Dynamics of kink clusters for scalar fields in dimension $1+1$

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Abstract. We consider classical scalar fields in dimension $1+1$ with a self-interaction potential being a symmetric double-well. Such a model admits non-trivial static solutions called kinks and antikinks. A kink cluster is a solution approaching, for large positive times, a superposition of alternating kinks and antikinks whose velocities converge to 0 and mutual distances grow to infinity.

The aim of this note is to present results on asymptotic behaviour of kink clusters. Our results are partially inspired by the notion of "parabolic motions" in the Newtonian n-body problem. We present this analogy and mention its limitations. We also explain the role of kink clusters as universal profiles for formation of multi-kink configurations.

1. Scalar fields in dimension $1+1$

We study scalar field equations in dimension $1+1$, which are associated to the Lagrangian action

$$\mathscr{L}(\phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - U(\phi)\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

where the self-interaction potential $U : \mathbb{R} \to [0,+\infty)$ is a given smooth function. The unknown field $\phi = \phi(t,x)$ is assumed to be real-valued. The resulting Euler-Lagrange equation is

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U'(\phi(t,x)) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R},\ \phi(t,x) \in \mathbb{R}. \tag{1.1}$$

For simplicity, we set

$$U(\phi) := \frac{1}{8}(1-\phi^2)^2,$$

which yields the so-called ϕ^4 *model*, but almost all our results are true for any non-degenerate *double-well potential*, namely any U satisfying the following conditions:

- U is an even function,
- $U(\phi) > 0$ for all $\phi \in (-1,1)$,
- $U(-1) = U(1) = 0$ and $U''(1) = U''(-1) = 1$.

The zeros of U are called the *vacua*. Linearisation of (1.1) around each of the vacua 1 and -1, $\phi = \pm 1 + g$, yields the free linear Klein-Gordon equation of mass 1:

$$\partial_t^2 g_{\mathrm{L}}(t,x) - \partial_x^2 g_{\mathrm{L}}(t,x) + g_{\mathrm{L}}(t,x) = 0.$$

Apart from the ϕ^4 model, another well-known example of (1.1) satisfying the hypotheses above is the *sine-Gordon equation* obtained for $U(\phi) := \frac{1}{\pi^2}\big(1+\cos(\pi\phi)\big)$. Unlike the ϕ^4 equation, the sine-Gordon equation is *completely integrable*, which means that in principle the Cauchy problem can be solved explicitly.

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The equation (1.1) can be rewritten as a system of first order in t:

$$\partial_t \begin{pmatrix} \phi(t,x) \\ \dot\phi(t,x) \end{pmatrix} = \begin{pmatrix} \dot\phi(t,x) \\ \partial_x^2 \phi(t,x) - U'(\phi(t,x)) \end{pmatrix}. \tag{1.2}$$

The potential energy E_p, the kinetic energy E_k and the total energy E of a state are given by

$$\begin{aligned} E_p(\phi_0) &= \int_{-\infty}^{+\infty} \Big(\frac{1}{2}(\partial_x \phi_0(x))^2 + U(\phi_0(x))\Big)\,\mathrm{d}x, \\ E_k(\dot\phi_0) &= \int_{-\infty}^{+\infty} \frac{1}{2}(\dot\phi_0(x))^2\,\mathrm{d}x, \\ E(\phi_0, \dot\phi_0) &= \int_{-\infty}^{+\infty} \Big(\frac{1}{2}(\dot\phi_0(x))^2 + \frac{1}{2}(\partial_x \phi_0(x))^2 + U(\phi_0(x))\Big)\,\mathrm{d}x. \end{aligned}$$

Denoting $\boldsymbol{\phi}(t,x) := (\phi(t,x), \dot\phi(t,x))$, the system (1.2) can be reformulated in the Hamiltonian form as

$$\partial_t \boldsymbol{\phi}(t) = \boldsymbol{J}\mathrm{D}E(\boldsymbol{\phi}(t)), \tag{1.3}$$

where $\boldsymbol{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic form and D is the Fréchet derivative for the $L^2 \times L^2$ inner product. In particular, $E = E(\boldsymbol{\phi}) = E(\phi, \partial_t \phi)$ is a conserved quantity. We only consider fields of finite energy.

By elementary arguments, the set of finite energy states $\boldsymbol{\phi}_0 = (\phi_0, \dot\phi_0)$ is the union of the following four affine spaces:

$$\begin{aligned} \mathcal{E}_{1,1} &:= \{(\phi_0, \dot\phi_0) : E(\phi_0, \dot\phi_0) < \infty \text{ and } \lim_{x\to-\infty} \phi_0(x) = 1, \lim_{x\to\infty} \phi_0(x) = 1\}, \\ \mathcal{E}_{-1,-1} &:= \{(\phi_0, \dot\phi_0) : E(\phi_0, \dot\phi_0) < \infty \text{ and } \lim_{x\to-\infty} \phi_0(x) = -1, \lim_{x\to\infty} \phi_0(x) = -1\}, \\ \mathcal{E}_{1,-1} &:= \{(\phi_0, \dot\phi_0) : E(\phi_0, \dot\phi_0) < \infty \text{ and } \lim_{x\to-\infty} \phi_0(x) = 1, \lim_{x\to\infty} \phi_0(x) = -1\}, \\ \mathcal{E}_{-1,1} &:= \{(\phi_0, \dot\phi_0) : E(\phi_0, \dot\phi_0) < \infty \text{ and } \lim_{x\to-\infty} \phi_0(x) = -1, \lim_{x\to\infty} \phi_0(x) = 1\}. \end{aligned}$$

