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Dynamics of multi-solitons for wave equations

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Preface

> "He did not consider his picture better than all Raphael's, but he knew that what he wanted to express in that picture had never been expressed by anyone. Of that he was firmly convinced, and had long been so – ever since he had begun painting it; yet the opinion of others, whoever they might be, seemed to him of great importance, and disturbed him to the depths of his soul. (...) He always attributed to those judges a better understanding than his own, and always expected to hear from them something he had himself not noticed in his work."
>
> L. Tolstoy, *Anna Karenina*, transl. L. & A. Maude

My research during the last four years addressed the questions of long time behaviour of solutions to nonlinear wave equations. In particular, I studied the properties of multi-solitons, which are solutions close to a superposition of a finite number of localised objects, the *solitons*, preserving their shape as time passes. Seen from a large distance, a localised object resembles a point particle, and their superposition a system of n interacting point masses. The aim of the memoir is to contribute to a better understanding of the question of the utility of this intuition. In other words, are there situations where multi-solitons evolve indeed like a system of interacting particles and, conversely, can one find cases where their behaviour is drastically different from what such a simplistic viewpoint could suggest?

Two models are studied in this memoir: the scalar field equation in dimension $1+1$ with a double-well potential, and the wave maps equation in dimension $1+2$, with values in the two-dimensional sphere. The two models are introduced in Chapter 1. Solitons are induced by the topology of the space of states of finite energy, which is why they are named *topological solitons*. Next, I define *strongly interacting two-solitons*. Finally, I state the main results, which can be resumed as follows:

- strongly interacting two-solitons behave like a system of two particles for positive times,
- however, at least in the case of the wave maps equation, they have a very different behaviour for negative times, namely they "disappear" and become pure radiation.

Although the "case study" method is adopted in my research, I found relevant to explain in this memoir how general methods find their application in this study. To this end, in Chapter 2, I describe a toy-model which allows me to introduce two robust methods of Asymptotic Analysis: the *modulation* and the *Lyapunov-Schmidt reduction*. In particular, Section 2.1 indicates the importance of an appropriate change of variables, a key idea in the analysis of two-solitons. The problem which I consider is relatively simple, therefore I decided to provide a full solution. The fact that two-solitons can be analysed following the exact same main steps comes as an enjoyable surprise.

Chapter 3 is devoted to the problem of the asymptotic description of the evolution of a multi-soliton in one time direction. In this chapter, I present in detail my work with Michał Kowalczyk and Andrew Lawrie [1] on kink-antikink pairs for scalar fields in dimension $1 + 1$. I also mention the ongoing work and some perspectives for future research.

Chapter 4 focuses on the wave maps equation. I present my works with Andrew Lawrie [2, 3, 4] on two-bubble wave maps, leading to a complete classification and description of dynamical behaviour in both time directions of the two-solitons in the equivariant case, if the equivariance degree is four or higher.

Préface en français

Mes travaux des quatre dernières années concernent l'étude du comportement en temps long des solutions des équations d'onde non linéaires. J'ai étudié, en particulier, les propriétés de multi-solitons, c'est-à-dire de solutions proches d'une superposition d'un nombre fini d'objets localisés appelés les *solitons*, préservant leur forme au cours du temps. Vu de loin, un objet localisé ressemble à une particule ponctuelle, et leur superposition à un système de n corps ponctuels. L'objet du mémoire est de contribuer à la compréhension de la question de l'utilité de cette intuition. Autrement dit, y a-t-il des situations où les multi-solitons évoluent vraiment comme des systèmes de particules interagissant entre elles et, au contraire, trouve-t-on des cas où leur comportement diffère radicalement de ce que cette vision simpliste puisse suggérer ?

Deux modèles sont étudiés dans ce mémoire : l'équation des champs scalaires en dimension $1+1$ avec un double puits de potentiel, et l'équation d'applications d'onde (wave maps) en dimension $1+2$, à valeurs dans la sphère 2-dimensionnelle. Les deux modèles sont introduits dans le Chapitre 1. Les solitons apparaissent comme un effet de la topologie de l'espace des états d'énergie finie. Pour cette raison, on les qualifie de *solitons topologiques*. Je définis ensuite les *deux-solitons en forte interaction*. J'énonce enfin les principaux résultats, qui peuvent être résumés ainsi :

- les deux-solitons en forte interaction se comportent comme un système de deux particules pour les temps positifs,
- pourtant, au moins dans le cas de l'équation des applications d'onde, ils ont un comportement très différent pour les temps négatifs, notamment ils "disparaissent" en se transformant en pure radiation.

Bien que la méthode de "l'étude de cas" soit adoptée dans mes travaux, j'ai trouvé pertinent d'expliquer dans ce mémoire comment des méthodes générales y trouvent une application. À cet effet, dans le Chapitre 2, je décris un modèle-jouet qui me permet d'introduire deux méthodes robustes d'analyse asymptotique : la *modulation* et la *réduction de Lyapunov-Schmidt*. En particulier, la Section 2.1 indique l'importance d'un changement de variable approprié, une idée clé pour l'analyse de deux-solitons. Le problème que je considère est suffisamment simple pour que je puisse en donner une solution complète. Il m'a paru agréablement surprenant de constater que l'analyse de deux-solitons peut se faire en suivant presque exactement les mêmes "grandes" étapes.

Le Chapitre 3 est consacré au problème de la description asymptotique de l'évolution d'un multi-soliton dans une direction du temps. Dans ce chapitre, je présente en détail mon travail avec Michał Kowalczyk et Andrew Lawrie [1] sur les paires kink-antikink pour les champs scalaires classiques en dimension $1+1$. Je mentionne aussi des résultats en cours de rédaction et quelques perspectives pour la future recherche.

Le Chapitre 4 porte sur l'équation des applications d'onde. J'y présente mes travaux avec Andrew Lawrie [2, 3, 4] sur les applications d'onde de type deux-bulle, aboutissant à une classification complète et description du comportement dynamique dans les deux directions du temps des deux-solitons dans le cas équivariant, pour le degré d'équivariance plus grand que 4.

Chapter 1

General presentation

1.1 The models

The memoir is devoted to the study of two nonlinear wave equations: the scalar field equation in spacetime dimension $1+1$, and the critical wave maps equation from the Minkowski plane $\mathbb{R}^{1+2}$ to the two dimensional sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

1.1.1 Classical scalar fields in dimension $1+1$

Let U be a smooth function defined on the real line, bounded from below. I will study solutions of the following equation:

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U'(\phi(t,x)) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{CSF}$$

This equation is formally obtained as the Euler-Lagrange equation corresponding to the Lagrangian

$$\mathscr{L} := \iint \Big(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - U(\phi)\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

Here are some commonly studied examples:

- For $U \equiv 0$, (CSF) becomes the linear wave equation on the line. Its solutions are called *massless scalar fields* in physics.
- If I take $U(\phi) = \frac{1}{2}m^2\phi^2$ for some $m > 0$, then the resulting equation is the linear Klein-Gordon equation. Its solutions are called *massive scalar fields*.
- If $U(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{24}\lambda\phi^4$ for some $\lambda \neq 0$, then (CSF) is a nonlinear equation, corresponding to what is called the ϕ^4 theory in physics.
- Of particular interest to me will be the situation similar to the last example, but with the "wrong" sign of the quadratic part and a positive quartic part, that is

$$U(\phi) := -\mu^2\phi^2 + \frac{1}{2}\lambda\phi^4, \qquad \mu, \lambda > 0.$$

  Adding a constant to U does not alter the resulting equation, hence I can equivalently set

$$U(\phi) := \frac{\mu^4}{2\lambda} - \mu^2\phi^2 + \frac{1}{2}\lambda\phi^4 = \frac{\lambda}{2}(\phi^2 - \omega^2)^2, \qquad \omega := \frac{\mu}{\sqrt{\lambda}}.$$

In the sequel, this is what I will refer to as the “ϕ^4 model”.

- If I set
$$U(\phi) := \frac{\alpha}{\beta^2}(1 - \cos(\beta\phi)), \qquad \alpha, \beta > 0,$$
the corresponding equation (CSF) is called the “sine-Gordon equation”.

- In general, if U is non-quadratic, then (CSF) is a nonlinear equation and its solutions are referred to in physics as *self-interacting fields*.

Observe that I am dealing with a *natural Lagrangian*, that is the Lagrangian density is the difference of the kinetic energy density $\frac{1}{2}(\partial_t\phi)^2$ and the potential energy density $\frac{1}{2}(\partial_x\phi)^2 + U(\phi)$. Upon adding a constant to U, I can assume that $U \geq 0$, so that the potential energy density is positive.

A state of the system is described by a pair of functions $\phi_0, \dot\phi_0 : \mathbb{R} \to \mathbb{R}$. I set
$$\begin{aligned} E_k(\dot\phi_0) &:= \int_{\mathbb{R}} \frac{1}{2}(\dot\phi_0(x))^2 \,\mathrm{d}x, \\ E_p(\phi_0) &:= \int_{\mathbb{R}} \Big(\frac{1}{2}(\partial_x\phi_0(x))^2 + U(\phi_0)\Big)\,\mathrm{d}x, \end{aligned}$$
which are finite or infinite positive numbers, interpreted as the *kinetic energy* and the *potential energy*. The *total energy* is given by
$$E(\phi_0, \dot\phi_0) := E_k(\dot\phi_0) + E_p(\phi_0) = \int_{\mathbb{R}} \Big(\frac{1}{2}(\dot\phi_0(x))^2 + \frac{1}{2}(\partial_x\phi_0(x))^2 + U(\phi_0)\Big)\,\mathrm{d}x.$$
I always consider states such that $E(\phi_0, \dot\phi_0) < \infty$. I call them *finite energy states* and the corresponding solutions *finite energy solutions*.

The energy is a conserved quantity: if $\phi(t, x)$ is a smooth solution of (CSF) defined for $(t, x) \in (T_-, T_+) \times \mathbb{R}$ such that $(\phi(0), \partial_t\phi(0)) = (\phi_0, \dot\phi_0)$, then
$$E(\phi(t), \partial_t\phi(t)) = E(\phi_0, \dot\phi_0), \qquad \text{for all } t \in (T_-, T_+).$$
Another conserved quantity is the *momentum*, given by
$$P(\phi_0, \dot\phi_0) := -\int_{\mathbb{R}} \dot\phi_0(x)\partial_x\phi_0(x)\,\mathrm{d}x.$$

Observe that (CSF) is Lorentz-invariant. If $\phi(t, x)$ is a solution of (CSF) and $\alpha \in \mathbb{R}$, then so is $\psi(t, x)$ defined by
$$\psi(t, x) := \phi(t\cosh\alpha - x\sinh\alpha, -t\sinh\alpha + x\cosh\alpha).$$
One can build *quantum fields* from (CSF). The resulting theories were studied as toy models in the Quantum Field Theory, see for example [20, Chapter 6].

As I mentioned, I am only interested in finite energy solutions. By a solution $\phi(t, x)$ of (CSF), I always mean a strong solution in the energy space, that is a strong limit in the norm $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ of smooth solutions, locally uniformly in time. By standard arguments, the Cauchy problem for (CSF) is locally well-posed for finite energy initial data, and globally well-posed under additional assumptions on U, for instance if U is globally Lipschitz or if $\lim_{\phi\to\pm\infty} U(\phi) = \infty$.

The problem of a precise description of solutions of (CSF) as $t \to \infty$, even for initial data of small energy, remains open. Delort [27] proved modified scattering for smooth compactly supported initial data.

1.1.2 Equivariant wave maps

Wave maps are natural analogs of linear waves in the case of maps taking values in a Riemannian manifold, and not a Euclidean space. I only consider critical wave maps with values in a two-dimensional sphere. By definition, an application $\Psi : \mathbb{R}^{1+2} \to \mathbb{S}^2 \subset \mathbb{R}^3$ is a *wave map* if it is a critical point of the Lagrangian

$$\mathscr{L}(\Psi) = \frac{1}{2}\iint \big(|\partial_t \Psi|^2 - |\nabla_x \Psi|^2\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Integrating by parts, one finds that Ψ is a wave map if and only if

$$\iint (\Box\Phi)\cdot\Psi\,\mathrm{d}x\,\mathrm{d}t, \qquad \text{with} \quad \Box := \partial_t^2 - \Delta_x,$$

for all Φ smooth with compact support and such that $\Phi(t,x) \perp \Psi(t,x)$ for all (t,x), in other words

$$\Box\Psi(t,x) = \mu(t,x)\Psi(t,x), \qquad \mu(t,x) \in \mathbb{R}.$$

Differentiating twice the identity $\Psi\cdot\Psi = 1$, I obtain

$$\Psi\cdot(\Box\Psi) = -|\partial_t\Psi|^2 + |\nabla_x\Psi|^2,$$

so I can write the wave map equation $\mathbb{R}^{1+2} \to S^2$ as follows:

$$\Box\Psi(t,x) = -\big(|\partial_t\Psi(t,x)|^2 - |\nabla_x\Psi(t,x)|^2\big)\Psi(t,x). \tag{1.1}$$

Directly from the form of the Lagrangian, the following *total energy* is a conserved quantity for wave maps:

$$E(\Psi,\partial_t\Psi) = \frac{1}{2}\int_{\mathbb{R}^2}\big(|\partial_t\Psi|^2 + |\nabla_x\Psi|^2\big)\,\mathrm{d}x.$$

Equation (1.1) has an important property of *criticality*. Let $\lambda > 0$ and consider

$$\Psi_\lambda(t,x) := \Psi(t/\lambda, x/\lambda).$$

It is clear from (1.1) (or from the Lagrangian) that Ψ is a wave map if and only if Ψ_λ is a wave map. Moreover,

$$E(\Psi_\lambda) = E(\Psi).$$

For this reason, equation (1.1) is called *energy critical*, and its solutions *critical wave maps*.

Remark 1.1. In general, a problem is *subcritical* if it becomes a "small data problem" when rescaling (zooming) to a small region. It is called *supercritical* if such a zoom makes it large. It is called *critical* if the size of the data remains unchanged.

The general equation (1.1) was studied in particular in the works of Klainerman and Machedon [54, 55], Klainerman and Selberg [56, 57], Sterbenz and Tataru [97, 98], Tataru [107, 108] and Tao [105, 106]. I only study the simplified *equivariant case*, which is the case of data with a particular symmetry. Namely, I assume that

$$\Psi(t, r\cos\theta, r\sin\theta) = \big(\sin\psi(t,r)\cos(k\theta), \sin\psi(t,r)\sin(k\theta), \cos\psi(t,r)\big) \in \mathbb{S}^2,$$

where $k \in \{1, 2, \ldots\}$ is called the equivariance degree. I compute

$$\begin{aligned}
\partial_r \Psi &= \partial_r \psi(\cos\psi\cos(k\theta), \cos\psi\sin(k\theta), -\sin\psi), \\
\partial_r^2 \Psi &= \partial_r^2 \psi(\cos\psi\cos(k\theta), \cos\psi\sin(k\theta), -\sin\psi) - (\partial_r^2\psi)\Psi, \\
\partial_t^2 \Psi &= \partial_t^2 \psi(\cos\psi\cos(k\theta), \cos\psi\sin(k\theta), -\sin\psi) - (\partial_t^2\psi)\Psi, \\
\partial_\theta^2 \Psi &= -k^2(\sin\psi\cos(k\theta), \sin\psi\sin(k\theta), 0), \\
\Delta\Psi &= \Big(\partial_r^2\psi + \frac{1}{r}\partial_r\psi - \frac{k^2}{2r^2}\sin(2\psi)\Big)\big(\cos\psi\cos(k\theta), \cos\psi\sin(k\theta), -\sin\psi\big) \\
&\qquad - \big((\partial_r\psi)^2 + k^2(\sin\psi)^2\big)\Psi,
\end{aligned}$$

so that Ψ is a wave map if and only if the colatitude $\psi(t, r)$ satisfies the following equation:

$$\partial_t^2\psi(t,r) - \partial_r^2\psi(t,r) - \frac{1}{r}\partial_r\psi(t,r) + \frac{k^2}{2r^2}\sin(2\psi(t,r)) = 0, \qquad (t,r) \in \mathbb{R}\times(0,\infty). \tag{WM}$$

I will say in this case that ψ is a wave map (of equivariance degree or equivariance class k).

Remark 1.2. Some authors use the term "corotational" instead of "equivariant".

Observe that equation (WM) can be formally obtained as the Euler-Lagrange equation corresponding to the Lagrangian

$$\mathscr{L} := \pi \iint \Big((\partial_t\psi)^2 - (\partial_r\psi)^2 - \frac{k^2\sin(\psi)^2}{r^2}\Big)\, r\mathrm{d}r\,\mathrm{d}t.$$

The kinetic energy and the potential energy are

$$\begin{aligned}
E_k(\dot\psi_0) &:= \pi\int_0^\infty (\dot\psi_0(r))^2\, r\mathrm{d}r, \\
E_p(\psi_0) &:= \pi\int_0^\infty \Big((\partial_r\psi_0(r))^2 + \frac{k^2\sin(\psi_0(r))^2}{r^2}\Big)\, r\mathrm{d}r.
\end{aligned} \tag{1.2}$$

The total energy is given by

$$E(\psi_0, \dot\psi_0) := E_k(\dot\psi_0) + E_p(\psi_0) = \pi\int_0^\infty \Big((\dot\psi_0(r))^2 + (\partial_r\psi_0(r))^2 + \frac{k^2\sin(\psi_0)^2}{r^2}\Big)\, r\mathrm{d}r$$

and is a conserved quantity.

Related to the energy is the following *energy norm*:

$$\|(\psi_0, \dot\psi_0)\|_{\mathcal{E}}^2 := \|\dot\psi_0\|_{L^2}^2 + \|\psi_0\|_{\mathcal{H}}^2,$$

where the norms L^2 and $\mathcal{H}$ are defined by

$$\|\dot\psi_0\|_{L^2}^2 := \int_0^\infty \big(\dot\psi_0(r)\big)r\mathrm{d}r, \qquad \|\psi_0\|_{\mathcal{H}}^2 := \int_0^\infty \Big((\partial_r\psi_0(r))^2 + \frac{k^2(\psi_0(r))^2}{r^2}\Big)r\mathrm{d}r. \tag{1.3}$$

Note that if $\|\psi_0\|_{L^\infty}$ is small, then $E_p(\psi_0) \simeq \|\psi_0\|_{\mathcal{H}}^2$, but this is no longer true when $\|\psi_0\|_{L^\infty}$ approaches π.

I always consider strong solutions of finite energy, that is strong limits of sequences of smooth solutions in the energy norm, locally uniformly in time. Their existence and uniqueness for any finite energy initial data was obtained in [42, 92]. It can be deduced from Strichartz estimates for the wave equation, see for example [23, Section 2] in the case $k \in \{1, 2\}$. If $k \geq 3$ is large, Strichartz estimates from [83] can be applied, see [2, Section 2]. Unlike for (CSF), finite-energy solutions of (WM) are not guaranteed to exist for all time. I denote $(T_-, T_+) \subset \mathbb{R}$ the maximal time interval on which the solution exists.

As a by-product of the local existence theory, one obtains *scattering* for small energy initial data, which roughly means that the nonlinearity becomes negligible for large time. Set

$$L_0 := -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2}, \tag{1.4}$$

so that the linearisation of (WM) around $\psi = 0$ is

$$\partial_t^2 \psi_l(t, r) + L_0 \psi_l(t, r) = 0, \qquad (t, r) \in \mathbb{R} \times (0, \infty). \tag{1.5}$$

The small data scattering result can be formulated as follows.

Proposition 1.3. *There exists $\eta > 0$ such that if $\|\psi_0\|_{\mathcal{H}} + \|\dot\psi_0\|_{L^2} \leq \eta$, then the solution of* (WM) *corresponding to the initial data $(\psi(0), \partial_t\psi(0)) = (\psi_0, \dot\psi_0)$ exists for all $t \in \mathbb{R}$. Moreover, there exists a solution ψ_l of the corresponding linear equation* (1.5) *such that*

$$\lim_{t\to\infty} \big(\|\psi(t) - \psi_l(t)\|_{\mathcal{H}} + \|\partial_t\psi(t) - \partial_t\psi_l(t)\|_{L^2}\big) = 0.$$

An analogous result is true for $t \to -\infty$. □

One could say that the result above provides a "complete understanding" of the dynamical behaviour of small energy solutions, though of course one could further study the properties of the scattering. In this memoir, I am interested in the long-time behaviour of solutions of *large* energy.

Remark 1.4. Minor modifications would allow to analyse the more general equation

$$\partial_t^2\psi(t, r) - \partial_r^2\psi(t, r) - \frac{1}{r}\partial_r\psi(t, r) + \frac{k^2}{r^2} f(\psi(t, r)) = 0, \qquad (t, r) \in \mathbb{R} \times (0, \infty),$$

where $f(\psi) = g(\psi)g'(\psi)$ and $g : \mathbb{R} \to \mathbb{R}$ is a smooth odd function such that $g'(0) = 1$. The Lagrangian is given by

$$\mathscr{L} := \pi \iint \Big((\partial_t\psi)^2 - (\partial_r\psi)^2 - \frac{k^2 g(\psi)^2}{r^2}\Big)\, r\mathrm{d}r\,\mathrm{d}t,$$

and the energy by

$$E(\psi_0, \dot\psi_0) = \pi \int_0^\infty \Big((\dot\psi_0)^2 + (\partial_r\psi_0)^2 + \frac{k^2 g(\psi_0)^2}{r^2}\Big)\, r\mathrm{d}r.$$

For the sake of simplicity, I only consider the case $g(\psi) = \sin(\psi)$, corresponding to (WM).

Remark 1.5. I normalised the energy so that its values be consistent with the non-equivariant setting.

1.2 Topological solitons

Almost all of this memoir is devoted to the study of special solutions of the models introduced above, arising as an effect of the topological properties of the configuration space.

1.2.1 Kinks

Let me consider the model (CSF) with U being a *double-well potential*, that is a smooth non-negative even function satisfying the following conditions for some $\omega > 0$:

$$U(-\omega) = U(\omega) = 0, \tag{1.6}$$

$$U(\phi) > 0 \text{ for all } \phi \notin \{-\omega, \omega\}, \tag{1.7}$$

$$U''(-\omega) = U''(\omega) > 0, \tag{1.8}$$

$$\lim_{|\phi|\to\infty} U(\phi) = \infty. \tag{1.9}$$

An example is given by the previously mentioned case of the ϕ^4 model, $U(\phi) := \frac{\lambda}{2}(\phi^2 - \omega^2)^2$. By rescaling, I will assume without loss of generality that

$$\omega = 1 \quad \text{and} \quad U''(1) = 1. \tag{1.10}$$

I claim that if $E_p(\phi_0) < \infty$, then $\lim_{x\to\infty} \phi_0(x)$ exists and equals 1 or -1. Indeed, assumption (1.9) implies that ϕ_0 is bounded. Thus, the other assumptions imply that $(U'(\phi(x)))^2 \leq CU(\phi(x))$ for some $C > 0$ and all $x \in \mathbb{R}$, hence

$$|\partial_x(U(\phi_0(x)))| = |\partial_x\phi_0(x)||U'(\phi_0(x))| \lesssim |\partial_x\phi_0(x)|^2 + U(\phi_0(x)),$$

which is integrable. I conclude that $\lim_{x\to\infty} U(\phi_0(x))$ exists. Using again $E_p(\phi_0) < \infty$, this limit has to equal 0, and $\lim_{x\to\infty} \phi_0(x) \in \{-1, 1\}$. Analogously, $\lim_{x\to-\infty} \phi_0(x) \in \{-1, 1\}$.

