Dispersive partial differential equations
A very brief introduction

Jacek Jendrej
CNRS & Université Sorbonne Paris Nord

Lectures given at the University of Warsaw in March and April 2024

Chapter 1

The wave equation

The *wave equation* is
$$\partial_t^2 u(t,x) - c^2 \Delta u(t,x) = 0, \tag{1.0.1}$$
where $t \in \mathbb{R}$ is interpreted as a moment time, $x \in \mathbb{R}^d$ as a position in the d-dimensional Euclidean space, $c > 0$ is a constant parameter (the propagation speed of the waves) and $u(t,x)$ is a real or complex scalar. We will always denote $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2 + \ldots + \partial_{x_d}^2$ the Laplace operator with respect to the spatial variables.

1.1 Mechanical derivations of the wave equation

1.1.1 String vibration using Newton's second law

Recall how the wave equation is derived in elementary dynamics. Suppose a string is described by the graph of a function u of a single variable $x \in \mathbb{R}$. We will assume that the string is almost at rest, so that $|\partial_x u(t,x)|$ is small for all t and x. Under this assumption, it is reasonable to assume that each part of the string is subject to a force of constant magnitude $F > 0$ along the direction of the string.

Fix a time t and consider a part of the string contained between x and $x + dx$. Using the small slope assumption again, we realize that the vertical force acting on it is at main order given by
$$F\partial_x u(t, x + dx) - F\partial_x u(t,x) \sim F\partial_x^2 u(t,x)dx.$$
Since the acceleration in the vertical direction is given by $\partial_t^2 u(t,x)$, denoting μ the mass density per unit length and applying Newton's 2nd law, we obtain
$$\mu\partial_t^2 u(t,x)dx = F\partial_x^2 u(t,x)dx,$$
which is (1.0.1) with $c := \sqrt{F/\mu}$.

1.1.2 String vibration using a Lagrangian

We fix again t and consider the part of the string contained between x and $x + dx$. The kinetic energy of this part is given by $\frac{1}{2}\mu(\partial_t u(t,x))^2\,dx$. The potential energy is given by F multiplied by the extension of the string. By Pythagore's theorem, this extension equals at main order

$$dx\big(\sqrt{1+(\partial_x u(t,x))^2}-1\big) \sim \frac{1}{2}(\partial_x u(t,x))^2 dx.$$

We thus obtain the Lagrangian density $\frac{1}{2}\mu(\partial_t u(t,x))^2 - \frac{1}{2}(\partial_x u(t,x))^2$. The corresponding Euler-Lagrange equation is (1.0.1) with $c := \sqrt{F/\mu}$.

1.1.3 Drum vibration using Newton's second law

Analogous reasoning can be done for $d \geq 2$. In this case, we assume that every part of the drum is subject to a force directed in the normal direction, whose magnitude per $(d-1)$-dimensional volume is constant and equal to F.

Fix a time t and consider a part of the drum in the ball $B(x, dx)$. The force acting on it is at main order given by

$$F\int_{\partial B(x,dx)} \partial_n u(t,y)\sigma(dy) = F\int_{B(x,dx)} \Delta u(t,y)dy \sim \mathrm{vol}(B(x,dx))F\Delta u(t,x).$$

The rest of the argument is exactly the same as for $d = 1$.

1.1.4 Drum vibration using a Lagrangian

The area extension of the part of the drum in the ball $B(x, dx)$ is given at main order by

$$\mathrm{vol}(B(x,dx))\big(\sqrt{1+|\nabla u(t,x)|^2}-1\big) \sim \frac{1}{2}|\partial_x u(t,x)|^2\,\mathrm{vol}(B(x,dx)).$$

The rest of the argument is the same as for $d = 1$.

Remark 1.1.1. The physical theory where the wave equation appears the most frequently is probably the classical theory of the electromagnetism.

1.2 The Cauchy problem

In the remaining part of the lectures, we let $c := 1$. Intuitively, the movement of a vibrating string or drum should be uniquely determined by the initial positions and velocities of its parts. We are thus led to considering the so-called *Cauchy problem*

$$\begin{aligned} \partial_t^2 u(t,x) - \Delta u(t,x) = 0, \qquad & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) = u_0(x), \qquad \partial_t u(0,x) = \dot{u}_0(x), \qquad & x \in \mathbb{R}^d. \end{aligned} \tag{1.2.1}$$

the functions u_0 and $\dot{u}_0$ are called the *initial data* or *initial conditions*.

Remark 1.2.1. If the spatial domain is not the Euclidean space, then appropriate *boundary conditions* have to be imposed.

1.2.1 A solution using Fourier series

Assume that u_0 and $\dot{u}_0$ are trigonometric polynomials:

$$u_0(x) = (2\pi)^{-d} \sum_{|k|\le n} a_0(k) e^{ik\cdot x},$$
$$\dot{u}_0(x) = (2\pi)^{-d} \sum_{|k|\le n} \dot{a}_0(k) e^{ik\cdot x}$$

(we use the standard notation $|k| := \sum_{j=1}^{d} |k_j|$). We look for a solution of (1.2.1) of the form

$$u(t,x) = (2\pi)^{-d} \sum_{|k|\le n} a(t,k) e^{ik\cdot x}. \tag{1.2.2}$$

Substituting into (1.2.1) and comparing the coefficients of the polynomials on both sides, we obtain for all $k \in \mathbb{Z}^d$ such that $|k| \le n$

$$\partial_t^2 a(t,k) + |k|^2 a(t,k) = 0, \qquad \text{for all } t$$
$$a(0,k) = a_0(k), \quad \partial_t a(0,k) = \dot{a}_0(k).$$

The solution is given by

$$a(t,k) = a_0(k)\cos(|k|t) + \dot{a}_0(k)\frac{\sin(|k|t)}{|k|}. \tag{1.2.3}$$

Plugging into (1.2.2), we have solved the Cauchy problem.

The last result is frequently written in a "functional" way as follows. We introduce the differential operator $D_j := \frac{1}{i}\partial_{x_j}$. We write $D = (D_1, D_2, \ldots, D_d)$, which is a d-tuple of differential operators. Then

$$u(t) = \cos(|D|t)u_0 + \frac{\sin(|D|t)}{|D|}\dot{u}_0. \tag{1.2.4}$$

Let us explain where this notation comes from. First, we use the standard convention that $u(t)$ should be interpreted as the function $x \mapsto u(t,x)$. Next, denote P_n the linear space of trigonometric polynomials of degree at most n and observe that $D_j : P_n \to P_n$ is a linear operator whose matrix is diagonal in the basis $\{e^{ik\cdot x} : |k| \le n\}$, with the corresponding eigenvalues equal to k_j. It follows from (1.2.3) that

$$u(t) = S(t)u_0 + \dot{S}(t)\dot{u}_0,$$

where $S(t)$ and $\dot{S}(t)$ are linear maps $P_n \to P_n$. The matrix of $S(t)$ is diagonal in the basis $\{e^{ik\cdot x} : |k| \le n\}$, with the numbers $\cos(|k|t)$ on the diagonal. By the standard notation of Linear Algebra, we can thus write $S(t) = \cos(|D|t)$. Analogously, $\dot{S}(t) = \frac{\sin(|D|t)}{|D|}$.

Remark 1.2.2. Let $m : \mathbb{Z}^d \to \mathbb{C}$, and for any trigonometric polynomial $v(x) = \sum_{|k|\le n} b(k)e^{ik\cdot x}$ set

$$(Tv)(x) := \sum_{|k|\le n} m(k)b(k)e^{ik\cdot x}.$$

The operator T is called a *Fourier multiplication operator* and m the *Fourier multiplier*. As justified by the discussion above, we write in this case $T = m(D)$. The operators $S(t)$ and $\dot{S}(t)$ are thus Fourier multiplication operators.

It will be important later to generalize (1.2.1) by allowing a non-zero *forcing term* and consider the problem

$$\begin{aligned} \partial_t^2 u(t,x) - \Delta u(t,x) = f(t,x), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \qquad \partial_t u(0,x) = \dot{u}_0(x), \qquad x \in \mathbb{R}^d. \end{aligned} \tag{1.2.5}$$

If $f(t)$ is a trigonometric polynomial of degree $\leq n$ for all t:

$$f(t,x) = (2\pi)^{-d} \sum_{|k| \leq n} c(t,k) e^{ik \cdot x},$$

then proceeding as above we find

$$\begin{aligned} \partial_t^2 a(t,k) + |k|^2 a(t,k) = c(t,k), \qquad \text{for all } t \\ a(0,k) = a_0(k), \quad \partial_t a(0,k) = \dot{a}_0(k). \end{aligned}$$

If we assume that for every k the function $t \mapsto c(t,k)$ is locally integrable, then the solution is given by the *variation of constants formula* (or the *Duhamel formula*)

$$a(t,k) = a_0(k) \cos(|k|t) + \dot{a}_0(k) \frac{\sin(|k|t)}{|k|} + \int_0^t c(s,k) \frac{\sin(|k|(t-s))}{|k|} \, ds,$$

which is usually written as

$$u(t) = \cos(|D|t) u_0 + \frac{\sin(|D|t)}{|D|} \dot{u}_0 + \int_0^t \frac{\sin(|D|(t-s))}{|D|} f(s) \, ds. \tag{1.2.6}$$

If, instead of assuming that the initial data are given by trigonometric polynomials, we assume instead that u_0 and $\dot{u}_0$ are functions which are 2π-periodic in each variable, then they can be expanded in *Fourier series*

$$\begin{aligned} u_0(x) = (2\pi)^{-d} \sum_{k \in \mathbb{Z}^d} a_0(k) e^{ik \cdot x}, \\ \dot{u}_0(x) = (2\pi)^{-d} \sum_{k \in \mathbb{Z}^d} \dot{a}_0(k) e^{ik \cdot x}, \end{aligned}$$

and we can *define* the solution of (1.2.2) by the formula

$$u(t,x) = (2\pi)^{-d} \sum_{k \in \mathbb{Z}^d} a(t,k) e^{ik \cdot x}, \tag{1.2.7}$$

where $a(t,k)$ is given by (1.2.3). Some conditions have to be imposed on u_0 and $\dot{u}_0$ in order for the series (1.2.7) to converge in some sense, or one should interpret (1.2.7) in the sense of periodic distributions. We will not enter into these considerations. Whenever (1.2.7) is meaningful, by analogy with the case of trigonometric polynomials we express the solution using the notation (1.2.4).