Each of them is parallel to the *energy space*

$$\mathcal{E} := H^1(\mathbb{R}) \times L^2(\mathbb{R}).$$

2. Kinks, antikinks, multi-kink configurations and kink clusters

Equation (1.3) admits static solutions. They are the critical points of the potential energy. The trivial ones are the vacuum fields $\phi(t,x) = \pm 1$. The solution $\phi(t,x) = 1$ (resp. $\phi(t,x) = -1$) has zero energy and is the ground state in $\mathcal{E}_{1,1}$ (resp. $\mathcal{E}_{-1,-1}$).

There are also non-constant static solutions $\phi(t,x)$ connecting the two vacua, that is

$$\lim_{x\to-\infty} \phi(t,x) = \mp 1, \quad \lim_{x\to\infty} \phi(t,x) = \pm 1. \tag{2.1}$$

One can easily find all these solutions: they are given by the formula $\phi(t,x) = \pm H(x-a)$ for some $a \in \mathbb{R}$, where $H(x) := \arctan(x/2)$. The translates of H are called the *kinks* and are the ground states in $\mathcal{E}_{-1,1}$. The translates of $-H$ are called the *antikinks* and are the ground states in $\mathcal{E}_{1,-1}$. Thanks to this variational characterisation, one obtains orbital stability of kinks and antikinks (up to translations), see [15].

Proposition 2.1. *The vacuum solutions, the kinks and the antikinks are the only static solutions of* (1.1).

Remark 2.2. Even if we will not use them explicitly, we note that moving kinks can be constructed by means of the Lorentz transformation. If $\beta \in (-1,1)$ and $a \in \mathbb{R}$, then

$$\phi(t,x) := H(\gamma_\beta(x - a - \beta t)), \qquad \text{where } \gamma_\beta := (1-\beta^2)^{-1/2},$$

is a solution of (1.1), a *travelling wave* whose velocity equals β.

The condition (2.1) defines a *topological class*, since for any continuous path of finite energy states, either none or all of them satisfy (2.1). In general, minimizers of the energy in a topological class that does not contain vacua are called topological solitons. Topological solitons were introduced in the physics literature by Skyrme as candidates for particles in classical field theories; see [49, 33]. Kinks and antikinks are one dimensional examples, and in higher dimensions examples include vortices, harmonic maps, monopoles, Skyrmions, and instantons.

Our main object of study are solutions which resemble (in a sense to be specified) a *superposition* of a finite number of stationary states. We thus introduce the so-called *multi-kink configurations*, which are defined as follows (see Figure 1). For $\vec{a} = (a_1, \ldots, a_n)$ such that $a_1 \leqslant \ldots \leqslant a_n$, we denote

$$H(\vec{a}) := 1 + \sum_{k=1}^{n} (-1)^k \big(H(\cdot - a_k) + 1\big)$$

(we chose the "additive ansatz", see [51, Section 1.7] for a comparison with a different "product ansatz", which we could also use without introducing any changes in the statement of our results below). It will always be assumed that $a_{k+1} - a_k$ is sufficiently large for all $k \in \{1, \ldots, n-1\}$. Note that the vacuum 1 is obtained for $n = 0$, and the antikinks for $n = 1$. From Proposition 2.1, we see that for $n \geqslant 2$ the multi-kink configurations are not static states, which is due to the nonlinear character of the equation (1.1).

By the variational characterisation of H and its translates as the ground states in $\mathcal{E}_{-1,1}$, one can informally view them as the transitions between the two vacua -1 and 1 having the minimal possible energy $E = E_p(H)$. Given a natural number n, we are interested in solutions of (1.1) containing, asymptotically as $t \to \infty$, n such transitions. Since energy $E_p(H)$ is needed for each transition, we necessarily have $E(\phi, \partial_t\phi) \geqslant nE_p(H)$. We call *kink clusters* the solutions for which equality holds.