Thanks to the claim proved above, with every function ϕ_0 such that $E_p(\phi_0) < \infty$ one can associate the numbers

$$\phi_0 \mapsto \big(\lim_{x\to-\infty} \phi_0(x), \lim_{x\to\infty} \phi_0(x)\big) \in \{-1, 1\} \times \{-1, 1\}.$$

This mapping divides the set of states of finite potential energy into four disjoint classes, which I denote $\mathcal{H}_{-1,-1}$, $\mathcal{H}_{-1,1}$, $\mathcal{H}_{1,-1}$ and $\mathcal{H}_{1,1}$, and I call them *topological classes*. Note that the equation itself is not necessary in order to define them, knowing E_p is sufficient. If the initial data belongs to one of the four classes, the solution stays in it for all time, by continuity of the flow with respect to t.

The minimum of E_p on $\mathcal{H}_{-1,-1}$ and on $\mathcal{H}_{1,1}$ is attained by functions whose value is constant and equal to one of the vacua. It is slightly more complicated to understand the minima on $\mathcal{H}_{-1,1}$ and $\mathcal{H}_{1,-1}$. The following simple fact is crucial.

Lemma 1.6 (Bogomolny trick)**.** *For any function ϕ_0 and any $\zeta_1 < \zeta_2$*

$$\int_{\zeta_1}^{\zeta_2} \Big(\frac{1}{2}(\partial_x\phi_0(x))^2 + U(\phi_0(x))\Big)\,\mathrm{d}x \geq \Big|\int_{\phi_0(\zeta_1)}^{\phi_0(\zeta_2)} \sqrt{2U(\phi)}\,\mathrm{d}\phi\Big|$$

with equality holding if and only if $\partial_x\phi_0(x) = \sqrt{2U(\phi_0(x))}$ for all $x \in (\zeta_1, \zeta_2)$ or $\partial_x\phi_0(x) = -\sqrt{2U(\phi_0(x))}$ for all $x \in (\zeta_1, \zeta_2)$.

Proof. Assume, without loss of generality, that $\phi_0(\zeta_1) \le \phi_0(\zeta_2)$ (the other case follows by considering $-\phi_0$ instead of ϕ_0 and exploiting the symmetry of U). By completing to a square, we have

$$\begin{aligned}
&\int_{\zeta_1}^{\zeta_2} \Big(\frac{1}{2}(\partial_x\phi_0(x))^2 + U(\phi_0(x))\Big)\,\mathrm{d}x \\
&\quad= \int_{\zeta_1}^{\zeta_2} \sqrt{2U(\phi_0(x))}\partial_x\phi_0(x)\,\mathrm{d}x + \frac{1}{2}\int_{\zeta_1}^{\zeta_2} \big(\partial_x\phi_0(x) - \sqrt{2U(\phi_0(x))}\big)^2\,\mathrm{d}x \\
&\quad= \int_{\phi_0(\zeta_1)}^{\phi_0(\zeta_2)} \sqrt{2U(\phi)}\,\mathrm{d}\phi + \frac{1}{2}\int_{\zeta_1}^{\zeta_2} \big(\partial_x\phi_0(x) - \sqrt{2U(\phi_0(x))}\big)^2\,\mathrm{d}x,
\end{aligned}$$

thus we obtain the desired inequality, with equality if and only if the second term of the last line vanishes, which is equivalent to $\partial_x\phi_0(x) = \sqrt{2U(\phi_0(x))}$ for all $x \in [\zeta_1, \zeta_2]$. ☐

I define the function $H : \mathbb{R} \to (-1,1)$ as the solution of $\partial_x H(x) = \sqrt{2U(H(x))}$ with the initial condition $H(0) = 0$. By elementary analysis, $H \in \mathcal{H}_{-1,1}$. I call H and its translates in x the *kinks.*

Corollary 1.7. *If $\phi_0 \in \mathcal{H}_{-1,1}$, then $E_p(\phi_0) \ge E_p(H) = \int_{-1}^{1}\sqrt{2U(\phi)}\,\mathrm{d}\phi$. Equality holds if and only if ϕ_0 is a kink.*

Proof. The inequality follows from Lemma 1.6, by letting $\zeta_1 \to -\infty$ and $\zeta_2 \to \infty$. Suppose equality holds. Then Lemma 1.6 yields $\partial_x\phi_0(x) = \sqrt{2U(\phi_0(x))}$ for all $x \in \mathbb{R}$. By continuity, there exists $\xi_0 \in \mathbb{R}$ such that $\phi_0(\xi_0) = 0$, implying $\phi_0(x) = H(x - \xi_0)$ for all $x \in \mathbb{R}$. ☐

The last result can be restated as follows: the set of minimisers of E_p on $\mathcal{H}_{-1,1}$ is the one-dimensional manifold given by the space translates of H.

The case of $\mathcal{H}_{1,-1}$ is completely analogous. The minimisers of E_p in this class are the *antikinks*, that is $-H$ and its space translates.

Proposition 1.8. *The constant functions $\phi_0(x) = \pm 1$, the kinks and the antikinks are the only critical points of E_p, thus the only static solutions of* (CSF)*.*

Proof. If ϕ_0 is a function such that $E_p(\phi_0) < \infty$ and $\mathrm{D}E_p(\phi_0) = 0$, then ϕ_0 satisfies the equation

$$-\partial_x^2\phi_0(x) + U'(\phi_0(x)) = 0.$$

In particular, $\phi_0 \in C^\infty$. Multiplying the equation by $\partial_x\phi_0$, I obtain

$$-\frac{1}{2}(\partial_x\phi_0(x))^2 + U(\phi_0(x)) = \mathrm{const} = 0, \tag{1.11}$$

where the last equality follows from $E_p(\phi_0) < \infty$.

If there exists x_0 such that $\phi_0(x_0) = 1$ or $\phi_0(x_0) = -1$, then (1.11) and Gronwall inequality yields $\phi_0 = \mathrm{const}$. In the opposite case, (1.11) and smoothness of ϕ_0 imply

$$\partial_x\phi_0(x) = \sqrt{2U(\phi_0(x))}\ \text{ for all } x \in \mathbb{R}, \quad \text{or } \partial_x\phi_0(x) = -\sqrt{2U(\phi_0(x))}\ \text{ for all } x \in \mathbb{R},$$

hence ϕ_0 is a kink or an antikink. ☐

Remark 1.9. The asymptotic stability of the kink for the ϕ^4 model is a well-known open problem. It was solved for odd initial data by Kowalczyk, Martel and Muñoz [58].

By means of the Lorentz transformation, one can construct moving kinks. If $\beta \in (-1,1)$ and $\xi_0 \in \mathbb{R}$, then

$$\phi(t,x) := H(\gamma_\beta(x - \xi_0 - \beta t)), \qquad \text{where } \gamma_\beta := (1-\beta^2)^{-1/2}, \tag{1.12}$$

is a solution of (CSF), a *travelling wave* whose velocity equals β.

1.2.2 Bubbles

A similar discussion can be made for the equation (WM). Recall that E_p is given by (1.2). I claim that if $E_p(\psi_0) < \infty$, then $\psi_0 \in C((0,\infty))$ and there exist $m, n \in \mathbb{Z}$ such that

$$\lim_{r\to 0}\psi_0(r) = m\pi, \qquad \lim_{r\to\infty}\psi_0(r) = n\pi.$$

In order to see this, it is convenient to consider the change of variable $r^k = \mathrm{e}^x$. Set $\phi_0(x) := \psi_0(\mathrm{e}^{x/k})$. Then $\partial_x\phi_0(x) := \frac{1}{k}\mathrm{e}^{x/k}\partial_r\psi_0(\mathrm{e}^{x/k})$ and $r\mathrm{d}r = \frac{1}{k}(\mathrm{e}^{x/k})^2\,\mathrm{d}x$, so I obtain,

$$E_p(\psi_0) = k\pi\int_{\mathbb{R}}\big((\partial_x\phi_0(x))^2 + (\sin\phi_0(x))^2\big)\,\mathrm{d}x.$$

By the argument from the previous section, $\lim_{x\to-\infty}\phi_0(x) = m\pi$ and $\lim_{x\to\infty}\phi_0(x) = n\pi$ for some $m, n \in \mathbb{Z}$. It is also clear that ϕ_0 is continuous. Returning to the original variable r, I obtain the claim. With every function ψ_0 such that $E_p(\psi_0) < \infty$ one can thus associate the numbers

$$\psi_0 \mapsto \big(\lim_{r\to 0}\psi_0(r), \lim_{r\to\infty}\psi_0(r)\big) \in \pi\mathbb{Z}\times\pi\mathbb{Z}.$$

This mapping divides the set of states of finite potential energy into an infinite number of disjoint classes. I call them *topological classes* and denote them $\mathcal{H}_{m,n}$ for $m, n \in \mathbb{Z}$.

Remark 1.10. I should mention that these topological classes do *not* exactly correspond to classes indexed by the homotopy degree of ψ_0, viewed as a map in the non-equivariant setting. In fact, all the maps in $\mathcal{H}_{m,n}$ with $m-n$ even have the homotopy degree 0, and all the maps in $\mathcal{H}_{m,n}$ with $m-n$ odd have the homotopy degree k.

The variational structure is not very different from the case of the kinks. The global minimisers of E_p on $\mathcal{H}_{m,m}$ are the constant functions $\psi_0(r) = m\pi$. In order to find minimisers on $\mathcal{H}_{m,n}$ for $m \neq n$ we use the following fact.

Lemma 1.11 (Bogomolny trick)**.** *For any function $\psi_0 : (0,\infty) \to \mathbb{R}$ and any $0 < \rho_1 < \rho_2 < \infty$ the following bound is true:*

$$\pi\int_{\rho_1}^{\rho_2}\Big((\partial_r\psi_0(r))^2 + \frac{k^2\sin(\psi_0(r))^2}{r^2}\Big)r\mathrm{d}r \geq 2k\pi\big|\cos(\psi_0(\rho_1)) - \cos(\psi_0(\rho_2))\big|,$$

with equality holding if and only if $\partial_r\psi_0(r) = \frac{k}{r}\sin(\psi_0(r))$ for all $r \in (\rho_1,\rho_2)$ or $\partial_r\psi_0(r) = -\frac{k}{r}\sin(\psi_0(r))$ for all $r \in (\rho_1,\rho_2)$.

Proof. The proof is similar to the proof of Lemma 1.6, so I skip it. □

I define the function $Q : (0,\infty) \to (0,\pi)$ as the solution of the equation $\partial_r Q(r) = \frac{k}{r}\sin(Q(r))$ with the initial condition $Q(1) = \frac{\pi}{2}$. One can solve this equation explicitly and obtain $Q(r) = 2\arctan(r^k)$. I also define the rescaled version $Q_\lambda(r) := 2\arctan\big((r/\lambda)^k\big)$ for any $\lambda > 0$, in particular $Q_1 = Q$. It is easy to see that $Q_\lambda \in \mathcal{H}_{0,1}$. I call the functions Q_λ the *bubbles*. In other topological classes there are similar objects, namely $m\pi + Q_\lambda \in \mathcal{H}_{m,m+1}$ and $m\pi - Q_\lambda \in \mathcal{H}_{m,m-1}$, for any $m \in \mathbb{Z}$. I will call the functions $2n\pi \pm Q_\lambda$ the *bubbles* and the functions $(2n+1)\pi \pm Q_\lambda$ the *anti-bubbles*, which is in accordance with the usual terminology in the non-equivariant setting. However, I will often forget this distinction and simply call all these objects "bubbles".

Proposition 1.12. *The constant functions $\psi_0(r) = m\pi$, the bubbles and the anti-bubbles are the only critical points of E_p.*

Proof. The proof is similar to the proof of Proposition 1.8, so I skip it. $\square$

Proposition 1.13. *If $m \in \mathbb{Z}$ and $\psi_0 \in \mathcal{H}_{m,m+1}$ or $\psi_0 \in \mathcal{H}_{m,m-1}$, then $E_p(\psi_0) \geq E_p(Q) = 4k\pi$. Equality holds if and only if ψ_0 is a bubble or an anti-bubble.*

If $m, n \in \mathbb{Z}$ and $|m-n| > 1$, then

$$\inf_{\psi_0 \in \mathcal{H}_{m,n}} E_p(\psi_0) = 4k|m-n|\pi,$$

but the infimum is not attained.

Proof. The first part follows from Lemma 1.11, by letting $\rho_1 \to 0$ and $\rho_2 \to \infty$.

Let $n - m \geq 2$ and $\psi_0 \in \mathcal{H}_{m,n}$ (the case $m - n \geq 2$ is similar). By continuity, there exist ρ_1 such that $\psi(\rho_1) = (m+1)\pi$, $\ldots$, ρ_{n-m-1} such that $\psi(\rho_{n-m-1}) = (n-1)\pi$. Applying Lemma 1.11 on each of the intervals $(0, \rho_1)$, (ρ_1, ρ_2), $\ldots$, (ρ_{n-m-1}, ∞) and taking the sum, I obtain $E_p(\psi_0) \geq 4k(n-m)\pi$. If there was equality, then ψ_0 would be a critical point of E_p, which is impossible by Proposition 1.12. $\square$

Remark 1.14. In the non-equivariant setting, the stationary states, which are the critical points of the potential energy

$$E_p(\Psi) = \frac{1}{2}\int |\nabla_x \Psi|^2 \, \mathrm{d}x,$$

are the well known *harmonic maps* $\mathbb{R}^2 \to \mathbb{S}^2$. Remarkably, even in the general case they can be completely classified: they are in one-to-one correspondence with complex rational functions, see Eells and Wood [40].

1.2.3 Other topological solitons

The general concept can be described as follows, see [64, Section 4.1]. Suppose that for every t one can assign to $\phi(t)$ a *discrete* quantity in a continuous way. A typical example is assigning to a map its homotopy class (appropriately defined in each particular case). In such a way, the set of states is divided into topological classes invariant by the flow. If the potential energy, restricted to one of these classes, has a global minimum, then the corresponding stationary solution of (CSF) is called a *topological soliton.* An important example of a topological soliton which is not considered in this memoir is a *vortex*, see [64] for information on various types of topological solitons.

Of fundamental importance in the study of the dynamical role of topological solitons are their *coercivity properties*, related to their variational characterisation as the global minimisers of the potential energy in a given topological class. Namely, it is immediate that the Hessian of the potential energy at a topological soliton is positive semi-definite. It is frequently the case that it is *not* positive definite due to invariances of the problem, and it is necessary to precisely describe this lack of positivity. I will do it in the case of kinks and bubbles in later chapters, see Lemma 3.5 and Lemma 4.2.

1.3 Strongly interacting pure multi-solitons

In Section 1.2, I classified all the critical points of the potential energy for the two models under consideration (equivalently, all the stationary states of the system). A more complicated topic, which is going to occupy me in this memoir, is to describe solutions which resemble (in a sense to be specified in each case) a *superposition* of a finite number of stationary states. Let me enumerate some reasons to study such solutions:

- they are related to Palais-Smale sequences of the potential energy,
- equivalently, they are the *almost stationary states*, that is solutions whose kinetic energy asymptotically vanishes,
- they can be viewed as attractors of solutions leaving a small neighborhood of a configuration of widely separated stationary states,
- in other words, one can view them as stable/unstable manifolds of an ideal *critical point at infinity* corresponding to widely separated stationary states,
- they constitute *threshold elements*, that is solutions of the minimal (threshold) energy allowing for a new dynamical behaviour (excluded for lower energies),
- for this reason, they can be convenient objects for the study of *collisions of solitons* (in a non-perturbative regime).

I will clarify some of these points in the discussion of the concrete cases below, see Section 3.4 and Section 4.3.

1.3.1 Kink clusters and kink-antikink pairs

Definition 1.15. Let $n \in \{0, 1, \ldots\}$. I say that a solution ϕ of (CSF) is a *kink n-cluster in the forward time direction* if

- $E(\phi, \partial_t \phi) \leq n E_p(H)$,
- there exist real-valued functions $\zeta_0(t) \leq \zeta_1(t) \leq \ldots \leq \zeta_n(t)$ such that $\lim_{t\to\infty} \phi(t, \zeta_k(t)) = (-1)^k$ for $k \in \{0, 1, \ldots, n\}$.

The intuitive meaning of this definition is the following. A kink n-cluster is a solution which, asymptotically as $t \to \infty$, is a chain of n transitions between pairs of consecutive vacua $1 \rightsquigarrow -1 \rightsquigarrow \ldots \rightsquigarrow (-1)^n$, and has the minimal possible energy allowing for such a chain. Note that the kink 0-clusters are the constant solutions $\phi \equiv 1$ and the kink 1-clusters are the antikinks. We say that ϕ is a kink cluster if it is a kink n-cluster for some $n \in \{0, 1, \ldots\}$.

Since the energy is minimal possible, the shape of each transition in the chain has to be close to optimal, that is close to a kink or an antikink. The following lemma provides a precise formulation of this fact. For $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ such that $\xi_1 < \ldots < \xi_n$, I denote

$$H_n(\boldsymbol{\xi}) := \frac{1 + (-1)^n}{2} + \sum_{k=1}^{n} (-1)^k H(\cdot - \xi_k).$$

Lemma 1.16. *A solution ϕ of* (CSF) *is a kink n-cluster if and only if there exist $T_0 \in \mathbb{R}$ and continuous functions $\xi_1, \ldots, \xi_n : [T_0, \infty) \to \mathbb{R}$ such that $\lim_{t\to\infty}\big(\xi_{k+1}(t) - \xi_k(t)\big) = \infty$ for $k \in \{1, \ldots, n-1\}$ and*

$$\lim_{t\to\infty}\Big(\big\|\partial_t\phi(t)\big\|_{L^2} + \big\|\phi(t) - H_n(\boldsymbol{\xi}(t))\big\|_{H^1}\Big) = 0.$$

□

For $n \in \{0, 1\}$ it is rather easy to classify all the n-kinks:

- the only 0-kink is the vacuum solution $\phi(t, x) = 1$,
- the only 1-kinks are the antikinks $\phi(t, x) = -H(x - a)$ for some $a \in \mathbb{R}$.

The simplest non-trivial case is $n = 2$. I call a kink n-cluster with $n = 2$ a *strongly interacting kink-antikink pair.* Since I am not going to study weakly interacting kink-antikink pairs in this memoir, I usually abbreviate the name to "kink-antikink pairs".

1.3.2 Two-bubbles

For the sake of simplicity, I will not discuss general multi-bubbles and restrict my attention to *two-bubble solutions.* Analogous definitions could be formulated in the case of more bubbles, with some minor complications.

Definition 1.17. I say that a solution ψ of (WM) is a *two-bubble in the forward time direction* if

- $E(\psi, \partial_t\psi) \leq 8k\pi$,
- $\psi(t) \in \mathcal{H}_{0,0}$,
- there exists a positive function $\rho_0(t)$ such that $\lim_{t\to T_+}\psi(t, \rho_0(t)) = \pi$ (a positive two bubble), or $\lim_{t\to T_+}\psi(t, \rho_0(t)) = -\pi$ (a negative two-bubble).

One of the main consequences of the analysis presented in Chapter 4 is that I could equivalently define two-bubbles by requiring convergence for a sequence of times only, a result which is currently unavailable for the ϕ^4 model.

Remark 1.18. If $E_p(\psi_0) \leq 8k\pi$, then $\psi_0 \in \mathcal{H}_{0,0}$ is equivalent to $\deg\psi_0 = 0$, the topological degree of the corresponding map $\mathbb{R}^2 \to S^2$, seen as a map $S^2 \to S^2$ (using the stereographic projection).

By elementary arguments based on the Bogomolny trick, one can prove the following fact analogous to Lemma 1.16.

Lemma 1.19. *A solution ψ of* (WM) *is a positive two bubble in the forward time direction if and only if there exist $T_0 < T_+$ and continuous functions $\lambda, \mu : [T_0, T_+) \to (0, \infty)$ such that $\lim_{t\to T_+}\lambda(t)/\mu(t) = 0$ and*

$$\lim_{t\to T_+}\Big(\big\|\partial_t\psi(t)\big\|_{L^2} + \big\|\psi(t) - (Q_{\lambda(t)} - Q_{\mu(t)})\big\|_{\mathcal{H}}\Big) = 0.$$

□

1.4 Main results

The results presented in this memoir concern a precise description of strongly interacting pure multi-solitons for (CSF) and (WM).

1.4.1 Classification of kink-antikink pairs for scalar fields

Given U satisfying (1.6)–(1.10), I introduce the following explicit constants:

$$\kappa := \exp\bigg(\int_0^1 \Big(\frac{1}{\sqrt{2U(y)}} - \frac{1}{1-y}\Big)\,\mathrm{d}y\bigg), \tag{1.13}$$

and

$$A := \bigg(\int_0^1 \sqrt{2U(y)}\,\mathrm{d}y\bigg)^{-\frac{1}{2}}\kappa = \sqrt{2}\|\partial_x H\|_{L^2}^{-1}\kappa. \tag{1.14}$$

I can now state the main result of [1], in collaboration with Michał Kowalczyk and Andrew Lawrie.

Theorem 1. *There exist a C^1 function $\xi(t)$ and a solution $\boldsymbol{\phi}_{(2)}(t,x)$ of* (CSF) *such that for all $\epsilon > 0$ and all $t \geq T_0 = T_0(\epsilon)$,*

$$\big|\xi(t) - \log(At)\big| \leq t^{-2+\epsilon}, \qquad \big|\xi'(t) - t^{-1}\big| \leq t^{-3+\epsilon} \tag{1.15}$$

and

$$\begin{aligned}&\big\|\boldsymbol{\phi}_{(2)}(t) - \big(1 - H(\cdot + \xi(t)) + H(\cdot - \xi(t))\big)\big\|_{H^1} \\ &+ \big\|\partial_t\boldsymbol{\phi}_{(2)}(t) + \xi'(t)\big(\partial_x H(\cdot + \xi(t)) + \partial_x H(\cdot - \xi(t))\big)\big\|_{L^2} \leq t^{-2+\epsilon}.\end{aligned}$$

Moreover, $\boldsymbol{\phi}_{(2)}$ is the unique kink-antikink pair up to translation, i.e., if $\phi(t,x)$ is any kink-antikink pair, then there exist $t_0, x_0 \in \mathbb{R}$ so that

$$\phi(t,x) = \boldsymbol{\phi}_{(2)}(t - t_0, x - x_0).$$

1.4.2 Uniqueness of two-bubble solutions in high equivariance classes

I now present analogous results about existence and uniqueness of two-bubble solutions. The following fact was proved in [46].

Theorem 2. *For all $k \geq 2$ there exists an infinite-time two-bubble solution $\psi : [T_0, \infty) \to \mathcal{H} \times L^2$. As $t \to \infty$, the scale of the less concentrated bubble satisfies $\mu(t) \simeq 1$ and the scale of the more concentrated bubble satisfies $\lambda(t) \simeq t^{-2/(k-2)}$ if $k \geq 3$ and $\lambda(t) \simeq \mathrm{e}^{-4t/\sqrt{\pi}}$ if $k = 2$.*

The main result of the paper [4], in collaboration with Andrew Lawrie, can be stated as follows.

Theorem 3. *If $k \geq 4$, then the forward in time two-bubble solution is unique up to rescaling and translation in time. In other words, if ψ and $\widetilde{\psi}$ are two-bubble solutions, then there exist λ and t_0 such that*

$$\widetilde{\psi}(t,r) = \psi\big((t - t_0)/\lambda, r/\lambda\big), \qquad \text{for all } (t,r) \in [T_0, \infty) \times (0, \infty).$$

These two theorems provide an analog of Theorem 1 for equivariant wave maps.

Our proof scheme, explained in Section 4.2, requires much more information about the constructed two-bubble than what was available from [46]. These estimates were proved in a separate paper [3].

1.4.3 Annihilation of the two-bubble

Some questions left open for kink-antikink pairs can be answered for two-bubbles, most importantly the collision problem can be solved, which was achieved in [2], in collaboration with Andrew Lawrie.

Theorem 4. *If ψ is a solution of* (WM) *which approaches a two-bubble configuration for an increasing sequence of times, then ψ is a two-bubble in the forward time direction.*

I will make this, slightly informal, statement rigorous in Section 4.3.