1.2.2 A solution using Fourier transform

In the non-periodic case, a similar procedure is made possible using the *Fourier transform*, which we briefly recall.

Let μ be a complex-valued Borel measure on $\mathbb{R}^d$ of finite total variation. We define its Fourier transform:

$$(\mathcal{F}\mu)(\xi) = \widehat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\mu(dx), \qquad \forall \xi \in \mathbb{R}^d.$$

We see that $\widehat{\mu}$ is a bounded continuous function. If $f \in L^1(dx)$, we set $\mathcal{F}f := \mathcal{F}(f\,dx)$.

Definition 1.2.3. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the space of complex-valued functions $f \in C^\infty(\mathbb{R}^d)$ such that for any multi-indices $\alpha, \beta \in \mathbb{N}^d$

$$x^\alpha \partial^\beta f \in L^\infty(\mathbb{R}^d).$$

We say that a sequence $f_n \in \mathcal{S}(\mathbb{R}^d)$ converges to $f \in \mathcal{S}(\mathbb{R}^d)$ if for any multi-indices α, β

$$\lim_{n\to\infty} \|x^\alpha \partial^\beta (f_n - f)\|_{L^\infty} = 0.$$

We recall without proofs a few standard facts about the Fourier transform.

Proposition 1.2.4. *The Fourier transform* $\mathcal{F}$ *is continuous* $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ *and onto. For any* $f, f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$*:*

- *(Fourier transform and derivatives)*

$$(i\partial)^\alpha \widehat{f}(\xi) = \mathcal{F}(x^\alpha f)(\xi), \qquad \textit{for all } \alpha \in \mathbb{N}^d,$$
$$(i\xi)^\alpha \widehat{f}(\xi) = \mathcal{F}(\partial^\alpha f)(\xi), \qquad \textit{for all } \alpha \in \mathbb{N}^d,$$

- *(Fourier inversion formula)*

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi} \frac{d\xi}{(2\pi)^d}, \qquad \forall x \in \mathbb{R}^d,$$

- *(Plancherel's formula)*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}} \|f\|_{L^2(\mathbb{R}^d)},$$

- *(Fourier transform and convolutions)*

$$\mathcal{F}(f_1 * f_2) = (\mathcal{F}f_1)(\mathcal{F}f_2),$$

  where

$$(f_1 * f_2)(x) := \int_{\mathbb{R}^d} f_1(x-y) f_2(y)\, dy.$$

□

Remark 1.2.5. It is convenient to introduce the space of continuous linear functionals $\mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$, which is called the space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^d)$, and extend the Fourier transform to $\mathcal{S}'(\mathbb{R}^d)$. We will not discuss these topics here.

Assume that u_0 and $\dot{u}_0$ are in $\mathcal{S}(\mathbb{R}^d)$. We can thus write

$$u_0(x) = \int_{\mathbb{R}^d} a_0(\xi)e^{i\xi\cdot x}\frac{d\xi}{(2\pi)^d}, \quad \dot{u}_0(x) = \int_{\mathbb{R}^d} \dot{a}_0(\xi)e^{i\xi\cdot x}\frac{d\xi}{(2\pi)^d},$$

where $a_0 = \mathcal{F}u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $\dot{a}_0 = \mathcal{F}\dot{u}_0 \in \mathcal{S}(\mathbb{R}^d)$. For every $\xi \in \mathbb{R}^d$, let $t \mapsto a(t,\xi)$ be the solution of the ODE

$$\partial_t^2 a(t,\xi) + |\xi|^2 a(t,\xi) = 0, \qquad \text{for all } t$$
$$a(0,\xi) = a_0(\xi), \quad \partial_t a(0,\xi) = \dot{a}_0(\xi),$$

in other words

$$a(t,\xi) = a_0(\xi)\cos(|\xi|t) + \dot{a}_0(\xi)\frac{\sin(|\xi|t)}{|\xi|}.$$

By differentiating under the integral, we see that

$$u(t,x) = \int_{\mathbb{R}^d} a(t,\xi)e^{i\xi\cdot x}\frac{d\xi}{(2\pi)^d}$$

is a solution of the Cauchy problem. By analogy with the case of trigonometric polynomials, we express this solution in the form (1.2.4).

Remark 1.2.6. Let us check that the energy

$$E = \frac{1}{2}\int_{\mathbb{R}^d} |\partial_t u(t,x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx$$

is conserved. By Plancherel's theorem we have

$$\int_{\mathbb{R}^d} |\dot{u}_0(x)|^2\,dx = \int_{\mathbb{R}^d} |\dot{a}_0(\xi)|^2\frac{d\xi}{(2\pi)^d}, \qquad \int_{\mathbb{R}^d} |\nabla u_0(x)|^2\,dx = \int_{\mathbb{R}^d} |\xi|^2|a_0(\xi)|^2\frac{d\xi}{(2\pi)^d}$$

and

$$\int_{\mathbb{R}^d} |\partial_t u(t,x)|^2\,dx = \int_{\mathbb{R}^d} |\partial_t a(t,\xi)|^2\frac{d\xi}{(2\pi)^d}, \qquad \int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx = \int_{\mathbb{R}^d} |\xi|^2|a(t,\xi)|^2\frac{d\xi}{(2\pi)^d}.$$

For every ξ the function $t \mapsto \frac{1}{2}|\partial_t a(t,\xi)|^2 + \frac{1}{2}|\xi|^2|a(t,\xi)|^2$ is constant, hence the conservation of energy follows by integrating in ξ.

Remark 1.2.7. If we allow a forcing term, we get analogously the formula (1.2.6) for the solution of the Cauchy problem. In this case, instead of the energy conservation, one can similarly obtain the *energy inequality*

$$\sqrt{\|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2} \leq \sqrt{\|\dot{u}_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2} + \int_0^t \|f(s)\|_{L^2}\,ds. \tag{1.2.8}$$

Remark 1.2.8. Let $m \in \mathcal{S}(\mathbb{R}^d)$, and for any $v \in \mathcal{S}(\mathbb{R}^d)$ set

$$(Tv)(x) := \mathcal{F}^{-1}\big(\xi \mapsto m(\xi)(\mathcal{F}v)(\xi)\big).$$

The operator T is called a *Fourier multiplication operator* and m the *Fourier multiplier*. We write in this case $T = m(D)$.

It follows from Proposition 1.2.4 that any Fourier multiplication operator can be expressed as a convolution:

$$Tv = K * v, \qquad \text{where } K := \mathcal{F}^{-1}m.$$

1.3 A dispersive estimate

Intuitively, if $d \geq 2$ and the initial data are localized, then for $t \gg 1$ the wave should spread in space, and thus decay in amplitude. The goal of this section is to prove the following estimate quantifying this phenomenon.

Proposition 1.3.1. *There exists* $C \geq 0$ *such that for all* $v \in \mathcal{S}(\mathbb{R}^d)$ *such that* $\operatorname{supp}(\mathcal{F}v) \subset \{\frac{1}{4} \leq |\xi| \leq 4\}$ *and all* $t \in \mathbb{R}$

$$\| \cos(t|D|)v\|_{L^\infty} \leq C\langle t\rangle^{-\frac{d-1}{2}}\|v\|_{L^1}, \tag{1.3.1}$$

$$\|\frac{\sin(t|D|)}{|D|}v\|_{L^\infty} \leq C\langle t\rangle^{-\frac{d-1}{2}}\|v\|_{L^1}. \tag{1.3.2}$$

The bound (1.3.2) is proved similarly as (1.3.1), so we will only focus on (1.3.1) in the discussion below.

Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(\rho) = 1$ for $\frac{1}{4} \leq \rho \leq 4$ and $\chi(\rho) = 0$ for $\rho \leq \frac{1}{8}$ or $\rho \geq 8$. By our assumption, we have

$$\cos(|D|t)v = \cos(|D|t)\chi(|D|)v,$$

thus according to Remark 1.2.8 we can write

$$\big(\cos(|D|t)v\big)(x) = (K_t * v)(x),$$

where

$$K_t(x) := \int_{\mathbb{R}^d} e^{i\xi\cdot x}\cos(|\xi|t)\chi(|\xi|)\frac{d\xi}{(2\pi)^d}.$$

Hence, it suffices to show that

$$\|K_t\|_{L^\infty} \lesssim \langle t\rangle^{-\frac{d-1}{2}}. \tag{1.3.3}$$

Changing to polar coordinates, $\xi = \rho\eta$ with $\eta \in \mathbb{S}^{d-1}$, we find

$$K_t(x) = \int_0^\infty \cos(t\rho)\chi(\rho)\rho^{d-1}\int_{\mathbb{S}^{d-1}} e^{i\rho\eta\cdot x}\frac{\sigma(d\eta)}{(2\pi)^d}\,d\rho,$$

where σ is the surface measure of $\mathbb{S}^{d-1}$. Assume first that $r := |x| \leq \frac{1}{2}t$. We change the order of integration, express $\cos(t\rho)$ as $\frac{1}{2}(e^{it\rho} + e^{-it\rho})$, and obtain

$$K_t(x) = \frac{1}{2}\int_{\mathbb{S}^{d-1}}\left(\int_0^\infty (e^{i(t+\eta\cdot x)\rho} + e^{i(-t+\eta\cdot x)\rho})\chi(\rho)\rho^{d-1}\,d\rho\right)\frac{\sigma(d\eta)}{(2\pi)^d}.$$

In the inner integral, we recognize the inverse Fourier transform of the function $\chi(\rho)\rho^{d-1}$, which is of Schwartz class. Since $|\pm t + \eta\cdot x| \geq \frac{1}{2}t$, the inner integral decays faster than any power of t.