Definition 2.3 (Kink n-cluster)**.** Let $n \in \{0, 1, \ldots\}$. We say that a solution ϕ of (1.1) is a *kink n-cluster* if $E(\phi, \partial_t\phi) \leqslant nE_p(H)$ and there exist real-valued functions $x_0(t) \leqslant x_1(t) \leqslant \ldots \leqslant x_n(t)$ such that

$$\lim_{t\to\infty} \phi(t, x_k(t)) = (-1)^k \qquad \text{for all } k \in \{0, 1, \ldots, n\}.$$

Note that the kink 0-clusters are the constant solutions $\phi \equiv 1$ and the kink 1-clusters are the antikinks. The simplest non-trivial case is $n = 2$. We will call kink 2-clusters

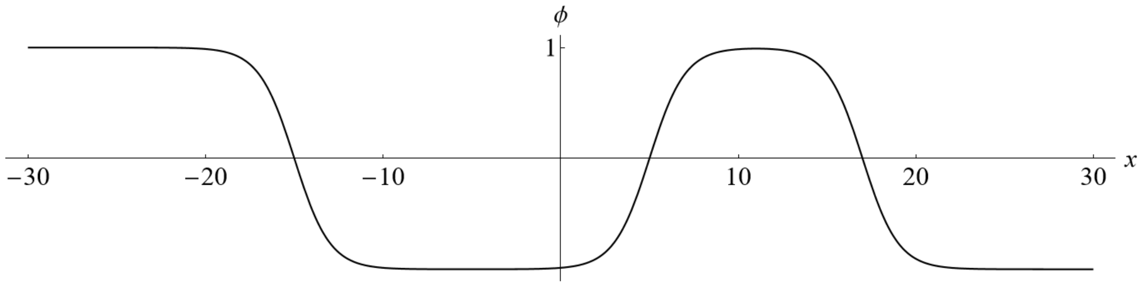

FIGURE 1. An example of a multi-kink configuration with $\vec{a} = (-15, 5, 17)$

kink-antikink pairs. We say that ϕ is a kink cluster if it is a kink n-cluster for some $n \in \{0, 1, \ldots\}$.

From the heuristic discussion above, the shape of each transition in a kink cluster has to be close to optimal, that is close to a kink or an antikink, so that the whole field is close to a multi-kink configuration. We have the following characterisation of kink n-clusters.

Proposition 2.4. *A solution ϕ of* (1.1) *is a kink n-cluster if and only if there exist continuous functions $a_1, \ldots, a_n : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \Big(\|\partial_t \phi(t)\|_{L^2}^2 + \|\phi(t) - H(\vec{a}(t))\|_{H^1}^2 + \sum_{k=1}^{n-1} \mathrm{e}^{-(a_{k+1}(t) - a_k(t))} \Big) = 0.$$

In other words, kink clusters can be equivalently defined as solutions approaching, as $t \to \infty$, a superposition of a finite number of alternating kinks and antikinks, whose mutual distances tend to ∞ and which travel with speeds converging to 0. In contrast to multikink solutions consisting of Lorentz-boosted kinks (travelling with asymptotically non-zero speed) constructed in [3], the dynamics of kink clusters are driven solely by interactions between the kinks and antikinks. Employing the term introduced by Martel and Raphaël in [35], we are dealing with multi-kinks in the regime of *strong interaction*. Proposition 2.4 implies in particular that the energy of a kink n-cluster equals $nE_p(H)$.

We also have the following characterisation of kink clusters as *asymptotically static solutions*, by which we mean solutions whose kinetic energy converges to 0 as $t \to \infty$.

Proposition 2.5. *A solution ϕ of* (1.1) *satisfies* $\lim_{t\to\infty} \|\partial_t \phi(t)\|_{L^2}^2 = 0$ *if and only if ϕ or $-\phi$ is a kink cluster.*

Remark 2.6. Let us stress that, according to our definition, kink clusters are solutions approaching multi-soliton configurations in the strong energy norm, in other words we address the question of interaction of solitons *in the absence of radiation.* Allowing for a radiation term seems to be currently out of reach, the question of the asymptotic stability of the kink being still unresolved, see for example [8, 28, 11, 13, 4, 31] for recent results on this and related problems.

Remark 2.7. We emphasize that our definition of kink clusters concerns *only one time direction*, and our study does not address the question of the behaviour of kink clusters as $t \to -\infty$, which goes by the name of the *kink collision problem.* We refer to [27] for an overview, and to [40, 41] for recent rigorous results in the case of the ϕ^6 model.

3. The n-body approximation

It is tempting to view the (anti)kink $\pm H(\cdot - a)$ as a *particle* whose position is given by $a \in \mathbb{R}$. From this viewpoint, a multi-kink configuration can be regarded as a set of interacting particles. Therefore, describing the evolution of such an object should bear an analogy with the n-body problem.

Remark 3.1. The particle-like character of solitons is a well-known phenomenon, see [33, Chapter 1] for a historical account. The question of justification that the positions of solitons in a field described by some wave equation satisfy an approximate n-body law of motion was considered for instance in [50, 12, 9, 46].

Remark 3.2. The justification of an approximation of solutions of some PDE by a system of point masses is a problem which appears in many contexts other than the wave equations. Let me mention the works on the Ginzburg-Landau gradient flow [25, 2], on the equations of fluid mechanics [1], on Bose-Einstein condensates [24, 26], but this list is of course far from being exhaustive.