Theorem 5. *If ψ is a two-bubble in the forward time direction, then ψ exists for all time and scatters as $t \to -\infty$.*

1.4.4 Comments on the results

One should see strongly interacting two-solitons as *threshold solutions*. Small energy data are topologically trivial and the corresponding evolution presents *oscillatory behaviour* (for example, scattering in the case of wave maps). When higher and higher energies are considered, when does a new type of dynamical behaviour appear? Of course solitons have a different behaviour, but they do not provide the correct energy threshold, because they are not topologically trivial. The correct energy threshold equals in fact *twice* the energy of the soliton, and strongly interacting two-solitons are topologically trivial solutions of lowest possible energy locally converging (up to translations/rescalings) to solitons.

From this perspective, uniqueness of strongly interacting two-solitons is an analog of the results of Merle [74] on uniqueness of minimal mass blow-up solutions of the mass-critical NLS, and the corresponding result of Raphaël and Szeftel [86] on non-homogeneous mass-critical NLS. Let me stress however that in [74, 86] the solution develops *one* bubble. The novelty of Theorems 1 and 3 with respect to these works is to consider solutions which are superpositions of *more than one* solitons. Similarly, Theorem 4 and Theorem 5 are inspired by no-return results for *one* soliton by Duyckaerts and Merle [37, 38], Nakanishi and Schlag [77, 78], and Krieger, Nakanishi and Schlag [59, 60]. The exponential instability considered in those works is replaced here by the attractive interaction between the solitons.

Historically, strongly interacting two-solitons were little studied. It is easy to check that the function $\phi(t, x) = 4 \arctan(t \operatorname{sech}(x))$ is a kink-antikink pair for the sine-Gordon equation

$$\partial_t^2 \phi(t, x) - \partial_x^2 \phi(t, x) + \sin(\phi(t, x)) = 0,$$

but the uniqueness part of Theorem 1 seems to be new even for this completely integrable model. Existence of strongly interacting 2-solitons and 3-solitons for the mKdV equation was observed by Wadati and Ohkuma [110]. For non-integrable models, the first rigorous construction is due to Martel and Raphaël [71], followed by [46, 79, 80], as well as [7]. Uniqueness results appear to be completely new.

Finally, I would like to mention that Theorem 5 was probably the first non-perturbative result on collision of solitons. Proofs of non-elastic soliton collision for quartic gKdV in perturbative regimes were obtained by Martel and Merle [67, 68].

1.4.5 Comments on the Soliton Resolution Conjecture

A major problem in the study of a wave equation involving solitons, sometimes viewed as the "final goal", is the *Soliton Resolution Conjecture* inspired by the analysis of completely integrable models, see [39]. It states that the solution of a dispersive equation corresponding to *any* initial data asymptotically, as time approaches the final time of existence (which can finite or infinite), resembles a superposition of solitons and a solution of the corresponding linear equation (the *radiation*). While the problem seems out of reach for (CSF), considerable progress has been achieved for (WM). In the discussion below, I will not distinguish between various equivariant classes k. Many papers consider only $k = 1$ or $k = 2$, but arguments can always be adapted to general k.

Fundamental results were obtained in [19, 18, 94, 95], see also [93, Chapter 8], the main conclusion being the *decay of energy on the light cone*, which in particular excludes self-similar blow-up, but also leads to *bubbling*: if ψ is a solution of (WM) which blows up in finite time T_+, then there exist sequences $t_n \to T_+$ and $0 < \lambda_n \ll T_+ - t_n$ such that

$$(\psi(t_n, \lambda_n\cdot), \lambda_n\partial_t\psi(t_n, \lambda_n\cdot)) \to n\pi \pm Q \qquad \text{in } \mathcal{H}_{\text{loc}} \times L^2_{\text{loc}}.$$

An analogous result is true for any target manifold which is a surface of revolution. As a consequence, global existence was obtained for target manifolds for which non-trivial harmonic maps are absent.

The bubbling also implies that a solution whose energy is smaller than the energy of Q cannot blow up. It was proved by Côte, Kenig, Lawrie and Schlag [23] that such a solution actually *scatters*, thus improving Proposition 1.3. Above this threshold energy, finite time blow-up can occur, as was proved by Krieger, Schlag and Tataru [61], and by Raphaël and Rodnianski [85]. Important progress toward the soliton resolution conjecture was obtained in [24]. *Sequential* soliton resolution, that is convergence to a superposition of solitons for a sequence of times, was proved by Côte [22] for $k \in \{1, 2\}$, and Jia and Kenig [51] for $k \geq 3$. Similar results in the non-radial case, but with a less precise description of the radiation, were obtained by Grinis [43].

For the closely related energy critical wave equation, scattering below the ground state energy threshold was proved by Kenig and Merle [53], establishing together with [52] the so-called *Kenig-Merle route map.* In the radially symmetric case, the soliton resolution conjecture was proved by Duyckaerts, Kenig and Merle in space dimension 3 in [33], and in any odd space dimension in [34, 35, 36], see also [30, 32, 31] by the same authors.

In the non-radial case, sequential solitons resolution was proved by Duyckaerts, Jia, Kenig and Merle [29].

Chapter 2

General methods

2.1 Modulation for a toy model

The method of "modulation" is a particular instance of a general idea of decomposing a problem into "slow" and "fast" variables in some asymptotic regime.

For the sake of illustration of the key ideas, let me analyse a simple system with 2 degrees of freedom. Let V be a smooth non-negative function defined on $(0, \infty)$ such that $V(1) = 0$ and $V''(1) > 0$, and W any smooth bounded real-valued function defined on $\mathbb{R}^2$, which we identify with $\mathbb{C}$ in the standard way. I consider the Newton equation of motion in the plane of a point unit mass under the influence the potential

$$U(x + iy) = U(r\mathrm{e}^{i\theta}) := \frac{1}{\epsilon^2}V(r) + W(r\mathrm{e}^{i\theta}), \qquad 0 < \epsilon \ll 1,$$

that is the system

$$x''(t) + iy''(t) = -\partial_x U(x(t) + iy(t)) - i\partial_y U(x(t) + iy(t)). \tag{2.1}$$

I am interested in describing the solutions of bounded energy (uniformly in ϵ) in the asymptotic regime $\epsilon \to 0$. The first term of U is a *constraining potential*, forcing solutions of bounded energy to stay at distance $O(\epsilon)$ from the unit circle. We can expect the solution to "slowly" move along the unit circle, while at the same time oscillating "fast" in the direction perpendicular to the circle.

Remark 2.1. The analogy between the evolution of multi-solitons and the movement of a particle in a strongly constraining potential is taken from the review paper by Stuart [101]. In the multi-soliton setting, the constraining potential is replaced by the coercivity properties of multi-solitons, and the weak potential W corresponds to the force between the solitons.

For the system I am considering, the modulation analysis boils down to decomposing

$$x(t) + iy(t) = \mathrm{e}^{i\theta(t)} + (r(t) - 1)\mathrm{e}^{i\theta(t)}.$$

The Chain Rule yields

$$\begin{aligned}
x'(t) + iy'(t) &= r'(t)\mathrm{e}^{i\theta(t)} + ir(t)\theta'(t)\mathrm{e}^{i\theta(t)}, \\
x''(t) + iy''(t) &= r''(t)\mathrm{e}^{i\theta(t)} + 2ir'(t)\theta'(t)\mathrm{e}^{i\theta(t)} + ir(t)\theta''(t)\mathrm{e}^{i\theta(t)} - r(t)(\theta'(t))^2\mathrm{e}^{i\theta(t)}.
\end{aligned} \tag{2.2}$$

Also, by the expression of the gradient in polar coordinates,

$$\partial_x U + i\partial_y U = \mathrm{e}^{i\theta}\partial_r U + i\mathrm{e}^{i\theta}\frac{1}{r}\partial_\theta U = \mathrm{e}^{i\theta}\Big(\frac{1}{\epsilon^2}\partial_r V(r) + \partial_r W(r)\Big) + i\mathrm{e}^{i\theta}\frac{1}{r}\partial_\theta W(r\mathrm{e}^{i\theta}),$$

thus (2.1) is equivalent to

$$\begin{aligned} r''(t)\mathrm{e}^{i\theta(t)} + 2ir'(t)\theta'(t)\mathrm{e}^{i\theta(t)} + ir(t)\theta''(t)\mathrm{e}^{i\theta(t)} - r(t)(\theta'(t))^2\mathrm{e}^{i\theta(t)} = \\ = -\mathrm{e}^{i\theta(t)}\Big(\frac{1}{\epsilon^2}\partial_r V(r(t)) + \partial_r W(r(t))\Big) - i\mathrm{e}^{i\theta(t)}\frac{1}{r(t)}\partial_\theta W(r(t)\mathrm{e}^{i\theta(t)}). \end{aligned}$$

Projecting the last equation on $i\mathrm{e}^{i\theta(t)}$, I obtain the *modulation equation*

$$r(t)\theta''(t) + 2r'(t)\theta'(t) = -\frac{1}{r(t)}\partial_\theta W(r(t)\mathrm{e}^{i\theta(t)}). \tag{2.3}$$

Note that V does not appear in this equation, because $\nabla V(r\mathrm{e}^{i\theta})$ is orthogonal to $i\mathrm{e}^{i\theta}$. This is precisely the reason why I chose $i\mathrm{e}^{i\theta(t)}$ as the direction on which to project (2.3).

If I artificially restricted the system (2.1) to the unit circle, then the evolution would be given by

$$\widetilde{\theta}''(t) = -\widetilde{W}'(\widetilde{\theta}(t)), \qquad \widetilde{W}(\theta) := W(\mathrm{e}^{i\theta}), \tag{2.4}$$

which is called the *formal modulation equation*. Note that it has the conserved energy

$$\widetilde{E}(\widetilde{\theta}(0), \widetilde{\theta}'(0)) = \widetilde{E}(\widetilde{\theta}(t), \widetilde{\theta}'(t)) := \frac{1}{2}(\widetilde{\theta}'(t))^2 + \widetilde{W}(\widetilde{\theta}(t)), \tag{2.5}$$

which allows to reduce (2.4) to a first-order equation and solve it. The question is if the evolution of the slow variable $\theta(t)$ for the original system (2.1) is asymptotically given by this formal equation.

The energy of the system (2.1) is given by

$$E := \frac{1}{2}\big((x'(t))^2 + (y'(t))^2\big) + U(x+iy),$$

which in the variables (r, θ) reads

$$E := \frac{1}{2}\big((r'(t))^2 + (r(t)\theta'(t))^2\big) + \frac{1}{\epsilon^2}V(r(t)) + W(r(t)\mathrm{e}^{i\theta(t)}). \tag{2.6}$$

Since we assume boundedness E uniformly in ϵ and $V''(1) > 0$, we have

$$|r(t) - 1| \lesssim \epsilon, \quad |r'(t)| \lesssim 1, \quad |\theta'(t)| \lesssim 1. \tag{2.7}$$

These simple *coercivity bounds* are not sufficient to directly exploit the modulation equation. Indeed, the second term of the left hand side of (2.3) could be of the same size as the right hand side.

The remedy is to consider the new variable

$$L(t) := r(t)^2\theta'(t),$$

and rewrite (2.3) as

$$L'(t) = -\partial_\theta W(r(t)\mathrm{e}^{i\theta(t)}).$$

I can now use the bounds (2.7) and write

$$|\theta'(t) - L(t)| \lesssim \epsilon, \qquad |L'(t) + \widetilde{W}'(\theta(t))| \lesssim \epsilon.$$

Note that L is the angular momentum, hence a conserved quantity in the absence of W.

Let $(\widetilde{\theta}(t), \widetilde{\theta}'(t))$ be the solution of (2.4) such that $(\widetilde{\theta}(0), \widetilde{\theta}'(0)) = (\theta(0), L(0))$. I have

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\theta} - \theta)\Big| \leq |\widetilde{\theta}' - L| + |L - \theta'| \leq |\widetilde{\theta}' - L| + C\epsilon,$$

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\theta}' - L)\Big| = |-\widetilde{W}'(\widetilde{\theta}) - L'| \leq |\widetilde{W}'(\widetilde{\theta}) - \widetilde{W}'(\theta)| + C\epsilon \leq C|\widetilde{\theta} - \theta| + C\epsilon,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(|\widetilde{\theta} - \theta| + |\widetilde{\theta}' - L|\big) \leq C\big(|\widetilde{\theta} - \theta| + |\widetilde{\theta}' - L| + \epsilon\big)$$

for almost all t. Let $1 \leq T \ll |\log \epsilon|$. Applying the Gronwall inequality, I conclude, using again $|\theta' - L| \lesssim \epsilon$, that

$$|\theta(t) - \widetilde{\theta}(t)| + |\theta'(t) - \widetilde{\theta}'(t)| \leq \epsilon\, C\mathrm{e}^{CT}, \qquad \text{for all } t \in [0, T] \tag{2.8}$$

for some $C > 0$ independent of ϵ.

To summarise, I have obtained that on a long time interval the evolution of the modulation parameter is close to the one given by the formal modulation equation. The estimates are sufficiently precise to account for the influence of the potential W. The crucial step is to introduce the new variable $L(t)$, related to a conservation law of the leading order part of the system. It is not necessary to assume that the energy of the fast oscillations is small as compared to the energy of the slow motion along the circle. In other words, I do not assume that the energy of the formal equation given by (2.5) is close to the total energy of the system given by (2.6).

Remark 2.2. The last sentence is related to the distinction, made by Rubin and Ungar [88], between the *tangential case* and the *non-tangential case*. In the tangential case, it is assumed that the initial data is almost tangent to the slow manifold. In this setting, even working with the original variables (θ, θ'), it is possible to obtain estimates sufficiently precise to account for the influence of W, which is the strategy adopted for example by Gustafson and Sigal [44], and Stuart [100], see also [82, 28].

Rubin and Ungar [88] treat both the tangential and the non-tangential case, using an appropriate change of variables (the *Routh variables*).

Remark 2.3. In the multi-soliton context, a change of variable analogous to the one presented in this section was used in the works [2, 1] presented in this memoir, and also in some of my other papers, namely [7, 8, 9]. The first use of a similar idea in the context of solitons that I am aware of is Proposition 4.3 of Raphaël and Szeftel [86].

2.2 Lyapunov-Schmidt reduction

In the previous section I analysed the evolution of slow variables for a simple toy model. Let me observe that this analysis used:

- the modulation equation (2.3),

- the conservation of energy, in order to obtain the coercivity bounds (2.7).

Not much care was devoted to the analysis of the direction of fast oscillations. Refining the analysis can proceed in various ways. Let me begin by completing the modulation equation with the *projected equation*, obtained by projecting (2.2) on the direction $\mathrm{e}^{i\theta(t)}$:

$$r''(t) - r(t)(\theta'(t))^2 = -\frac{1}{\epsilon^2}\partial_r V(r(t)) - \partial_r W(r(t)\mathrm{e}^{i\theta(t)}). \tag{2.9}$$

The system of equations (2.3), (2.9) is equivalent to the system (2.1). Projecting on $\mathrm{e}^{i\theta(t)}$ is mainly a question of convenience, one could consider another direction complementary to $i\mathrm{e}^{i\theta(t)}$.

The Lyapunov-Schmidt reduction is a general approach to a wide variety of problems of Asymptotic Analysis, consisting in a systematic elimination of some variables, in a specific order, using the Contraction Principle (or the Implicit Function Theorem) at each step. The book [17] contains a general discussion of the method in Section 2.4, and numerous examples.

If my goal is to reduce the full system to an equation involving only the slow variable θ, then I can proceed as follows:

- for any given function θ, find the solution $r = r(\theta)$ of the projected equation (2.9),
- inject $r = r(\theta)$ to the modulation equation, thus obtaining an equation for θ only.

This last equation is called the *bifurcation equation*. Some authors also use the name *determining equation*. It is, in general, non-local (in time). Directly from the procedure described above, (r, θ) is a solution if and only if θ solves the bifurcation equation and $r = r(\theta)$.

Remark 2.4. A natural and powerful method of the two-scale analysis that one could try to apply here is *averaging*. The general idea consists in solving (2.9) for *frozen* values of θ and θ', and computing average in time values of the fast variables r and r', denoted $\bar r(\theta, \theta')$ and $\bar r'(\theta, \theta')$. Then, these average values are injected into the modulation equation instead of r and r', yielding a local in time system for θ and θ' only.

This scheme could be difficult to apply in the multi-soliton setting. One usually uses ergodic theorems to compute average quantities, which is not immediate in the PDE context. Also, averaging leads to an improvement of the modulation equation, but remains an approximate method. The Lyapunov-Schmidt method leads to bifurcation equations which are exactly equivalent to the original system.

Let me present an application of the Lyapunov-Schmidt approach to the toy model (2.1). First, I claim that there exists a unique solution ϑ of the formal system (2.4) starting at $\vartheta(0) = 0$, moving counterclockwise, and after time 1 going back to $\vartheta(1) = 2\pi$. Indeed, let $\widetilde{E}_0 > \max_\theta \widetilde{W}(\theta)$ be the unique number satisfying

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{2(\widetilde{E}_0 - \widetilde{W}(\theta))}} = 1. \tag{2.10}$$

I set $\vartheta(0) := 0$ and $\vartheta'(0) := \sqrt{2(\widetilde{E}_0 - \widetilde{W}(0))}$, so that the corresponding solution ϑ has energy $\widetilde{E}_0$ and initially moves counterclockwise. Separating variables, one finds $\vartheta(1) = 2\pi$. Conversely, if ϑ is an increasing solution such that $\vartheta(1) = 2\pi$, then a separation of variables shows that its energy is determined by (2.10).

I denote $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ the "time circle". One-periodic functions of time are often considered as being defined on $\mathbb{T}$. Sobolev spaces on the circle are denoted $H^s(\mathbb{T})$.

Theorem 2.5. *For any $\delta > 0$ there exists $\eta > 0$ such that if ϵ is small enough and satisfies the* non-resonance condition

$$\min_{n\in\mathbb{Z}} \left|\epsilon^2 - (2\pi n)^{-2}\partial_r^2 V(1)\right| \geq \delta\epsilon^3, \tag{2.11}$$

then the system (2.1) *has exactly one* 1*-periodic solution* (r, θ) *making one turn along the unit circle in the counterclockwise direction whose energy does not exceed* $\widetilde{E}_0 + \eta$.

Moreover there exists C, depending on W and V but independent of ϵ and δ, such that

$$\|r - 1\|_{H^1(\mathbb{T})} + \|\theta - \vartheta\|_{H^2(\mathbb{T})} \leq C\epsilon^2. \tag{2.12}$$

This is a situation where the Lyapunov-Schmidt method is particularly well-suited. We are not only interested in the description of the asymptotic behaviour of the slow variable, but rather in the classification of all the solutions such that the slow variable and the energy satisfy some (natural) condition.

Remark 2.6. I do not know if Theorem 2.5 appears in the literature with the exact same formulation. A similar result was obtained by Shatah and Zeng [96], whose proof was also based on the Contraction Principle, but the elimination of variables was done in the reverse order. The authors considered a more general situation and assumed a weaker non-resonance condition, but obtained uniqueness only in a neighbourhood of the constructed solution whose size in the energy norm depends on ϵ (I am grateful to Chongchun Zeng for his help with understanding this issue.) Here, since the discussion in this section serves mainly as an illustration of the Lyapunov-Schmidt scheme, I chose the formulation which bears, in my opinion, the closest analogy with the multi-soliton setting presented in subsequent chapters.

Remark 2.7. Let $A(\delta) := \{\epsilon : (2.11) \text{ holds}\}$. I claim that there exists $C < \infty$ such that for ϵ_0 small enough

$$\epsilon_0^{-1}\big|[0, \epsilon_0] \cap A(\delta)\big| \geq 1 - C\delta, \tag{2.13}$$

where $|\cdot|$ denotes the Lebesgue measure. Thus, if δ is small, then "most" numbers are non-resonant.

In order to prove (2.13), consider the interval $\epsilon \in \big[(2\pi(n+1))^{-1}\sqrt{\partial_r^2 V(1)}, (2\pi n)^{-1}\sqrt{\partial_r^2 V(1)}\big]$. Let ϵ_1 and ϵ_2 be the smallest positive solutions of

$$\epsilon_1^2 - (2\pi(n+1))^{-2}\partial_r^2 V(1) = \delta\epsilon_1^3, \qquad \epsilon_2^2 - (2\pi n)^{-2}\partial_r^2 V(1) = -\delta\epsilon_2^3.$$

Then every $\epsilon \in [\epsilon_1, \epsilon_2]$ is non-resonant. By an elementary computation,

$$\epsilon_1 \leq (2\pi(n+1))^{-1}\sqrt{\partial_r^2 V(1)}(1 + C\delta n^{-1}), \quad \epsilon_2 \geq (2\pi n)^{-1}\sqrt{\partial_r^2 V(1)}(1 - C\delta n^{-1}),$$

which implies (modifying C if necessary)

$$(\epsilon_2 - \epsilon_1) \geq (1 - C\delta)\big((2\pi n)^{-1} - (2\pi(n+1))^{-1}\big)\sqrt{\partial_r^2 V(1)}.$$

Summing over n, I get (2.13).

Remark 2.8. Before treating the general case, it is instructive to consider the simpler situation $W \equiv 0$, which yields an integrable system, thus amenable to an explicit analysis.

Let $\epsilon > 0$ be small and r^* be the unique solution in a neighborhood of $r = 1$ of the equation $(r^*)^{-1}\partial_r V(r^*) = 4\pi^2\epsilon^2$. Then $(r(t), \theta(t)) = (r^*, 2\pi t)$ yields a 1-periodic solution of the system, regardless of the non-resonance condition. Note that $|r^* - 1| \lesssim \epsilon^2$.

The system can be integrated using the conservation of angular momentum $r(t)^2\theta'(t) = L_0 =$ const, which, inserted into (2.9), yields the following equation for r:

$$r''(t) = L_0^2 r(t)^{-3} - \epsilon^{-2}\partial_r V(r(t)).$$

By computing the fundamental period of this system, it can be checked that if $\eta > 0$ is small enough, and ϵ is small and non-resonant, then there are no 1-periodic solutions of energy $\leq 2\pi^2 + \eta$, except for the constant one found above. However, at larger energies or for resonant ϵ, 1-periodic solutions can exist (this can be confirmed by a rather lengthy but explicit computation).

Remark 2.9. I expect that, at the cost of some more computation, in (2.12) one could replace $H^1(\mathbb{T})$ and $H^2(\mathbb{T})$ by any Sobolev norms.

Lemma 2.10. *If (r,θ) is a solution satisfying the conditions of Theorem 2.5 and $\theta(0) = 0$, then*

$$\|\theta - \vartheta\|_{L^\infty} + \|\theta' - \vartheta'\|_{L^\infty} \lesssim \epsilon, \qquad \|r-1\|_{L^\infty} \lesssim \epsilon\sqrt{\eta} + \epsilon^{3/2}, \qquad \|r'\|_{L^\infty} \lesssim \sqrt{\eta} + \sqrt{\epsilon}. \tag{2.14}$$

Proof. As in the previous section, let $\widetilde\theta$ be the solution of the formal modulation equation (2.4) for the initial data $(\widetilde\theta(0), \widetilde\theta'(0)) = (0, r(0)^2\theta'(0))$. Let $\widetilde E_1 := \frac{1}{2}(\widetilde\theta'(0))^2 + \widetilde W(0)$ be its energy. Since $\theta(1) = 2\pi$, by (2.8) I have $|\widetilde\theta(1) - 2\pi| \lesssim \epsilon$. By a change of variables, I obtain $\widetilde E_1 \geq \max_{\theta\in[0,\widetilde\theta(1)]}\widetilde W(\theta)$ and

$$\int_0^{\widetilde\theta(1)} \frac{\mathrm{d}\theta}{\sqrt{2(\widetilde E_1 - \widetilde W(\theta))}} = 1.$$

Recalling (2.10), I thus have

$$\Big|\int_0^{\widetilde\theta(1)}\Big(\frac{1}{\sqrt{2(\widetilde E_1 - \widetilde W(\theta))}} - \frac{1}{\sqrt{2(\widetilde E_0 - \widetilde W(\theta))}}\Big)\mathrm{d}\theta\Big| \leq \Big|\int_{\widetilde\theta(1)}^{2\pi}\frac{\mathrm{d}\theta}{\sqrt{2(\widetilde E_0 - \widetilde W(\theta))}}\Big| \lesssim \epsilon,$$

which implies $|\widetilde E_1 - \widetilde E_0| \lesssim \epsilon$. Invoking smooth dependence on initial conditions, I have $\|\vartheta - \widetilde\theta\|_{L^\infty([0,1])} + \|\vartheta' - \widetilde\theta'\|_{L^\infty([0,1])} \lesssim \epsilon$. Note that $\theta - \vartheta$ is a 1-periodic function. The triangle inequality yields $\|\theta - \vartheta\|_{L^\infty} + \|\theta' - \vartheta'\|_{L^\infty} \lesssim \epsilon$.

