\epsilon \in \mathcal{E}_i$ of a control volume $K = K(\epsilon)$ is discretized by

$$meas(\epsilon)(p_\epsilon + \alpha u_\epsilon) = \sum_{j \neq i} \sum_{\epsilon_j \in \mathcal{E}_j} meas(\epsilon \cap \epsilon_j)(-p_{\epsilon_j} + \alpha u_{\epsilon_j}) \tag{11}$$

It will be useful in the error analysis to interpret (11) as a L^2 projection. Indeed, let $P^0(\partial\Omega_i \backslash \partial\Omega)$ be the set of functions from $\partial\Omega_i \backslash \partial\Omega$ into $\mathbb{R}$ which are piecewise constant on the interface faces. To any discrete values $(v_\epsilon)_{\epsilon \in \mathcal{E}_i}$, we associate its natural piecewise constant extrapolation $\pi_i((v_\epsilon)_{\epsilon \in \mathcal{E}_i}) \in P^0(\partial\Omega_i \backslash \partial\Omega)$:

$$\begin{aligned} \pi_i((v_\epsilon)_{\epsilon \in \mathcal{E}_i}) : \partial\Omega_i \backslash \partial\Omega &\rightarrow \mathbb{R} \\ x &\mapsto v_\epsilon \text{ if } x \in \epsilon \end{aligned}$$

The L^2 projection on $P^0(\partial\Omega_i \backslash \partial\Omega)$ is denoted by P_i. With these notations, (11) is equivalent to

$$\pi_i((p_\epsilon + \alpha u_\epsilon)_{\epsilon \in \mathcal{E}_i}) = P_i(\sum_{j \neq i} \pi_j((-p_\epsilon + \alpha u_\epsilon)_{\epsilon \in \mathcal{E}_j})\, 1_{[\partial\Omega_i \cap \partial\Omega_j]}) \tag{12}$$

For simplicity, (12) will (sometimes) be denoted

$$p_i + \alpha u_i = P_i(\sum_{j \neq i} (-p_j + \alpha u_j) 1_{[\partial\Omega_i \cap \partial\Omega_j]}) \tag{13}$$

4 Well posedness

The finite volume scheme on the non matching grids is well posed:

Theorem 4.1 *For $\alpha > 0$, the finite volume discretization defined by (6)-(8)-(9) -(10)-(11) is well-posed.*

The proof is based on

Lemma 4.2 *In each subdomain Ω_i, $1 \leq i \leq N$, let (u, p) satisfy the equations (6)-(8)-(9) -(10). Then the following estimate holds:*

$$-\frac{1}{4\alpha} \sum_{\epsilon \in \mathcal{E}_i} [(p_\epsilon + \alpha u_\epsilon)^2 - (-p_\epsilon + \alpha u_\epsilon)^2]\, meas(\epsilon)$$
$$+ \sum_{\epsilon \in \mathcal{E}_{iD}} \frac{u_{K(\epsilon)}^2}{d(y_\epsilon, x_{K(\epsilon)})} meas(\epsilon) + \sum_{\epsilon \in \mathcal{E}_i} d(x_{K(\epsilon)}, y_\epsilon) p_\epsilon^2 meas(\epsilon)$$

$$+\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}(\frac{u_K-u_{K'}}{d(x_K,x_{K'})})^2 d(x_K,x_{K'})meas([K,K'])$$
$$+\sum_{K\in\mathcal{T}_i}\eta u_K^2 meas(K)=\sum_{K\in\mathcal{T}_i}F_K u_K.$$

Proof The summation over $K\in\mathcal{T}_i$ of the equation (6) multiplied by u_K yields

$$\sum_{K\in\mathcal{T}_i}\left(\eta u_K^2 meas(K)+\sum_{K'\in\mathcal{N}_i(K)}-p_{KK'}u_K meas([K,K'])\right.$$
$$\left.-\sum_{\epsilon\in\mathcal{E}_{iD}(K)\cup\mathcal{E}_i(K)}p_\epsilon u_K meas(\epsilon)\right)=\sum_{K\in\mathcal{T}_i}F_K u_K.$$

We distinguish between the interface faces and the internal faces and use the relations (8) and (9),

$$\sum_{\epsilon\in\mathcal{E}_{iD}}\frac{u_K^2}{d(y_\epsilon,x_{K(\epsilon)})}meas(\epsilon)-\sum_{\epsilon\in\mathcal{E}_i}p_\epsilon meas(\epsilon)u_{K(\epsilon)}$$
$$+\sum_{K\in\mathcal{T}_i}\left(\sum_{K'\in\mathcal{N}_i(K)}\frac{u_K-u_{K'}}{d(x_K,x_{K'})}meas([K,K'])u_K+\eta u_K^2 meas(K)\right)$$
$$=\sum_{K\in\mathcal{T}_i}F_K u_K.$$

By using (10) and rewriting the second term, we get

$$\sum_{\epsilon\in\mathcal{E}_{iD}}\frac{u_K^2}{d(y_\epsilon,x_{K(\epsilon)})}meas(\epsilon)+\sum_{\epsilon\in\mathcal{E}_i}-p_\epsilon meas(\epsilon)u_\epsilon+d(y_\epsilon,x_{K(\epsilon)})p_\epsilon^2 meas(\epsilon)$$
$$+\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}(\frac{u_K-u_{K'}}{d(x_K,x_{K'})})^2 d(x_K,x_{K'})meas([K,K'])$$
$$+\sum_{K\in\mathcal{T}_i}\eta u_K^2 meas(K)=\sum_{K\in\mathcal{T}_i}F_K u_K.$$

and lemma 4.2 follows by noticing that

$$-p_\epsilon u_\epsilon=-\frac{1}{4\alpha}[(p_\epsilon+\alpha u_\epsilon)^2-(-p_\epsilon+\alpha u_\epsilon)^2]. \tag{14}$$

□

Proof of Theorem 4.1 The linear system arising from the finite volume discretization and the interface condition (11) is square so it suffices to prove that the solution is zero for $F_K=0$. By using the notations of § 3.1, the estimate of Lemma (4.2) reads

$$-\frac{1}{4\alpha}\int_{\partial\Omega_i}(p_i+\alpha u_i)^2-(-p_i+\alpha u_i)^2$$

$$+\sum_{\epsilon\in\mathcal{E}_{iD}}\frac{u_{K(\epsilon)}^2}{d(y_\epsilon,x_{K(\epsilon)})}meas(\epsilon)+\sum_{\epsilon\in\mathcal{E}_i}d(x_{K(\epsilon)},y_\epsilon)p_\epsilon^2 meas(\epsilon)$$
$$+\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}(\frac{u_K-u_{K'}}{d(x_K,x_{K'})})^2 d(x_K,x_{K'})meas([K,K'])$$
$$+\sum_{K\in\mathcal{T}_i}\eta u_K^2 meas(K)=0.$$

By using (13) and the fact that the L^2 projection P_i is a contraction, we get

$$-\frac{1}{4\alpha}\sum_{j\neq i}\int_{\partial\Omega_i\cap\partial\Omega_j}(-p_j+\alpha u_j)^2-(-p_i+\alpha u_i)^2$$
$$+\sum_{\epsilon\in\mathcal{E}_{iD}}\frac{u_{K(\epsilon)}^2}{d(y_\epsilon,x_{K(\epsilon)})}meas(\epsilon)+\sum_{\epsilon\in\mathcal{E}_i}d(x_{K(\epsilon)},y_\epsilon)p_\epsilon^2 meas(\epsilon)$$
$$+\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}(\frac{u_K-u_{K'}}{d(x_K,x_{K'})})^2 d(x_K,x_{K'})meas([K,K'])$$
$$+\sum_{K\in\mathcal{T}_i}\eta u_K^2 meas(K)\leq 0.$$