We now assume that $r \geq \frac{1}{2}t$.

<u>First method</u> (using the theory of Bessel functions). By the formula for the inverse Fourier transform of spherically symmetric functions, see for example [3, Theorem 3.3], we have

$$K_t(x) = (2\pi)^{-\frac{d}{2}}r^{-\nu}\int_0^\infty \cos(t\rho)\chi(\rho)J_\nu(r\rho)\rho^{\frac{d}{2}}\,d\rho,$$

where $\nu := \frac{d-2}{2} \geq 0$ and J_ν is the Bessel function of the first kind. It is well-known that J_ν decays like inverse square root, hence the result.

Second method (self-contained). If we parametrize $\mathbb{S}^{d-1}$ as $\eta = \frac{x}{r}\sin\phi + \zeta\cos\phi$, where $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$ and $\zeta \in \mathbb{S}^{d-2}$, we obtain

$$\int_{\mathbb{S}^{d-1}} e^{i\rho\eta\cdot x}\sigma(d\eta) = \mathrm{vol}(\mathbb{S}^{d-2}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\phi)^{d-2} e^{ir\sin\phi}\, d\phi.$$

If $k \in \mathbb{Z}$ and $k \geq 2$, then for any smooth function $a(\phi)$ an integration by parts yields

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\phi)^k a(\phi) e^{ir\sin\phi}\, d\phi = -\frac{1}{ir}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\phi)^{k-2}\widetilde{a}(\phi) e^{ir\sin\phi}\, d\phi.$$

where $\widetilde{a}(\phi) := -(k-1)a(\phi)\sin\phi + a'(\phi)\cos\phi$ is smooth.

If $d = 2\ell + 3$ is odd, then repeating this ℓ times, we arrive at

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\phi)^{d-2} e^{ir\sin\phi}\, d\phi = \frac{(-1)^\ell}{(ir)^\ell}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a(\phi)\cos\phi\, e^{ir\sin\phi}\, d\phi,$$

where a is a smooth function. We integrate one last time by parts and obtain the decay of order $r^{-\ell-1} = r^{-\frac{d-1}{2}}$.

If $d = 2\ell + 2$ is even, ℓ integrations by parts yield

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\phi)^{d-2} e^{ir\sin\phi}\, d\phi = \frac{(-1)^\ell}{(ir)^\ell}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a(\phi) e^{ir\sin\phi}\, d\phi,$$

where a is a smooth function. We are here in the setting of *stationary phase*: $\phi = -\frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$ are the critical points of the phase $\sin\phi$, preventing an integration by parts. The main contribution to the value of the integral comes from a small neighborhood of these points. We set $\epsilon := r^{-\frac{1}{2}}$ and separate the integration region into three intervals $[-\frac{\pi}{2}, -\frac{\pi}{2}+\epsilon]$, $[-\frac{\pi}{2}+\epsilon, 0]$, $[0, \frac{\pi}{2}-\epsilon]$ and $[\frac{\pi}{2}-\epsilon, \frac{\pi}{2}]$. The contribution of the 1st and 4th is clearly at most of order ϵ. The 2nd and 3rd are similar, so let us focus on the 3rd interval. An integration by parts yields

$$\begin{aligned}\int_0^{\frac{\pi}{2}-\epsilon} a(\phi) e^{ir\sin\phi}\, d\phi &= \frac{1}{ir}\int_0^{\frac{\pi}{2}-\epsilon} \frac{a(\phi)}{\cos(\phi)}\frac{d}{d\phi}(e^{ir\sin\phi})\, d\phi \\ &= \frac{1}{ir}\Big(\frac{a(\pi/2-\epsilon)e^{ir\sin(\phi/2-\epsilon)}}{\cos(\pi/2-\epsilon)} - a(0)\Big) - \frac{1}{ir}\int_0^{\frac{\pi}{2}-\epsilon} \frac{d}{d\phi}\Big(\frac{a(\phi)}{\cos(\phi)}\Big) e^{ir\sin\phi}\, d\phi.\end{aligned}$$

Since $\cos(\pi/2-\epsilon) \gtrsim \epsilon = r^{-\frac{1}{2}}$, the boundary term contributes at most $r^{-\frac{1}{2}}$. Finally, $\left|\frac{d}{d\phi}\left(\frac{a(\phi)}{\cos(\phi)}\right)\right| \lesssim \left(\frac{\pi}{2}-\phi\right)^{-2}$, so the last integral is bounded up to a constant by

$$\int_0^{\frac{\pi}{2}-\epsilon} \left(\frac{\pi}{2}-\phi\right)^{-2} d\phi \lesssim \epsilon^{-1} = \sqrt{r},$$

and again, after division by r, the contribution turns out to be at most of order $r^{-\frac{1}{2}}$.

Remark 1.3.2. One could resume the proof of (1.3.3) given above as follows: if $|x| \leq \frac{1}{2}t$, we use oscillations in the radial direction and ignore oscillations on the concentric spheres. If $|x| \geq \frac{1}{2}t$, to the contrary, we only exploit oscillations on the concentric spheres, and ignore the radial ones.

Remark 1.3.3. Using the stationary phase approximation is one of the typical ways of establishing the asymptotic behaviour of the Bessel functions, hence the two methods above are in fact not very different.

Chapter 2

Strichartz estimates

2.1 Results from real analysis

Lemma 2.1.1 (Three-line theorem, Phragmen-Lindelöf principle). *Let* $F(x+iy)$ *be bounded and continuous on the strip* $0 \le x \le 1$ *and analytic inside. If* $|F(iy)| \le M_1$ *and* $F(1+iy) \le M_2$ *for all* y*, then*

$$|F(x+iy)| \le M_1^{1-x} M_2^x, \qquad \text{for all } x \in [0,1] \text{ and } y \in \mathbb{R}.$$

Proof. By multiplying the function $F(z)$ by the analytic function $M_1^{z-1} M_2^{-z}$, we reduce the problem to the case $M_1 = M_2 = 1$. By considering the function $\widetilde{F}(z) := F(z)e^{\epsilon(z^2-1)}$, we reduce to the case $\lim_{y\to\infty} |F(z)| = 0$. The conclusion now follows from the Maximum Principle. □

Proposition 2.1.2 (Riesz-Thorin interpolation theorem). *Let* (X, μ) *and* $(\widetilde{X}, \widetilde{\mu})$ *be measure spaces. Let* $1 \le p_1, p_2 \le \infty$ *and assume that* $Y \subset L^{p_1}(X,\mu) \cap L^{p_2}(X,\mu)$ *is dense in both* $L^{p_1}(X,\mu)$ *and* $L^{p_2}(X,\mu)$*. Let* T *be a linear operator defined on* Y *taking its values in measurable functions on* $(\widetilde{X}, \widetilde{\mu})$ *and assume that* $1 \le q_1, q_2 \le \infty$, M_1, M_2 *are such that*

$$\|Tf\|_{L^{q_j}(\widetilde{X},\widetilde{\mu})} \le M_j \|f\|_{L^{p_j}(X,\mu)}, \qquad \text{for all } f \in Y \text{ and } j \in \{1,2\}.$$

Then for all $\theta \in [0,1]$

$$\|Tf\|_{L^q(\widetilde{X},\widetilde{\mu})} \le M_1^\theta M_2^{1-\theta} \|f\|_{L^p(X,\mu)} \qquad \text{for all } f \in Y,$$

where

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof. The conclusion is obvious if $\theta = 0$ or $\theta = 1$, so assume $0 < \theta < 1$. If $p_1 = p_2 = \infty$, then the theorem follows from the Hölder inequality, thus we may assume $p_1 < \infty$ or $p_2 < \infty$, which allows us to consider only f being a step function with finite set of values. Note that we can assume that Y contains such functions (extending T by density if needed; we could also assume that $Y = L^{p_1} \cap L^{p_2}$).

We need to estimate

$$\sup\{\langle Tf, g\rangle : \|f\|_{L^p} \le 1, \|g\|_{L^{q'}} \le 1\},$$

with the supremum taken over step functions with a finite set of values:

$$f = \sum_j a_j I_{A_j}, \qquad g = \sum_k b_k I_{B_k}.$$

For $0 \le \Re z \le 1$ we set

$$\frac{1}{p(z)} := \frac{1-z}{p_1} + \frac{z}{p_2}, \qquad \frac{1}{q'(z)} := \frac{1-z}{q_1'} + \frac{z}{q_2},$$

$$\phi(z) := \sum_j |a_j|^{\frac{p}{p(z)}} e^{i \arg a_j} I_{A_j}, \quad \psi(z) := \sum_k |b_k|^{\frac{q'}{q'(z)}} e^{i \arg b_k} I_{B_k}.$$

We apply the three-line theorem to the analytic function $z \mapsto \langle T\phi(z), \psi(z)\rangle$. ☐

Proposition 2.1.3 (Minkowski inequality). *If* (X, μ), (Y, ν) *measure spaces,* $1 \le p \le q \le \infty$ *and* $f : X \times Y \to \mathbb{R}_+$ *is measurable, then*

$$\big\| y \mapsto \|f(\cdot, y)\|_{L^p(X)} \big\|_{L^q(Y)} \le \big\| x \mapsto \|f(x, \cdot)\|_{L^q(Y)} \big\|_{L^p(X)}.$$

Proof. We can assume that $f \ge 0$ and, upon replacing f by f^p, also that $p = 1$. Let $g \in L^{q'}(Y)$. We have

$$\int_Y g(y) \int_X f(x, y)\, dx\, dy \le \int_X \|f(x, \cdot)\|_{L^q} \|g\|_{L^{q'}}\, dx$$

by Hölder inequality. ☐

Recall that for f, g functions on $\mathbb{R}^d$ we denote

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y) g(y)\, dy,$$

whenever this expression makes sense.