The last bound and (2.7) imply

$$\frac{1}{2}(r(t)\theta'(t))^2 + W(r(t)\mathrm{e}^{i\theta(t)}) = \frac{1}{2}(\vartheta'(t))^2 + W(\mathrm{e}^{i\vartheta(t)}) + O(\epsilon) = \widetilde E_0 + O(\epsilon).$$

Therefore, (2.6) and the energy constraint $E \leq \widetilde E_0 + \eta$ yield

$$\epsilon^{-2}|r(t) - 1|^2 + |r'(t)|^2 \lesssim \eta + \epsilon.$$

□

From now on, I will assume for the sake of simplicity that $\epsilon \leq \eta \leq \delta \leq 1$, which I can do because η is allowed to depend on δ and I always consider ϵ sufficiently small. Writing $r = 1 + \rho$ and $\theta = \vartheta + \xi$, (2.14) becomes

$$\|\xi\|_{L^\infty} + \|\xi'\|_{L^\infty} \lesssim \epsilon, \tag{2.15}$$

$$\|\rho\|_{L^\infty} \lesssim \epsilon\sqrt{\eta}, \qquad \|\rho'\|_{L^\infty} \lesssim \sqrt{\eta}, \tag{2.16}$$

and (2.9) becomes

$$\rho'' - (1+\rho)(\vartheta' + \xi')^2 = -\frac{1}{\epsilon^2}\partial_r V(1+\rho) - \partial_r W((1+\rho)\mathrm{e}^{i(\vartheta+\xi)}). \tag{2.17}$$

I can rewrite the modulation equation (2.3) as follows:

$$\vartheta'' + \xi'' = -2(1+\rho)^{-1}\rho'(\vartheta' + \xi') - \frac{1}{(1+\rho)^2}\partial_\theta W((1+\rho)\mathrm{e}^{i(\vartheta+\xi)}). \tag{2.18}$$

In particular, the bounds (2.15) and (2.16), together with the fact that $\vartheta'' = -\widetilde{W}'(\vartheta)$, imply

$$\|\xi\|_{H^2(\mathbb{T})} \lesssim \sqrt{\eta}. \tag{2.19}$$

Lemma 2.11. *Assume that ϵ satisfies the non-resonance condition* (2.11)*.*

- *For any $s \geq 0$ and for $\epsilon > 0$ small enough, the equation*

$$\rho'' = -\epsilon^{-2}\partial_r^2 V(1)\rho + f \tag{2.20}$$

defines a bounded linear operator $H^s(\mathbb{T}) \ni f \mapsto \rho \in H^s(\mathbb{T})$, and

$$\|\rho\|_{H^s} \lesssim \delta^{-1}\epsilon\|f\|_{H^s}. \tag{2.21}$$

- *For any $s \geq 0$, $s' > s+1$ and $\epsilon > 0$ small enough, the equation* (2.20) *defines a bounded linear operator $H^{s'}(\mathbb{T}) \ni f \mapsto \rho \in H^s(\mathbb{T})$ and*

$$\|\rho\|_{H^s} \lesssim \epsilon^2\|f\|_{H^{s'}}. \tag{2.22}$$

Proof. Taking the Fourier transform, $\rho(t) = \sum_{n\in\mathbb{Z}} \widehat{\rho}(n)\mathrm{e}^{2\pi int}$ and $f(t) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\mathrm{e}^{2\pi int}$, I obtain

$$\widehat{\rho}(n) = \frac{-1}{4\pi^2 n^2 - \epsilon^{-2}\partial_r^2 V(1)}\widehat{f}(n).$$

Condition (2.11) implies $|4\pi^2 n^2 - \epsilon^{-2}\partial_r^2 V(1)| \gtrsim \delta\epsilon^{-1}$ for all $n \in \mathbb{Z}$, hence (2.21) follows from the Plancherel theorem.

In order to prove (2.22), it suffices to check that

$$\inf_{n\in\mathbb{Z}} \left[(1+n^2)^{\frac{s'-s}{2}}\left|4\pi^2 n^2 - \epsilon^{-2}\partial_r^2 V(1)\right|\right] \gtrsim \epsilon^{-2}, \qquad \text{for } \epsilon \text{ small enough.} \tag{2.23}$$

If $4\pi^2 n^2 \leq \frac{1}{2}\epsilon^{-2}\partial_r^2 V(1)$, then the bound is clear. In the opposite case, one has $(1+n^2)^{\frac{s'-s}{2}} \gtrsim \epsilon^{-(s'-s)}$, thus, using the non-resonance condition,

$$(1+n^2)^{\frac{s'-s}{2}}\left|4\pi^2 n^2 - \epsilon^{-2}\partial_r^2 V(1)\right| \gtrsim \delta\epsilon^{-(s'-s+1)},$$

which implies (2.23) provided that $\epsilon \leq \delta^{\frac{1}{s'-s-1}}$. □

Lemma 2.12. *For any $\delta > 0$ and $\eta > 0$ small enough (depending on δ) the following is true. If ϵ is small enough and satisfies* (2.11)*, then for any ξ satisfying* (2.19)*, the equation* (2.17) *has a unique solution $\rho = \rho(\xi) \in H^1(\mathbb{T})$ satisfying* (2.16)*. Moreover, the following bounds hold:*

$$\|\rho(0)\|_{H^1} \lesssim \epsilon^2 \tag{2.24}$$

and, for all $\xi, \xi^\sharp$ satisfying (2.19)*,*

$$\|\rho(\xi^\sharp) - \rho(\xi)\|_{H^1} \lesssim \delta^{-1}\epsilon\|\xi^\sharp - \xi\|_{H^2}. \tag{2.25}$$

Proof. I reformulate the statement as a fixed point problem. For $\rho \in H^s(\mathbb{T})$ and $\xi \in H^{s+1}(\mathbb{T})$, set

$$f(\rho, \xi) := (1+\rho)(\vartheta' + \xi')^2 - \partial_r W((1+\rho)\mathrm{e}^{i(\vartheta+\xi)}) - \epsilon^{-2}\partial_r V(1+\rho) + \epsilon^{-2}\partial_r^2 V(1)\rho.$$

Then (2.17) is equivalent to

$$\rho'' = -\epsilon^{-2}\partial_r^2 V(1)\rho + f(\rho, \xi). \tag{2.26}$$

I denote, only in this proof, $f_2(\rho) := \partial_r V(1+\rho) - \partial_r^2 V(1)\rho$, thus f_2 is a smooth function such that $f_2(0) = f_2'(0) = 0$. Let me recall a standard method of estimating $f_2(\rho)$ in the Sobolev norms. I will check that

$$\big\|f_2(\rho)\big\|_{H^1} \lesssim \|\rho\|_{L^\infty}\|\rho\|_{H^1}. \tag{2.27}$$

Indeed, by Taylor's theorem I have, for all $t \in \mathbb{T}$,

$$f_2(\rho(t)) = \rho(t)\int_0^1 f_2'(\rho(t)\sigma)\,\mathrm{d}\sigma.$$

Since $|f_2'(\rho)| \lesssim |\rho|$, it is clear that $\big\|f_2(\rho)\big\|_{L^2} \lesssim \|\rho\|_{L^\infty}\|\rho\|_{L^2}$. Using the Product Rule and differentiating under the sign of the integral, I have

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}f_2(\rho(t))\Big| = \Big|\rho'(t)\int_0^1 f_2'(\rho(t)\sigma)\,\mathrm{d}\sigma + \rho(t)\rho'(t)\int_0^1 \sigma f_2''(\rho(t)\sigma)\,\mathrm{d}\sigma\Big| \lesssim |\rho(t)||\rho'(t)|,$$

and I deduce (2.27).

Let $\rho, \rho^\sharp \in H^s(\mathbb{T})$ such that $\|\rho\|_{H^1} + \|\rho^\sharp\|_{H^1} \lesssim \sqrt{\eta}$. In order to bound $f_2(\rho^\sharp) - f_2(\rho)$, I write

$$f_2(\rho^\sharp(t)) - f_2(\rho(t)) = (\rho^\sharp(t) - \rho(t))\int_0^1 f_2'\big(\rho(t)(1-\sigma) + \rho^\sharp(t)\sigma\big)\,\mathrm{d}\sigma.$$

The Product Rule yields

$$\big\|f_2(\rho^\sharp(t)) - f_2(\rho(t))\big\|_{H^1} \lesssim \|\rho^\sharp(t) - \rho(t)\|_{H^1}\Big\|\int_0^1 f_2'\big(\rho(t)(1-\sigma) + \rho^\sharp(t)\sigma\big)\,\mathrm{d}\sigma\Big\|_{H^1}.$$

Differentiating under the integral sign, I get

$$\Big\|\int_0^1 f_2'\big(\rho(t)(1-\sigma) + \rho^\sharp(t)\sigma\big)\,\mathrm{d}\sigma\Big\|_{H^1} \lesssim \|\rho\|_{H^1} + \|\rho^\sharp\|_{H^1},$$

thus

$$\|f_2(\rho^\sharp) - f_2(\rho)\|_{H^1} \lesssim \|\rho^\sharp - \rho\|_{H^1}\big(\|\rho\|_{H^1} + \|\rho^\sharp\|_{H^1}\big). \tag{2.28}$$

I denote, in the remaining part of this proof,

$$f_1(t,\rho,\xi,\dot\xi) := (1+\rho)(\vartheta'(t)+\dot\xi)^2 - \partial_r W((1+\rho)\mathrm{e}^{i(\vartheta(t)+\xi)}),$$

thus f_1 is a smooth function of $(t,\rho,\xi,\dot\xi)\in\mathbb{R}^4$. The following identity holds:

$$\begin{aligned}
&f_1(t,\rho^\sharp,\xi^\sharp,\dot\xi^\sharp) - f_1(t,\rho,\xi,\dot\xi) \\
&= (\rho^\sharp-\rho)\int_0^1 \partial_\rho f_1(t,\rho(1-\sigma)+\rho^\sharp\sigma,\xi(1-\sigma)+\xi^\sharp\sigma,\dot\xi(1-\sigma)+\dot\xi^\sharp\sigma)\,\mathrm{d}\sigma \\
&+ (\xi^\sharp-\xi)\int_0^1 \partial_\xi f_1(t,\rho(1-\sigma)+\rho^\sharp\sigma,\xi(1-\sigma)+\xi^\sharp\sigma,\dot\xi(1-\sigma)+\dot\xi^\sharp\sigma)\,\mathrm{d}\sigma \\
&+ (\dot\xi^\sharp-\dot\xi)\int_0^1 \partial_{\dot\xi} f_1(t,\rho(1-\sigma)+\rho^\sharp\sigma,\xi(1-\sigma)+\xi^\sharp\sigma,\dot\xi(1-\sigma)+\dot\xi^\sharp\sigma)\,\mathrm{d}\sigma.
\end{aligned}$$

Applying the Product Rule, differentiating under the integral sign and using the fact that $\|\rho\|_{H^1}+\|\rho^\sharp\|_{H^1}+\|\xi\|_{H^2}+\|\xi^\sharp\|_{H^2}\lesssim\sqrt\eta\lesssim 1$, I get

$$\big\|f_1(t,\rho^\sharp(t),\xi^\sharp(t),\xi'^\sharp(t)) - f_1(t,\rho(t),\xi(t),\xi'(t))\big\|_{H^1} \lesssim \|\rho^\sharp-\rho\|_{H^1}+\|\xi^\sharp-\xi\|_{H^2}. \tag{2.29}$$

Combining the last bound with (2.28), I have

$$\|f(\rho^\sharp,\xi^\sharp)-f(\rho,\xi)\|_{H^1}\lesssim \|\rho^\sharp-\rho\|_{H^1}\big(1+\epsilon^{-2}\big(\|\rho\|_{H^1}+\|\rho^\sharp\|_{H^1}\big)\big)+\|\xi^\sharp-\xi\|_{H^2}. \tag{2.30}$$

The bound (2.29), combined with (2.27), yields

$$\|f(\rho,\xi)-f(0,0)\|_{H^1}\lesssim \|\rho\|_{H^1}+\epsilon^{-2}\|\rho\|_{L^\infty}\|\rho\|_{H^1}+\|\xi\|_{H^2}. \tag{2.31}$$

Let ρ be a solution of (2.26) satisfying (2.16). Let ρ_1 be the solution of $\rho_1''=-\epsilon^{-2}\partial_r^2V(1)\rho_1+f(0,0)$ and ρ_2 the solution of $\rho_2''=-\epsilon^{-2}\partial_r^2V(1)\rho_2+f(\rho,\xi)-f(0,0)$, so that $\rho=\rho_1+\rho_2$. The bound (2.22), applied with $s=1$ and any $s'>2$, yields $\|\rho_1\|_{H^1}\lesssim\epsilon^2$. The bound (2.31), together with (2.16) and (2.19), yields $\|f(\rho,\xi)-f(0,0)\|_{H^1}\lesssim\sqrt\eta+\epsilon^{-1}\sqrt\eta\|\rho\|_{H^1}$. Let $c_0>0$. By (2.21), I get

$$\|\rho_2\|_{H^1}\lesssim \delta^{-1}\epsilon\big(\eta^{1/2}+\epsilon^{-1}\eta^{1/2}(\|\rho_2\|_{H^1}+\epsilon^2)\big)\ \Rightarrow\ \|\rho\|_{H^1}\le c_0\delta\epsilon, \tag{2.32}$$

as long as I assume $\eta\ll\delta^4$, which is allowed. Note that the last bound is a significant improvement with respect to (2.16).

I claim that if $c_0>0$ is sufficiently small, then (2.26) defines a contraction on the ball $\|\rho\|_{H^1}\le c_0\delta\epsilon$. The invariance of the ball follows from the computation above. Let ρ and $\rho^\sharp$ be two elements of the ball. The bound (2.30) yields

$$\|f(\rho^\sharp,\xi)-f(\rho,\xi)\|_{H^1}\lesssim c_0\delta\epsilon^{-1}\|\rho^\sharp-\rho\|_{H^1},$$

and it suffices to apply (2.21).

This finishes the proof of existence and uniqueness of $\rho=\rho(\xi)$.

In the case $\xi=0$, (2.31) yields $\|f(\rho,0)-f(0,0)\|_{H^1}\lesssim\|\rho\|_{H^1}+\epsilon^{-2}\|\rho\|_{L^\infty}\|\rho\|_{H^1}$, thus instead of (2.32), I have

$$\|\rho_2\|_{H^1}\lesssim\delta^{-1}\eta^{1/2}(\|\rho_2\|_{H^1}+\epsilon^2)\ \Rightarrow\ \|\rho_2\|_{H^1}\le c_0\delta\epsilon^2\ \Rightarrow\ \|\rho\|_{H^1}\lesssim\epsilon^2,$$

which is (2.24).

Let ξ and $\xi^\sharp$ satisfy (2.19), and set

$$\rho := \rho(\xi), \quad \rho^\sharp := \rho(\xi^\sharp), \quad \rho^\flat := \rho^\sharp - \rho,$$

so that

$$(\rho^\flat)'' = -\epsilon^{-2}\partial_r^2 V(1)\rho^\flat + (f(\rho^\sharp, \xi^\sharp) - f(\rho, \xi)).$$

Since $\|\rho\|_{H^1} \le c_0\delta\epsilon$ and $\|\rho^\sharp\|_{H^1} \le c_0\delta\epsilon$, (2.30) yields

$$\|f(\rho^\sharp, \xi^\sharp) - f(\rho, \xi)\|_{H^1} \lesssim (1 + c_0\delta\epsilon^{-1})\|\rho^\flat\|_{H^1} + \|\xi^\sharp - \xi\|_{H^2},$$

and (2.21) allows me to conclude, for ϵ small enough, that

$$\|\rho^\flat\|_{H^1} \lesssim c_0\|\rho^\flat\|_{H^1} + \delta^{-1}\epsilon\|\xi^\sharp - \xi\|_{H^2},$$

hence (2.25). $\square$

Let $\vartheta(t)$ be the solution of the formal modulation equation (2.4) such that ϑ is strictly increasing, $\vartheta(0) = 0$ and $\vartheta(1) = 2\pi$. By elementary properties of systems with one degree of freedom, these conditions determine ϑ uniquely. I decompose

$$\theta(t) = \vartheta(t) + \xi(t).$$

My objective is to find all the 1-periodic functions ξ such that $\xi(0) = 0$, solving the bifurcation equation, and such that the energy constraint is satisfied.

The bifurcation equation will be solved using the Contraction Principle. To this end, I first consider the corresponding linear non-homogeneous problem.

Lemma 2.13. *For any $f \in L^2(\mathbb{T})$, the equation*

$$(L\xi)(t) := \xi''(t) + \widetilde{W}''(\vartheta(t))\xi(t) = f(t) + a\vartheta'(t) \tag{2.33}$$

has a unique solution $(\xi, a) \in H^2(\mathbb{T}) \times \mathbb{R}$ such that $\xi(0) = 0$. The mapping $f \mapsto \xi$ defines a bounded linear operator $L^2(\mathbb{T}) \to H^2(\mathbb{T})$.

Proof. First, note that the presence of the term $a\vartheta'(t)$ is forced by the fact that $\ker L = \operatorname{span}(\vartheta')$. Indeed, differentiating with respect to t the identity $\vartheta''(t) + \widetilde{W}'(\vartheta(t)) = 0$, one obtains $L(\vartheta') = 0$. Thus, the right hand side of (2.33) has to be orthogonal to ϑ'. The number a is thus determined by the condition

$$\int_0^1 \big(f(t) + a\vartheta'(t)\big)\vartheta'(t)\,\mathrm{d}t = 0 \;\Leftrightarrow\; a = -\bigg(\int_0^1 (\vartheta'(t))^2\,\mathrm{d}t\bigg)^{-1}\int_0^1 \vartheta'(t)f(t)\,\mathrm{d}t,$$

in particular $|a| \lesssim \|f\|_{L^2}$. The function ϑ' is positive, and standard arguments (variation of constants) imply that 0 is a simple eigenvalue. Thus, the operator L, restricted to the orthogonal complement in $L^2(\mathbb{T})$ of ϑ', which is its invariant subspace, is invertible. I denote $\zeta = L^{-1}f$ the function $\zeta \in L^2(\mathbb{T})$ such that

$$\zeta''(t) + \widetilde{W}''(\vartheta(t))\zeta(t) = f(t) + a\vartheta'(t), \qquad \int_0^1 \vartheta'(t)\zeta(t)\,\mathrm{d}t = 0,$$

thus $\|\zeta\|_{L^2} \lesssim \|f\|_{L^2}$. From the differential equation I have $\|\zeta\|_{H^2} \lesssim \|f\|_{L^2}$, in particular $|\zeta(0)| \lesssim \|f\|_{L^2}$. The function ξ is given by $\xi = \zeta + b\vartheta'$, where b is the unique number such that $\xi(0) = 0$. $\square$

Proof of Theorem 2.5. I formulate the result as a fixed-point problem. With given $\xi \in H^2(\mathbb{T})$ such that $\|\xi\|_{H^2} \lesssim \sqrt{\eta} \ll 1$, I associate $\zeta = Z(\xi) \in H^2(\mathbb{T})$, the solution of

$$\zeta'' + \widetilde{W}''(\mathrm{e}^{i\vartheta})\zeta = f(\xi) + a\vartheta',$$

where

$$f(\xi) := -\frac{2(\rho(\xi))'}{1+\rho(\xi)}(\vartheta' + \xi') - \left(\frac{1}{(1+\rho(\xi))^2}\partial_\theta W\big((1+\rho(\xi))\mathrm{e}^{i(\vartheta+\xi)}\big) - \widetilde{W}'(\mathrm{e}^{i\vartheta}) - \widetilde{W}''(\mathrm{e}^{i\vartheta})\xi\right),$$

so that ξ solves (2.18) if and only if it is a fixed point and $a = 0$. I claim that

$$\|f(0)\|_{L^2} \lesssim \epsilon^2, \qquad \|f(\xi^\sharp) - f(\xi)\|_{L^2} \lesssim \|\xi^\sharp - \xi\|_{H^2}\big(\delta^{-1}\epsilon + \|\xi\|_{H^2} + \|\xi^\sharp\|_{H^2}\big). \tag{2.34}$$

The first bound follows from (2.24). In order to prove the second bound, I set $\rho := \rho(\xi)$, $\rho^\sharp := \rho(\xi^\sharp)$,

$$\begin{aligned}
f_1 &:= -\frac{2\rho'}{1+\rho}(\vartheta'(t) + \xi'), \\
f_2 &:= \frac{1}{(1+\rho)^2}\partial_\theta W\big((1+\rho)\mathrm{e}^{i(\vartheta(t)+\xi)}\big) - \widetilde{W}'(\mathrm{e}^{i\vartheta(t)}) - \widetilde{W}''(\mathrm{e}^{i\vartheta(t)})\xi,
\end{aligned}$$

and analogously $f_1^\sharp$, $f_2^\sharp$, with $(\xi^\sharp, \rho^\sharp)$ instead of (ξ, ρ). From the Taylor formula and the triangle inequality, one gets

$$\|f_1^\sharp - f_1\|_{L^2} \lesssim \|\rho^\sharp - \rho\|_{H^1} + \big(\|\rho\|_{H^1} + \|\rho^\sharp\|_{H^1}\big)\|(\xi^\sharp)' - \xi'\|_{L^\infty},$$

and

$$\|f_2^\sharp - f_2\|_{L^2} \lesssim \|\rho^\sharp - \rho\|_{L^\infty} + \big(\|\rho\|_{L^\infty} + \|\rho^\sharp\|_{L^\infty} + \|\xi\|_{L^\infty} + \|\xi^\sharp\|_{L^\infty}\big)\|\xi^\sharp - \xi\|_{L^\infty}.$$

Taking the sum and using (2.25), I obtain the second inequality in (2.34).

Let ξ be the unique fixed point of Z satisfying (2.19). The first bound in (2.34) implies that Z is a contraction on a ball of radius $C\epsilon^2$ for C large enough, so in fact $\|\xi\|_{H^2} \lesssim \epsilon^2$, which, by (2.24), implies $\|\rho(\xi)\|_{H^1} \lesssim \epsilon^2$. Let $\theta := \vartheta + \xi$, $r := 1 + \rho(\xi)$. Thus, by construction of r and θ, (2.9) holds and

$$r(t)\theta''(t) + 2r'(t)\theta'(t) = -\frac{1}{r(t)}\partial_\theta W(r(t)\mathrm{e}^{i\theta(t)}) + ar(t)\vartheta'(t).$$

Define $E(t)$ by the formula (2.6). One finds, by a direct computation, $E'(t) = ar(t)^2\theta'(t)\vartheta'(t)$. Since both θ and ϑ are increasing functions, and E is periodic, necessarily $a = 0$, which means that (r, θ) solves (2.3), hence is a 1-periodic solution of the system. $\square$

A few concluding remarks are in order. For the example considered here, a part of the conclusion can be deduced, at least informally, from the KAM theory. Indeed, for $W = 0$ the system is completely integrable. Since it has two degrees of freedom, the invariant tori form nested families on each energy surface, see for example [10, p. 402]. It can be checked that the periodic orbit corresponds to the invariant torus degenerating to a circle, lying inside all of them. By the KAM theory, "most" tori are preserved under the perturbation by $W \neq 0$. Each of them delimits an invariant set, hence one gets a nested family of invariant sets, and the desired invariant circle is obtained as their intersection. The non-resonance condition is not required.