These equations are summed up over i. The terms

$$-\frac{1}{4\alpha}\int_{\partial\Omega_i\cap\partial\Omega_j}(-p_{\epsilon_j}+\alpha u_{\epsilon_j})^2-(-p_{\epsilon_i}+\alpha u_{\epsilon_i})^2$$

cancel out and the following sum of positive terms satisfies :

$$\sum_{\epsilon\in\mathcal{E}_{iD}}\frac{u_{K(\epsilon)}^2}{d(y_\epsilon,x_{K(\epsilon)})}meas(\epsilon)+\sum_i\sum_{\epsilon\in\mathcal{E}_i}d(x_{K(\epsilon)},y_\epsilon)p_\epsilon^2 meas(\epsilon)$$
$$+\sum_i\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}(\frac{u_K-u_{K'}}{d(x_K,x_{K'})})^2 d(x_K,x_{K'})meas([K,K'])$$
$$+\sum_i\sum_{K\in\mathcal{T}_i}\eta u_K^2 meas(K)\leq 0.$$

This proves that $u_K=0$ for any $K\in\mathcal{T}_i$ and any i, and that $p_\epsilon=0$ for any face $\epsilon\in\mathcal{E}_i$ and any i. From relation (9) we have $u_\epsilon=0$, $\forall\epsilon\in\mathcal{E}_i$. □

5 Error analysis (part I)

For the error analysis, we need the following additional assumption on the interface faces:

Assumption 5.1 *a) For any i and any $\epsilon\in\mathcal{E}_i$, y_ϵ is the barycenter of ϵ.*
b) For any i,j, $\partial\Omega_i\cap\partial\Omega_j=\cup_{\{\epsilon:\epsilon\in\mathcal{E}_i,\epsilon\subset\partial\Omega_j\}}\epsilon$, i.e. $\partial\Omega_i\cap\partial\Omega_j$ can be written as the union of faces of $\mathcal{E}_i$ and of $\mathcal{E}_j$.

Remark 5.2 *The above assumption 5.1a) is relaxed in § 6.*

Let $\tilde{u}_K^i = u(x_K)$ for any control cell K in $\mathcal{T}(\Omega_i)$ and for any $\epsilon \in \mathcal{E}_{iD}$, let $\tilde{u}_\epsilon^i = u(y_\epsilon) = 0$ and $\tilde{p}_\epsilon^i = \frac{\partial u}{\partial n_i}(y_\epsilon)$. For any interface face $\epsilon \in \mathcal{E}_i$, let $\tilde{p}_\epsilon^i$ (resp. $\tilde{u}_\epsilon^i$) be the mean value of $\frac{\partial u}{\partial n_i}$ (resp. u) on ϵ. With the notations of § 3.1, this can be rewritten as

$$\pi_i((\tilde{p}_\epsilon^i)_{\epsilon\in\mathcal{E}_i}) = P_i(\frac{\partial u}{\partial n_i})$$

and

$$\pi_i((\tilde{u}_\epsilon^i)_{\epsilon\in\mathcal{E}_i}) = P_i(u).$$

The solution of the finite volume discretization with F_K as a right hand-side is denoted by u_K^i for any control cell K in $\mathcal{T}(\Omega_i)$ and p_ϵ^i on any face $\epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}$ and u_ϵ^i for any face $\epsilon \in \mathcal{E}_i$. We shall estimate the discrete errors $e_K^i = u_K^i - \tilde{u}_K^i$, $e_\epsilon^i = u_\epsilon^i - \tilde{u}_\epsilon^i$ and $q_\epsilon^i = p_\epsilon^i - \tilde{p}_\epsilon^i$.

Theorem 5.3 *We assume that the solution u of (1) is in $C^2(\bar{\Omega})$. Let us consider a family of admissible meshes $\mathcal{T}_{1\le i\le N}$ which satisfy Assumption 5.1 and:*

$$\exists C' > 0 \ s.t.\ \forall i, \forall \epsilon \in \mathcal{E}_i \tag{15}$$

$$d(y_\epsilon, x_{K(\epsilon)}) \ge C'\, diam(\epsilon)^2. \tag{16}$$

We take $\alpha = \dfrac{C'}{\max_{\epsilon \in \mathcal{E}_i, 1\le i\le N} diam(\epsilon)^\gamma}$ *with $C' > 0$ and $\gamma \ge 0$. Then, $\exists C$ s.t.*

$$\sum_{\epsilon\in\mathcal{E}_{iD}} \frac{e_{K(\epsilon)}^2}{d(y_\epsilon, x_{K(\epsilon)})} meas(\epsilon) + \sum_i \sum_{\epsilon\in\mathcal{E}_i} d(y_\epsilon, x_{K(\epsilon)}) \frac{1}{2} q_\epsilon^2 meas(\epsilon) \tag{17}$$

$$+ \sum_i \frac{1}{2} \sum_{K\in\mathcal{T}_i} \sum_{K'\in\mathcal{N}_i(K)} (\frac{e_K - e_{K'}}{d(x_K, x_{K'})})^2 d(x_K, x_{K'}) meas([K, K']) \tag{18}$$

$$+ \sum_i \sum_{K_i\in\mathcal{T}_i} \eta e_K^2 meas(K_i) \le C h^{2-\gamma} \tag{19}$$

where $h = \sup_{1\le i\le N}\{diam(K),\ K \in \mathcal{T}_i\}$.

Proof By Taylor expansions, it is easy to check that for all i, $1 \le i \le N$ and for all $K \in \mathcal{T}_i$

$$- \sum_{\epsilon\in\mathcal{E}_i(K)\cup\mathcal{E}_{iD}(K)} q_\epsilon^i\, meas(\epsilon) - \sum_{K'\in\mathcal{N}_i(K)} \frac{e_{K'} - e_K}{d(x_{K'}, x_K)} meas([K, K'])$$

$$+\eta e_K meas(K) = \sum_{\epsilon\in\mathcal{E}_{iD}(K)} R_\epsilon + \sum_{K'\in\mathcal{N}_i(K)} R_{K,K'} + S_K. \tag{20}$$

where $R_\epsilon = meas(\epsilon) O(h)$, $S_K = meas(K)\, O(h)$, $R_{K,K'} = meas([K, K'])O(h)$ and $R_{K,K'} = -R_{K',K}$.

Lemma 4.2 cannot be applied directly to the discrete errors because equation (10) is not satisfied by $e^i_{K(\epsilon)}$, e^i_ϵ and q^i_ϵ, $\epsilon \in \mathcal{E}_i$. From Assumption 5.1 it can be checked by using Taylor expansions that for any face $\epsilon \in \mathcal{E}_i$

$$q_\epsilon = \frac{e_\epsilon - e_{K(\epsilon)}}{d(y_\epsilon, x_{K(\epsilon)})} + O(h) + O(\frac{diam(\epsilon)^2}{d(y_\epsilon, x_{K(\epsilon)})}). \tag{21}$$

Indeed, let

$$T = \tilde{p}_\epsilon - \frac{\tilde{u}_\epsilon - \tilde{u}_K(\epsilon)}{d(y_\epsilon, x_{K(\epsilon)})}.$$

By a Taylor expansion, $\tilde{u}_K = u(y_\epsilon) - \frac{\partial u}{\partial n}(y_\epsilon) d(y_\epsilon, x_{K(\epsilon)}) + O(d(y_\epsilon, x_{K(\epsilon)})^2)$. Hence,

$$T = [\frac{1}{meas(\epsilon)} \int_\epsilon \frac{\partial u}{\partial n} - \frac{\partial u}{\partial n}(y_\epsilon)] - \frac{1}{d(y_\epsilon, x_{K(\epsilon)})}[\frac{1}{meas(\epsilon)} \int_\epsilon u - u(y_\epsilon)] \\ + O(d(y_\epsilon, x_{K(\epsilon)}).$$

Since y_ϵ is located at the barycenter of the face ϵ, the first two terms in brackets are $O(diam(\epsilon)^2)$.