In view of the mass-energy equivalence, it is reasonable to define the *mass of the kink*

$$M := E_p(H) = \frac{2}{3}. \tag{3.1}$$

The potential energy of a multi-kink configuration $H(\vec{a})$ is given by

$$\widetilde{E}_p(\vec{a}) := E_p(H(\vec{a})).$$

One can check that

$$\Big|\widetilde{E}_p(\vec{a}) - nM + 8\sum_{k=1}^{n-1} \mathrm{e}^{-(a_{k+1}-a_k)}\Big| \lesssim \max_{1\leqslant k<n}(a_{k+1}-a_k)\mathrm{e}^{-2(a_{k+1}-a_k)},$$

which allows to identify $-8\sum_{k=1}^{n-1}\mathrm{e}^{-(a_{k+1}-a_k)}$ as the main term of the interaction energy between the n particles. We denote

$$\rho(\vec{a}) := \sum_{k=1}^{n-1} \mathrm{e}^{-(a_{k+1}-a_k)}, \tag{3.2}$$

which measures the size of the interaction energy. According to the principles of Classical Mechanics, we obtain the force acting on the k-th particle:

$$F_k(\vec{a}) = -\partial_{a_k}\widetilde{E}_p(\vec{a}) = 8\big(\mathrm{e}^{-(a_{k+1}-a_k)} - \mathrm{e}^{-(a_k-a_{k-1})}\big) + \ldots, \tag{3.3}$$

where by convention $a_0 = -\infty$ and $a_{n+1} = \infty$. Recalling (3.1) and applying Newton's second law, we derive the following n-body problem with attractive nearest-neighbor exponential interactions:

$$a_k''(t) = 12\big(\mathrm{e}^{-(a_{k+1}(t)-a_k(t))} - \mathrm{e}^{-(a_k(t)-a_{k-1}(t))}\big). \tag{3.4}$$

According to the heuristics presented above, this system should be relevant for the evolution of states close to multi-kink configurations.

Observe the similarity of (3.4) with the well-known *Toda system*. The essential difference lies in the sign of the interactions, which are attractive in (3.4) and repulsive in the Toda system. Hénon [14] found n independent conservation laws, both for the Toda system and for (3.4). Our arguments do not explicitly rely on the conservation laws related to the complete integrability, and we expect that part of the analysis should be applicable also in the cases where the modulation equations are not related to any completely integrable system of ODEs.

Remark 3.3. In their work on blow-up for nonlinear waves, Merle and Zaag [38] obtained a system of ODEs with exponential terms like in (3.4), but which was a *gradient flow* and not an n-body problem. The dynamical behaviour of solutions of this system was described by Côte and Zaag [7].

4. Classification of kink-antikink pairs

In the case $n = 2$, one easily finds all the solutions of (3.4) such that the distance between the two particles converges to ∞ and their velocities converge to 0. They are given by

$$(a_1(t), a_2(t)) = \big(a_0 - \log(2\sqrt{3}(t-t_0)), a_0 + \log(2\sqrt{3}(t-t_0))\big).$$

where a_0 and t_0 are arbitrary real numbers. It turns out that the set of kink-antikink pairs has a similar structure. The following theorem is the main result of [20], obtained in collaboration with Kowalczyk.

Theorem 1. *There exist a C^1 function $a(t)$ and a solution $\phi_{(2)}(t,x)$ of (1.1) such that for all $\epsilon > 0$ and all $t \geqslant T_0 = T_0(\epsilon)$*

$$\left|a(t) - \log(2\sqrt{3}t)\right| \leqslant t^{-2+\epsilon}, \qquad \left|a'(t) - t^{-1}\right| \leqslant t^{-3+\epsilon}$$

and

$$\begin{aligned}&\left\|\phi_{(2)}(t) - \big(1 - H(\cdot + a(t)) + H(\cdot - a(t))\big)\right\|_{H^1} \\ &+ \left\|\partial_t\phi_{(2)}(t) + a'(t)\big(\partial_x H(\cdot + a(t)) + \partial_x H(\cdot - a(t))\big)\right\|_{L^2} \leqslant t^{-2+\epsilon}.\end{aligned}$$

Moreover, $\phi_{(2)}$ is the unique kink-antikink pair up to translation, i.e., if $\phi(t,x)$ is any kink-antikink pair, then there exist $t_0, a_0 \in \mathbb{R}$ so that

$$\phi(t,x) = \phi_{(2)}(t - t_0, x - a_0) \qquad \text{for all } (t,x) \in \mathbb{R}^2.$$

It is easy to check that the function $\phi(t,x) = \frac{4}{\pi}\arctan(t\,\mathrm{sech}(x)) - 1$ is a kink-antikink pair for the sine-Gordon equation

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) - \frac{1}{\pi}\sin(\pi\phi(t,x)) = 0.$$

For the mKdV equation, existence of solutions converging to a superposition of 2 or 3 solitons with asymptotically vanishing velocities was observed by Wadati and Ohkuma [52]. For a non-integrable model, the first construction of a two-soliton solution with trajectories having asymptotically vanishing velocities was obtained by Krieger, Martel and Raphaël [29], see also Martel and Raphaël [35], followed by [19, 44, 45], as well as [16]. The work [20] was the first to consider the question of uniqueness.