Let me observe, however, that this argument has the drawback of not being applicable to higher dimensions, and of not providing uniqueness or any other precise information on the constructed solution. In this degenerate situation, the Lyapunov-Schmidt method presented above seems more appropriate. Quoting Bourgain [15], "The Liapounov-Schmidt method is more flexible, however, than KAM, and works also in certain degenerate cases."

I would also like to make a side remark about a difference in the intuition related to both approaches. In the KAM approach, the "reference system" is the system containing *only* the constraining potential (that is, $W = 0$), and the full system is treated as its perturbation. In the problem of evolution under a strong constraining potential, it is preferable to think of the system restricted to the zero-set of the constraining potential (but with $W \neq 0$) as the reference system, and treat the full system as its perturbation in some sense. As the bifurcation equations are closely related to the restricted system, the Lyapunov-Schmidt approach better corresponds to this intuition.

Finally, I expect that existence of a 1-periodic solution, even without the non-resonance condition, could be obtained by a variational argument, though I have not checked this. Such an argument would present the same disadvantages of not clarifying the questions of uniqueness or asymptotic behaviour. In general, when applicable, methods based on the Contraction Principle are the most efficient.

2.3 Profile decomposition

As is clear from Section 1.2, variational arguments in the presence of a non-compact symmetry group are going to play an important role in the study of topological solitons. A very powerful tool in such situations is the celebrated *profile decomposition method* due to Lions [62, 63], Brezis and Coron [16], Gérard [41], Schindler and Tintarev [91], Bahouri and Gérard [12], and Merle and Vega [75]. Since the theory is nowadays very well-known, I will not discuss it here.

Closely related is another rather general method, the *Kenig-Merle route map* [52, 53], which plays a crucial role in the proofs of Theorem 4 and Theorem 5, see Section 4.3.

Chapter 3

The n-soliton problem for scalar fields

It has been observed that solitons have many properties of what one usually requires from a *particle*. One can often define their position, momentum, mass and energy. From this viewpoint, a multi-soliton can be regarded as a set of interacting particles. Therefore, describing the evolution of such an object should bear an analogy with the n-body problem.

I would like to stress that an approximation by an n-body problem is not always possible or useful. For instance, one should not expect any useful approximation unless the relativistic effects are small, because of "no interaction" theorems in Special Relativity. Nevertheless, it seems reasonable to explore the analogy further.

One can propose the following scheme of analysing widely separated multi-solitons, consisting of three almost independent ingredients:

- derive and solve the corresponding n-body problem,
- prove that the full system behaves at main order like the corresponding n-body problem,
- verify that the stable/unstable manifold structure of the n-body problem survives in the full problem.

In accomplishing the first step, one can take inspiration from the study of the gravitational n-body problem, resumed in Section 3.A. In the second step, the modulation method described in Section 2.1 comes into play. The third step can be attacked in various ways, one natural approach being to adapt the method of Lyapunov-Schmidt presented in the previous chapter.

Remark 3.1. The justification of an approximation of solutions of some PDE by a system of point masses is a problem which appears in many contexts other than the wave equations. Let me mention the works on the Ginzburg-Landau gradient flow [48, 49, 14], on the equations of fluid mechanics [13], on Bose-Einstein condensates [47, 50], but this list is of course far from being exhaustive.

3.1 Formal evolution of a multi-kink

I would like to derive a *formal modulation equation* for the evolution of the centres of kinks and antikinks under the evolution driven by their interactions. Let me recall that for the toy model analysed in Section 2.1, this was achieved by restricting the system to the "slow manifold", which led to the equation (2.4). I will apply the same technique here. In the multi-soliton context, it is called the *restriction method*.

I wish to consider the configurations consisting of kinks separated by a large distance plus a rest of small energy. Even though I will not study the effect of the "radiation", some of the results, as I explain in the sequel, are relevant for the general phase portrait of finite energy solutions.

For given $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, denote

$$H(\boldsymbol{\xi}, \boldsymbol{\beta}) := \frac{1 + (-1)^n}{2} + \sum_{k=1}^{n} (-1)^k H(\gamma_{\beta_k}(\cdot - \xi_k)) \in H^1(\mathbb{R}),$$

which is a superposition of moving kinks, see (1.12). The admissible domain for the positions $\boldsymbol{\xi}$ is

$$\Sigma = \Sigma_\Delta := \{\boldsymbol{\xi} \in \mathbb{R}^n : \xi_{k+1} - \xi_k > \Delta \text{ for all } 1 \leq k \leq n-1\}, \qquad \text{where } \Delta \gg 1.$$

As for the velocities, it is natural to consider $\boldsymbol{\beta} \in (-1, 1)^n$. It would be tempting to derive a differential system for $(\boldsymbol{\xi}(t), \boldsymbol{\beta}(t))$, but in fact in doing so I would encounter important difficulties, not only of technical nature. Any canonical system describing the evolution of $(\boldsymbol{\xi}(t), \boldsymbol{\beta}(t))$ should be invariant by Lorentz transformations, and it is known that under some natural conditions such systems do not exist, see for instance [109] for a discussion of this and related issues. My current understanding is that the best one can do is to assume *a priori* a division of the kink configuration into subsystems of kinks moving asymptotically with the same velocity, like in the Newtonian n-body problem (briefly considered below in Section 3.A) and then reduce each subsystem (for which the relativistic effects can be neglected) to an evolution of point masses. Probably, in accordance with the general principle that "in relativistic theories, interactions are transmitted by a field", one should not expect that the influence of distinct subsystems on each other can be sensibly expressed by an ordinary differential equation.

A reasonable direction of study is, I believe, to consider the setting of just one subsystem, which means that all the kinks move asymptotically with the same velocity. Upon applying a suitable Lorentz transform, I can assume this limit velocity equals 0. In such a situation, relativistic effects can be neglected and there is hope that the interactions between the kinks are correctly reflected by the formal n-body system.

In order to derive this system, I write

$$H(\boldsymbol{\xi}) := H(\boldsymbol{\xi}, 0) = \frac{1 + (-1)^n}{2} + \sum_{k=1}^{n} (-1)^k H(\cdot - \xi_k), \qquad \text{for } \boldsymbol{\xi} \in \Sigma_\Delta,$$

and I proceed to restricting the flow to the manifold

$$\mathcal{M} := \big\{ H(\boldsymbol{\xi}) \; : \; \boldsymbol{\xi} \in \Sigma_\Delta \big\}, \tag{3.1}$$

which is the slow manifold of the problem. Viewing $\boldsymbol{\xi}$ as a trajectory in Σ_Δ, I define the *reduced Lagrangian*

$$\widetilde{\mathscr{L}}(\boldsymbol{\xi}) := \mathscr{L}(H(\boldsymbol{\xi})).$$

Let me compute the corresponding reduced Lagrangian in the case $n = 2$, that is for a kink-antikink pair. I denote $a := \xi_2 - \xi_1$ the distance between the two kinks. It will be convenient to introduce the following notation for the L^2 inner product of functions of x:

$$\langle g, h \rangle := \int_{\mathbb{R}} g(x) h(x) \, \mathrm{d}x.$$

I set

$$F(a) := -\big\langle \partial_x H, U'(1 - H + H(\cdot - a)) + U'(H) - U'(H(\cdot - a))\big\rangle \tag{3.2}$$

and

$$E_p(a) := E_p(1 - H + H(\cdot - a)) - 2E_p(H),$$

which is the interaction energy of a kink and an antikink, placed at the distance a from each other. The following identity, which can be obtained either by integrating by parts or by a change of variable $x \mapsto a - x$, is worth noting:

$$F(a) = -\big\langle \partial_x H(\cdot - a), U'(1 - H + H(\cdot - a)) + U'(H) - U'(H(\cdot - a))\big\rangle.$$

By a direct computation one has, see [1, Lemma 2.9],

$$F(a) = \frac{\mathrm{d}}{\mathrm{d}a} E_p(a).$$

The asymptotic behaviour of F can be computed explicitly and is given by

$$|F(a) - 2\kappa^2 \mathrm{e}^{-a}| \lesssim a\mathrm{e}^{-2a}, \tag{3.3}$$

where κ is the constant defined by (1.13). Integration in a, together with the fact that $\lim_{a\to\infty} E_p(a) = 0$, yields

$$|E_p(a) + 2\kappa^2 \mathrm{e}^{-a}| \lesssim a\mathrm{e}^{-2a}. \tag{3.4}$$

Denote

$$H_j(t, x) := H(x - \xi_j(t)), \qquad j \in \{1, 2\}.$$

The reduced Lagrangian, after subtracting the (irrelevant) constant, is

$$\widetilde{\mathscr{L}}(\xi_1, \xi_2) = \frac{1}{2}\big((\xi_1')^2 + (\xi_2')^2\big)\|\partial_x H\|_{L^2}^2 - \xi_1'\xi_2'\langle \partial_x H_1, \partial_x H_2\rangle - E_p(\xi_2 - \xi_1),$$

and the corresponding Euler-Lagrange equations yield the following formal modulation equations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\partial_x H\|_{L^2}^2 \xi_1' - \langle \partial_x H_1, \partial_x H_2\rangle \xi_2'\big) &= \xi_1'\xi_2'\langle \partial_x^2 H_1, \partial_x H_2\rangle + F(\xi_2 - \xi_1),\\
\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\partial_x H\|_{L^2}^2 \xi_2' - \langle \partial_x H_1, \partial_x H_2\rangle \xi_1'\big) &= \xi_1'\xi_2'\langle \partial_x H_1, \partial_x^2 H_2\rangle - F(\xi_2 - \xi_1).
\end{aligned}$$

Observing that $\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_x H_1, \partial_x H_2\rangle = -\xi_1'\langle \partial_x^2 H_1, \partial_x H_2\rangle - \xi_2'\langle \partial_x H_1, \partial_x^2 H_2\rangle$, I obtain, equivalently,

$$\begin{aligned}
\|\partial_x H\|_{L^2}^2 \xi_1'' - \langle \partial_x H_1, \partial_x H_2\rangle \xi_2'' &= -(\xi_2')^2\langle \partial_x^2 H_1, \partial_x H_2\rangle + F(\xi_2 - \xi_1),\\
\|\partial_x H\|_{L^2}^2 \xi_2'' - \langle \partial_x H_1, \partial_x H_2\rangle \xi_1'' &= -(\xi_1')^2\langle \partial_x H_1, \partial_x^2 H_2\rangle - F(\xi_2 - \xi_1).
\end{aligned} \tag{3.5}$$

In both equations, the second term of the left hand side and the first term of the right hand side turn out to be negligible. Erasing them and using (3.3) leads to

$$\begin{aligned}
\|\partial_x H\|_{L^2}^2 \xi_1'' &\simeq 2\kappa^2 \mathrm{e}^{\xi_2 - \xi_1},\\
\|\partial_x H\|_{L^2}^2 \xi_2'' &\simeq -2\kappa^2 \mathrm{e}^{\xi_2 - \xi_1}.
\end{aligned}$$

These equations have a solution

$$\xi_1(t) = -\log(At), \qquad \xi_2(t) = \log(At), \qquad A := \sqrt{2}\|\partial_x H\|_{L^2}^{-1}\kappa,$$

in accordance with (1.15). Any other solution such that the separation tends to infinity and the velocities tend to 0 is its space-time translate.

The formal modulation equation for any $n \geq 2$ can be found by the same method and is given by

$$\|\partial_x H\|_{L^2}^2 \xi_j'' \simeq 2\kappa^2\big(\mathrm{e}^{\xi_{j+1}-\xi_j} - \mathrm{e}^{\xi_j - \xi_{j-1}}\big), \qquad j \in \{1, \ldots, n\}, \tag{3.6}$$

with the convention $\xi_0 = -\infty$ and $\xi_{n+1} = \infty$. Observe the similarity with the well-known *Toda system*. The essential difference lies in the sign of the interactions, which are attractive in (3.6) and repulsive in the Toda system.

Remark 3.2. In the general case of several subsystems, as explained above, I have to neglect the contribution of (exponentially decaying in time) cross terms between distinct subsystems, so the reduced Lagrangian can be computed for each subsystem separately, and the results summed up. The solutions of the Euler-Lagrange equation resemble a concatenation of subsystems, each with logarithmic distances found above. We can expect that, in the absence of radiation, multi-kinks behave in the same way.

In the next section, I give a detailed presentation of a joint work with Kowalczyk and Lawrie [2], where we found all the kink-antikink pairs. A generalisation to any n is under preparation, in a joint work with Andrew Lawrie.

3.2 Modulation analysis for kink-antikink pairs

Let me now explain how the modulation analysis, presented in Section 2.1 for a toy model, can be carried out in the case of multikinks. For the sake of simplicity, I will only consider the case of two kinks, in other words the kink-antikink pairs, even though all the analysis in this section can be adapted to the case of a general multikink, which is going to be clarified in a paper under preparation.

Let ϕ be a kink 2-cluster, according to Definition 1.15 with $n = 2$. By Lemma 1.16, ϕ is of the form

$$\phi(t, x) = 1 - H(x - \xi_1(t)) + H(x - \xi_2(t)) + g(t, x), \quad \text{with } \lim_{t\to\infty}\big(\|g(t)\|_{H^1} + \|\partial_t\phi(t)\|_{L^2}\big) = 0.$$

As in the previous section, I denote $H_j(t, x) := H(x - \xi_j(t))$ for $j \in \{1, 2\}$, so that $\phi(t) = 1 - H_1(t) + H_2(t) + g(t)$. The slow manifold, defined in general by the formula (3.1), in the case of kink-antikink pairs is given by

$$\mathcal{M} := \{1 - H(\cdot - \xi_1) + H(\cdot - \xi_2) : \xi_2 - \xi_1 \gg 1\}.$$

The set $\mathcal{M}$ is a two-dimensional smooth manifold of functions of Schwartz class, on which the pair (ξ_1, ξ_2) constitutes a natural system of coordinates. The purpose of the change of unknowns $\phi \mapsto (\xi_1, \xi_2, g)$ is to decompose the motion into a sum of a motion on $\mathcal{M}$ and a motion transversal to $\mathcal{M}$. This decomposition is obviously non-unique, and the standard way of correcting such an inconvenience is to impose *orthogonality conditions*. My choice is probably the most obvious one, namely to require g to be L^2-orthogonal to the tangent space to $\mathcal{M}$ at $1 - H(\cdot - \xi_1) + H(\cdot - \xi_2)$, in other words I require

$$\int_{\mathbb{R}} \partial_x H_1(t) g(t) \, \mathrm{d}x = \int_{\mathbb{R}} \partial_x H_2(t) g(t) \, \mathrm{d}x = 0. \tag{3.7}$$

Proposition 3.3. *There exists $\eta > 0$ such that the following is true. Let I be a finite or infinite time interval and ϕ a strong solution of* (CSF) *such that*

$$\operatorname{dist}_{H^1}(\phi(t), \mathcal{M}) \le \eta, \qquad \textit{for all } t \in I.$$

Then there exist unique twice continuously differentiable functions $\xi_1, \xi_2 : I \to \mathbb{R}$ such that $\xi_2(t) - \xi_1(t) \ge \eta^{-1}$ for all $t \in I$ and the rest term g defined by $\phi(t) = 1 - H_1(t) + H_2(t) + g(t)$ satisfies, for all $t \in I$, $\|g(t)\|_{H^1} \le 2\eta$ and (3.7). □

Proofs of similar results are contained for example in the articles by Gustafson and Sigal [44, Proposition 3], Merle and Zaag [76, Proposition 3.1], and in my paper [45, Lemma 3.3]. They are based on a quantitative version of the Implicit Function Theorem.

I have thus identified the slow variables of the problem. Note that (ξ_1, ξ_2, g) live on a codimension-2 submanifold of $\mathbb{R}^2 \times H^1$ defined by

$$\langle H(\cdot - \xi_1), g \rangle = \langle H(\cdot - \xi_2), g \rangle = 0,$$

which is a condition linear in g, but nonlinear in ξ_1 and ξ_2.

In order to rewrite the equation (CSF) in terms of the new unknowns, I compute

$$\begin{aligned}
\partial_t \phi(t) &= \xi_1'(t) \partial_x H_1(t) - \xi_2'(t) \partial_x H_2(t) + \partial_t g(t), \\
\partial_t^2 \phi(t) &= \xi_1''(t) \partial_x H_1(t) - (\xi_1'(t))^2 \partial_x^2 H_1(t) - \xi_2''(t) \partial_x H_2(t) + (\xi_2'(t))^2 \partial_x^2 H_2(t) + \partial_t^2 g(t), \\
\partial_x \phi(t) &= -\partial_x H_1(t) + \partial_x H_2(t) + \partial_x g(t), \\
\partial_x^2 \phi(t) &= -\partial_x^2 H_1(t) + \partial_x^2 H_2(t) + \partial_x^2 g(t),
\end{aligned}$$

thus (CSF) becomes

$$\begin{aligned}
\partial_t^2 g + \xi_1'' \partial_x H_1 - (\xi_1')^2 \partial_x^2 H_1 - \xi_2'' \partial_x H_2 + (\xi_2')^2 \partial_x^2 H_2 \\
+ \partial_x^2 H_1 - \partial_x^2 H_2 - \partial_x^2 g + U'(1 - H_1 + H_2 + g) = 0,
\end{aligned}$$

or, using $\partial_x^2 H_j = U'(H_j)$,

$$\begin{aligned}
\partial_t^2 g + \xi_1'' \partial_x H_1 - (\xi_1')^2 \partial_x^2 H_1 - \xi_2'' \partial_x H_2 + (\xi_2')^2 \partial_x^2 H_2 \\
- \partial_x^2 g + U'(1 - H_1 + H_2 + g) + U'(H_1) - U'(H_2) = 0.
\end{aligned} \tag{3.8}$$

A convenient method of deriving the modulation equations is to differentiate in time the orthogonality conditions. For example, differentiating the first orthogonality condition yields

$$0 = -\xi_1' \langle \partial_x^2 H_1, g \rangle + \langle \partial_x H_1, \partial_t g \rangle.$$

One more differentiation yields

$$\langle \partial_x H_1, \partial_t^2 g \rangle = \xi_1'' \langle \partial_x^2 H_1, g \rangle - (\xi_1'(t))^2 \langle \partial_x^3 H_1, g \rangle + 2\xi_1' \langle \partial_x^2 H_1, \partial_t g \rangle.$$

Using this information, I can project (3.8) on $\partial_x H_1$ and obtain

$$\begin{aligned}
&\xi_1''\big(\|\partial_x H\|_{L^2}^2 + \langle \partial_x^2 H_1, g\rangle\big) - \xi_2''\langle \partial_x H_1, \partial_x H_2\rangle \\
&\quad = (\xi_1')^2\langle \partial_x^3 H_1, g\rangle - 2\xi_1'\langle \partial_x^2 H_1, \partial_t g\rangle - (\xi_2')^2\langle \partial_x H_1, \partial_x^2 H_2\rangle \\
&\quad - \langle \partial_x H_1, U'(1 - H_1 + H_2 + g) + U'(H_1) - U'(H_2) - U''(H_1)g\rangle.
\end{aligned}$$

Note that I have employed the identities $\langle \partial_x H_1, \partial_x^2 H_1\rangle = 0$ and $\langle \partial_x H_1, -\partial_x^2 g + U''(H_1)g\rangle = 0$. Similarly, projecting on $\partial_x H_2$ yields

$$\begin{aligned}
&\xi_2''\big(\|\partial_x H\|_{L^2}^2 - \langle \partial_x^2 H_2, g\rangle\big) - \xi_1''\langle \partial_x H_1, \partial_x H_2\rangle = \\
&\quad = -(\xi_2')^2\langle \partial_x^3 H_2, g\rangle + 2\xi_2'\langle \partial_x^2 H_2, \partial_t g\rangle - (\xi_1')^2\langle \partial_x H_2, \partial_x^2 H_1\rangle \\
&\quad + \langle \partial_x H_2, U'(1 - H_1 + H_2 + g) + U'(H_1) - U'(H_2) - U''(H_2)g\rangle.
\end{aligned}$$

If $\|g\|_{H^1}$ is small, then the last two equations form a linear system for ξ_1'' and ξ_2'' which can be solved, yielding the desired *modulation equations*.

Remark 3.4. Let me observe that plugging $g = \partial_t g = 0$ into the modulation equations one obtains precisely the formal modulation equations (3.5). Such an accordance is not a surprise, since imposing $g = 0$ precisely means artificially restricting the flow to $\mathcal{M}$.

The key to obtain first, rough estimates on the oscillatory component g is, just as for the toy model, the conservation of energy. The coercivity properties of the kinks and antikinks, which I state below, play the role similar to a strongly constraining potential.

Lemma 3.5. *Let $L := -\partial_x^2 + U''(H)$. There exists a constant $c_0 > 0$ (depending on U) such that for all $v \in H^1(\mathbb{R})$*

$$\langle v, Lv\rangle \geq c_0\|v\|_{H^1}^2 - c_0^{-1}\langle \partial_x H, v\rangle^2.$$

□

I skip the proof, which is an easy application of the Sturm-Liouville theory. Note that differentiating the identity $\partial_x^2 H(x - \xi_0) = U'(H(x - \xi_0))$ with respect to ξ_0 at $\xi_0 = 0$ yields $L(\partial_x H) = 0$, which is expected since $\langle v, Lv\rangle$ is the Hessian of E_p at H. Since E_p attains its minimum at a one-parameter family, the Hessian has to be degenerate in the direction tangential to the family of minimisers. Lemma 3.5 states that it is positive definite in any transversal direction.

It turns out that coercivity can be *localised in space*, yielding the following result on coercivity near a kink-antikink pair.

Lemma 3.6. *Let $\xi_1 < \xi_2$ and $L := -\partial_x^2 + U''(1 - H(\cdot - \xi_1) + H(\cdot - \xi_2))$. There exists $c_0 > 0$ such that if $\xi_2 - \xi_1 \geq c_0^{-1}$, then for all $v \in H^1(\mathbb{R})$*

$$\langle v, Lv\rangle \geq c_0\|v\|_{H^1}^2 - c_0^{-1}\big(\langle \partial_x H(\cdot - \xi_1), v\rangle^2 + \langle \partial_x H(\cdot - \xi_2), v\rangle^2\big).$$

□

The proof consists in applying the previous Lemma for v multiplied by suitable cut-off functions.

Corollary 3.7. *If $\phi = 1 - H_1 + H_2 + g$ is a kink-antikink pair, then for all t sufficiently large*

$$\|g(t)\|_{H^1} + \|\partial_t \phi(t)\|_{L^2} \lesssim \mathrm{e}^{-\frac{1}{2}(\xi_2(t) - \xi_1(t))}.$$

Sketch of a proof. The 2nd order Taylor expansion of the energy yields

$$E(\phi, \partial_t\phi) = \frac{1}{2}\|\partial_t\phi\|_{L^2}^2 + E_p(1 - H_1 + H_2) + \langle \mathrm{D}E_p(1 - H_1 + H_2), g\rangle + \frac{1}{2}\langle g, Lg\rangle + o(\|g\|_{H^1}^2).$$

If ϕ is a kink-antikink pair, then $E(\phi, \partial_t\phi) = 2E_p(H)$, thus (3.4) and Lemma 3.6 yield

$$\|g\|_{H^1}^2 + \|\partial_t\phi\|_{L^2}^2 \lesssim \mathrm{e}^{-(\xi_2 - \xi_1)} + |\langle \mathrm{D}E_p(1 - H_1 + H_2), g\rangle|,$$

and one can check that the last term is absorbed by the others. □

Recall that the easy coercivity bounds (2.7) were not sufficient to directly analyse the toy model in the non-tangential case. Here as well, I need an appropriate change of variables, based on approximate conservation laws. I define the *localised momenta*

$$\begin{aligned} p_1(t) &:= \langle \partial_x(H_1(t) - \chi_1(t)g(t)), \partial_t\phi\rangle, \\ p_2(t) &:= \langle -\partial_x(H_2(t) + \chi_2(t)g(t)), \partial_t\phi(t)\rangle. \end{aligned} \tag{3.9}$$

The functions χ_1 and χ_2 are cut-offs defined as follows. Let $\chi \in C^\infty$ be a decreasing function such that $\chi(x) = 1$ for $x \leq \frac{1}{3}$ and $\chi(x) = 0$ for $x \geq \frac{2}{3}$. I set

$$\chi_1(t, x) := \chi\Big(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\Big), \quad \chi_2 := 1 - \chi_1,$$

so that $\chi_j(t, x) = 1$ whenever $|x - x_j(t)| \leq \frac{1}{3}|x_2(t) - x_1(t)|$.