We proceed now as for Lemma 4.2 by multiplying (20) by e_K. The only modification compared to that lemma is the interface term. Hence we get,

$$\begin{aligned} &- \sum_{\epsilon \in \mathcal{E}_i} q_\epsilon e_{K(\epsilon)} meas(\epsilon) - \sum_{\epsilon \in \mathcal{E}_{iD}} q_\epsilon e_{K(\epsilon)} meas(\epsilon) \\ &+ \frac{1}{2} \sum_{K \in \mathcal{T}_i} \sum_{K' \in \mathcal{N}_i(K)} (\frac{e_K - e_{K'}}{d(x_K, x_{K'}})^2 d(x_K, x_{K'}) meas([K, K']) \\ &+ \sum_{K \in \mathcal{T}(\Omega)} \eta e_K^2 meas(K) = \sum_{K \in \mathcal{T}_i} (\sum_{\epsilon \in \mathcal{E}_{iD}(K)} R_\epsilon + \sum_{K' \in \mathcal{N}_i(K)} R_{K,K'} + S_K)\, e_K. \end{aligned} \tag{22}$$

Using $R_{K,K'} = -R_{K',K}$, the right-hand side may be rewritten as

$$\sum_{\epsilon \in \mathcal{E}_{iD}} R_\epsilon e_{K(\epsilon)} + \sum_{K \in \mathcal{T}_i} S_K\, e_K + \frac{1}{2} \sum_{K \in \mathcal{T}_i} \sum_{K' \in \mathcal{N}_i(K)} R_{K,K'} d(x_K, x_{K'}) \frac{e_K - e_{K'}}{d(x_K, x_{K'})}. \tag{23}$$

By using (21), the first term of (22) can be estimated:

$$\begin{aligned} &- \sum_{\epsilon \in \mathcal{E}_i} q_\epsilon e_{K(\epsilon)} meas(\epsilon) \\ &= - \sum_{\epsilon \in \mathcal{E}_i} q_\epsilon \left[e_\epsilon - d(y_\epsilon, x_{K(\epsilon)}) \left[q_\epsilon + O(h) + O(\frac{diam(\epsilon)^2}{d(y_\epsilon, x_{K(\epsilon)})}) \right] \right] meas(\epsilon) \end{aligned}$$

By using (16) and $ab \leq \frac{C}{2} a^2 + \frac{1}{2C} b^2$, we have

$$\begin{aligned} - \sum_{\epsilon \in \mathcal{E}_i} q_\epsilon e_{K(\epsilon)} meas(\epsilon) &\geq -\frac{1}{4\alpha} \sum_{\epsilon \in \mathcal{E}_i} [(q_\epsilon + \alpha e_\epsilon)^2 - (-q_\epsilon + \alpha e_\epsilon)^2] meas(\epsilon) \\ &+ \sum_{\epsilon \in \mathcal{E}_i} d(y_\epsilon, x_{K(\epsilon)}) \frac{1}{2} q_\epsilon^2 meas(\epsilon) - O(h^2) \end{aligned} \tag{24}$$

As for the second term of (22), we have by using $e_\epsilon = 0$ for $\epsilon \in \mathcal{E}_{iD}$ that

$$q_\epsilon = \frac{0 - e_{K(\epsilon)}}{d(x_{K(\epsilon)}, \epsilon)} + O(h). \tag{25}$$

Hence, we have

$$\begin{aligned} -\sum_{\epsilon \in \mathcal{E}_{iD}} q_\epsilon e_{K(\epsilon)} meas(\epsilon) &= \sum_{\epsilon \in \mathcal{E}_{iD}} (q_\epsilon^2 d(x_{K(\epsilon)}, \epsilon) + q_\epsilon d(x_{K(\epsilon)}, \epsilon) O(h)) meas(\epsilon) \\ &\geq \sum_{\epsilon \in \mathcal{E}_{iD}} q_\epsilon^2 d(x_{K(\epsilon)}, \epsilon) meas(\epsilon) - O(h^2). \end{aligned} \tag{26}$$

By using (23), (24) and (26) in (22), we get

$$-\frac{1}{4\alpha} \sum_{\epsilon \in \mathcal{E}_i} [(q_\epsilon + \alpha e_\epsilon)^2 - (-q_\epsilon + \alpha e_\epsilon)^2] meas(\epsilon) \tag{27}$$

$$\begin{aligned} &+ \sum_{\epsilon \in \mathcal{E}_i} d(y_\epsilon, x_{K(\epsilon)}) \frac{1}{2} q_\epsilon^2 meas(\epsilon) + \sum_{\epsilon \in \mathcal{E}_{iD}} \frac{q_\epsilon^2}{2} d(x_{K(\epsilon)}, \epsilon)\, meas(\epsilon) \\ &+ \frac{1}{2} \sum_{K \in \mathcal{T}_i} \sum_{K' \in \mathcal{N}_i(K)} (\frac{e_K - e_{K'}}{d(x_K, x_{K'}})^2 d(x_K, x_{K'}) meas([K, K']) \\ &+ \sum_{K \in \mathcal{T}(\Omega)} \eta e_K^2 meas(K) \\ &\leq \sum_{\epsilon \in \mathcal{E}_{iD}} R_\epsilon e_{K(\epsilon)} + \sum_{K \in \mathcal{T}_i} S_K\, e_K \\ &+ \frac{1}{2} \sum_{K \in \mathcal{T}_i} \sum_{K' \in \mathcal{N}_i(K)} R_{K,K'} d(x_K, x_{K'}) \frac{e_K - e_{K'}}{d(x_K, x_{K'})} + O(h^2). \end{aligned} \tag{28}$$

We now have to work on the term (27) in order to prove the error estimate. On the interface $\partial\Omega_i \cap \partial\Omega_j$, we introduce a new grid $\mathcal{E}_{ij}$ composed of the union of the grids $\mathcal{E}_i$ and $\mathcal{E}_j$:

$$\mathcal{E}_{ij} = \{\epsilon_i \cap \epsilon_j :\ \epsilon_i \in \mathcal{E}_i,\ \epsilon_j \in \mathcal{E}_j\}.$$

The L^2 projection on piecewise constant functions on $\mathcal{E}_{ij}$ is denoted by P_{ij}. Since the grid $\mathcal{E}_{ij}$ is finer than the grids $\mathcal{E}_i$ and $\mathcal{E}_j$, we have

$$P_i P_{ij} = P_{ij} P_i = P_i \text{ and } P_j P_{ij} = P_{ij} P_j = P_j. \tag{29}$$