Proposition 2.1.4 (Young's inequality). *Let* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. *If*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

then

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. If $q = 1$, this follows from Minkowski inequality. If $q = p'$ and $r = \infty$, this follows from Hölder inequality. The remaining cases follow from Proposition 2.1.2. ☐

Corollary 2.1.5 (Bernstein's inequalities). *If* $f \in \mathcal{S}(\mathbb{R}^d)$ *is such that* $\mathrm{supp}(\mathcal{F} f) \subset B(0, 4)$ *and* $1 \le p \le \widetilde{p} \le \infty$, *then* $\|f\|_{L^{\widetilde{p}}} \lesssim \|f\|_{L^p}$.

Proof. We taken the convolution with the inverse Fourier transform of a smooth function equal 1 on $B(0, 4)$ and with support in $B(0, 8)$. ☐

We will need the following result, which we give without proof.

Proposition 2.1.6 (Hardy-Littlewood-Sobolev inequality). *If* $\alpha \in (0, d)$ *and* $(p, r) \in (1, \infty)$ *satisfy*

$$\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r},$$

then

$$\| |\cdot|^{-\alpha} * f \|_{L^r} \leq C \|f\|_{L^p}.$$

□

Remark 2.1.7. If $|x|^{-\alpha}$ belonged to $L^{\frac{d}{\alpha}}$, then the Hardy-Littlewood-Sobolev inequality would follow from Young's inequality. Of course, $|x|^{-\alpha}$ is not in $L^{\frac{d}{\alpha}}$ due to logarithmic divergences.

2.2 Sobolev and Besov norms

Definition 2.2.1. For any $s \in \mathbb{R}$, the *homogeneous Sobolev norm* $\dot{H}^s$ is defined by

$$\|v\|_{\dot{H}^s}^2 := \int_{\mathbb{R}^d} |\xi|^{2s} |(\mathcal{F}f)(\xi)|^2 \, d\xi, \qquad v \in \mathcal{S}(\mathbb{R}^d).$$

Remark 2.2.2. The same argument as in Remark 1.2.6 shows that if u is a solution of (1.2.1), then for any s the quantity

$$\|\partial_t u(t)\|_{\dot{H}^{s-1}}^2 + \|u(t)\|_{\dot{H}^s}^2$$

is constant in time.

Proposition 2.2.3 (Sobolev embedding). *For any* $0 \leq s < \frac{d}{2}$ *there exists* C *such that*

$$\|v\|_{L^{\frac{2d}{d-2s}}} \leq C \|v\|_{\dot{H}^s} \qquad \textit{for all } v \in \mathcal{S}(\mathbb{R}^d). \tag{2.2.1}$$

Remark 2.2.4. The Lebesgue exponent $p := \frac{2d}{d-2s}$ is the only one for which (2.2.1) can hold. To see this, we can use the *scaling argument*. Let $0 \neq v \in \mathcal{S}(\mathbb{R}^d)$, $\lambda > 0$ and $v_\lambda(x) := v(x/\lambda)$. We then have

$$\|v_\lambda\|_{\dot{H}^s} = \lambda^{\frac{d}{2}-s} \|v\|_{\dot{H}^s}, \qquad \|v_\lambda\|_{L^p} = \lambda^{\frac{d}{p}} \|v\|_{L^p}.$$

If $\frac{d}{2} - s \neq \frac{d}{p} \Leftrightarrow p \neq \frac{2d}{d-2s}$, then we cannot have $\|v_\lambda\|_{L^p} \leq C\|v_\lambda\|_{\dot{H}^s}$ both for small and large λ.

Proof. By duality, (2.2.1) is equivalent to

$$\|w\|_{\dot{H}^{-s}} \leq C \|w\|_{L^{\frac{2d}{d+2s}}} \qquad \text{for all } w \in \mathcal{S}(\mathbb{R}^d).$$

The principle of the proof is straightforward. Based on the formula for the Fourier transform of spherically symmetric functions, see [3, Theorem 3.3], one can expect that

$$\mathcal{F}(x \mapsto |x|^{-d+s}) = \xi \mapsto C|\xi|^{-s},$$

the rigorous meaning of the formula being unclear for now. Assuming we can apply Proposition 1.2.4, we obtain

$$\|w\|_{\dot{H}^{-s}}^2 = \| |\xi|^{-s} \mathcal{F} w \|_{L^2}^2 = \|\mathcal{F}(C^{-1}|x|^{-d+s} * w)\|_{L^2}^2 = C^{-2}(2\pi)^d \| |x|^{-d+s} * w \|_{L^2}^2,$$

and we conclude by invoking Proposition 2.1.6, since $\frac{d+2s}{2d} + \frac{d-s}{d} = 1 + \frac{1}{2}$.

We leave to the Reader the task of making this argument rigorous, by using tempered distributions or a regularization argument. □

We will need some elements of the *Littlewood-Paley theory*.

2.3 Littlewood-Paley theory

Lemma 2.3.1 (Partition of unity over a geometric scale). *There exists a radial nonnegative function* $\psi \in C^\infty(\mathbb{R}^d)$ *such that* $\operatorname{supp}\psi \subset \{\frac{1}{2} \leq x \leq 2\}$ *and*

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1, \qquad \forall x \neq 0.$$

Proof. We take $\chi \in C^\infty$ a radial non-increasing cut-off function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. We set $\psi(x) := \chi(x) - \chi(2x)$. □

Definition 2.3.2. For $j \in \mathbb{Z}$ we define the *homogeneous dyadic block* $\dot{\Delta}_j$ and the *homogeneous low-frequency cut-off operator* $\dot{S}_j$:

$$\dot{\Delta}_j u := \psi(2^{-j}D)u := \mathcal{F}^{-1}(\psi(2^{-j}\xi)\widehat{u}(\xi)) = 2^{jd}\int_{\mathbb{R}^d} (\mathcal{F}^{-1}\psi)(2^j y)u(x-y)\,dy,$$

$$\dot{S}_j u := \sum_{j'<j} \dot{\Delta}_{j'}u = \mathcal{F}^{-1}(\chi(2^{-j}\xi)\widehat{u}(\xi)) = 2^{jd}\int_{\mathbb{R}^d} (\mathcal{F}^{-1}\chi)(2^j y)u(x-y)\,dy.$$

Lemma 2.3.3. *The operators* $\dot{\Delta}_j$ *and* $\dot{S}_j$ *are bounded* $L^p \to L^p$ *for all* $p \in [1,\infty]$*, with bounds independent of* j*.*

Proof. We take the convolution with the inverse Fourier transform of a cut-off function and use the Young inequality for convolutions. We leave the details to an interested reader and refer to [1, Section 2.1] for a detailed exposition. □

Note that $\dot{\Delta}_j$ and $\dot{S}_j$ are Fourier multiplication operators, and as such they commute with other Fourier multiplication operators, like convolutions, derivatives, ...

The *formal* homogeneous Littlewood-Paley decomposition is

$$\mathrm{Id} = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j,$$

but in what sense the series converges is, for now, unclear.

Definition 2.3.4 (Homogeneous Besov norms). We denote $\mathcal{S}_0(\mathbb{R}^d)$ the set of functions $u \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\widehat{u} \subset \mathbb{R}^d \setminus \{0\}$. Let $s \in \mathbb{R}$ and $p, r \in [1,\infty]$. For any $u \in \mathcal{S}_0(\mathbb{R}^d)$ we define

$$\|u\|_{\dot{B}^s_{p,r}} := \Big(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j u\|_{L^p}^r\Big)^{\frac{1}{r}}.$$

We call $\|\cdot\|_{\dot{B}^s_{p,r}}$ the *homogeneous Besov norm.*

Remark 2.3.5. We can think of the homogeneous Besov norms as follows. For each $j \in \mathbb{Z}$, take the L^p norm of $\dot{\Delta}_j u$, multiply it by 2^{js} and take the ℓ^r norm of the resulting sequence.

Remark 2.3.6. One can check that, up to a constant, the definition of the Besov norm does not depend on the choice of the function ψ.

Proposition 2.3.7. *For any* $p \in [2, \infty)$ *there exists* C_p *such that for all* $u \in \mathcal{S}_0(\mathbb{R}^d)$

$$\|u\|_{L^p} \leq C_p \|u\|_{\dot{B}^0_{p,2}}.$$

For any $p \in (1, 2]$ *there exists* C_p *such that for all* $u \in \mathcal{S}_0(\mathbb{R}^d)$

$$\|u\|_{\dot{B}^0_{p,2}} \leq C_p \|u\|_{L^p}.$$

Proof of Proposition 2.3.7. We have to skip it. See [1, Theorem 2.40] for an elementary proof, or [2, Problem 8.8] for a proof using a fundamental but difficult result in Harmonic Analysis called the Littlewood-Paley theorem. □

2.4 The TT^* method

In this section, we prove general Strichartz estimates. We follow [1, Section 8.2].

For $f \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ and $p, q \in [1, \infty]$, we define

$$\|f\|_{L^p L^q} := \left(\int_{\mathbb{R}} \|f(t, \cdot)\|^p_{L^q}\, dt \right)^{\frac{1}{p}}.$$

Lemma 2.4.1. *Let* $(p_j, q_j) \in [1, \infty]^2$ *and* $\theta_j \geq 0$ *with* $\sum_{j=1}^m \theta_j = 1$. *Suppose that*

$$\frac{1}{p} = \sum_{j=1}^m \frac{\theta_j}{p_j}, \qquad \frac{1}{q} = \sum_{j=1}^m \frac{\theta_j}{q_j}.$$

Then

$$\|f\|_{L^p L^q} \leq \prod_{j=1}^m \|f\|^{\theta_j}_{L^{p_j} L^{q_j}}, \qquad \forall f \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d).$$

Proof. Exercise. □

Definition 2.4.2. Let $\sigma > 0$. We say that a pair (p, q) is σ-*admissible* if

$$\frac{1}{p} + \frac{\sigma}{q} = \frac{\sigma}{2}, \qquad (p, q, \sigma) \neq (2, \infty, 1).$$

If σ is known from the context, we can call such a pair *admissible*.