Note that if I make H_2 "disappear" by letting $\xi_2 \to \infty$, then in the limit I have $\phi = -H_1 + g$, whereas p_1 becomes $\langle \partial_x(H_1 - g), \partial_t\phi\rangle = -\langle \partial_x\phi, \partial_t\phi\rangle$, which is an exact conservation law (the momentum). The situation is completely analogous to the toy model case, where L becomes an exact conservation law in the absence of W. Analogously, if $\xi_1 \to -\infty$, then p_2 formally converges to the momentum.

Lemma 3.8. *If ϕ is a strongly interacting kink-antikink pair, then there exist $C, T_0 > 0$ such that $p_j \in C^1([T_0, \infty))$ and for all $t \geq T_0$*

$$\begin{aligned} \big|\xi_j'(t)\|\partial_x H\|_{L^2}^2 - p_j(t)\big| &\leq C\mathrm{e}^{-(\xi_2(t) - \xi_1(t))}, \\ \big|p_j'(t) + (-1)^j F(\xi_2(t) - \xi_1(t))\big| &\leq C(\xi_2(t) - \xi_1(t))^{-1}\mathrm{e}^{-(\xi_2(t) - \xi_1(t))}, \end{aligned} \tag{3.10}$$

where F is defined by (3.2). □

The proof is rather direct, but too long to present here, see Section 3 of [1]. These differential inequalities determine the main order of the asymptotic behaviour of the separation between the kinks, which leads to the following result concluding the modulation analysis of kink-antikink pairs.

Proposition 3.9. *Let A be the constant defined by* (1.14)*. If ϕ is a strongly interacting kink-antikink pair, then there exist $C, T_0 > 0$ such that for all $t \geq T_0$*

$$\begin{gathered} 2t^{-1} - C(t\log t)^{-1} \leq \xi_2'(t) - \xi_1'(t) \leq 2t^{-1} + C(t\log t)^{-1}, \\ 2\log(At) - C(\log t)^{-1} \leq \xi_2(t) - \xi_1(t) \leq 2\log(At) + C(\log t)^{-1}, \\ \|g(t)\|_{H^1} + \|\partial_t g(t)\|_{L^2} \leq Ct^{-1}(\log t)^{-1/2}. \end{gathered}$$

□

3.3 Lyapunov-Schmidt approach to kink-antikink pairs

I am interested in the problem of the classification of all the kink-antikink pairs. By analogy with the problem of uniqueness of 1-periodic solutions of the toy model under an energy constraint, analysed in Section 2.2, the Lyapunov-Schmidt method is particularly well-suited to approach it.

The general scheme is the same as in Section 2.2, one notable difference being that energy estimates for a wave equation with slowly moving potentials are used instead of the Fourier transform in time. Let me indicate the main steps.

The projected equation is obtained, as in Section 2.2, by projecting the equation (CSF) on some direction transversal to the slow manifold. Again I make the most obvious choice of projecting on the space defined by the orthogonality conditions (3.7). As the result of this operation, I obtain the equation

$$\partial_t^2\phi(t,x) = \partial_x^2\phi(t,x) + U'(\phi(t,x)) + \lambda_1(t)\partial_x H_1(t,x) + \lambda_2(t)\partial_x H_2(t,x), \tag{3.11}$$

where λ_1 and λ_2 are unknown functions. Written in terms of g, the equation (3.11) becomes

$$\begin{aligned}\partial_t^2 g + \xi_1''\partial_x H_1 - (\xi_1')^2\partial_x^2 H_1 - \xi_2''\partial_x H_2 + (\xi_2')^2\partial_x^2 H_2 \\ - \partial_x^2 g + U'(1 - H_1 + H_2 + g) + U'(H_1) - U'(H_2) = \lambda_1\partial_x H_1 + \lambda_2\partial_x H_2.\end{aligned} \tag{3.12}$$

My objective is to solve for $(g, \lambda_1, \lambda_2)$, given a pair of trajectories (ξ_1, ξ_2) satisfying

$$2\log t - C_0 \leq \xi_2(t) - \xi_1(t) \leq 2\log t + C_0, \tag{3.13}$$

$$\xi_2'(t) - \xi_1'(t) \geq 0, \tag{3.14}$$

$$|\xi_1'(t)| + |\xi_2'(t)| \leq C_0 t^{-1}, \tag{3.15}$$

$$|\xi_1''(t)| + |\xi_2''(t)| \leq C_0 t^{-2} \tag{3.16}$$

for some $T_0 > 0$ and all $t \geq T_0$ (these bounds follow quite easily from Proposition 3.9). I call *admissible* any such pair of trajectories (ξ_1, ξ_2).

Let me introduce some relevant function spaces. Let $\gamma, \beta, \alpha \in \mathbb{R}$, $T_0 > 0$ and $z : [T_0, \infty) \to \mathbb{R}$ a continuous function. I set

$$\|z\|_{N_\gamma} := \sup_{t \geq T_0} t^\gamma |z(t)|,$$

$$\|z\|_{W_{\alpha,\beta}} := \sup_{\tau \geq t \geq T_0} t^{\beta-\alpha}\Big|\int_t^\tau s^\alpha z(s)\,\mathrm{d}s\Big|.$$

For $\gamma > 2$, I set

$$\mathbf{W}_\gamma := \bigcap_{\alpha \in \{-1,0,1,2\}} W_{\alpha,\gamma}.$$

If z is twice continuously differentiable, I set

$$\|z\|_{S_\gamma} := \|z\|_{N_\gamma} + \|z'\|_{N_{\gamma+1}} + \|z''\|_{N_{\gamma+1}}.$$

Note that I am using the same time weight for z' and z''.

If z is a continuous function from $[T_0, \infty)$ to some Banach space E, I denote

$$\|z\|_{N_\gamma(E)} := \big\|t \mapsto \|z(t)\|_E\big\|_{N_\gamma}.$$

If z is twice continuously differentiable function from $[T_0, \infty)$ to E, I denote

$$\|z\|_{S_\gamma(E)} := \|z\|_{N_\gamma(E)} + \|z'\|_{N_{\gamma+1}(E)} + \|z''\|_{N_{\gamma+1}(E)}.$$

If the space E is clear from the context, I write N_γ instead of $N_\gamma(E)$ and S_γ instead of $S_\gamma(E)$.

The analog of Lemma 2.12 for kink-antikink pairs can be formulated as follows. In the statement, I am using the formal attraction force F defined by (3.2).

Proposition 3.10. *For any $C_0 > 0$ there exist $T_0 > 0$ and $\delta > 0$ such that the following is true. For any admissible $\xi_1, \xi_2 : [T_0, \infty) \to \mathbb{R}$, the equation* (3.12) *has a unique solution $(\lambda_1, \lambda_2, g) = \big(\lambda_1(\xi_1, \xi_2), \lambda_2(\xi_1, \xi_2), g(\xi_1, \xi_2)\big)$ such that $\|(g, \partial_t g)\|_{N_1(H^1\times L^2)} \leq \delta$. For all $\gamma \in [1, 2)$ there exist $C = C(\gamma)$ and $T_0 = T_0(\gamma)$ such that this solution satisfies*

$$\sum_{j=1}^{2} \|\lambda_j + (-1)^j \xi_j'' + F(\xi_2 - \xi_1)\|_{N_{\gamma+1}\cap \mathbf{W}_{\gamma+1}} + \|(g, \partial_t g)\|_{N_\gamma(H^1\times L^2)} \leq 1. \tag{3.17}$$

Moreover, for all $\nu > 1$ and $\beta \in (2, \nu + 2)$ there exist $C = C(\nu, \beta) > 0$ and $T_0 = T_0(\nu, \beta)$ such that

$$\begin{aligned} &\big\|\lambda_j(\xi_1^\sharp, \xi_2^\sharp) - \lambda_j(\xi_1, \xi_2) + (-1)^j\big((\xi_j^\sharp)'' - \xi_j''\big) + \big(F(\xi_2^\sharp - \xi_1^\sharp) - F(\xi_2 - \xi_1)\big)\big\|_{N_\beta\cap\mathbf{W}_\beta} \\ &+\|(g(\xi_1^\sharp, \xi_2^\sharp) - g(\xi_1, \xi_2), \partial_t(g(\xi_1^\sharp, \xi_2^\sharp) - g(\xi_1, \xi_2)))\|_{N_{\beta-1}(H^1\times L^2)} \leq C\|(\xi_1^\sharp, \xi_2^\sharp) - (\xi_1, \xi_2)\|_{S_\nu}, \end{aligned}$$

where (ξ_1, ξ_2) and $(\xi_1^\sharp, \xi_2^\sharp)$ are any two pairs of admissible trajectories satisfying $\xi_j^\sharp - \xi_j \in S_\nu$.

The key to proving this result is to obtain bounds for the corresponding non-homogeneous linear problems.

Lemma 3.11. *For any $\gamma > 1$ and $\beta \in (2, \gamma + 1)$ there exists $C = C(\beta, \gamma) > 0$ and $T_0 = T_0(\beta, \gamma)$ such that the following holds. For all admissible (ξ_1, ξ_2), and all $f \in N_{\gamma+1}(L^2)$, the system*

$$\partial_t^2 h - \partial_x^2 h + U''(1 - H_1 + H_2)h = f + \lambda_1 \partial_x H_1 + \lambda_2 \partial_x H_2, \tag{3.18}$$
$$\langle \partial_x H_1, h\rangle = \langle \partial_x H_2, h\rangle = 0 \tag{3.19}$$

has a unique solution $(h, \lambda_1, \lambda_2)$ such that $(h, \partial_t h) \in N_\gamma(H^1 \times L^2)$. Moreover, this solution satisfies

$$\|(h, \partial_t h)\|_{N_\gamma(H^1\times L^2)} + \sum_{j=1}^{2} \big\|\lambda_j + \|\partial_x H\|_{L^2}^{-2}\langle \partial_x H_j, f\rangle\big\|_{\mathbf{W}_\beta\cap N_{\gamma+1}} \leq C\|f\|_{N_{\gamma+1}(L^2)}. \tag{3.20}$$

If $\gamma = 1$, the same result holds without the inclusion of the $\mathbf{W}_\beta$ norm on the left-hand side of (3.20).

□

Lemma 3.12. *For any $\gamma > 1$ and $\beta \in (2, \gamma + 1)$ there exists $C = C(\beta, \gamma) > 0$ and $T_0 = T_0(\beta, \gamma)$ such that for all admissible (ξ_1, ξ_2), and for all $f \in N_\gamma(L^2)$ such that $\partial_t f \in N_{\gamma+1}(L^2)$, the system* (3.18)–(3.19) *has a unique solution $(h, \lambda_1, \lambda_2)$ and*

$$\|(h, \partial_t h)\|_{N_\gamma(H^1\times L^2)} + \sum_{j=1}^{2} \big\|\lambda_j + \|\partial_x H\|_{L^2}^{-2}\langle \partial_x H_j, f\rangle\big\|_{\mathbf{W}_\beta\cap N_{\gamma+1}} \leq C\big(\|f\|_{N_\gamma(L^2)} + \|\partial_t f\|_{N_{\gamma+1}(L^2)}\big).$$

If $\gamma = 1$, the same result holds without the inclusion of the $\mathbf{W}_\beta$ norm on the left-hand side of (3.20).

□

Remark 3.13. Lemma 3.11 is an analog of the bound (2.21), whereas Lemma 3.12, exploiting the additional regularity in time of the forcing term, is an analog of (2.22).

Let me spend a few paragraphs explaining how to prove these results, because it is here that an important new idea, with respect to the toy model studied in Chapter 2, is needed.

One of the common ways to invert a wave operator like (3.18) is to use an energy estimate. Slightly simplifying the problem, and pretending there is just one kink, I am essentially dealing with a wave equation with a slowly moving potential:

$$\partial_t^2 h(t,x) - \partial_x^2 h(t,x) + V(x-\xi_1(t))h(t,x) = f(t,x),$$

where $|\xi_1'(t)| \simeq t^{-1}$. A naive idea would be to ignore the time-dependence of the potential and differentiate in time the usual energy functional

$$I(t) := \int_{\mathbb{R}} \Big(\frac{1}{2}(\partial_t h(t,x))^2 + \frac{1}{2}(\partial_x h(t,x))^2 + \frac{1}{2}V(x-\xi_1(t))h(t,x)^2\Big)\,\mathrm{d}x.$$

An integration by parts yields

$$I'(t) = \int_{\mathbb{R}} f(t,x)\partial_t h(t,x)\,\mathrm{d}x - \frac{\xi_1'(t)}{2}\int_{\mathbb{R}} \partial_x V(x-x_1(t))h(t,x)^2\,\mathrm{d}x, \tag{3.21}$$

thus

$$\begin{aligned}|I'(t)| &\lesssim \|\partial_t h(t)\|_{L^2}\|f(t)\|_{L^2} + |\xi_1'(t)|\|h(t)\|_{L^2}^2 \\ &\lesssim t^{-2\gamma-1}\big(\|(h,\partial_t h)\|_{N_\gamma(H^1\times L^2)}\|f\|_{N_\gamma(L^2)} + \|(h,\partial_t h)\|_{N_\gamma(H^1\times L^2)}^2\big),\end{aligned}$$

where γ is any positive number. Integrating in time, the best I can get is

$$\begin{aligned}\|(h,\partial_t h)\|_{N_\gamma(H^1\times L^2)}^2 &\simeq \sup_{t\geq T_0}\big(t^{2\gamma}\|(h(t),\partial_t h(t))\|_{H^1\times L^2}^2\big) \lesssim \sup_{t\geq T_0} t^{2\gamma} I(t) \\ &\lesssim \|(h,\partial_t h)\|_{N_\gamma(H^1\times L^2)}\|f\|_{N_{\gamma+1}(L^2)} + \|(h,\partial_t h)\|_{N_\gamma(H^1\times L^2)}^2,\end{aligned}$$

which is a trivial and useless inequality. Even though the potential moves with speed $t^{-1}\to 0$, its time dependence still cannot be neglected in the regime of polynomial in time decay. The remedy is to use a *mixed energy-momentum functional*

$$\widetilde{I}(t) := I(t) - \xi_1'(t)\int_{\mathbb{R}} \partial_t h(t,x)\partial_x h(t,x)\,\mathrm{d}x,$$

and idea going back to the works of Martel, Merle and Tsai [69, 70]. The added term, sometimes called the *corrective term*, is designed to cancel the "bad" second term of the right hand side of (3.21), and indeed an integration by parts yields

$$\widetilde{I}'(t) = \int_{\mathbb{R}} f(t,x)\partial_t h(t,x)\,\mathrm{d}x - \xi_1''(t)\int_{\mathbb{R}} \partial_t h(t,x)\partial_x h(t,x)\,\mathrm{d}x.$$

Now the situation is much better, since $|\xi_1''(t)| \simeq t^{-2}$ thus, assuming coercivity of the energy functional, I am in a position to prove the desired bound $\|(h,\partial_t h)\|_{N_\gamma(H^1\times L^2)} \lesssim \|f\|_{N_{\gamma+1}(L^2)}$.

Remark 3.14. If $\xi_1'' = 0$ and $f = 0$, then $\widetilde{I}$ is an exact conservation law, which can also be seen from the Lorentz invariance.

Let me finally comment on what happens in the presence of two (or more) slowly moving potentials,

$$\partial_t^2 h(t,x) - \partial_x^2 h(t,x) + V(x-\xi_1(t))h(t,x) + V(x-\xi_2(t))h(t,x) = f(t,x).$$

The problem is of course the dependence of the correction term on the speed of the potential, and its solution is to *localise* each correction term,

$$\begin{aligned}\widetilde{I}(t) := I(t) &- \xi_1'(t)\int_{\mathbb{R}} \chi(x-\xi_1(t))\partial_t h(t,x)\partial_x h(t,x)\,\mathrm{d}x \\ &- \xi_2'(t)\int_{\mathbb{R}} \chi(x-\xi_2(t))\partial_t h(t,x)\partial_x h(t,x)\,\mathrm{d}x.\end{aligned}$$

Here, χ is a cut-off function such that $\chi(x) = 1$ for $|x| \le R$, $\chi(x) = 0$ for $|x| \ge 3R$ and $|\chi'(x)| \le R^{-1}$ for all $x \in \mathbb{R}$, and $R \gg 1$. Such a functional allows to arrive at the same conclusion, namely $\|(h, \partial_t h)\|_{N_\gamma(H^1\times L^2)} \lesssim \|f\|_{N_{\gamma+1}(L^2)}$.

Remark 3.15. The correction terms are in fact the same as the ones used to construct the localised momenta (3.9).

Once the estimates for the non-homogeneous problem are obtained, Proposition 3.10 is proved in a standard manner, using the Contraction Principle.

With the help of Proposition 3.10, I am ready to solve the bifurcation equations. In terms of the functions λ_j, these equations are simply

$$\lambda_1(t) = 0, \qquad \lambda_2(t) = 0, \qquad \text{for all } t \in [T_0, \infty). \tag{3.22}$$

Lemma 3.16. *Let $1 < \gamma < 2$. If ξ_1 and ξ_2 are trajectories satisfying* (3.13)–(3.16) *and solving the equations* (3.22)*, then there exist unique $\xi_0, t_0 \in \mathbb{R}$ such that*

$$\xi_1(t) - \xi_0 + \log(A(t-t_0)) \in S_\gamma, \qquad \xi_1(t) - \xi_0 - \log(A(t-t_0)) \in S_\gamma. \tag{3.23}$$

Sketch of a proof. Indeed, the claim follows from (3.17). Using the bounds $\|(-1)^j\xi_j'' + F(\xi_2 - \xi_1)\|_{N_{\gamma+1}\cap \mathbf{W}_{\gamma+1}} \lesssim 1$, it is not difficult to bootstrap (3.13)–(3.16) to (3.23). □

In order to finish the proof of Theorem 1, it remains to show that there exist unique trajectories ξ_1, ξ_2 which solve (3.23), such that $\xi_1(t) + \log(At) \in S_\gamma$, $\xi_2(t) - \log(At) \in S_\gamma$ for some $\gamma \in (1,2)$. I follow the usual "contraction" scheme. I write

$$\xi_1(t) = -\log(At) + \eta_1(t), \qquad \xi_2(t) = \log(At) + \eta_2(t),$$

and express (3.22) as a fixed point problem for $\eta_j \in S_\gamma$: if

$$\widetilde{\eta}_1'' = -t^{-2}(\widetilde{\eta}_2 - \widetilde{\eta}_1) - \lambda_1(-\log(At) + \eta_1, \log(At) + \eta_2) + \eta_1'' + t^{-2}(\eta_2 - \eta_1), \tag{3.24}$$

$$\widetilde{\eta}_2'' = t^{-2}(\widetilde{\eta}_2 - \widetilde{\eta}_1) + \lambda_2(-\log(At) + \eta_1, \log(At) + \eta_2) + \eta_2'' - t^{-2}(\eta_2 - \eta_1), \tag{3.25}$$

then (3.22) is equivalent to $\widetilde{\eta}_j = \eta_j$. Note that $-t^{-2}(\eta_2 - \eta_1)$ is simply the linearisation of $F(\xi_2 - \xi_1) = F(2\log(At) + (\eta_2 - \eta_1))$. The linear part of the system (3.24)–(3.25) can be diagonalised explicitly, and one checks that the system is a contraction on $S_\gamma \times S_\gamma$, in fact for any $\gamma \in (1,2)$.

3.4 Dynamical role of the kink clusters

In this section, I develop the discussion initiated at the beginning of Section 1.3, arguing that the kink clusters play an important role in the phase portrait of solutions of (CSF). The content of this section will appear in a paper under preparation, in collaboration with Andrew Lawrie.

Definition 3.17. A sequence of functions ϕ_n is a *Palais-Smale sequence* of the potential energy E_p if

- $\sup_n E_p(\phi_n) < \infty$,
- $\lim_{n\to\infty} \|\mathrm{D}E_p(\phi_n)\|_{H^{-1}} = 0$.

Proposition 3.18. *Let ϕ be a solution of* (CSF) *such that, for any sequence $t_n \to \infty$, $\phi(t_n)$ is a Palais-Smale sequence of E_p. Then ϕ is a kink cluster.* $\square$

In the presence of more than one kink, these Palais-Smale sequences do not converge in the energy space. They converge to an ideal "critical point at infinity", corresponding to some number of widely separated kinks. The kink clusters are the solutions converging to this critical point at infinity, and it is natural to ask to what extent they have properties similar to the usual stable manifolds of hyperbolic stationary states.

Let me restrict my attention to kink-antikink pairs. I formally define the distance from a state $\boldsymbol{\phi}_0 = (\phi_0, \dot\phi_0)$ to the critical point at infinity $\boldsymbol{H}_{(2)}$ by

$$\|\boldsymbol{\phi}_0 - \boldsymbol{H}_{(2)}\|_{H^1\times L^2} := \inf_{\xi_1<\xi_2} \big((\xi_2-\xi_1)^{-1} + \|\phi_0 - (1 - H(\cdot - \xi_1) + H(\cdot - \xi_2))\|_{H^1} + \|\dot\phi_0\|_{L^2}\big).$$

I also set $\mathcal{M}_s := \{\boldsymbol{\phi}_{(2)}(t_0, \cdot - \xi_0) : t_0 \gg 1, \xi_0 \in \mathbb{R}\}$, which is a forward invariant two-dimensional manifold. In this language, the uniqueness part of Theorem 1 can be formulated as follows:

$$\phi \text{ solution of (CSF) and } \lim_{t\to\infty} \|\boldsymbol{\phi}(t) - \boldsymbol{H}_{(2)}\|_{H^1\times L^2} = 0 \quad \Leftrightarrow \quad \boldsymbol{\phi}(t) \in \mathcal{M}_s \text{ for } t \text{ large enough.}$$

Can $\mathcal{M}_s$ be viewed as the stable manifold of $\boldsymbol{H}_{(2)}$? The kink-antikink pairs are unstable: initial data at a small but nonzero distance from $\mathcal{M}_s$ in general lead to solutions leaving a neighbourhood of $\mathcal{M}_s$. Such a behaviour is expected from a stable manifold, whose role is rather to indicate universal behaviour of the solutions *entering* a small neighbourhood of a stationary state. (Equivalently, by time reversal, of the solutions exiting a neighbourhood of a stationary state.) In the case of kink-antikink pairs, $\mathcal{M}_s$ plays the same role, which is the content of the statement which follows.

Proposition 3.19. *Let $\eta > 0$ be sufficiently small. For any $\epsilon > 0$ there exists $\delta > 0$ such that the following holds. Let ϕ be a solution of* (CSF) *such that $\|\boldsymbol{\phi}(t_0) - \boldsymbol{H}_{(2)}\|_{H^1\times L^2} \le \delta$ for some $t_0 \in \mathbb{R}$, and set $t_- := \inf\{t : \|\boldsymbol{\phi}(\tau) - \boldsymbol{H}_{(2)}\|_{H^1\times L^2} \le \eta$ for all $\tau \in [t, t_0]\}$. If $t_- > -\infty$, then $\mathrm{dist}_{H^1\times L^2}(\boldsymbol{\phi}(t_-), \mathcal{M}_s) \le \epsilon$.* $\square$

In other words, a solution entering a small neighborhood of the critical point at infinity has to be close to $\mathcal{M}_s$ *while it is still far from* $\boldsymbol{H}_{(2)}$.