On this grid, we introduce four discrete auxiliary unknowns p_{ij}^i, p_{ij}^j and u_{ij}^i, u_{ij}^j (see Lemma 5.4) defined by

$$u_{ij}^i = u_{ij}^j = \frac{1}{2\alpha} P_{ij}(-p_i - p_j + \alpha(u_i + u_j)) \tag{30}$$

$$p_{ij}^i = -p_{ij}^j = \frac{1}{2} P_{ij}(-p_i + p_j + \alpha(u_i - u_j)) \tag{31}$$

Lemma 5.4 *The quantities defined by (30)-(31) satisfy*

$$p_i + \alpha u_i = P_i(-p^i_{ij} + \alpha u^i_{ij}) \tag{32}$$
$$p^i_{ij} + \alpha u^i_{ij} = P_{ij}(-p_i + \alpha u_i) \tag{33}$$
$$p_j + \alpha u_j = P_j(-p^j_{ij} + \alpha u^j_{ij}) \tag{34}$$
$$p^j_{ij} + \alpha u^j_{ij} = P_{ij}(-p_j + \alpha u_j) \tag{35}$$
$$p^i_{ij} = -p^j_{ij}, \;\; u^i_{ij} = u^j_{ij} \tag{36}$$

Proof It is obvious that (33), (35) and (36) are satisfied. Let us check (32). Assumption 5.1.b) enables a localization of (13) on every interface $\partial\Omega_i \cap \partial\Omega_j$. Hence,

$$\begin{aligned} P_i(-p^i_{ij} + \alpha u^i_{ij}) &= P_i(p^j_{ij} + \alpha u^j_{ij}), \\ &= P_i P_{ij}(-p_j + \alpha u_j) = P_i(-p_j + \alpha u_j), \\ &= p_i + \alpha u_i \end{aligned}$$

Equation (34) can be checked in the same manner.

Let $\tilde{p}^i_{ij} = -\tilde{p}^j_{ij} = P_{ij}(-\frac{\partial u}{\partial n_i})$ and $\tilde{u}^i_{ij} = \tilde{u}^j_{ij} = P_{ij}(u)$ on $\partial\Omega_i \cap \partial\Omega_j$. Similarly, we introduce $q^i_{ij} = p^i_{ij} - \tilde{p}^i_{ij}$, $q^j_{ij} = p^j_{ij} - \tilde{p}^j_{ij}$, $e^i_{ij} = u^i_{ij} - \tilde{u}^i_{ij}$ and $e^j_{ij} = u^j_{ij} - \tilde{u}^j_{ij}$. Let us notice that

$$q^j_{ij} = -q^i_{ij} \text{ and } e^j_{ij} = e^i_{ij} \tag{37}$$

On the interface $\partial\Omega_i \cap \partial\Omega_j$, we have by using Assumption 5.1

$$\begin{aligned} (q_i + \alpha e_i) &= (p_i + \alpha u_i) - (\tilde{p}_i + \alpha \tilde{u}_i) \\ &= P_i((-p^i_{ij} + \alpha u^i_{ij}) - P_i(\frac{\partial u}{\partial n_i} + \alpha u) \\ &= P_i(-p^i_{ij} + \alpha u^i_{ij}) - P_i(P_{ij}(\frac{\partial u}{\partial n_i} + \alpha u)) \\ &= P_i(-q^i_{ij} + \alpha e^i_{ij}). \end{aligned} \tag{38}$$

Similarly, we have

$$\begin{aligned} q^j_{ij} + \alpha e^j_{ij} &= p^j_{ij} + \alpha u^j_{ij} - P_{ij}(\frac{\partial u}{\partial n_j} + \alpha u) \\ &= P_{ij}(-p_j + \alpha u_j) - P_{ij}P_j(\frac{\partial u}{\partial n_j} + \alpha u) \qquad (39) \\ &\quad +(P_{ij}P_j - P_{ij})(\frac{\partial u}{\partial n_j} + \alpha u) \\ &= (-q_j + \alpha e_j) + P_{ij}(P_j - Id)(\frac{\partial u}{\partial n_j} + \alpha u) \qquad (40) \end{aligned}$$

The contribution of (27) on $\partial\Omega_i \cap \partial\Omega_j$ can be rewritten as

$$
\begin{aligned}
-\frac{1}{4\alpha}\int_{\partial\Omega_i\cap\partial\Omega_j} & \big[(q_i+\alpha e_i)^2 - (-q^i_{ij}+\alpha e^i_{ij})^2 \\
& +(-q^i_{ij}+\alpha e^i_{ij})^2 - (q^j_{ij}+\alpha e^j_{ij})^2 \\
& +(q^j_{ij}+\alpha e^j_{ij})^2 - (-q_i+\alpha e_i)^2\big]
\end{aligned} \tag{41}
$$

By (37), the contribution of (41) is in fact null. By using (38) and (40), the contribution of (27) on $\partial\Omega_i \cap \partial\Omega_j$ can be rewritten as

$$
\begin{aligned}
-\frac{1}{4\alpha}\int_{\partial\Omega_i\cap\partial\Omega_j} & \Big[(P_i(-q^i_{ij}+\alpha e^i_{ij}))^2 - (-q^i_{ij}+\alpha e^i_{ij})^2 \\
+(P_{ij}(-q_j+\alpha e_j) &+ P_{ij}(P_j - Id)(\frac{\partial u}{\partial n_j}+\alpha u_j))^2 - (-q_i+\alpha e_i)^2\Big].
\end{aligned}
$$

By expanding the square terms, and using the definitions of the projectors $P_{i,\,j\text{ or }ij}$ and noticing that $-q_j+\alpha e_j = P_j(-q_j+\alpha e_j)$,

$$
\begin{aligned}
-\frac{1}{4\alpha}\int_{\partial\Omega_i\cap\partial\Omega_j} & \Big[\big[P_i(-q^i_{ij}+\alpha e^i_{ij})\big]^2 - (-q^i_{ij}+\alpha e^i_{ij})^2 + (-q_j+\alpha e_j)^2 \\
& +2(-q_j+\alpha e_j)\, P_j P_{ij}(P_j - Id)(\frac{\partial u}{\partial n_j}+\alpha u)) \\
& +(P_{ij}(P_j - Id)(\frac{\partial u}{\partial n_j}+\alpha u))^2 - (-q_i+\alpha e_i)^2\Big]
\end{aligned}
$$

By (29), we have $P_j P_{ij}(P_j - Id) = P_j^2 - P_j = 0$. Moreover, since $\|P_j - Id\|_{H^1\to L^2} = O(h)$ and P_i is a contraction, the contribution of (27) on $\partial\Omega_i\cap\partial\Omega_j$ can be estimated:

$$
\begin{aligned}
& -\frac{1}{4\alpha}\int_{\partial\Omega_i\cap\partial\Omega_j} (q_i+\alpha e_i)^2 - (-q_i+\alpha e_i)^2 \\
\geq & -\frac{1}{4\alpha}\left(\int_{\partial\Omega_i\cap\partial\Omega_j} (-q_j+\alpha e_j)^2 - (-q_i+\alpha e_i)^2\right) - O(h^{2-\gamma})
\end{aligned}
$$