One could expect that kink-antikink pairs could be *threshold solutions* (at least for some choices of the self-interaction potential U) in the following sense. Small energy data are topologically trivial and the corresponding evolution presents *oscillatory behaviour* (scattering or modified scattering). When higher and higher energies are considered, at some energy threshold a new type of dynamical behavior can appear. Of course the kinks have a non-oscillatory behavior, but they do not provide the correct energy threshold, because they are not topologically trivial. The correct threshold could equal in fact *twice* the energy of the kink, and kink-antikink pairs would be topologically trivial solutions of lowest possible energy locally converging (up to translations) to kinks.

From this perspective, uniqueness of strongly interacting two-solitons is an analog of the results of Merle [37] on uniqueness of minimal mass blow-up solutions of the mass-critical NLS, and the corresponding result of Raphaël and Szeftel [48] on non-homogeneous mass-critical NLS. Let me stress however that in [37, 48] the solution develops *one* bubble. The novelty of Theorem 1 with respect to these works is to consider solutions which are superpositions of *more than one* kink.

The role of kink-antikink pairs as threshold elements remains an open problem for the ϕ^4 model. For the equivariant critical wave maps equation, see [21, 22] for the problem of existence, uniqueness and threshold behaviour of two-bubble solutions, which are to some extent analogous to kink-antikink pairs.

5. Asymptotic behavior of kink clusters

Theorem 1 yields in particular the asymptotic behavior of any kink-antikink pair at the main order. In [23], we determined the leading order of any kink n-cluster *for any* $n \in \mathbb{N}$. Before we state the main result of [23], we note that the system (3.4) has the following explicit solution such that the distances between the particles converge to ∞ and their velocities converge to 0:

$$a_{k+1}(t) - a_k(t) = 2\log(2t) - \log\frac{k(n-k)}{3}, \qquad a_k'(t) = \frac{n+1-2k}{t} \tag{5.1}$$

(these formulas determine $a_k(t)$ up to an arbitrary choice of the mass center).

Remark 5.1. In the study of the gravitational n-body problem, the solutions such that the distances between the bodies grow to ∞ and the velocities of the bodies converge to 0 are called *parabolic motions*.

Theorem 1 from [23] affirms that any kink n-cluster has quantitatively the same behavior as $t \to \infty$.

Theorem 2. *If ϕ is a kink n-cluster, then there exist continuously differentiable functions $a_1, \ldots, a_n : \mathbb{R} \to \mathbb{R}$ such that $g(t) := \phi(t) - H(\vec{a}(t))$ satisfies*

$$\lim_{t\to\infty}\bigg(\max_{1\leqslant k<n}\Big|\big(a_{k+1}(t)-a_k(t)\big)-\Big(2\log(2t)-\log\frac{k(n-k)}{3}\Big)\Big| + \max_{1\leqslant k\leqslant n}|ta_k'(t)+(n+1-2k)| + t\|\partial_t g(t)\|_{L^2} + t\|g(t)\|_{H^1}\bigg) = 0.$$

Remark 5.2. The bound $\|\partial_t g(t)\|_{L^2} + \|g(t)\|_{H^1} \ll t^{-1}$ allows to treat the nonlinear terms perturbatively and thus in principle improve the bounds above to any precision by standard perturbation theory methods.

6. Modulation analysis for kink clusters

The starting point for proving Theorem 2 is the *modulation method*, frequently employed in the study of solitons and multi-solitons. We present below this important tool.

The decomposition $\phi(t) = H(\vec{a}(t)) + g(t)$ used in the statement above is clearly not unique. It is convenient to use a specific choice of $\vec{a}(t)$ determined by the *orthogonality conditions*

$$\int_{-\infty}^{\infty}\partial_x H(x-a_k(t))g(t,x)\,\mathrm{d}x = 0, \qquad \text{for all } k\in\{1,\ldots,n\}. \tag{6.1}$$

This way, whenever $\phi(t)$ is close to a multi-kink configuration, the uniquely determined number $a_k(t)$ indicates the "position" of the k-th kink. Before we give the precise statement which guarantees the existence of of $\vec{a}(t)$ satisfying (6.1), we introduce the following notion.

Definition 6.1 (Distance to a multi-kink configuration)**.** For all $(\phi_0, \dot{\phi}_0) \in \mathcal{E}_{1,(-1)^n}$, the distance from $(\phi_0, \dot{\phi}_0)$ to the set of multi-kink configurations is defined by

$$\delta(\phi_0,\dot{\phi}_0) := \inf_{\vec{b}\in\mathbb{R}^n}\big(\|\dot{\phi}_0\|_{L^2}^2 + \|\phi_0 - H(\vec{b})\|_{H^1}^2 + \rho(\vec{b})\big),$$

where $\rho(\vec{b})$ is given by (3.2).