Remark 2.4.3. It is easy to see that in the case $\sigma = 0$ we do not obtain anything interesting. We would be forced to admit $(\infty, 2)$ is the only 0-admissible pair.

Theorem 2.4.4. *Let* $U(t)$ *be a family of continuous operators, bounded for the* $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *norm, such that*

$$\|U(t)U^*(s)u_0\|_{L^\infty} \leq C|t - s|^{-\sigma}\|u_0\|_{L^1}, \qquad \forall t, s \in \mathbb{R},\ u_0 \in \mathcal{S}(\mathbb{R}^d). \tag{2.4.1}$$

Let $\chi : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function such that $|\chi(t,s)| \le 1$ for all t, s. Then for all σ-admissible pairs (p,q), (p_1,q_1), (p_2,q_2), $u_0 \in \mathcal{S}(\mathbb{R})$ and $f \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$

$$\|U(t)u_0\|_{L^pL^q} \le C\|u_0\|_{L^2}, \tag{2.4.2}$$

$$\left\| \int_{\mathbb{R}} \chi(t,s)U(t)U^*(s)f(s)\,ds \right\|_{L^{p_1}L^{q_1}} \le C\|f\|_{L^{p_2'}L^{q_2'}}, \tag{2.4.3}$$

with C independent of χ.

Remark 2.4.5. We can think of $U(t)$ as the forward evolution operator (we will construct in the next section an appropriate operator for the wave equation). Often $U(t)$ is a unitary operator, hence $U^*(t)$ is the backward evolution. If we take χ to be the indicator function of $\{0 \le s \le t\}$, then we can recognize the Duhamel term in (2.4.3).

Remark 2.4.6. All the functions can be vector-valued.

Proof of Theorem 2.4.4 in the non-endpoint case. We will only treat the so-called *non-endpoint case* $p_1, p_2 > 2$, which is considerably easier than the endpoint case and will be sufficient in these lectures.

Step 1. For $f, g \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$ we define

$$T_\chi(f,g) := \int_{\mathbb{R}^2} \chi(t,s)\langle U(t)U^*(s)f(s), g(t)\rangle\,dt\,ds,$$

where $\langle\cdot,\cdot\rangle$ is the L^2 inner product. By duality, (2.4.3) is equivalent to

$$|T_\chi(f,g)| \le C\|f\|_{L^{p_2'}L^{q_2'}}\|g\|_{L^{p_1'}L^{q_1'}}. \tag{2.4.4}$$

Step 2. We show (2.4.4) with $(p_2,q_2) = (p_1,q_1)$. Interpolating between (2.4.1) and the $L^2 \to L^2$ bound we have

$$\|U(t)U^*(s)f(s)\|_{L^q} \le |t-s|^{-\sigma\left(1-\frac{2}{q}\right)}\|f(s)\|_{L^{q'}},$$

thus

$$\langle U(t)U^*(s)f(s), g(t)\rangle \le C|t-s|^{-\sigma\left(1-\frac{2}{q}\right)}\|f(s)\|_{L^{q'}}\|g(t)\|_{L^{q'}} = C|t-s|^{-\frac{2}{p}}\|f(s)\|_{L^{q'}}\|g(t)\|_{L^{q'}},$$

and we conclude using Hardy-Littlewood-Sobolev inequality, using the fact that $2 < p < \infty$.

Step 3. We prove that

$$\left\| \int_{\mathbb{R}} U^*(t)f(t)\,dt \right\|_{L^2} \le C\|f\|_{L^{p'}L^{q'}}. \tag{2.4.5}$$

Denote $T = T_\chi$ with $\chi(t,s) = 1$ for all t, s. Directly from the definition of T_χ we obtain

$$T(f,f) = \left\| \int_{\mathbb{R}} U^*(t)f(t)\,dt \right\|_{L^2}^2,$$

so (2.4.5) follows from Step 1.

Step 4. We prove (2.4.4) for any σ-admissible pairs (p_1,q_1) and (p_2,q_2). By symmetry, without loss of generality we can assume $q_1 \le q_2$. Fixing t and using (2.4.5) with s instead of t and $\chi(t,s)f(s)$ instead of $f(t)$ we get

$$\left\| \int_{\mathbb{R}} \chi(t,s)U(t)U^*(s)f(s)\,ds \right\|_{L^\infty L^2} \le C\|f\|_{L^{p_2'}L^{q_2'}}.$$

Lemma 2.4.1 and (2.4.3) for $(p_1,q_1)=(p_2,q_2)$ thus imply (2.4.3) in the general case.
Step 5. The bound (2.4.2) follows from (2.4.5) by duality. □

2.5 Strichartz estimates for the wave equation

Definition 2.5.1. We say that a pair (p,q) is *wave-admissible* if there exists $2 \le \widetilde{q} \le q$ such that

$$\frac{2}{p} + \frac{d-1}{\widetilde{q}} = \frac{d-1}{2}, \qquad (p,\widetilde{q},d) \ne (2,\infty,3).$$

Theorem 2.5.2. *Suppose that* (p,q) *and* (a,b) *are wave-admissible,* $\nu > 0$ *and*

$$\frac{1}{p} + \frac{d}{q} = \frac{1}{a'} + \frac{d}{b'} - 2 = \frac{d}{2} - \nu. \tag{2.5.1}$$

Let u *be the solution of* (1.2.5). *Then*

$$\|u\|_{L^pL^q} \le C\big(\|u_0\|_{\dot{H}^\nu} + \|\dot{u}_0\|_{\dot{H}^{\nu-1}} + \|f\|_{L^{a'}L^{b'}}\big).$$

Remark 2.5.3. The condition of wave-admissibility is related to the dispersion of the equation under the assumption that the Fourier support of the data belongs to an annulus, and it frequently happens in the applications that $\widetilde{q} < q$. The condition (2.5.1) is related to the scaling invariance of the equation, see Remark 2.2.4 for a similar argument.

Proof. We first prove that the theorem is true if all the functions involved have spatial Fourier transforms contained in $\{\frac{1}{2} \le |\xi| \le 2\}$. In order to use Theorem 2.4.4, we need to define $U(t)$. We define it as acting on functions with values in $\mathbb{R}^2$ (or $\mathbb{C}^2$ if complex-valued fields are considered) in the following way:

$$U(t)\begin{pmatrix} w_0 \\ \dot{w}_0 \end{pmatrix} = \begin{pmatrix} \cos(|D|t)\chi(|D|)w_0 + \sin(|D|t)\chi(|D|)\dot{w}_0 \\ -\sin(|D|t)\chi(|D|)w_0 + \cos(|D|t)\chi(|D|)\dot{w}_0 \end{pmatrix}, \tag{2.5.2}$$

where $\chi \in C^\infty(\mathbb{R})$ equals 1 on $[\frac{1}{2},2]$ and 0 outside of $[\frac{1}{4},4]$. Notice that the Fourier multipliers are smooth thanks to the term $\chi(|D|)$. The adjoint $U^*(t)$ is given by

$$U^*(t)\begin{pmatrix} w_0 \\ \dot{w}_0 \end{pmatrix} = \begin{pmatrix} \cos(|D|t)\chi(|D|)w_0 - \sin(|D|t)\chi(|D|)\dot{w}_0 \\ \sin(|D|t)\chi(|D|)w_0 + \cos(|D|t)\chi(|D|)\dot{w}_0 \end{pmatrix}.$$

Assume that $\mathcal{F}u_0$, $\mathcal{F}\dot{u}_0$ and $\mathcal{F}(f(t))$ for all t have their supports contained in $[\frac{1}{2},2]$. Comparing (2.5.2) with the solution formula (1.2.6), we notice that if u is given by (1.2.6), then

$$\begin{aligned}\begin{pmatrix} |D|u(t) \\ \partial_t u(t) \end{pmatrix} &= U(t)\begin{pmatrix} |D|u_0 \\ \dot{u}_0 \end{pmatrix} + \int_0^t U(t-s)\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \\ &= U(t)\begin{pmatrix} |D|u_0 \\ \dot{u}_0 \end{pmatrix} + \int_0^t U(t)U^*(s)\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds.\end{aligned}$$

Theorem 2.4.4 thus yields

$$\|t \mapsto |D|u(t)\|_{L^p L^{\widetilde{q}}} \lesssim \||D|u_0\|_{L^2} + \|\dot{u}_0\|_{L^2} + \|f\|_{L^{a'} L^{\widetilde{b}'}},$$

where $\widetilde{q}$ and $\widetilde{b}$ are given by Definition 2.5.1 for the pairs (p, q) and (a, b) respectively. Using the assumption about the Fourier supports and Corollary 2.1.5, we obtain

$$\|t \mapsto u(t)\|_{L^p L^q} \lesssim \|u_0\|_{H^\nu} + \|\dot{u}_0\|_{H^{\nu-1}} + \|f\|_{L^{a'} L^{b'}}. \tag{2.5.3}$$

By scaling invariance, this implies that the conclusion holds if all the functions involved have spatial Fourier transforms contained in $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ for some $j \in \mathbb{Z}$.