Hence, the estimate (27)-(28) becomes

$$
\begin{aligned}
& -\frac{1}{4\alpha}\sum_{j\neq i}\int_{\partial\Omega_i\cap\partial\Omega_j} (-q_j+\alpha e_j)^2 - (-q_i+\alpha e_i)^2 \\
& + \sum_{\epsilon\in\mathcal{E}_i\cup\mathcal{E}_{iD}} d(y_\epsilon, x_{K(\epsilon)})\frac{1}{2}q_\epsilon^2 meas(\epsilon) \\
+\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)} & (\frac{e_K - e_{K'}}{d(x_K, x_{K'})})^2 d(x_K, x_{K'}) meas([K, K'])
\end{aligned}
$$

$$+\sum_{K\in\mathcal{T}(\Omega_i)}\eta e_K^2 meas(K)\leq\sum_{\epsilon\in\mathcal{E}_{iD}}R_\epsilon e_{K(\epsilon)}+\sum_{K\in\mathcal{T}_i}S_K\,e_K$$

$$+\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}(R_{K,K'}d(x_K,x_{K'})\frac{e_K-e_{K'}}{d(x_K,x_{K'})}+C\,h^2+C\,h^{2-\gamma}.$$

By summing up over i the above estimate, the terms $\int_{\partial\Omega_i\cap\partial\Omega_j}(-q_j+\alpha e_j)^2-(-q_i+\alpha e_i)^2$ cancel. Since $R_\epsilon = meas(\epsilon)O(h)$, $S_K = meas(K)\,O(h)$ and $R_{K,K'} = meas([K,K'])O(h)$, we get by using (25)

$$\sum_i\sum_{\epsilon\subset\partial\Omega_i}d(y_\epsilon,x_{K(\epsilon)})\frac{1}{2}q_\epsilon^2 meas(\epsilon)$$

$$+\sum_i\frac{1}{2}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}_i(K)}(\frac{e_K-e_{K'}}{d(x_K,x_{K'})})^2 d(x_K,x_{K'})meas([K,K'])$$

$$+\sum_i\sum_{K\in\mathcal{T}(\Omega_i)}\eta e_K^2 meas(K)\leq C\,h^{2-\gamma}$$

□

6 Error Estimate (Part II): Relaxing assumption 5.1

The assumptions 3.2 and 5.1:a) are verified if triangles with acute angles or rectangles are used. However, they are not satisfied by very simple and usual examples. Even with triangles, such a condition is too strong if one wants to generalize the method to an elliptic equation of the type

$$-\nabla.A\nabla u = f.$$

The scheme FV9 has been proposed (see e.g. [13]) to cope with more general meshes inside each subdomain. At the level of the interfaces, we introduce now a slight modification of the scheme inspired from the same ideas, and allowing for somewhat weaker geometrical assumption. We give it as an example but we are conscious that it could also be formulated in many other ways. The analysis inside each subdomain is performed in [23]- [13] so we focus our attention on the modification at the interface. We assume that Assumption 3.2 holds but only Assumption 5.1:a) is relaxed and replaced by Assumption 6.1 below.

In order to simplify the explanation, we consider a 2D situation where each inner vertex is associated to exactly 4 quadrangles. We keep the model equation (1). We replace the assumption 5.1:a) by the following:

Assumption 6.1 *For a face $\epsilon\in\mathcal{E}_i$, (see Figure 1), we call y_ϵ the barycenter of ϵ. We call ϵ' and ϵ'' if it exists, the neighboring faces in $\mathcal{E}_i$ aligned with ϵ. We take the straight lines (one or two) issued from $x(K(\epsilon))$ and joining $x_{K(\epsilon')}$ and possibly $x_{K(\epsilon'')}$. We assume that they intersect the straight line containing y_ϵ*

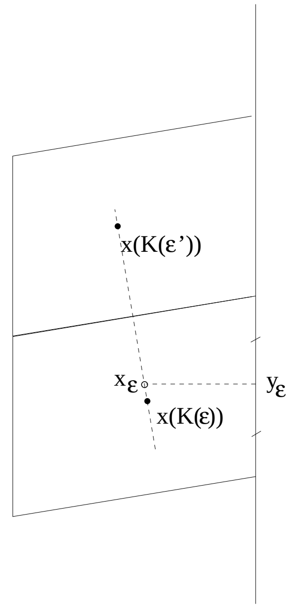


Figure 1: Construction of x_ϵ

and perpendicular to ϵ at points denoted by $z(\epsilon')$ (and possibly $z(\epsilon'')$). We call x_ϵ one of these points. We shall moreover assume that there exists $\beta < 2$ such that $\forall \epsilon \in \mathcal{E}_i$, and x_ϵ defined for example with the choice $x_\epsilon = z(\epsilon')$,

$$\frac{meas(\epsilon)}{d(x_{K(\epsilon)}, x_{K(\epsilon')})} \frac{d^2(x_{K(\epsilon)}, x_\epsilon)}{d(y_\epsilon, x_\epsilon) meas[K(\epsilon), K(\epsilon')]} \leq \beta \tag{42}$$

Now the only change in the scheme is the new definition of p_ϵ, for $\epsilon \in \mathcal{E}_i$: assume that we have made the choice $x_\epsilon = z(\epsilon')$, we introduce p_ϵ and u_ϵ related by the relation

$$p_\epsilon = \frac{u_\epsilon - \bar{u}_\epsilon}{d(y_\epsilon, x_\epsilon)}, \tag{43}$$

where $\bar{u}_\epsilon$ is given by the interpolation

$$\bar{u}_\epsilon = u_{K(\epsilon)} + \frac{u_{K(\epsilon')} - u_{K(\epsilon)}}{d(x(K(\epsilon)), x(K(\epsilon')))} d(x(K(\epsilon)), x_\epsilon). \tag{44}$$

6.1 Well posedness

As in the § 4, the existence and the uniqueness for (6)-(8)-(9) -(43)-(44)-(11) follows from the contracting properties of the L^2 projectors in the transmission conditions and from the following stability result valid for each subdomain.

Lemma 6.2 *If assumption 6.1 is satisfied and if the solution of (6)-(8)-(9) -(43)-(44)-(11) exists, then we have in domain Ω_i, $1 \le i \le N$*

$$-\frac{1}{4\alpha}\sum_{\epsilon\in\mathcal{E}_i}[(p_\epsilon+\alpha u_\epsilon)^2-(-p_\epsilon+\alpha u_\epsilon)^2]meas(\epsilon)$$

$$+\sum_{\epsilon\in\mathcal{E}_{iD}}\frac{u^2_{K(\epsilon)}}{d(y_\epsilon,x_{K(\epsilon)})}meas(\epsilon)+\sum_{\epsilon\in\mathcal{E}_{iD}}d(x_{K(\epsilon)},y_\epsilon)p^2_\epsilon meas(\epsilon)$$

$$+\frac{2-\beta}{4}\sum_{\epsilon\in\mathcal{E}_i}d(x_\epsilon,y_\epsilon)p^2_\epsilon meas(\epsilon)$$

$$+\frac{2-\beta}{2(\beta+2)}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}(K)}(\frac{u_K-u_{K'}}{d(x_K,x_{K'})})^2 d(x_K,x_{K'})meas([K,K'])$$

$$+\sum_{K\in\mathcal{T}_i}\eta u^2_K meas(K)\le\sum_{K\in\mathcal{T}_i}F_K u_K.$$