Note that, by Proposition 2.4, if ϕ is a kink n-cluster, then $\lim_{t\to\infty}\delta(\phi(t),\partial_t\phi(t)) = 0$. We have the following *static modulation lemma.*

Lemma 6.2. *There exist $\eta_0, \eta_1, C_0 > 0$ having the following property. For all $(\phi_0, \dot{\phi}_0) \in \mathcal{E}_{1,(-1)^n}$ such that $\delta(\phi_0, \dot{\phi}_0) < \eta_0$ there exists unique $\vec{a} = \vec{a}(\phi_0, \dot{\phi}_0) \in \mathbb{R}^n$ such that*

$$\|\dot{\phi}_0\|_{L^2} + \|\phi_0 - H(\vec{a})\|_{\mathcal{E}}^2 + \rho(\vec{a}) < \eta_1$$

and

$$\langle\partial_x H(\cdot - a_k), \phi_0 - H(\vec{a})\rangle = 0 \qquad \textit{for all } k\in\{1,\ldots,n\}.$$

It satisifes

$$\begin{aligned}\|\dot{\phi}_0\|_{L^2}^2 + \|\phi_0 - H(\vec{a})\|_{H^1}^2 + \rho(\vec{a}) &\leqslant C_0\delta(\phi_0,\dot{\phi}_0),\\ \|\dot{\phi}_0\|_{L^2}^2 + \|\phi_0 - H(\vec{a})\|_{H^1}^2 &\leqslant C_0(\rho(\vec{a}) + E(\phi_0,\dot{\phi}_0) - nM).\end{aligned} \tag{6.2}$$

Moreover, the map $\mathcal{E}_{1,(-1)^n} \ni (\phi_0, \dot{\phi}_0) \mapsto \vec{a}(\phi_0, \dot{\phi}_0) \in \mathbb{R}^n$ is of class C^1.

Proofs of similar results are contained for example in the articles by Gustafson and Sigal [12, Proposition 3], Merle and Zaag [39, Proposition 3.1], and in my paper [18, Lemma 3.3]. They are based on a quantitative version of the Implicit Function Theorem.

The map $(\phi_0, \dot\phi_0) \mapsto (\vec a, g := \phi_0 - H(\vec a), \dot\phi_0)$ given by the lemma above is a diffeomorphism from a neighborhood of the set of widely separated multi-kink configurations to a manifold in $\mathbb{R}^n \times \mathcal{E}$ of codimension n determined by the orthogonality conditions.

Remark 6.3. We stress that the position parameters determined by the orthogonality conditions (6.1) do not necessarily achieve the infimum in Definition 6.1 but, as a consequence of Lemma 6.2, they do achieve it up to a constant.

Let ϕ be a kink n-cluster, according to Definition 2.3. Lemma 6.2 yields $\vec a(t) \in \mathbb{R}^n$ defined for all t large enough, such that $g(t) := \phi(t) - H(\vec a(t))$ satisfies (6.1). It can be checked that $t \mapsto \vec a(t)$ is of class C^1. By differentiating in time (6.1) and using the differential equation, one arrives at a coupled system of differential equations for $\vec a(t)$ and $g(t)$. In general, one tries to decouple this system as much as possible, in order to reduce the dynamics to an ODE for $\vec a(t)$. The *coercivity bound* (6.2) is very useful in such a reduction, since it allows to bound the size of the remainder $(g(t), \partial_t\phi(t))$ in terms of $\vec a(t)$.

The crucial step in obtaining an approximate ODE for the position vector $\vec a(t)$ is to introduce an appropriate notion of *momentum* of each kink. We follow an idea used in a similar context in [17] and introduce *localised momenta*, see also [48, Proposition 4.3].

Let $\chi \in C^\infty$ be a decreasing function such that $\chi(x) = 1$ for all $x \leqslant \frac13$ and $\chi(x) = 0$ for all $x \geqslant \frac23$.

Definition 6.4 (Localised momenta)**.** Let ϕ be a kink n-cluster and let $\vec a(t)$ be the positions of the kinks defined above (for t large enough). We set

$$\begin{aligned}
\chi_1(t,x) &:= \chi\Big(\frac{x - a_1(t)}{a_2(t) - a_1(t)}\Big), \\
\chi_k(t,x) &:= \chi\Big(\frac{x - a_k(t)}{a_{k+1}(t) - a_k(t)}\Big) - \chi\Big(\frac{x - a_{k-1}(t)}{a_k(t) - a_{k-1}(t)}\Big), \qquad \text{for } k \in \{2, \ldots, n-1\}, \\
\chi_n(t,x) &:= 1 - \chi\Big(\frac{x - a_{n-1}(t)}{a_n(t) - a_{n-1}(t)}\Big).
\end{aligned}$$