The final step is to "glue the Littlewood-Paley pieces". Let $u_0, \dot{u}_0 \in \mathcal{S}(\mathbb{R}^d)$ and $f \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$, with no condition on the Fourier support. Let $u(t)$ be given by (1.2.6). Since $\dot{\Delta}_j$ commutes with $\cos(|D|t)$ and $\frac{\sin(|D|t)}{|D|}$, (2.5.3) yields

$$\|t \mapsto \dot{\Delta}_j u(t)\|_{L^p L^q} \lesssim \|\dot{\Delta}_j u_0\|_{H^\nu} + \|\dot{\Delta}_j \dot{u}_0\|_{H^{\nu-1}} + \|\dot{\Delta}_j f\|_{L^{a'} L^{b'}}. \tag{2.5.4}$$

For fixed t we can write

$$\|u(t)\|^2_{\dot{B}^0_{q,2}} = \sum_{j \in \mathbb{Z}} \|\dot{\Delta}_j u(t)\|^2_{L^q},$$

so the Minkowski inequality (used twice, both for the left and the right hand side) together with (2.5.4) yield

$$\|u(t)\|_{L^p \dot{B}^0_{q,2}} \lesssim \|u_0\|_{\dot{H}^\nu} + \|\dot{u}_0\|_{\dot{H}^{\nu-1}} + \|f\|_{L^{a'} \dot{B}^0_{b',2}}.$$

Finally, we use Proposition 2.3.7 on both sides. □

Chapter 3

Cauchy theory for wave equations

3.1 Cauchy problem for the cubic wave equation

In this chapter we are interested in equations of the form

$$\partial_t^2 u(t,x) = \Delta u(t,x) + f(x,u(t,x)), \qquad (t,x) \in \mathbb{R}^{1+d}.$$

In order to illustrate two common techniques, we consider two cases:

- $d = 3$ and $f(x,u) := \pm u^3$,
- $d = 4$ and $f(x,u) := \pm u^3$.

As it often happens in the study of nonlinear PDEs, it is not immediate to construct smooth solutions, even for smooth initial data. One generally constructs non-smooth solutions and the study of their regularity is a separate question. The first problem which we have to face is thus to define what it means for a non-smooth function u to be a solution of the Cauchy problem

$$\begin{aligned} \partial_t^2 u(t,x) &= \Delta u(t,x) + f(x,u(t,x)), \qquad (t,x) \in \mathbb{R}^{1+d}, \\ u(0,x) = u_0, \quad \partial_t u(0,x) &= \dot u_0(x), \qquad x \in \mathbb{R}^d. \end{aligned} \tag{3.1.1}$$

There are several possibilities and we present one of them.

Let $0 \in I \subset \mathbb{R}$. It follows from (1.2.8) that the formula (1.2.6), seen as a linear operator $(u_0, \dot u_0, f) \mapsto u$, can be extended by density as a continuous linear operator $\dot H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times L^1(I, L^2(\mathbb{R}^d)) \to C(I, \dot H^1(\mathbb{R}^d))$.

Definition 3.1.1. We say that a measurable function $u : I \times \mathbb{R}^d \to \mathbb{R}$ solves (3.1.1) if $(t,x) \mapsto f(x,u(t,x))$ belongs to $L^1(I, L^2(\mathbb{R}^d))$ and

$$u(t) = \cos(|D|t)u_0 + \frac{\sin(|D|t)}{|D|}\dot u_0 + \int_0^t \frac{\sin(|D|(t-s))}{|D|} f(\cdot, u(s))\,ds$$

for all $t \in I$.

Theorem 3.1.2. *Let* $d = 3$ *and* $f(x,u) = u^3$. *For any* $(u_0, \dot u_0) \in \dot H^1 \times L^2$ *there exists* $T > 0$ *and a unique solution of* (3.1.1) *on the time interval* $[-T, T]$.

Proof. We will not need Strichartz estimates here. The so-called *energy method* is sufficient. Denote $I := [-T, T]$, $X := L^1(I; L^2(\mathbb{R}^d))$ and consider the map $\Phi : X \to X$ defined as follows. For a given $f \in X$, we define $\Phi(f) := u^3$, where u is given by (1.2.6). We claim that Φ is a contraction on the ball $B(0, R) \subset X$ for $R := \sqrt{\|\dot{u}_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2}$ and T sufficiently small.

Let $f \in B(0, R)$ and $t \in I$. The energy inequality (1.2.8) yields

$$\|\nabla u(t)\|_{L^2} \leq R + \|f\|_X \leq 2R.$$

By the Sobolev embedding (2.2.1), we have

$$\|u(t)\|_{L^6}^3 \leq CR^3.$$

An integration in time yields

$$\|\Phi(u)\|_X^3 \leq 2TCR^3,$$

hence $\Phi(u) \in B(0, R)$ provided that $T \leq \frac{1}{2CR^2}$.

Let now $f, f^\sharp \in B(0, R)$. □

Theorem 3.1.3. *Let* $d = 3$ *and* $f(x, u) = u^3$. *For any* $(u_0, \dot{u}_0) \in \dot{H}^1 \times L^2$ *there exists* $T > 0$ *and a unique solution of* (3.1.1) *on the time interval* $[-T, T]$.

Proof. The method of the previous proof fails here. We consider the same map Φ as in the previous proof, but this time we will need a Strichartz estimate in order to prove that it is a contraction. □

Chapter 4

Cauchy theory for equivariant wave maps

4.1 Equivariant wave maps

We will study wave maps $\psi : \mathbb{R}^{1+2} \to \mathbb{S}^2 \subset \mathbb{R}^3$. Recall that they are critical points for the Lagrangian

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\Big(\frac{1}{2}|\partial_t\psi|^2 - \frac{1}{2}|\nabla_x\psi|^2\Big)\,dx.$$

The equation can be written explicitly:

$$\partial_t^2\psi - \Delta\psi = (|\partial_t\psi|^2 - |\nabla\psi|^2)\psi.$$

This equation is difficult to study. We will consider a particular class of solutions. Take $k \in \{1, 2, \ldots\}$ and consider initial data of the form

$$\psi_0(r\cos\theta, r\sin\theta) = \big(\sin(u_0(r))\cos(k\theta), \sin(u_0(r))\sin(k\theta), \cos(u_0(r))\big).$$

The evolution preserves this particular form of initial data and we obtain a simple equation for the scalar-valued function $u(t, r)$:

$$\partial_t^2 u = \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{k^2\sin(2u)}{2r^2}. \tag{4.1.1}$$

The subsequent chapters of these lectures will be exclusively devoted to the study of (4.1.1).

There is the conserved energy given by

$$E(\mathbf{u}_0) := 2\pi\int_0^\infty\Big(\frac{1}{2}(\dot{u}_0)^2 + \frac{1}{2}(\partial_r u_0)^2 + \frac{k^2\sin(u_0)^2}{2r^2}\Big)r\,dr.$$

We see that our problem is energy-critical.

For $m, n \in \mathbb{Z}$ we define

$$\mathcal{E}_{m,n} := \{\mathbf{u}_0 : E(\mathbf{u}_0) < \infty,\ \lim_{r\to 0} u(r) = m\pi,\ \lim_{r\to\infty} u(r) = n\pi\}.$$

Exercise 4.1.1. Prove that if $E(\mathbf{u}_0) < \infty$, then there exist $m, n \in \mathbb{Z}$ such that $\lim_{r\to 0} u_0(r) = m\pi$ and $\lim_{r\to\infty} u_0(r) = n\pi$.

The sets $\mathcal{E}_{m,n}$ are affine spaces and play the role of the critical space. They are parallel to the linear space $\mathcal{E} := \mathcal{E}_{0,0}$. We define the critical norm:

$$\|\mathbf{u}_0\|_{\mathcal{E}}^2 := \int_0^\infty \Big(\dot{u}_0^2 + (\partial_r u_0)^2 + \frac{k^2}{r^2}u_0^2\Big) r\,dr.$$

We also denote the part corresponding to the potential energy

$$\|u_0\|_{\mathcal{H}} := \int_0^\infty \Big((\partial_r u_0)^2 + \frac{k^2}{r^2}u_0^2\Big) r\,dr.$$

4.1.1 Strichartz estimates for the linearized problem

The key to solving (4.1.1) will be to have appropriate Strichartz estimates for the following problem analogous to (1.2.5):

$$\begin{aligned} \partial_t^2 u(t,r) - \partial_r^2 u(t,r) - \frac{1}{r}\partial_r u(t,r) + \frac{k^2}{r^2}u(t,r) = f(t,r), \qquad (t,r) \in \mathbb{R}\times(0,\infty), \\ u(0,r) = u_0(r), \qquad \partial_t u(0,r) = \dot{u}_0(r), \qquad r \in (0,\infty). \end{aligned} \tag{4.1.2}$$

Here, in general $k \in \mathbb{Z}$, but we as mentioned above we assume $k \geq 1$. It is worth noting that (4.1.2) appears by considering a Fourier series decomposition in the angular variable of solutions of (1.2.5) in dimension $d = 2$. Namely, if (u, f) is a complex-valued solution of (1.2.5) and we decompose

$$u(t, re^{i\theta}) = \sum_{k\in\mathbb{Z}} u^{(k)}(t,r)e^{ik\theta}, \qquad f(t, re^{i\theta}) = \sum_{k\in\mathbb{Z}} f^{(k)}(t,r)e^{ik\theta},$$

then $(u^{(k)}, f^{(k)})$ solves (4.1.2).

We introduce the following notation for the solution operator related to (4.1.2). Let $\mathbf{u}_0 = (u_0, \dot{u}_0) \in \mathcal{E}$ and let u be the solution of (4.1.2) with $f = 0$. We then denote

$$U(t)\mathbf{u}_0 := u(t), \qquad \dot{U}(t)\mathbf{u}_0 := \partial_t u(t), \qquad \mathbf{U}(t)\mathbf{u}_0 := \mathbf{u}(t) = (u(t), \partial_t u(t)).$$

The conservation of energy for the wave equation in space dimension 2 implies that $\mathbf{U}(t)$ is a unitary operator from $\mathcal{E}$ to itself. Note that the operators $U(t)$ and $\mathbf{U}(t)$ are different than the one introduced by (2.5.2).