Proof

We proceed as in § 4. The only change lies in estimating the term

$$T=-\sum_{\epsilon\in\mathcal{E}_i}p_\epsilon meas(\epsilon)u_{K(\epsilon)}$$

Thanks to (43)-(44), we write it as

$$\begin{aligned}
&T=\\
&-\sum_{\epsilon\in\mathcal{E}_i}p_\epsilon meas(\epsilon)\left(u_\epsilon-d(y_\epsilon,x_\epsilon)p_\epsilon-d(x(K(\epsilon)),x_\epsilon)\frac{u_{K(\epsilon')}-u_{K(\epsilon)}}{d(x(K(\epsilon)),x(K(\epsilon')))}\right)\\
&=-\sum_{\epsilon\in\mathcal{E}_i}meas(\epsilon)u_\epsilon p_\epsilon+\sum_{\epsilon\in\mathcal{E}_i}meas(\epsilon)d(y_\epsilon,x_\epsilon)p^2_\epsilon\\
&+\textstyle\sum_{\epsilon\in\mathcal{E}_i}meas(\epsilon)d(x(K(\epsilon)),x_\epsilon)p_\epsilon\frac{u_{K(\epsilon')}-u_{K(\epsilon)}}{d(x(K(\epsilon)),x(K(\epsilon')))}
\end{aligned} \tag{45}$$

The first term is taken care of exactly as in (14). We just have to estimate the third term. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\sum_{\epsilon\in\mathcal{E}_i}meas(\epsilon)d(x(K(\epsilon)),x_\epsilon)p_\epsilon\frac{u_{K(\epsilon')}-u_{K(\epsilon)}}{d(x(K(\epsilon)),x(K(\epsilon')))}\\
&\le\gamma\sum_{\epsilon\in\mathcal{E}_i}meas(\epsilon)d(y_\epsilon,x_\epsilon)p^2_\epsilon\\
&+\frac{1}{4\gamma}\sum_{\epsilon\in\mathcal{E}_i}meas(\epsilon)\frac{d^2(x(K(\epsilon)),x_\epsilon)}{d(y_\epsilon,x_\epsilon)}\frac{(u_{K(\epsilon')}-u_{K(\epsilon)})^2}{d^2(x(K(\epsilon)),x(K(\epsilon')))}
\end{aligned}$$

But thanks to (42),

$$\frac{1}{4\gamma}\sum_{\epsilon\in\mathcal{E}_i} meas(\epsilon)\frac{d^2(x(K(\epsilon)),x_\epsilon)}{d(y_\epsilon,x_\epsilon)}\frac{(u_{K(\epsilon')}-u_{K(\epsilon)})^2}{d^2(x(K(\epsilon)),x(K(\epsilon')))}$$
$$\leq \tfrac{\beta}{4\gamma}\sum_{K\in\mathcal{T}_i}\sum_{K'\in\mathcal{N}(K)}(\tfrac{u_K-u_{K'}}{d(x_K,x_{K'})})^2 d(x_K,x_{K'})meas([K,K']).$$

We conclude the proof by choosing $\gamma=\frac{\beta+2}{4}$.

6.2 Error Estimate

The error estimate is done exactly as in section § 5, and the only difference is the estimate of q_ϵ, see (21) for $\epsilon\in\mathcal{E}_i$. A Taylor expansion shows that q_ϵ is of order $O(h+\frac{diam(\epsilon)^2}{d(x_\epsilon,y_\epsilon})$, so we have an analogous result as in Theorem 5.3, provided assumption 6.1 is satisfied.

7 Advection-diffusion problems

We consider now the advection diffusion problem with a continuous velocity field a and a viscosity ν :

$$\begin{array}{rcll} \eta u + a.\nabla u - \nu\Delta u &=& f & \text{in } \Omega,\\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{46}$$

Equation (8) is changed in

$$p_{K,K'}=\nu\,\frac{u_{K'}-u_K}{d(x_{K'},x_K)}, \tag{47}$$

(9) in

$$p_{\epsilon,D}=\nu\,\frac{0-u_K}{d(y_\epsilon,x_K)} \tag{48}$$

and (10) in

$$p_\epsilon=\nu\,\frac{u_\epsilon-u_K}{d(y_\epsilon,x_K)}. \tag{49}$$

For simplicity we restrict ourselves to the case

$$\nabla\cdot a=0,$$

which implies the identity $\nabla\cdot(ua)=a.\nabla u$. We could also handle the case where $\|\nabla\cdot a\|_\infty<\eta$.

We are going to use a first order upwind scheme to deal with convection. For that, we need the following notations in addition to those previously defined:

for a given control volume $K(\subset\Omega_i$ for example), we denote by $\mathcal{N}(K)$ the set of all the control volumes neighboring K, *i.e.* the set of all the control volumes L (not only in the subdomain Ω_i) such that $meas(K\cap L)>0$. For $L\in\mathcal{N}(K)$, we denote by $[K,L]$ the piece of face $[K,L]=K\cap L$.

We also introduce for $L \in \mathcal{N}(K)$,

$$a_{KL} = \int_{[K,L]} (a \cdot n_K)^+, \tag{50}$$

where f^+ (resp. f^-) stands for the positive (resp. negative) part of a function $f\,:\, f = f^+ - f^-$, $|f| = f^+ + f^-$.

For $\epsilon \in \mathcal{E}_{iD}$, we also denote by $a^+_{\epsilon,D}$ and $a^-_{\epsilon,D}$ the quantities

$$a^+_{\epsilon,D} = \int_\epsilon (a \cdot n_K(\epsilon))^+, \quad a^-_{\epsilon,D} = \int_\epsilon (a \cdot n_K(\epsilon))^-. \tag{51}$$

Since $\nabla.a = 0$, it is easy to check that for a volume K such that $K \subset \Omega$

$$\sum_{L\in\mathcal{N}(K)} a_{KL} - a_{LK} = 0. \tag{52}$$

and that, more generally, for a volume $K \subset \overline{\Omega}_i$,

$$\sum_{L\in\mathcal{N}(K)} a_{KL} - a_{LK} = -\sum_{\epsilon\in\mathcal{E}_{iD}(K)} \int_\epsilon (a \cdot n_K). \tag{53}$$

We introduce the finite volume discretization of (46): for K a control volume in Ω_i,

$$\begin{array}{l} \eta meas(K) u_K - \sum_{L\in\mathcal{N}_i(K)} meas([K,L]) p_{K,L} \\ - \displaystyle\sum_{\epsilon\in\mathcal{E}_{iD}(K)} meas(\epsilon) p_{\epsilon,D} - \sum_{\epsilon\in\mathcal{E}_i(K)} meas(\epsilon) p_\epsilon \\ + \displaystyle\sum_{L\in\mathcal{N}(K)} a_{KL} u_K - a_{LK} u_L + \sum_{\epsilon\in\mathcal{E}_{iD}(K)} a^+_{\epsilon,D} u_K = F_K, \end{array} \tag{54}$$

where $p_{K,L}$, $p_{\epsilon,D}$ and p_ϵ are respectively defined by (47), (48), (49) and satisfy the transmission condition (11).