We define $\vec p = (p_1, \ldots, p_n) : I \to \mathbb{R}^n$ by

$$p_k(t) := \langle -(-1)^k \partial_x H_k(t) + \chi_k(t)\partial_x g(t), \dot g(t)\rangle.$$

We can now state the *dynamical modulation lemma.*

Lemma 6.5. *There exists C_0 such that for any kink n-cluster ϕ the following bounds hold for all $k \in \{1, \ldots, n\}$ and t large enough:*

$$|Ma_k'(t) - p_k(t)| \leqslant C_0\rho(\vec a(t)), \tag{6.3}$$

$$|p_k'(t) - F_k(\vec a(t))| \leqslant \frac{C_0\rho(\vec a(t))}{-\log\rho(\vec a(t))}, \tag{6.4}$$

where M and F_k are defined by (3.1) *and* (3.3)*.*

Remark 6.6. If $\vec a(t)$ is given by (5.1), then $|\vec p(t)| \simeq t^{-1}$, $|\vec F(\vec a(t))| \simeq t^{-2}$ and $\rho(\vec a(t)) \simeq t^{-2}$, hence the bounds (6.3) and (6.4) are reasonable in the sense that the bound of the error is much smaller than the terms appearing on the left hand side.

Once Lemma 6.5 is established, the proof of Theorem 2 relies on an ODE-type analysis partly inspired by classical techniques developed for the gravitational n-body problem, see for instance [47].

7. Existence of kink clusters

Our next result concerns the problem of existence of kink n-clusters. We prove that, for any choice of n points on the line sufficiently distant from each other, there exists a kink n-cluster such that the initial positions of the (anti)kinks are given by the n chosen points. The result is inspired by the work of Maderna and Venturelli [32] on the gravitational n-body problem.

Theorem 3. *There exist $C_0, L_0 > 0$ such that the following is true. If $L \geqslant L_0$ and $\vec{a}_0 \in \mathbb{R}^n$ satisfies $a_{0,k+1} - a_{0,k} \geqslant L$ for all $k \in \{1, \ldots, n-1\}$, then there exists $(g_0, \dot{g}_0) \in \mathcal{E}$ satisfying $\|(g_0, \dot{g}_0)\|_{\mathcal{E}}^2 \leqslant C_0 \mathrm{e}^{-L}$ and the orthogonality conditions*

$$\int_{-\infty}^{\infty} \partial_x H(x - a_{0,k}) g_0(x)\,\mathrm{d}x = 0 \qquad \text{for all } k \in \{1, \ldots, n\}$$

such that the solution of (1.3) *corresponding to the initial data*

$$(\phi(0), \partial_t \phi(0)) := \big(H(\vec{a}_0) + g_0, \dot{g}_0\big)$$

is a kink cluster and satisfies $\delta(\phi(t), \partial_t \phi(t)) \leqslant C_0/(\mathrm{e}^L + t^2)$ for all $t \geqslant 0$.

Remark 7.1. We expect that for a given choice of $\vec{a}_0$ there is actually a *unique* $(g_0, \dot{g}_0)$ in a small ball of $\mathcal{E}$ leading to a kink n-cluster. This is clearly true for $n = 1$. In the case $n = 2$, uniqueness of $(g_0, \dot{g}_0)$ can be obtained as a consequence of Theorem 1. Partial uniqueness results for $n > 2$ will be proved in forthcoming work.

The overall proof scheme is taken from Martel [34], see also the earlier work of Merle [36], and contains two steps:

- for any $T > 0$, prove existence of a solution ϕ satisfying the conclusions of Theorem 3, but only on the finite time interval $t \in [0, T]$,
- take a sequence $T_m \to \infty$ and consider a weak limit of the solutions ϕ_m obtained in the first step with $T = T_m$.

The first step relies on a novel application of the Poincaré-Miranda theorem, which is essentially a version of Brouwer's fixed point theorem. We choose data close to a multi-kink configuration at time $t = T$ and control how it evolves backwards in time. It could happen that the multi-kink collapses before reaching the time $t = 0$. For this reason, we introduce an appropriately defined "exit time" T_1. The mapping which assigns the positions of the (anti)kinks at time T_1 to their positions at time T turns out to be continuous and, for topological reasons, surjective in the sense required by Theorem 3.

Remark 7.2. In the second step, it is crucial to dispose of some *uniform* estimate on the sequence ϕ_m. In our case, the relevant inequality is $\delta(\phi_m(t), \partial_t \phi_m(t)) \lesssim (\mathrm{e}^L + t^2)^{-1}$ with a universal constant. The existence of such a uniform bound is related to what we would call the "ejection property" of the system. Intuitively, once $\delta(\phi_m(t), \partial_t \phi_m(t))$ starts to grow, it has to continue growing at a definite rate until the multi-kink configuration collapses.

Remark 7.3. Brouwer's theorem was previously used in constructions of multi-solitons, but for a rather different purpose, namely in order to avoid the growth of linear unstable modes, see [5, 6].