For any $I \subset \mathbb{R}$, we define the following "Strichartz norm" adapted to the study of (4.1.1):

$$\|u\|_{S(I)} := \bigg(\int_I \Big(\int_0^\infty \frac{u(t,r)^6}{r^3}\,dr\Big)^{\frac{1}{2}}\bigg)^{\frac{1}{3}}.$$

We write S instead of $S(\mathbb{R})$.

Lemma 4.1.2. *Let* u *be a solution of* (4.1.2)*. Then*

$$\|\mathbf{u}\|_{L^\infty(I;\mathcal{E})} + \|u\|_{S(I)} \leq C\big(\|\mathbf{u}_0\|_{\mathcal{E}} + \|f\|_{L^1(I;L^2)}\big).$$

Proof. The energy bound follows from the energy inequality (1.2.8) in space dimension 2.

In order to get the Strichartz bound, we use a somewhat artificial trick. Let $P : \mathbb{R}^4 \to \mathbb{R}$ be any homogeneous harmonic polynomial of degree $k-1$ and consider

$$v(t,x) := u(t,r)r^{-k}P(x), \qquad g(t,x) := f(t,r)r^{-k}P(x)$$

where $r := |x|$ and $x \in \mathbb{R}^4$. A computation shows that

$$\partial_t^2 v - \Delta v = g,$$
$$\|\mathbf{u}_0\|_{\mathcal{E}} \simeq \|\mathbf{v}_0\|_{\dot{H}^1 \times L^2(\mathbb{R}^4)},$$
$$\|f\|_{L^1(I;L^2)} \simeq \|g\|_{L^1(I;L^2(\mathbb{R}^4))},$$

and it suffices to apply Theorem 2.5.2. □

Remark 4.1.3. It is likely that one could prove directly Strichartz estimates by adapting the material presented in Chapter 2 and using the Hankel transform instead of the Fourier transform. This would perhaps require a considerable amount of work, since one would have to define an analogue of Besov spaces and prove the relevant embeddings.

The following estimate will also be useful.

Lemma 4.1.4. *Let* u *be a solution of* (4.1.2) *with* $f = 0$. *Then*

$$\|u\|_S \leq C\|\mathbf{u}_0\|_{\mathcal{E}}^{\frac{11}{12}} \|u\|_{L^\infty L^\infty}^{\frac{1}{12}}.$$

Proof. The Hölder inequality yields

$$\int \frac{u^6}{r^3}\,dr \leq \|u\|_{L^\infty}^{\frac{1}{2}} \Big(\int \frac{u^{20/3}}{r^{11/3}}\,dr\Big)^{\frac{3}{4}} \Big(\int \frac{u^2}{r}\,dr\Big)^{\frac{1}{4}}.$$

We take the square root and integrate in t. The proof of the previous lemma with the pair of exponents $(5/2, 20/3)$ instead of $(3,6)$ yields

$$\int_{\mathbb{R}} \Big(\int \frac{u^{20/3}}{r^{11/3}}\,dr\Big)^{\frac{3}{8}} \lesssim \|\mathbf{u}_0\|_{\mathcal{E}}^{\frac{5}{2}}.$$

□

4.1.2 Local well-posedness

The case of the initial data in $\mathcal{E}$ is easier, so we discuss it first. By analogy with the case of a power nonlinearity, we say that a measurable function $u : I \times (0,\infty) \to \mathbb{R}$ is a solution of (4.1.1) if the function

$$f(t,r) := \frac{k^2 u}{r^2} - \frac{k^2 \sin(2u)}{2r^2}$$

belongs to $L^1(I; L^2(r\,dr))$ and (4.1.2) holds. It is convenient to denote $Z(u) := (2k^2u - k^2\sin(2u))/(2u^3)$, which is an analytic function, bounded for $u \in \mathbb{R}$. We then have $f(t,r) = Z(u(t,t))u(t,r)^3 r^{-2}$, in particular $u \in S(I)$ implies $f \in L^1(I; L^2)$.

Proposition 4.1.5. *For any $\mathbf{u}_0 \in \mathcal{E}$ there exists a unique solution. It satisfies $(u, \partial_t u) \in C(I; \mathcal{E})$.*

Proof. □

Lemma 4.1.6. *For all $M > 0$ there exist $\eta_0 = \eta_0(M) > 0$ and $C = C(M) > 0$ which have the following property. Let $0 \in I \subset \mathbb{R}$. Let $\|\mathbf{u}_0\|_{\mathcal{E}} \le M$ with $\|\mathbf{U}(\cdot)\mathbf{u}_0\|_{S(I)} = \eta \le \eta_0$. Then there is a unique strong solution $u \in L^\infty(I; \mathcal{E})$ of the problem (4.1.1). This solution satisfies*

$$\sup_{t \in I} \|\mathbf{u} - \mathbf{u}_L\|_{\mathcal{E}} + \|u - u_L\|_{S(I)} \le C\eta^3,$$

where $\mathbf{u}_L(t) = \mathbf{U}(t)\mathbf{u}_0$.

Proof. For $\rho \ge 0$, let

$$B_\rho := \{u : \|(u, \partial_t u) - (u_L, \partial_t u_L)\|_{L^\infty(I;\mathcal{E})} + \|u - u_L\|_{S(I)} \le \rho\}.$$

Denote $L_k := -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2}$. We consider the map F which to $u \in B_\rho$ associates $v = F(u)$, the solution of the problem

$$\partial_t^2 v + L_k v = Z(u)\frac{u^3}{r^2}, \qquad (v(0), \partial_t v(0)) = (u_0, \dot{u}_0),$$

so $w := v - u_L$ solves the problem

$$\partial_t^2 w + L_k w = Z(u)\frac{u^3}{r^2}, \qquad (w(0), \partial_t w(0)) = (0, 0),$$

By Lemma 4.1.2,

$$\|(w, \partial_t w)\|_{L^\infty(I;\mathcal{E})} + \|w\|_{S(I)} \lesssim \eta^3 + \rho^3,$$

so if we take $\rho = C\eta^3$ with C a sufficiently large constant, the ball $B(\rho)$ will be invariant.

Similarly, for η_0 sufficiently small, F is a contraction on B_ρ, for the norm $L^\infty(I; \mathcal{E}) + S(I)$. Let $u, \widetilde{u} \in B_\rho$, $v := F(u)$, $\widetilde{v} := F(\widetilde{u})$, $w := \widetilde{v} - v$. Then

$$\partial_t^2 w + L_k w = f(u, \widetilde{u}) := Z(\widetilde{u})\frac{\widetilde{u}^3}{r^2} - Z(u)\frac{u^3}{r^2}, \qquad (w(0), \partial_t w(0)) = (0, 0).$$

We see that

$$\|f(u, \widetilde{u})\|_{L^1 L^2} \lesssim (\|\widetilde{u} - u\|_{L^\infty \mathcal{E}} + \|\widetilde{u} - u\|_S)(\eta^2 + \eta^3),$$

and we conclude using Lemma 4.1.2.

By Picard's theorem there exists the unique application $u \in B_\rho$ such that $F(u) = u$. Since $C(I; \mathcal{E})$ is complete, we see that $u \in C(I; \mathcal{E})$. □

Proposition 4.1.7. *If $\mathbf{u}_0 \in \mathcal{E}$, then there exists a unique strong solution $\mathbf{u} \in L^\infty(I_{\max}; \mathcal{E})$ of the equation (4.1.1), defined on the maximal interval of existence $I_{\max} = I_{\max}(u) := (T_-(u), T_+(u))$ such that $\mathbf{u}(0) = \mathbf{u}_0$. It has the following properties.*

- $\mathbf{u} \in C(I_{\max}; \mathcal{E})$,

- *for any compact interval* $J \subset I_{max}$, u *is continuous with respect to the initial data* $\mathbf{u}_0$*, for topologies* $\mathcal{E} \to L^\infty(J;\mathcal{E})$*,*

- *for any compact interval* $J \subset I_{max}$ *we have*

$$\|u\|_{S(J)} < \infty,$$

- *if*

$$\|u\|_{S([0,T_+))} < \infty,$$

then $T_+ = \infty$ *and* $\mathbf{u}$ *scatters as* $t \to \infty$*, which means that there exists* $\mathbf{u}_+ \in \mathcal{E}$ *such that*

$$\|\mathbf{u}(t) - \mathbf{U}(t)\mathbf{u}_+\|_{\mathcal{E}} \to 0 \qquad \text{as } t \to \infty,$$

- *conversely, if* u *scatters for* $t \to \infty$*, then*

$$\|u\|_{S([0,\infty))} < \infty,$$

- *an analogous statement for negative times.*

Proof. The first part is a direct consequence of the preceding lemma.

The second point results from the following general principle for the linear equation. If $f \in L^1L^2([0,T_+))$ and u is solution of $\partial_t^2 u + L_k u = f$, then

- in the case $T_+ < \infty$, the strong limit in $\mathcal{E}$, $\lim_{t\to T_+}(u(t), \partial_t u(t))$, exists,
- in the case $T_+ = \infty$, u scatters when $t \to \infty$.