Remark 3.20. In the context of dispersive equations, similar observations played a role in the study of centre-stable manifolds of ground states for various nonlinear wave equations, see the already cited works [77, 78, 59, 60].

3.A Gravitational n-body problem in one dimension

A lot of the intuition in the study of multi-solitons comes from imagining the individual solitons as moving particles. Although I have mentioned the limitations of such a viewpoint, I would like to describe briefly in this additional section the conjectural behaviour of multi-solitons based on it.

The n-body problem which was studied the most intensively, or at least for the longest time, is probably the problem of describing the evolution of n point masses under the Newton's law of universal gravitation. I restrict my discussion to the collinear case, since

- a complete answer is known,
- it resembles the situation of a chain of alternating kinks and antikinks.

More general results can be found in the works of Saari [89], and Saari and Hulkower [90], building on earlier ideas of Sundman [102, 103], Pollard [84] and others authors.

Let $m_1, \ldots, m_n > 0$. I consider (the negation of) the potential energy

$$U(x_1, \ldots, x_n) := \sum_{1 \leq j < k \leq n} \frac{m_j m_k}{x_k - x_j}, \qquad x_j \in \mathbb{R}, \quad x_1 < \ldots < x_n$$

(I adopt the traditional sign convention). I consider the corresponding evolution equation, which is the Newton's gravitation law for the point masses $(m_1, \ldots, m_n)$ located on the real line, at the points $\boldsymbol{x} = (x_1, \ldots, x_n)$. It is assumed that the units are chosen in such a way that the gravitational constant equals 1. If I denote $\boldsymbol{M} := \mathrm{diag}(m_1, \ldots, m_n) \in \mathbb{R}^{n \times n}$, the equation is

$$\boldsymbol{M}\boldsymbol{x}''(t) = \nabla U(\boldsymbol{x}(t)), \qquad t \in \mathbb{R},\ \boldsymbol{x}(t) \in \mathbb{R}^n. \tag{3.26}$$

The total energy is given by

$$E(\boldsymbol{x}, \boldsymbol{x}') = T(\boldsymbol{x}') - U(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}' \cdot \boldsymbol{M}\boldsymbol{x}' - \sum_{1 \leq j < k \leq n} \frac{m_j m_k}{x_k - x_j},$$

and is a conserved quantity.

One can ask the following question.

Problem. *What are the solutions such that no two masses collide for sufficiently large positive time?*

Remark 3.21. Recall that two-body collisions can be canonically continued as an elastic bounce. I could thus allow two-body collisions and ask for a description of all the solutions such that, after some time, no *three* masses collide. Such a question would be much more difficult to answer, in fact a full solution is currently unknown. In the world of multi-solitons, this would amount to allowing collisions. Certainly, colliding solitons cannot be considered as particles, at least for non-integrable models (for example, unlike particles, they can collide in a non-elastic way), so even if one succeeded in solving the problem, the answer could be of no help in the study of multi-solitons.

Until the end of this section $\boldsymbol{x}(t)$ is always a solution of (3.26) such that $x_1(t) < \ldots < x_n(t)$ for all t large enough.

Lemma 3.22. *If $\boldsymbol{x}(t)$ is a solution such that $x_1(t) < \ldots < x_n(t)$ for all $t \geq t_0$, then there exist $v_1, \ldots, v_n \in \mathbb{R}$ such that $v_1 \leq \ldots \leq v_n$ and $\lim_{t\to\infty} x_k'(t) = v_k$ for $k = 1, \ldots, n$.*

Proof. For $k \in 1, \ldots, n$, let

$$y_k(t) := \sum_{j=1}^{k} m_j x_j(t).$$

From the equations of motion, $y_k''(t) \geq 0$, hence $\lim_{t\to\infty} y_k'(t) \in (-\infty, \infty]$ exists. Since $x_n''(t) \leq 0$ for all t, I have $y_k(t) \leq x_n(t) \sum_{k=1}^{n} m_k \lesssim t$, so the limit is finite. Thus, $v_k := \lim_{t\to\infty} x_k'(t) = m_k^{-1} \lim_{t\to\infty}(y_k'(t) - y_{k-1}'(t))$ exists and is finite. Clearly, $x_k(t) < x_{k+1}(t)$ for all t large enough implies $v_k \leq v_{k+1}$. $\square$

The sequence of the limit velocities allows to divide the system into *subsystems*. I can define the sequence $0 = s_0 < s_1 < \ldots < s_{L-1} < s_L = n$ be the conditions $v_{s_l+1} > v_{s_l}$ for $l = 1, \ldots, L-1$ and $v_{s_{l-1}+1} = v_{s_l}$ for $l = 1, \ldots, L$. For $l \in \{1, \ldots, L\}$, I set $I_l := \{s_{l-1}+1, \ldots, s_l\}$ and call I_l the l-th subsystem. There is the partition $\{1, \ldots, n\} = I_1 \cup \ldots \cup I_L$ and, for each $l \in \{1, \ldots, L\}$, all the masses in the l-th subsystem have the same limit velocity v_{s_l}. Any two masses belonging to distinct subsystems have distinct limit velocities. I denote $\boldsymbol{x}_l := (x_{s_{l-1}+1}, \ldots, x_{s_l}) \in \mathbb{R}^{|I_l|}$, $\boldsymbol{M}_l := \mathrm{diag}(m_{s_{l-1}+1}, \ldots, m_{s_l}) \in \mathbb{R}^{|I_l|\times|I_l|}$, $V_l := v_{s_l} \in \mathbb{R}$ (the common limit velocity of the masses belonging to the l-th subsystem) and $\boldsymbol{V}_l := (V_l, \ldots, V_l) \in \mathbb{R}^{|I_l|}$.

It is clear that the distance between subsystems grows linearly with time, and the masses in each subsystem remain at distance $o(t)$. In fact, one can be much more precise.

Lemma 3.23. *For all $j, k \in I_l$ with $j < k$ there exist $c > 0$ and $t_0 \in \mathbb{R}$ such that*

$$x_k(t) - x_j(t) \geq c t^{\frac{2}{3}} \qquad \textit{for all } t \geq t_0.$$

For all $k \in I_l$ there exist $C > 0$ and $t_0 \in \mathbb{R}$ such that

$$|x_k(t) - V_l t| \leq C t^{\frac{2}{3}} \qquad \textit{for all } t \geq t_0.$$

$\square$

I skip the proof, which can be obtained using virial-type identities or similar differential inequalities. Interestingly, these bounds can be substantially improved, which is what I explain next.

I fix $l \in \{1, \ldots, L\}$ and restrict my attention to the l-th subsystem. I set

$$\boldsymbol{q}_l(t) := \boldsymbol{x}_l(t) - \boldsymbol{V}_l t.$$

Since the influence of distant bodies (that is, from another subsystem) is $\lesssim t^{-2}$, I have

$$\boldsymbol{M}_l \boldsymbol{q}_l''(t) = \nabla U_l(\boldsymbol{q}_l(t)) + O(t^{-2}).$$

Consider now the *rescaled* (or *self-similar*) *coordinates*

$$\boldsymbol{r}_l(t) := t^{-\frac{2}{3}} \boldsymbol{q}_l(t).$$

The conclusion of Lemma 3.23 is that $\boldsymbol{r}_l(t)$ and $U_l(\boldsymbol{r}_l(t))$ remain bounded as $t \to \infty$. I define

$$W_l(\boldsymbol{r}_l) := \frac{1}{9} \boldsymbol{r}_l \cdot \boldsymbol{M}_l \boldsymbol{r}_l + U_l(\boldsymbol{r}_l).$$

Lemma 3.24. *For each subsystem, the rescaled coordinate $\boldsymbol{r}_l(t)$ satisfies*

$$\lim_{t\to\infty} \nabla W_l(\boldsymbol{r}_l(t)) = 0.$$

□

Again I skip the proof. Intuitively, the rescaling introduces a *damping effect*, and a damped Newton equation has to converge to an equilibrium.

Critical points of W_l are called *central configurations*. In general, the set of central configurations can have a complicated structure, but in the collinear case considered here the situation is very simple. The function W_l, defined on the set $\{\boldsymbol{r}_l \in \mathbb{R}^{|I_l|} : -\infty < r_{s_{l-1}+1} < \ldots < r_{s_l} < \infty\}$, is strictly convex and converges to ∞ at the boundary of its domain, so it has exactly one critical point $\boldsymbol{z}_l \in \mathbb{R}^{|I_l|}$. Note that if $|I_l| = 1$, then $\boldsymbol{z}_l = 0$ is the central configuration.

Corollary 3.25. *For each subsystem, the rescaled coordinate $\boldsymbol{r}_l$ satisfies*

$$\lim_{t\to\infty} \boldsymbol{r}_l(t) = \boldsymbol{z}_l,$$

where $\boldsymbol{z}_l$ is the unique central configuration in $\mathbb{R}^{|I_l|}$.

Remark 3.26. Note the crucial role of the *central configurations* in this analysis. They give rise to self-similar solutions of the n-body problem. The analogs of central configurations are just as important in the study of n-body problems related to multi-soliton dynamics in the strong interaction regime.

Remark 3.27. The analysis of dynamics close to central configurations was essential in the breakthrough work of McGehee [72] on continuation after collision for the three-body problem.

Remark 3.28. I described the main order of the asymptotic behaviour of any solution avoiding collisions for large enough times. The configuration decomposes into subsystems of masses having the same limit speed, and each subsystem approaches, up to rescaling, the unique central configuration. It turns out that this preliminary description allows to use the Contraction Principle in order to give a precise description, up to order $o(t^{2/3})$, of the set of all the solutions avoiding collisions for large enough times. They form smooth manifolds, see [90].

Note the similarity with the problems previously discussed in this memoir: once the main order of the dynamics is extracted using modulation, the Contraction Principle becomes applicable. For example, in Section 3.3, a preliminary estimate up to order $o(t^{-1})$ in the energy space was necessary in order to apply the Contraction Principle.

Remark 3.29. In the previous sections I restricted my attention to kink-antikink pairs, corresponding to the rather trivial situation $n = 2$. The case of a (strongly interacting) n-kink corresponds to an evolution of n point masses forming just one subsystem. Proving results for multi-kinks corresponding to the results presented in this section is work in progress.

Another extreme case is an evolution of n masses forming asymptotically n subsystems, that is when all the masses have asymptotically distinct velocities. The corresponding multi-kinks can be called *weakly interacting*. They are relatively easy to construct, using the Lorentz transformation, but uniqueness is an open problem even for $n = 2$. A similar problem for the generalised KdV equation was completely solved by Martel [65], see also [73, 66, 25, 26].

The remaining cases (several subsystems containing more than one element) are completely open in the multikink setting.

Chapter 4

Two-bubbles for wave maps

Similar results as the ones presented in the last chapter can also be obtained for two-bubble solutions. In fact, the picture is currently more complete in the latter case, at least for equivariance degree $k \geq 4$.

4.1 Modulation analysis of two-bubble solutions

The evolution of two-bubbles can be predicted similarly as in the case of multi-kinks presented in Section 3.1, the discussion below follows in fact exactly the same path.

In this chapter, L^2 and $\mathcal{H}$ are spaces defined by (1.3), and $\langle \cdot, \cdot \rangle$ denotes the corresponding L^2 inner product, $\langle g_1, g_2 \rangle := \int_0^\infty g_1(r) g_2(r)\, r\mathrm{d}r$.

I consider a positive two-bubble $Q(\lambda, \mu) := Q_\lambda - Q_\mu$ for $0 < \lambda \ll \mu$ and define the reduced Lagrangian

$$\widetilde{\mathscr{L}}(\lambda, \mu) := \mathscr{L}(Q(\lambda, \mu)).$$

An important difference with respect to the case of kinks is that here the relevant invariance is the *rescaling* (and not translations). I denote Λ the generator of the energy-critical rescaling, that is

$$\Lambda \psi_0(r) := -\partial_\lambda \big(\psi_0(r/\lambda)\big)\Big|_{\lambda=1} = r\partial_r \psi_0(r),$$

and Λ_0 the generator of the L^2 rescaling, that is

$$\Lambda_0 \psi_1(r) := -\partial_\lambda \big(\lambda^{-1}\psi_1(r/\lambda)\big)\Big|_{\lambda=1} = (1 + r\partial_r)\psi_1(r).$$

I denote $\sigma := \lambda/\mu \ll 1$ the ratio of the scales of the bubbles and I set

$$F(\sigma) := \Big\langle \Lambda Q, \frac{k^2}{2r^2}\big(\sin(2(Q_\sigma - Q)) - \sin(2Q_\sigma) + \sin(2Q)\big)\Big\rangle$$

and, abusing somewhat the notation,

$$E_p(\sigma) := E_p(Q_\sigma - Q) - 2E_p(Q).$$

(the interaction energy). The following identity holds

$$F(\sigma) := \Big\langle \Lambda Q_\sigma, \frac{k^2}{2r^2}\big(\sin(2(Q_\sigma - Q)) - \sin(2Q_\sigma) + \sin(2Q)\big)\Big\rangle,$$

and by a direct computation one can check that

$$F(\sigma) = -\sigma \frac{\mathrm{d}}{\mathrm{d}\sigma} E_p(\sigma)$$

(the sign is different than for the kink-antikink pairs). One can also check that

$$\big|F(\sigma) - 8k^2\sigma^k\big| \le C\sigma^{2k}|\log\sigma|. \tag{4.1}$$

The reduced Lagrangian, after subtracting the (irrelevant) constant, is

$$\widetilde{\mathscr{L}}(\lambda, \mu) = \frac{1}{2}\big((\lambda')^2 + (\mu')^2\big)\|\Lambda Q\|_{L^2}^2 - \lambda'\mu'\langle\lambda^{-1}\Lambda Q_\lambda, \mu^{-1}\Lambda Q_\mu\rangle - E_p(\lambda/\mu),$$

and the corresponding Euler-Lagrange equations yield the following formal modulation equations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Lambda Q\|_{L^2}^2\lambda' - \langle\lambda^{-1}\Lambda Q, \mu^{-1}\Lambda Q_\mu\rangle\mu'\big) &= \lambda'\mu'\lambda^{-1}\langle\lambda^{-1}\Lambda_0\Lambda Q_\lambda, \mu^{-1}\Lambda Q_\mu\rangle + \mu^{-1}(\lambda/\mu)^{-1}F(\lambda/\mu), \\
\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Lambda Q\|_{L^2}^2\mu' - \langle\lambda^{-1}\Lambda Q, \mu^{-1}\Lambda Q_\mu\rangle\lambda'\big) &= \lambda'\mu'\mu^{-1}\langle\lambda^{-1}\Lambda Q_\lambda, \mu^{-1}\Lambda_0\Lambda Q_\mu\rangle - \lambda\mu^{-2}(\lambda/\mu)^{-1}F(\lambda/\mu),
\end{aligned}$$

equivalently,

$$\begin{aligned}
\|\Lambda Q\|_{L^2}^2\lambda'' - \langle\lambda^{-1}\Lambda Q_\lambda, \mu^{-1}\Lambda Q_\mu\rangle\mu'' &= -(\mu')^2\mu^{-1}\langle\lambda^{-1}\Lambda Q_\lambda, \mu^{-1}\Lambda_0\Lambda Q_\mu\rangle + \lambda^{-1}F(\lambda/\mu), \\
\|\Lambda Q\|_{L^2}^2\mu'' - \langle\lambda^{-1}\Lambda Q_\lambda, \mu^{-1}\Lambda Q_\mu\rangle\lambda'' &= -(\lambda')^2\lambda^{-1}\langle\lambda^{-1}\Lambda_0\Lambda Q_\lambda, \mu^{-1}\Lambda Q_\mu\rangle - \mu^{-1}F(\lambda/\mu).
\end{aligned}$$

In both equations, the second term of the left hand side and the first term of the right hand side turn out to be negligible. Erasing them and using (4.1) leads to

$$\begin{aligned}
\|\Lambda Q\|_{L^2}^2\lambda'' &\simeq 8k^2\frac{\lambda^{k-1}}{\mu^k}, \\
\|\Lambda Q\|_{L^2}^2\mu'' &\simeq -8k^2\frac{\lambda^k}{\mu^{k+1}}.
\end{aligned}$$

For $k \ge 3$, the only solutions (λ, μ) of this system such that $(\lambda/\mu) + |\lambda'| + |\mu'| \to 0$ as $t \to \infty$ are given at main order by

$$\mu(t) \simeq \mu_0, \quad \lambda(t) \simeq \mu_0 q_k t^{-\frac{2}{k-2}}, \quad \mu_0 \in (0, \infty), \quad q_k := \Big(\frac{k-2}{2}\Big)^{-\frac{2}{k-2}}\Big(\frac{8k}{\pi}\sin\Big(\frac{\pi}{k}\Big)\Big)^{-\frac{1}{k-2}}. \tag{4.2}$$

For $k = 2$, the solutions satisfy

$$\mu(t) \simeq \mu_0, \qquad \lambda(t) \simeq \mu_0 \mathrm{e}^{-q(t-t_0)}, \qquad \mu_0 \in (0, \infty), \qquad q := \frac{4}{\sqrt{\pi}}.$$

The explicit constants q_k and q are obtained using $\|\Lambda Q\|_{L^2}^2 = \frac{2\pi}{\sin(\pi/k)}$; they are not essential.

Remark 4.1. For $k = 1$ the computations above are not valid, because $\Lambda Q \notin L^2$. One of the results of Rodriguez [87] is that for $k = 1$ two-bubbles do not exist.

In the sequel, I always assume $k \geq 4$, so I expect to find two-bubbles with one "stationary" bubble and the other one concentrating at a polynomial rate $t^{-\frac{2}{k-2}}$. Note that one can assume $\mu_0 = 1$, the other solutions being obtained by a simple rescaling.

The coercivity properties of bubbles and multi-bubbles closely resemble the corresponding properties of kinks and multi-kinks, and in fact can even be rigorously deduced from the latter by means of the change of variable $r^k = \mathrm{e}^x$. Let me state the relevant facts and skip the proofs, which can be found in [46] and [2].

Lemma 4.2. *Let $V(r) := \cos(2Q) - 1 = -\frac{8r^{2k}}{(1+r^{2k})^2}$, let L_0 be defined by (1.4) and $L := L_0 + \frac{k^2}{r^2}V(r)$. There exists a constant $c_0 > 0$ such that for all $v \in \mathcal{H}$*

$$\langle v, Lv\rangle \geq c_0\|v\|_{\mathcal{H}}^2 - c_0^{-1}\langle \Lambda Q, v\rangle^2.$$

□

The two-bubble version can be formulated as follows.

Lemma 4.3. *Let $\lambda < \mu$ and $L := L_0 + \frac{k^2}{r^2}(V_\lambda(r) + V_\mu(r))$. There exists $c_0 > 0$ such that if $\lambda/\mu \leq c_0$, then for all $v \in \mathcal{H}$*

$$\langle v, Lv\rangle \geq c_0\|v\|_{\mathcal{H}}^2 - c_0^{-1}\big(\langle \lambda^{-2}\Lambda Q_\lambda, v\rangle^2 + \langle \mu^{-2}\Lambda Q_\mu, v\rangle^2\big).$$

□

Corollary 4.4. *If $\psi = 1 - H_1 + H_2 + g$ is a two-bubble solution, then for all t sufficiently large*

$$\|g(t)\|_{\mathcal{H}} + \|\partial_t g(t)\|_{L^2} \lesssim \lambda(t)^{\frac{k}{2}}. \tag{4.3}$$

□

In order to perform the modulation analysis, I decompose

$$\psi(t) = Q_{\lambda(t)} - Q_{\mu(t)} + g(t), \qquad 0 < \lambda(t) \ll \mu(t),$$

with the orthogonality conditions

$$\langle \Lambda Q_{\lambda(t)}, g(t)\rangle = 0, \qquad \langle \Lambda Q_{\mu(t)}, g(t)\rangle = 0.$$

Note that the formulas above require $\Lambda Q \in \mathcal{H}^*$, which is true only if $k \geq 3$. For $k = 2$, and especially for $k = 1$, the failure of this condition introduces additional difficulties.

The modulation equations, which can be obtained in the same way as for kink-antikink pairs, cannot be solved directly using these coercivity bounds. The relevant change of variables is obtained using a *localised virial functional* introduced in [46]. I set

$$b(t) := -\big\langle \lambda(t)^{-1}\Lambda Q_{\lambda(t)}, \partial_t\psi(t)\big\rangle - \big\langle \partial_t\psi(t), \mathcal{A}_0(\lambda(t))\psi(t)\big\rangle, \tag{4.4}$$

where $\mathcal{A}_0(\lambda)$ can be thought of as a version of $\lambda^{-1}\Lambda$, localised to the scale λ, in other words an operator which acts like $\lambda^{-1}\Lambda$ for $0 \leq r \lesssim \lambda$ and which is 0 for $r \gg \lambda$. A precise construction of $\mathcal{A}_0(\lambda)$ is done in [46, Section 4.6].

Lemma 4.5. *If ψ is a two-bubble in the forward time direction, then for all $\epsilon > 0$ there exist $C, T_0 > 0$ such that $b \in C^1([T_0, T_+))$ and for all $t \geq T_0$*

$$\big|\lambda'(t)\|\Lambda Q\|_{L^2}^2 - b(t)\big| \leq C\lambda(t)^k, \tag{4.5}$$

$$b'(t) \geq (8k^2 - \epsilon)\lambda(t)^k. \tag{4.6}$$

□

A major difference with respect to the case of (CSF) is that the scaling invariance of (WM), with which the virial identity is associated, is not really an invariance in the sense of Hamiltonian systems, and thus, unlike translations for (CSF), does not correspond to a conservation law. This explains why in (4.6) we only obtain a one-sided bound, whereas in (3.10) an approximate equality was available.

Nevertheless, the estimate (4.6) is in the "favourable" direction and allows to prove the following claim, finishing the modulation analysis of the two-bubbles.

Lemma 4.6. *If ψ is a two-bubble in the forward time direction, then $T_+ = \infty$ and the modulation parameters $(\lambda(t), \mu(t))$ satisfy (4.2).* □

Remark 4.7. Instead of assuming that ψ is a two-bubble, one can also consider ψ which has energy $2E_p(Q)$ and is close to a two-bubble configuration on some finite or infinite open time interval $I \subset \mathbb{R}$. By the same proof, one obtains that (4.5) and (4.6) hold for all $t \in I$.

4.2 Existence and uniqueness of two-bubble wave maps

Existence of two-bubble solutions in any equivariance class $k \geq 2$ was proved in [46], but the question of uniqueness remained unanswered. The problem can be solved by means of the Lyapunov-Schmidt approach, similarly as for kink-antikink pairs, which will be clarified in a paper in preparation, with Andrew Lawrie. In the preprints [3, 4], a different proof scheme was adopted, exploiting the role of two-bubble solutions as "threshold elements". Let me explain this scheme, which could be carried out, let me recall, only in high equivariance classes $k \geq 4$.

The main idea of [4] is to treat the *constructed* two-bubbles as a two-dimensional manifold, around which any presumed two-bubble is modulated. The result can be thought of as a kind of weak-strong uniqueness, and indeed some parts of it resemble the proof of weak-strong uniqueness for wave maps by Struwe [99]. The first and main step is to obtain refined asymptotic behaviour of the two-bubble solutions constructed in [46], which is done in [3]. Here, I only state the main result of this rather technical paper.