8. **Kink clusters as profiles of kink formation/collapse**

Finally, we discuss the role of the kink clusters as universal profiles for the formation/collapse of a multi-kink configuration. In this section, it will be convenient to use the Hamiltonian formulation (1.3). We use boldface letters $\boldsymbol{\phi}_0 = (\phi_0, \dot{\phi}_0)$ to denote elements of the phase space.

Theorem 4. *Let $\eta > 0$ be sufficiently small and let $\boldsymbol{\phi}_m$ be a sequence of solutions of* (1.3) *defined on time intervals $[0, T_m]$ satisfying the following assumptions:*

(i) $\lim_{m\to\infty} \delta(\boldsymbol{\phi}_m(T_m)) = 0$,

(ii) $\delta(\boldsymbol{\phi}_m(t)) \leqslant \eta$ *for all* $t \in [0, T_m]$,

(iii) $\delta(\boldsymbol{\phi}_m(0)) = \eta$.

Then, after extraction of a subsequence, there exist $0 = n^{(0)} < n^{(1)} < \ldots < n^{(\ell)} = n$, finite energy states $\boldsymbol{P}_0^{(1)}, \ldots, \boldsymbol{P}_0^{(\ell)}$ and sequences of real numbers $(X_m^{(1)})_m, \ldots, (X_m^{(\ell)})_m$ such that

(i) for all $j \in \{1, \ldots, \ell\}$, the solution $\boldsymbol{P}^{(j)}$ of (1.3) *for the initial data $\boldsymbol{P}^{(j)}(0) = \boldsymbol{P}_0^{(j)}$ is a cluster of $n^{(j)} - n^{(j-1)}$ kinks,*

(ii) for all $j \in \{1, \ldots, \ell - 1\}$, $\lim_{m\to\infty} \big(X_m^{(j+1)} - X_m^{(j)}\big) = \infty$,

(iii) $\lim_{m\to\infty} \Big\| \boldsymbol{\phi}_m(0) - \Big(\mathbf{1} + \sum_{j=1}^{\ell} (-1)^{n^{(j-1)}} \big(\boldsymbol{P}_0^{(j)}(\cdot - X_m^{(j)}) - \mathbf{1}\big)\Big)\Big\|_{\mathcal{E}} = 0.$

Theorem 4 can be understood to mean that kink clusters have properties similar to the stable/unstable manifolds of a hyperbolic stationary state. This analogy is most easily understood in the case $n = 2$, which we explain here.

If we artificially extended the phase space by a state $\boldsymbol{H}^\infty$ corresponding to the limit of $(H(a_1, a_2), 0)$ as $a_2 - a_1 \to \infty$, then the function δ gives a distance to $\boldsymbol{H}^\infty$ and the 2-kink clusters satisfy $\lim_{t\to\infty} \delta(\boldsymbol{\phi}(t)) = 0$, in other words they form the *stable manifold* of $\boldsymbol{H}^\infty$. In this language, Theorem 4 characterizes the trajectories in the phase space that *enter* (or in reverse time, *exit*) a small neighbourhood of the "critical point" $\boldsymbol{H}^\infty$, by affirming that a such a trajectory, while still far away from the critical point, must be close to its (un)stable manifold. For hyperbolic critical points, this property is a consequence of the Hartman-Grobman theorem. In our setting, the soliton interactions play an analogous role as exponential (in)stability in the hyperbolic case.

The analogy described above carries over to $n > 2$, but is slightly more complicated, since at the "exit" time $t = 0$ the solution $\boldsymbol{\phi}_m(0)$ is close to a superposition of well-separated kink clusters, rather than to a single one. Intuitively, for $n > 2$ it can happen that only some of the neighbouring kinks "collapse", while the distances between other neighbouring kinks remain large.

Let us finish by sketching a proof of Theorem 4. The identification of the clusters presents no difficulty: the positions of any two consecutive (anti)kinks at time $t = 0$, after taking a subsequence in m, either remain at a bounded distance or separate with their distance growing to infinity as $m \to \infty$. This dichotomy determines whether they fall into the same cluster or to distinct ones. The next step is to again make use of the ejection property, see Remark 7.2, in order to obtain bounds on $\delta(\boldsymbol{\phi}(t))$ independent of m, for any $t \geqslant 0$. By standard localisation techniques involving the finite speed of propagation, these bounds are inherited by each of the clusters. We mention that the proof of strong convergence in Theorem 4 (iii) is based on a novel application of the well-known principle from the Calculus of Variations affirming that, for a strictly convex functional $\mathcal{F}$, if $\boldsymbol{g}_m \rightharpoonup \boldsymbol{g}$ and $\mathcal{F}(\boldsymbol{g}_m) \to \mathcal{F}(\boldsymbol{g})$, then $\boldsymbol{g}_m \to \boldsymbol{g}$.

Remark 8.1. Determining universal profiles of soliton collapse played an important role in several works on dispersive equations related to the problem of Soliton Resolution. We mention the study of centre-stable manifolds of ground states for various nonlinear wave equations, see for instance [42, 43, 30], as well as the earlier work [10].

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