Remark 7.1 *As one could expect, the discretization of the convection terms on the interface is much more natural than that of the diffusion terms, because the finite volume method was designed for that, see [17], [13].*

7.1 Stability

Let us denote by X the space of the vectors U of the degrees of freedom

1. u_K, $K \in \mathcal{T}_i, 1 \le i \le N$,
2. u_ϵ, $\epsilon \in \mathcal{E}_i, 1 \le i \le N$,

(the unknowns p_ϵ can be readily computed from U), satisfying furthermore the transmission conditions (11) at the interfaces of the subdomains. We denote by

$\mathcal{A}$ the bilinear form on X defined by

$$
\begin{aligned}
&\mathcal{A}(U,V) = \sum_{i=1}^{N} \sum_{K\in\mathcal{T}_i} \eta meas(K) u_K v_K \\
&- \sum_{i=1}^{N} \sum_{K\in\mathcal{T}_i} v_K \left(\sum_{L\in\mathcal{N}_i(K)} meas([K,L]) p_{K,L} - \sum_{\epsilon\in\mathcal{E}_{iD}(K)} meas(\epsilon) p_{\epsilon,D} \right) \\
&- \sum_{i=1}^{N} \sum_{K\in\mathcal{T}_i} v_K \left(- \sum_{\epsilon\in\mathcal{E}_i(K)} meas(\epsilon) p_\epsilon \right) \\
&+ \sum_{i=1}^{N} \sum_{K\in\mathcal{T}_i} \left(\sum_{L\in\mathcal{N}(K)} (a_{KL} u_K - a_{LK} u_L) v_K + \sum_{\epsilon\in\mathcal{E}_{iD}(K)} a^+_{\epsilon,D} u_K v_K \right).
\end{aligned} \tag{55}
$$

In order to analyze this method, we need a stability estimate on $\mathcal{A}$. When there is no advection, this estimate has already been proved. Thus, we shall only discuss the modifications caused by the new convection terms. The new ingredients in the stability proof are given in [13]. We repeat them for completeness.

Lemma 7.2 *We have the estimate*

$$
\begin{aligned}
\mathcal{A}(U,U) \geq \sum_{i=1}^{N} \sum_{\epsilon\in\mathcal{E}_{iD}} \frac{u^2_{K(\epsilon)}}{d(y_\epsilon, x_{K(\epsilon)})} meas(\epsilon) \\
+ \sum_{i=1}^{N} \sum_{\epsilon\in\mathcal{E}_i\cup\mathcal{E}_{iD}} \frac{1}{2} d(x_{K(\epsilon)}, y_\epsilon) p_\epsilon^2 meas(\epsilon) \\
+ \sum_{i=1}^{N} \frac{1}{2} \sum_{K\in\mathcal{T}_i} \sum_{K'\in\mathcal{N}_i(K)} \nu \left(\frac{u_K - u_{K'}}{d(x_K, x_{K'})}\right)^2 d(x_K, x_{K'}) meas([K,K']) \\
+ \sum_{i=1}^{N} \sum_{K\in\mathcal{T}_i} \eta u_K^2 meas(K) \\
+ \frac{1}{2} \sum_{K,L} 1_{u_K > u_L} (a_{KL} + a_{LK})(u_K - u_L)^2 + \sum_{K} \sum_{\epsilon\in\mathcal{E}_D(K)} \frac{1}{2} (a^+_{\epsilon D} + a^-_{\epsilon D}) u_K^2.
\end{aligned}
$$

Proof The stability is obtained by multiplying (54) by u_K and by summing over all the volumes. With respect to the stability analysis used in the proof of Theorem 5.3 , the only new term is

$$
T = \sum_{K} \sum_{L\in\mathcal{N}(K)} a_{KL} u_K^2 - a_{LK} u_K u_L + \sum_{K} \sum_{\epsilon\in\mathcal{E}_D(K)} a^+_{\epsilon D} u_K^2
$$

Thanks to the identities (52) and (53), this sum may be rewritten

$$
T = -\sum_{K} \sum_{L\in\mathcal{N}(K)} a_{LK} (u_K u_L - u_K^2) + \sum_{K} \sum_{\epsilon\in\mathcal{E}_D(K)} a^-_{\epsilon D} u_K^2,
$$

or

$$T = \sum_{K,L} 1_{u_K > u_L} \left[a_{KL}(u_L^2 - u_K u_L) - a_{LK}(u_K u_L - u_K^2) \right] + \sum_K \sum_{\epsilon \in \mathcal{E}_D(K)} a_{\epsilon D}^- u_K^2.$$

One can check easily that

$$\begin{aligned} T = &-\frac{1}{2} \sum_{K,L} 1_{u_K > u_L} (a_{KL} - a_{LK})(u_K^2 - u_L^2) \\ &+\frac{1}{2} \sum_{K,L} 1_{u_K > u_L} (a_{KL} + a_{LK})(u_K - u_L)^2 + \sum_K \sum_{\epsilon \in \mathcal{E}_D(K)} a_{\epsilon D}^- u_K^2. \end{aligned}$$

But from (52)-(53),

$$\begin{aligned} \sum_{K,L} 1_{u_K > u_L} \frac{1}{2} (a_{KL} - a_{LK})(u_K^2 - u_L^2) &= \frac{1}{2} \sum_K \sum_{L \in \mathcal{N}(K)} (a_{KL} - a_{LK}) u_K^2 \\ &= -\sum_K \sum_{\epsilon \in \mathcal{E}_D(K)} (a_{\epsilon D}^+ - a_{\epsilon D}^-) u_K^2. \end{aligned}$$

Therefore

$$T = \frac{1}{2} \sum_{K,L} 1_{u_K > u_L} (a_{KL} + a_{LK})(u_K - u_L)^2 + \sum_K \sum_{\epsilon \in \mathcal{E}_D(K)} \frac{1}{2} (a_{\epsilon D}^+ + a_{\epsilon D}^-) u_K^2.$$

The stability result yields readily the existence and uniqueness of the solution of the finite volume problem.

7.2 Error analysis

Theorem 7.3 *Under the same assumptions as in theorem 5.3, and provided that $a \in C(\bar{\Omega})$ satisfies $\nabla \cdot a = 0$, we have the error estimate*

$$\begin{aligned} &\sum_{i=1}^N \sum_{\epsilon \in \mathcal{E}_{iD}} \nu^2 \frac{e_{K(\epsilon)}^2}{d(y_\epsilon, x_{K(\epsilon)})} meas(\epsilon) + \sum_{i=1}^N \sum_{\epsilon \in \mathcal{E}_i} \frac{1}{2} d(x_{K(\epsilon)}, y_\epsilon) q_\epsilon^2 meas(\epsilon) \\ &\quad + \sum_{i=1}^N \frac{\nu}{2} \sum_{K \in \mathcal{T}_i} \sum_{K' \in \mathcal{N}_i(K)} (\frac{e_K - e_{K'}}{d(x_K, x_{K'})})^2 d(x_K, x_{K'}) meas([K, K']) \\ &\quad + \sum_{i=1}^N \sum_{K \in \mathcal{T}_i} \eta e_K^2 meas(K) \\ &+\frac{1}{4} \sum_{K,L} 1_{e_K > e_L} (a_{KL} + a_{LK})(e_K - e_L)^2 + \frac{1}{4} \sum_K \sum_{\epsilon \in \mathcal{E}_D(K)} (a_{\epsilon D}^+ + a_{\epsilon D}^-) e_K^2 \\ &\leq C(h^{2-\gamma} + h). \end{aligned}$$

Proof We proceed as in the proof of Theorem 5.3. The two new terms are

$$T_1 = \sum_{K,L} 1_{e_K > e_L} \left(\int_{[K,L]} a.n_K u - (a_{KL} u(x_K) - a_{LK} u_(x_L)) \right) (e_K - e_L)$$
$$= \sum_{K,L} 1_{e_K > e_L} (e_K - e_L) \left(\int_{[K,L]} a.n_K \left[1(a.n_K > 0)(u - u(x_K)) \right] \right)$$
$$+ \sum_{K,L} 1_{e_K > e_L} (e_K - e_L) \left(\int_{[K,L]} a.n_K \left[1(a.n_K < 0)(u - u(x_L)) \right] \right),$$

and

$$T_2 = -\sum_K \sum_{\epsilon \in \mathcal{E}_D(K)} a^+_{\epsilon,D} u_{x(K)} e_K.$$