I need to define *profiles* refining the two-bubble ansatz. I define $C^\infty(0,\infty)$ functions $A, B, \widetilde{B}$ as the unique solutions to the equations

$$\begin{aligned}
LA &= -\Lambda_0\Lambda Q, & 0 &= \langle A, \Lambda Q\rangle,\\
LB &= \gamma_k\Lambda Q - 4r^{k-2}(\Lambda Q)^2, & 0 &= \langle B, \Lambda Q\rangle,\\
L\widetilde{B} &= -\gamma_k\Lambda Q + 4r^{-k-2}(\Lambda Q)^2, & 0 &= \langle \widetilde{B}, \Lambda Q\rangle,
\end{aligned}$$

where L has the same meaning as in Lemma 4.2 and $\gamma_k := \frac{4k^2}{\pi}\sin(\pi/k)$ is the number making the right hand sides L^2-orthogonal to ΛQ. A standard ODE analysis shows that

$$|A(r)| \simeq r^k, \quad |B(r)| \simeq r^k, \quad |\widetilde{B}(r)| \simeq r^k|\log r| \quad \text{as } r \to 0,$$
$$|A(r)| \simeq r^{-k+2}, \quad |B(r)| \simeq r^{-k+2}, \quad |\widetilde{B}(r)| \simeq r^{-k+2}, \quad \text{as } r \to \infty.$$

It is convenient to work in the phase space $\mathcal{E} := \mathcal{H} \times L^2$. I denote $\boldsymbol{\psi} = (\psi, \dot{\psi})$ the elements of this space. Given a time interval $J \subset \mathbb{R}$ and C^1 functions $(\mu(t), \lambda(t), a(t), b(t))$ on J, I define the 2-bubble ansatz,

$$\boldsymbol{\Psi}(\mu(t), \lambda(t), a(t), b(t), r) = (\Psi(\mu(t), \lambda(t), a(t), b(t), r), \dot{\Psi}(\mu(t), \lambda(t), a(t), b(t), r))$$

by

$$\begin{aligned} \Psi(\mu, \lambda, a, b) &:= (Q_\lambda + b^2 A_\lambda + \sigma^k B_\lambda) - (Q_\mu + a^2 A_\mu + \sigma^k \widetilde{B}_\mu) \\ \dot{\Psi}(\mu, \lambda, a, b) &:= \lambda^{-1}\big(b\Lambda Q_\lambda + b^3 \Lambda A_\lambda - 2\gamma_k b\sigma^k A_\lambda + b\sigma^k \Lambda B_\lambda - kb\sigma^k B_\lambda - ka\sigma^{k+1} B_\lambda\big) \\ &\quad + \mu^{-1}\big(a\Lambda Q_\mu + a^3 \Lambda A_\mu + 2\gamma_k a\sigma^k A_\mu + a\sigma^k \Lambda \widetilde{B}_\mu + kb\sigma^{k-1}\widetilde{B}_\mu + ka\sigma^k \widetilde{B}_\mu\big), \end{aligned} \tag{4.7}$$

where $\sigma := \lambda/\mu$. The reason of the restriction $k \geq 4$ is that the proof requires $\dot{\Psi} \in L^2$.

Before stating the main result from [3], I need to define a function space corresponding to H^2 regularity. Note that $\|\psi_0\|_{\mathcal{H}} = \langle \psi_0, L_0 \psi_0 \rangle$, thus it is natural to set $\|\psi_0\|_{\mathcal{H}^{(2)}} := \langle \psi_0, L_0^2 \psi_0 \rangle$ and $\|\boldsymbol{\psi}_0\|^2_{\mathcal{E}^{(2)}} := \|\psi_0\|^2_{\mathcal{H}^{(2)}} + \|\dot{\psi}_0\|^2_{\mathcal{H}}$. Finally, for $\boldsymbol{\psi}_0 = (\psi_0, \dot{\psi}_0)$, I denote $\boldsymbol{\Lambda}\boldsymbol{\psi}_0 := (\Lambda\psi_0, \Lambda_0 \dot{\psi}_0)$.

Theorem 4.8 (A refined two-bubble construction)**.** *Let $k \geq 4$. There exists a global-in-time solution $\boldsymbol{\psi}_c(t) \in \mathcal{E}$ of* (WM) *which is a two-bubble in forward time and has the following additional properties:*

- $\boldsymbol{\psi}_c(t) \in \mathcal{E}^{(2)}$ *and* $\boldsymbol{\Lambda}\boldsymbol{\psi}_c(t) \in \mathcal{E}$,
- *There exist $T_0 > 0$, $C^1([T_0, \infty))$ functions $(\mu_c(t), \lambda_c(t), a_c(t), b_c(t))$, and $\boldsymbol{w}_c(t) \in \mathcal{E}$ such that on the time interval $[T_0, \infty)$ the solution $\boldsymbol{\psi}_c(t)$ decomposes as*

$$\boldsymbol{\psi}_c(t) = \boldsymbol{\Psi}(\mu_c(t), \lambda_c(t), a_c(t), b_c(t)) + \boldsymbol{w}_c(t),$$

where $\boldsymbol{\Phi}$ is defined in (4.7)*, and the functions $(\mu_c(t), \lambda_c(t), a_c(t), b_c(t))$ satisfy*

$$\begin{aligned} \lambda_c(t) &= q_k t^{-\frac{2}{k-2}} + O(t^{-\frac{6}{k-2}+\epsilon}), \\ \mu_c(t) &= 1 - \frac{kq_k^2}{2(k+2)} t^{-\frac{4}{k-2}} + O(t^{-\frac{6}{k-2}+\epsilon}), \\ b_c(t) &= \frac{2q_k}{k-2} t^{-\frac{k}{k-2}} + O(t^{-\frac{k+4}{k-2}+\epsilon}), \\ a_c(t) &= \frac{2kq_k^2}{(k-2)(k+2)} t^{-\frac{k+2}{k-2}} + O(t^{-\frac{k+6}{k-2}+\epsilon}) \end{aligned}$$

as $t \to \infty$, where $\epsilon > 0$ is any small number. Moreover,

$$\begin{aligned} |\lambda_c'(t) + b_c(t)| &\lesssim t^{-\frac{4k-2}{k-2}}, \quad \text{as } t \to \infty, \\ |\mu_c'(t) - a_c(t)| &\lesssim t^{-\frac{4k-2}{k-2}}, \quad \text{as } t \to \infty. \end{aligned}$$

Finally, $\boldsymbol{w}_c(t)$ satisfies

$$\begin{aligned} \|\boldsymbol{w}_c(t)\|^2_{\mathcal{E}} &\lesssim \lambda_c(t)^{3k-2}, \\ \|\boldsymbol{w}_c(t)\|^2_{\mathcal{E}^{(2)}} &\lesssim \lambda_c(t)^{3k-4}, \\ \|\boldsymbol{\Lambda}\boldsymbol{w}_c(t)\|^2_{\mathcal{E}} &\lesssim \lambda_c(t)^{2k-2}. \end{aligned}$$

□

Until the end of this section, $\boldsymbol{\psi}_c, \lambda_c, \mu_c$ and $\boldsymbol{w}_c$ are the objects obtained in the theorem above. I define the manifold

$$\mathcal{M}_s := \{(\boldsymbol{\psi}_c(t_0))_{\mu_0} : t_0 \gg 1, \mu_0 \in (0, \infty)\}.$$

A convenient parametrisation of $\mathcal{M}_s$ is given by

$$\boldsymbol{U}(\mu, \sigma) = \big(\boldsymbol{\psi}_c(\lambda_c^{-1}(\sigma))\big)_\mu, \qquad \mu \in (0, \infty),\ \sigma \in [0, \lambda_c(T_0)).$$

This way, $\boldsymbol{U}(\mu, \sigma) \simeq (Q_{\mu\sigma} - Q_\mu, 0)$. Let $\boldsymbol{\psi} = (\psi, \dot{\psi})$ be any positive two-bubble in the forward time direction. We define the *modulation parameters* $(\mu(t), \sigma(t))$ and the rest $\boldsymbol{g}(t)$ by

$$\boldsymbol{\psi}(t) = \boldsymbol{U}(\mu(t), \sigma(t)) + \boldsymbol{g}(t),$$
$$\langle \Lambda Q_{\mu(t)\sigma(t)}, g(t)\rangle = \langle \Lambda Q_{\mu(t)}, g(t)\rangle = 0.$$

Existence of $\mu(t)$ and $\sigma(t)$ can be proved similarly as in the case of the usual modulation analysis, but Theorem 4.8 is already necessary. Theorem 3 is now equivalent to $\boldsymbol{g}(t) = 0$ (for some, equivalently for all, t large enough).

The conservation of energy yields

$$\begin{aligned} 2E_p(Q) &= E(\boldsymbol{U}(\mu(t), \sigma(t))) = E(\boldsymbol{u}(t)) = E(\boldsymbol{U}(\mu(t), \sigma(t)) + \boldsymbol{g}(t)) \\ &= E(\boldsymbol{U}(\mu(t), \sigma(t))) + \langle \mathrm{D}E(\boldsymbol{U}(\mu(t), \sigma(t))), \boldsymbol{g}(t)\rangle + \langle \mathrm{D}^2E(\boldsymbol{U}(\mu(t), \sigma(t)))\boldsymbol{g}(t), \boldsymbol{g}(t)\rangle + o(\|\boldsymbol{g}\|_{\mathcal{H}}^2). \end{aligned}$$

Applying Lemma 4.3, one can show that

$$\langle \mathrm{D}^2E(\boldsymbol{U}(\mu(t), \sigma(t)))\boldsymbol{g}(t), \boldsymbol{g}(t)\rangle \gtrsim \|\boldsymbol{g}\|_{\mathcal{E}}^2,$$

thus

$$-\langle \mathrm{D}E(\boldsymbol{U}(\mu(t), \sigma(t))), \boldsymbol{g}(t)\rangle \gtrsim \|\boldsymbol{g}(t)\|_{\mathcal{E}}^2. \tag{4.8}$$

This inequality induces an "instability" and prevents $\boldsymbol{g}(t)$ from converging to 0, unless $\boldsymbol{g}(t) = 0$. The instability can be captured by considering the function

$$b(t) := -\frac{1}{\rho_k \sigma(t)^{\frac{k}{2}}}\langle \mathrm{D}E(\boldsymbol{U}(\mu(t), \sigma(t))), \boldsymbol{g}(t)\rangle - \langle \mathcal{A}_0(\mu(t)\sigma(t))g(t), \dot{g}(t)\rangle.$$

The bound (4.8) implies

$$b(t) \gtrsim \sigma(t)^{-k/2}\|\boldsymbol{g}(t)\|_{\mathcal{E}}^2 \gg \|\boldsymbol{g}(t)\|_{\mathcal{E}}^2 > 0. \tag{4.9}$$

Let me stress that $b(t)$ is not the same function as the one defined in (4.4), but it uses the same localised scaling operator $\mathcal{A}_0$. Crucially, the following bound, analogous to (4.6), is true.

Proposition 4.9. *Let $\boldsymbol{\psi}(t) \in \mathcal{H}$ be a two-bubble in forward time. For any $c_0 > 0$ there exists $T_0 > 0$ such that for all $t \geq T_0$*

$$b'(t) \geq \frac{k\rho_k}{2\mu(t)\sigma(t)}\sigma(t)^{\frac{k}{2}}b(t) - c_0\frac{1}{\mu(t)\sigma(t)}\Big(b(t)\sigma(t)^{\frac{k}{2}} + \|\boldsymbol{g}(t)\|_{\mathcal{H}}^2\Big). \tag{4.10}$$

□

The lower bound (4.9) and the differential inequality (4.10) lead to a growth of $b(t)$, unless $b(t) = \boldsymbol{g}(t) = 0$, which finishes the proof of Theorem 3.

Remark 4.10. It is, a priori, possible to consider more bubbles, concentrating at rates $0 < \lambda_1 \ll \lambda_2 \ll \ldots \ll \lambda_n$, but for equivariant wave maps even constructions are not yet available. For the energy-critical *heat* equation in high dimensions, solutions of this type were constructed by del Pino, Musso and Wei [81].

4.3 No-return analysis and the collision problem

I would like to explain now how to prove, following [2], that a wave map which approaches a superposition of two bubbles for a sequence of times, is in fact a two-bubble in one time direction.

Like in the discussion of the dynamical role of kink-antikink pairs in Section 3.4, I introduce two formal critical points at infinity $\boldsymbol{Q}_+$ and $\boldsymbol{Q}_-$. The distance from a given state $\boldsymbol{\psi}_0 = (\psi_0, \dot{\psi}_0) \in \mathcal{E}$ to these ideal points is defined by

$$\|\boldsymbol{\psi}_0 - \boldsymbol{Q}_+\|_{\mathcal{E}}^2 := \inf_{\lambda,\mu>0} \big(\|(\psi_0 - (Q_\lambda - Q_\mu), \dot{\psi}_0)\|_{\mathcal{E}}^2 + (\lambda/\mu)^k\big),$$
$$\|\boldsymbol{\psi}_0 - \boldsymbol{Q}_-\|_{\mathcal{E}}^2 := \inf_{\lambda,\mu>0} \big(\|(\psi_0 + (Q_\lambda - Q_\mu), \dot{\psi}_0)\|_{\mathcal{E}}^2 + (\lambda/\mu)^k\big).$$

Thus, $\boldsymbol{Q}_+$ represents a positive two-bubble which goes from 0 at $r = 0$ to π, and then back to 0 at $r \to \infty$, whereas $\boldsymbol{Q}_-$ represents a negative two-bubble going from 0 at $r = 0$ to $-\pi$, and then back to 0 at $r \to \infty$. I denote

$$\mathbf{d}(\boldsymbol{\psi}_0) := \min\{\|\boldsymbol{\psi}_0 - \boldsymbol{Q}_+\|_{\mathcal{E}}, \|\boldsymbol{\psi}_0 - \boldsymbol{Q}_+\|_{\mathcal{E}}\},$$

so that $\mathbf{d}(\boldsymbol{\psi}_0)$ is small if and only if the state $\boldsymbol{\psi}_0$ is close to a two-bubble configuration, either positive or negative.

The following fact is an elementary consequence of the Bogomolny trick:

Lemma 4.11. *For all $M > 0$ there exist $\epsilon_1 = \epsilon_1(M)$ and $\epsilon_2 = \epsilon_2(M)$ such that*

$$\mathbf{d}(\boldsymbol{\psi}_0) \leq \epsilon_1 \ \Rightarrow\ \|\boldsymbol{\psi}_0\|_{\mathcal{E}} \geq M,$$
$$E(\boldsymbol{\psi}_0) \leq 8k\pi \text{ and } \|\boldsymbol{\psi}_0\|_{\mathcal{E}} \geq M \ \Rightarrow\ \mathbf{d}(\boldsymbol{\psi}_0) \leq \epsilon_2.$$

□

Following the *Kenig-Merle route map*, one can show that any solution bounded in $\mathcal{E}$ has to scatter. Combined with the above lemma, one deduces the following result, essentially proved in [23].

Theorem 4.12. *Let $\boldsymbol{\psi} : (T_-, T_+) \to \mathcal{E}$ be a wave map of energy $E(\psi, \partial_t\psi) \leq 8k\pi$. If $\boldsymbol{\psi}$ does not scatter in the positive time direction, then there exists a sequence $t_n \to T_+$ such that*

$$\lim_{n\to\infty} \mathbf{d}(\boldsymbol{\psi}(t_n)) = 0.$$

□

The possibility of $\boldsymbol{\psi}(t)$ coming arbitrarily close to a two-bubble configuration for a sequence of times, but leaving its neighbourhood for another sequence of times, is for the moment not excluded. Such a result is called a *no-return lemma*.

Its proof based on the *virial identity*. I fix a cut-off function $\chi \in C_0^\infty([0,\infty[)$ such that

$$\chi(r) = 1 \text{ for } r \leq 1, \quad \chi(r) = 0 \text{ for } r \geq 3, \quad |\chi'(r)| \leq 1 \text{ for all } r \geq 0.$$

For any $R > 0$, I define

$$\chi_R(r) := \chi(r/R)$$

By a direct computation, one obtains the following claim.

Lemma 4.13. *If $\boldsymbol{\psi} : I \to \mathcal{E}$ is a solution of* (WM)*, then for all $t \in I$ and $R > 0$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \chi_R\, \partial_t\psi\, r\partial_r\psi\, r\mathrm{d}r = -\int_0^\infty (\partial_t\psi)^2\, r\mathrm{d}r + O(\mathcal{E}_R(\boldsymbol{\psi})),$$

where

$$\mathcal{E}_R(\boldsymbol{\psi}(t)) := \int_R^\infty \Big((\partial_t\psi)^2 + (\partial_r\psi)^2 + k^2\frac{\sin^2\psi}{r^2}\Big)\, r\,\mathrm{d}r.$$

□

Heuristically, if $T_- < T_1 < T_2 < T_+$ and $\mathbf{d}(\boldsymbol{\psi}(T_j)) \ll 1$ for $j \in \{1,2\}$, then in particular $\|\partial_t\psi(T_j)\|_{L^2} \ll 1$, thus

$$\Big|\int_0^\infty \chi_R\partial_t\psi(T_j)\, r\partial_r\psi(T_j)\, r\mathrm{d}r\Big| \ll R.$$

In order to make the cut-off errors small, R has to be much larger than the scale of the less concentrated bubble. If $\boldsymbol{\psi}$ is far from a two-bubble configuration, thus away from a critical point, then it is an oscillating wave, so one can expect that

$$\int_{T_1}^{T_2} |\partial_t\psi|_{L^2}^2\, \mathrm{d}t \simeq T_2 - T_1,$$

which indeed turns out to be true. Moreover, the *compactness property* of Kenig-Merle allows to absorb the error term $\mathcal{E}_R(\psi(t), \partial_t\psi(t))$ for states far from a two-bubble configuration (I have to skip the details of this argument, which would require a full discussion of the Kenig-Merle route map). Hence, the no-return lemma would be proved if I knew that:

- each time a two-bubble is annihilated, the solution stays far from a two-bubble configuration during a time interval whose length is comparable to the scale of the least concentrated bubble,
- the cut-off error $\mathcal{E}_R(\psi(t), \partial_t\psi(t))$ can be absorbed for states close to a two-bubble configuration.

Fortunately, both points can be checked using the modulation analysis from Section 4.1. In order to explain the main idea, let me assume $\mu(t) \equiv 1$. If $\boldsymbol{\psi}$ is close to a two-bubble configuration on a finite time interval I, but the bubbles enter into a collision at the endpoints of the interval, then $\lambda(t)$ attains its minimum value λ_0 at some $t_0 \in I$. The inequalities (4.5) and (4.6) provide a lower bound on the growth of λ in both time directions:

$$\lambda(t) \gtrsim \big(\lambda_0^{-\frac{k-2}{2}} - c_0|t-t_0|\big)^{-\frac{2}{k-2}}, \qquad \text{for some } c_0 > 0 \text{ and all } |t - t_0| \ll \lambda^{-\frac{k-2}{2}}.$$

The crucial point is that, regardless of how small $\mathbf{d}(\boldsymbol{\psi}(t_0))$ is, $\int_I \lambda(t)^{\frac{k}{2}}\,\mathrm{d}t \lesssim 1$. The coercivity estimate (4.3) implies that $\mathcal{E}_R(\boldsymbol{\psi}(t)) \lesssim \lambda(t)^{\frac{k}{2}}$, with a constant independent of R. Combining the two bounds, one can absorb the cut-off error $\mathcal{E}_R$.

Remark 4.14. In the language of the formal critical points $\boldsymbol{Q}_\pm$, I could rephrase the content of this section by saying that there is no homoclinic orbit connecting $\boldsymbol{Q}_-$ to itself, no homoclinic orbit connecting $\boldsymbol{Q}_+$ to itself, and no heteroclinic orbit connecting $\boldsymbol{Q}_-$ to $\boldsymbol{Q}_+$.

Remark 4.15. Let me stress again that no no-return lemma is available for kink-antikink pairs.

4.4 Multi-bubbles for an energy-critical wave equation

In the last section, I present the results with Martel [5], which do not concern the wave maps, but rather solutions of the energy-critical wave equation with a power nonlinearity, sometimes viewed as a simplified version of the critical wave maps.

I consider the following equation:

$$\partial_t^2 u(t,x) - \Delta u(t,x) - f(u(t,x)) = 0, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^5, \tag{NLW}$$

where

$$f(u) := |u|^{4/3}u.$$

The scaling of the equation is given by $u_\lambda(t,x) := \lambda^{-3/2}u(t/\lambda, x/\lambda)$, meaning that u is a solution if and only if u_λ is a solution as well. The energy is given by

$$E(u_0,\dot u_0) := \int_{\mathbb{R}^5}\Big(\frac12(\dot u_0(x))^2 + \frac12|\nabla u_0(x)|^2 + F(u_0(x))\Big)\,\mathrm{d}x,$$

where $F(u) := \frac{3}{10}|u|^{10/3}$. The energy space is $\mathcal{E} := H^1(\mathbb{R}^5)\times L^2(\mathbb{R}^5)$. Since the energy of u_λ is the same as the energy of u, the equation can be called energy-critical.

Unlike for wave maps, the classification of all the stationary states is unknown. I only consider the *ground states*, which are given by

$$Q\big((x-z_0)/\lambda\big), \quad Q(x) := (1+|x|^2/15)^{-3/2}, \quad (x_0,\lambda)\in\mathbb{R}^5\times(0,\infty).$$

The function Q is the maximiser of the critical Sobolev embedding, see [11, 104].

I fix distinct points $z_1,\ldots,z_n\in\mathbb{R}^5$, and set

$$Q(\lambda_1,\ldots,\lambda_n) := \sum_{j=1}^n Q\big((x-z_j)/\lambda_j\big), \quad (\lambda_1,\ldots,\lambda_n)\in(0,\infty)^n.$$

If $\lambda_1,\ldots,\lambda_n \ll 1$, then $Q(\lambda_1,\ldots,\lambda_n)$ is a superposition of n bubbles centered at $z_1,\ldots,z_n$. The reduced Lagrangian and the formal modulation equations can be computed explicitly. At main order, one obtains

$$\lambda_j''(t) \simeq -B_j(\lambda_1(t),\ldots,\lambda_n(t)), \tag{4.11}$$

where

$$B_j(\lambda_1,\ldots,\lambda_n) := -\frac{128\sqrt{15}}{7\pi}\lambda_j^{1/2}\sum_{l\neq j}\lambda_l^{3/2}|z_l-z_j|^{-3}.$$

It is natural to seek self-similar solutions of (4.11), that is solutions of the form $\lambda_j(t) = c_j t^{-2}$. The vector $(c_1,\ldots,c_n)$, the *central configuration* of our problem, has to satisfy

$$B_j(c_1,\ldots,c_n) = -6c_j, \qquad \text{for all } j\in\{1,\ldots,n\}.$$

As for the gravitational n-body problem, central configurations have a variational characterisation. In fact, it is rather easy to see that $(c_1,\ldots,c_n)$ is proportional to a critical point of the functional

$$V(\theta_1,\ldots,\theta_n) := \sum_j\sum_{l<j}\theta_l^{3/2}\theta_j^{3/2}|z_j-z_l|^{-3}, \qquad (\theta_1,\ldots,\theta_n)\in\mathbb{S}^{n-1}, \theta_j\geq 0.$$

One can check that the global minimum of V is attained at a point $(\theta_1,\ldots,\theta_n)$ such that $\theta_j > 0$ for all j, yielding the desired central configuration.

Theorem 4.16. *Let $z_1, \ldots, z_n$ be distinct points in $\mathbb{R}^5$. There exist numbers $c_j > 0$ and a solution u of* (NLW) *such that*

$$\lim_{t\to\infty} \left(\big\|u(t) - Q(c_1 t^{-2}, \ldots, c_n t^{-2})\big\|_{\dot H^1} + \|\partial_t u(t)\|_{L^2}\right) = 0.$$

The main novelty of the paper [5] is the stability analysis of the self-similar solutions to (4.11) found above. Let me just stress that the variational structure plays an important role. As testified by the analysis of the gravitational n-body problem in Section 3.A, such a variational structure is a common feature of problems of similar kind.

Remark 4.17. An analogous problem for the wave maps equation, which would require to consider the full non-equivariant system, is completely open.

Remark 4.18. A somewhat similar construction in the parabolic case was obtained by Cortazar, del Pino and Musso [21].

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