Consider first T_1, since $|u - u(x_K)| \leq C diam(K)$ and $|u - u(x_L)| \leq C diam(L)$, we have

$$|(e_K - e_L)| \left| \int_{[K,L]} a.n_K (1(a.n_K > 0)(u - u(x_K)) + 1(a.n_K < 0)(u - u(x_L))) \right|$$
$$\leq C(diam(L) + diam(K))(a_{KL} + a_{LK})|e_K - e_L|.$$

Finally

$$T_1 \leq C(\sum_{K,L} 1_{e_K > e_L} (a_{KL} + a_{LK})(e_K - e_L)^2)^{\frac{1}{2}}$$
$$\times \ (\sum_{K,L} 1_{e_K > e_L} (a_{KL} + a_{LK})(diam(L) + diam(K))^2)^{\frac{1}{2}}$$
$$\leq C h^{\frac{1}{2}} (\sum_{K,L} 1_{e_K > e_L} (a_{KL} + a_{LK})(e_K - e_L)^2)^{\frac{1}{2}}.$$

As for T_2, we have

$$T_2 \leq \sum_K \sum_{\epsilon \in \mathcal{E}_D(K)} a^+_{\epsilon,D} (u_{x(K)} - 0)^2 + \frac{1}{4} \sum_K \sum_{\epsilon \in \mathcal{E}_D(K)} a^+_{\epsilon,D} e_K^2.$$

The first term is $O(h^2)$ since $u = 0$ on Ω.

8 Numerical results

In order to illustrate the use of Robin interface conditions on non matching grids, 2-D test problems were performed with the finite volume scheme analyzed above. We solve the following problem:

$$cu + a.\nabla u - \nu \Delta u = f \text{ in } \Omega = (0,1) \times (0,1),$$
$$u = g \text{ on } \partial\Omega,$$

with c and ν positive constants and g is a given function. The stopping criterion of the algorithm is that the max norm of the jump of the Robin interface condition is smaller than 10^{-8}.
The numerical solution is compared to an exact solution. We choose $u(x, y) = x^3\, y^2 + sin(xy)$. The right-hand side is obtained by applying the operator to u and using an approximate integration formula, the Gauss quadrature rule with four by four nodes per control volume. Thus the error due to inexact integration will be small with respect to the scheme error. A decomposition into 2×2 subdomains is considered. The L_∞ and discrete H^1_h (cf. Theorem 7.3) norms are given for a decomposition into four subdomains and successively refined grids:

- Case 1: diffusive case $c = 1$, $a = (0, 0)$, $\nu = 1$. Initial grid: 9×9 - 8×8 - 7×7 - 6×6.
- Case 2: convective case $c = 1$, $a = (y, -x)$, $\nu = 1e - 2$. Initial grid: 9×9 - 8×8 - 7×7 - 6×6.
- Case 3: diffusive case with nested grids $c = 1$, $a = (0, 0)$, $\nu = 1$. Initial grid: 8×8 - 4×4 - 8×8 - 4×4.

	1/h	$\Vert\ \Vert_\infty$	$\Vert\ \Vert_{H^1_h}$	H^1_h error reduction
Case 1 $\alpha = 1$	8	0.00429947	0.0104186	
	16	0.00189652	0.0069332	1.50
	32	0.00081337	0.0042162	1.64
	64	0.00032407	0.0022791	1.85
	128	0.00011640	0.0010857	2,10

Table 1: Error vs. mesh refinement – No convection

	1/h	$\Vert\ \Vert_\infty$	$\Vert\ \Vert_{H^1_h}$	H^1_h error reduction
Case 1 $\alpha = 1/h$	8	0.00164963	0.00609823	
	16	0.000828163	0.00389768	1.79
	32	0.0004912	0.00266351	1.46
	64	0.000273028	0.00186123	1.43
	128	0.000147225	0.00130971	1,42

Table 2: Error vs. mesh refinement – No convection

For case 1 and $\alpha = 1$ (i.e. $\gamma = 0$, table 1), the first order in the discrete H^1_h norm as expected from the theory is attained only for rather fine meshes.

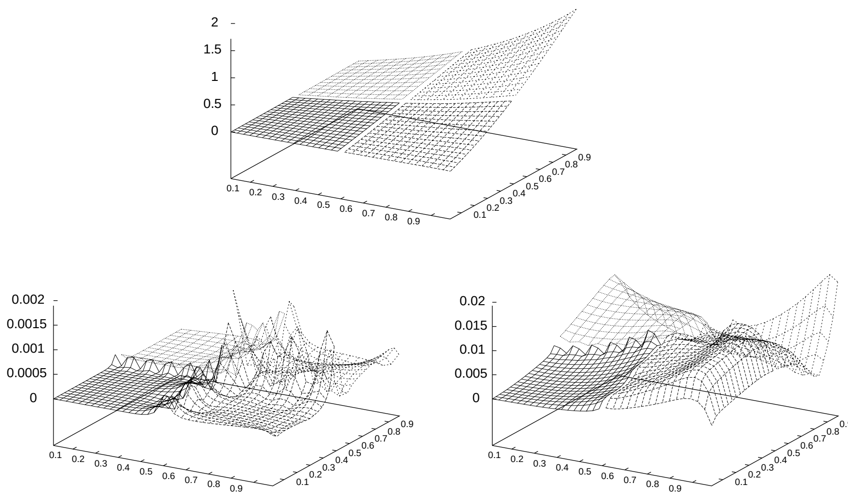
	1/h	$\| \|_\infty$	$\| \|_{H_h^1}$	H_h^1 error reduction
	8	0.0363194	0.0208039	
Case 2	16	0.0192991	0.0103208	2.02
$\alpha = 1$	32	0.0089670	0.0051218	2.02
	64	0.0046442	0.0199572	1.96
	128	0.0023960	0.0013520	1.92

Table 3: Error vs. mesh refinement – Convection

	1/h	$\| \|_\infty$	$\| \|_{H_h^1}$	H_h^1 error reduction
	8	0.00461566	0.00986891	
	16	0.00136263	0.00341835	2.88
Case 3	32	0.000389296	0.00117091	2.92
$\alpha = 1$	64	0.000109016	0.00040139	2.92
	128	0.00003011	0.00013925	2.88

Table 4: Error vs. mesh refinement. Nested grids – No Convection

The error in the L_∞ norm is also improved with mesh refinement and seems better than $O(h)$ since the error reduction factor between two successive levels seems better than 2. For case 1 and $\alpha = 1/h$ (i.e. $\gamma = 1$, table 2), the error is an $O(h^{1/2})$ as expected since the error reduction factor between two successive levels seems to converge to $\sqrt{2}$. In case 2 and $\alpha = 1$ (i.e. $\gamma = 0$, table 3) the error reduction factor between two successive levels deteriorates slightly and is close to 2 as the mesh is refined. The error is better than the expected value $O(h^{1/2})$. In Case 3 (diffusive case with nested grids on the interface), the error seems better than the theoretical value $O(h)$ since the error reduction factor between two successive levels is close to 3. This is related to the superconvergence of the finite volume scheme on a regular mesh that does not seem to be lost when using nested grids on the interface.



above: Computed solution, bottom left : error for Case 1, bottom right: error for Case 2
Note the **different** scales

In the diffusive case (Case 1), note that the error is very small (roughly 0.2%) but not smooth on the interface. It is about three times what would be obtained with matching grids. In the convective case (Case 2), the error is smoother but larger (roughly 2%).

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