Let v_τ be the solution of $\partial_t^2 v_\tau + L_k v_\tau = 0$ with initial data $(v_\tau(\tau), \partial_t v_\tau(\tau)) = (u(\tau), \partial_t u(\tau))$, and $sigma \geq \tau$. Then

$$\begin{aligned}\|(v_\sigma(0), \partial_t v_\sigma(0)) - (v_\tau(0), \partial_t v_\tau(0))\|_{\mathcal{E}} &= \|(v_\sigma(\tau), \partial_t v_\sigma(\tau)) - (v_\tau(\tau), \partial_t v_\tau(\tau))\|_{\mathcal{E}} \\ &= \|((v_\sigma - u)(\tau), \partial_t(v_\sigma - u)(\tau))\|_{\mathcal{E}} \leq \|f\|_{L^1L^2([\tau,\sigma])} \to 0\end{aligned}$$

when $\sigma, \tau \to T_+$, which implies that the limit $(v_0, v_1) := \lim_{\tau\to\infty}(v_\tau(0), \partial_t v_\tau(0))$ exists, and we obtain

$$\lim_{t\to T_+} \|(u(t), \partial_t u(t)) - S(t)(v_0, v_1)\|_{\mathcal{E}} = 0.$$

Finally, if u scatters for $t \to \infty$, then we see that if we take T_0 to be large and $u_L(t) := S(t - T_0)(u(T_0), \partial_t u(T_0))$, then $\|u_L\|_S < \eta_0$, so we can apply the previous lemma to the interval $[T_0, \infty)$. □

Remark 4.1.8. The same proof shows that we can solve the Cauchy problem at infinity. More precisely, in the case $I = [T_0, \infty)$, under the same hypotheses, there exists a single strong solution $u \in L^\infty(I;\mathcal{E})$ of the equation (4.1.1) such that

$$\lim_{t\to\infty} \|(u(t), \partial_t u(t)) - (u_L, \partial_t u_L)\|_{\mathcal{E}} = 0.$$

To do this, simply define $v = F(u)$ as the solution to the problem

$$\partial_t^2 v + L_k v = Z(u)\frac{u^3}{r^2}, \qquad \lim_{t\to\infty} \|(v(t), \partial_t v(t)) - S(t)(u_0, \dot{u}_0)\|_{\mathcal{E}} = 0.$$

In the same way, we also solve the Cauchy problem for $t \to -\infty$.

To conclude this chapter, we will prove a *perturbation lemma.*

Lemma 4.1.9. *For all $M > 0$ there exists $\varepsilon_0 = \varepsilon_0(M)$ and $C = C(M)$ with the following property. Let $I = [0, T]$ or $I = [0, +\infty)$, and let v be a function defined on $I \times (0, \infty)$ such that*

$$\|(v, \partial_t v)\|_{L^\infty(I;\mathcal{E})} + \|v\|_{S(I)} \le M$$

solving the problem

$$\partial_t^2 v + L_k v = Z(v)\frac{v^3}{r^2} + h, \quad (t, r) \in I \times (0, \infty),$$
$$(v(0), \partial_t v(0)) = (v_0, \dot{v}_0) \in \mathcal{E}.$$

Let $(u_0, \dot{u}_0) \in \mathcal{E}$ and $\varepsilon \in (0, \varepsilon_0]$. Assume that

$$\|(u_0, \dot{u}_0) - (v_0, \dot{v}_0)\|_{\mathcal{E}} + \|h\|_{L^1L^2} \le \varepsilon.$$

Then the solution u of the problem (4.1.1) with initial data $(u_0, \dot{u}_0)$ is defined on I and

$$\|(u(t), \partial_t u(t)) - (v(t), \partial_t v(t))\|_{L^\infty(I;\mathcal{E})} + \|u - v\|_{S(I)} \le C(M)\varepsilon.$$

Remark 4.1.10. Due to the time reversibility, an analogous theoorem is true for negative times.

Remark 4.1.11. The meaning of this lemma is as follows. If we have an approximate solution v, then the true solution u exists as long as v and remains close to it.

Proof. We first prove the result under the additional hypothesis that $\|v\|_{S(I)} \le \eta_0$ is sufficiently small. In this case, the conclusion is obtained by a continuity argument. Let $I' \subset I$ an interval on which u is defined and let's consider $w := u - v$, $(w_0, \dot{w}_0) := (u_0 - v_0, \dot{u}_0 - \dot{v}_0)$. The equation for w is as follows:

$$\partial_t^2 w + L_k w = f := Z(v + w)\frac{(v + w)^3}{r^2} - Z(v)\frac{v^3}{r^2} - h.$$

We see that

$$\|f\|_{L^1L^2(I')} \lesssim (\|w\|_{L^\infty(I';\mathcal{E})} + \|w\|_{S(I')})(\|v\|_{S(I')}^2 + \|w\|_{S(I')}^2 + \|w\|_{S(I')}^3) + \|h\|_{L^1L^2(I')}.$$

Applying Lemma 4.1.2, we obtain

$$\|w\|_{L^\infty(I';\mathcal{E})} + \|w\|_{S(I')} \lesssim \varepsilon + (\|w\|_{L^\infty(I';\mathcal{E})} + \|w\|_{S(I')})(\|v\|_{S(I')}^2 + \|w\|_{S(I')}^2 + \|w\|_{S(I')}^3).$$

If $\|w\|_{S(I')}$ is small enough, then we can absorb the second term and conclude that $\|w\|_{S(I')} \lesssim \varepsilon$. The desired estimate is obtained by progressively enlarging I'.

Now let $n := \left[\frac{M}{\eta_0}\right] + 1$. Then there exists $0 = t_0 < t_1 < \ldots < t_n = T$ a sequence such that $\|v\|_{S(J)} \le \eta_0$ for $J = [t_j, t_{j+1}]$, $j = 0, \ldots, n-1$. We repeat the argument on each interval $[t_j, t_{j+1}]$. □

Chapter 5

Profile decomposition

In this chapter we present the method of profile decomposition, developed in the work of several mathematicians, in particular Lions, Brézis and Coron, Gérard, Merle and Vega, Bahouri and Gérard, ... The abstract framework presented here was proposed by Schindler and Tintarev. Add references.

5.1 Théorie abstraite

Let H be a separable Hilbert space and G a metric group acting on H through isomorphisms (a unitary representation),

$$G \ni g \mapsto T_g \in U(H).$$

Remark 5.1.1. We can easily deal with the case where G does not act by isometries, but only in a bounded manner, i.e.

$$G \ni g \mapsto T_g \in \mathcal{L}(H), \qquad \sup_{g\in G} \|T_g\|_{\mathcal{L}(H)} < \infty.$$

Indeed, replacing the $\|u\|_H$ norm by the equivalent $\sup_{g\in G} \|T_g u\|_H$ norm, we are reduced to the case of of unitary applications. In practice, we are almost always dealing with a unitary representation.

We will usually write gu instead of $T_g u$.

Definition 5.1.2. For a sequence $g_n \in G$ we write $g_n \to \infty$ or $\lim_{n\to\infty} g_n = \infty$ if for any compact set $K \subset G$ we have $g_n \notin K$ for n large enough. Two sequences g_n and $\widetilde{g}_n$ are said to be *orthogonal* if $g_n^{-1}\widetilde{g}_n \to \infty$.

Definition 5.1.3. We say that a sequence $u_n \in H$ converges to 0 *weakly with concentration* if

$$g_n u_n \rightharpoonup 0, \qquad \forall g_n \in G.$$

We write in this case

$$u_n \rightharpoonup_G 0.$$

We observe that the topology of weak convergence with concentration on a ball of H is metrizable in the following way. Let ϕ_k be a dense sequence in the unit ball of H. We define

$$\|u\|_W := \Big(\sum_{k=1}^{\infty} \frac{\langle \phi_k, u\rangle^2}{2^k}\Big)^{\frac{1}{2}},$$
$$\|u\|_G := \sup_{g\in G} \|gu\|_W.$$

Lemma 5.1.4. *If* $u_n \in H$ *a bounded sequence, then after extraction of a sub-sequence there exists a sequence* $g_n \in G$ *such that*

$$g_n u_n \rightharpoonup u \qquad \text{and} \qquad \|u\|_H \geq \limsup_{n\to\infty} \|u_n\|_G.$$

Proof. By extracting a sub-sequence, there exists a sequence $g_n \in G$ such that

$$\sum_{k=1}^{\infty} 2^{-k} la\phi_k, g_n u_n\rangle^2 \to \limsup_{n\to\infty} \|u_n\|_G^2.$$

After a new extraction of a sub-sequence, $g_n u_n \rightharpoonup u \in H$, and we see that

$$\sum_{k=1}^{\infty} 2^{-k} la\phi_k, g_n u_n\rangle^2 \to \sum_{k=1}^{\infty} 2^{-k} la\phi_k, u\rangle^2 \leq \|u\|_H^2,$$

which completes the proof. □

Proposition 5.1.5. *Let* $u_n \in H$ *be a bounded sequence. Then*

$$u_n \rightharpoonup_G 0 \Leftrightarrow \|u_n\|_G \to 0.$$

Proof. The implication $\Rightarrow$ is a direct consequence of the definitions and the preceding lemma.

Conversely, if $\|u_n\|_G \to 0$, then for all k we have $\sup_{g_n\in G}\langle \phi_k, g_n u_n\rangle \to 0$. Let $g_n \in G$ be any sequence. Since the sequence $g_n u_n$ is bounded in H, we obtain $\langle \phi, g_n u_n\rangle \to 0$. for all $\phi \in H$, so $u_n \rightharpoonup_G 0$. □

We make two more assumptions about the action of the group G:

- $g_n \to \infty$ implies $\langle u, g_n v\rangle \to 0$ for all $u, v \in H$ (decay of matrix coefficients),
- for all $u \in H$ the application $G \ni g \mapsto gu \in H$ is continuous (strong continuity).

Definition 5.1.6. Let $u_n \in H$ be a bounded sequence. We say that u_n admits a *profile decomposition* with the *profiles* $U^{(j)} \in H$, the *displacements* $g_n^{(j)} \in G$ and the *remainders* $w_n^{(J)}$ if the sequences $g_n^{(j)}$ and $g_n^{(k)}$ are orthogonal for $j \neq k$,

$$u_n = \sum_{j=1}^{J} g_n^{(j)} U^{(j)} + w_n^{(J)}, \qquad \lim_{J\to\infty} \limsup_{n\to\infty} \|w_n^{(J)}\|_G = 0. \tag{5.1.1}$$

Proposition 5.1.7. *If* u_n *a bounded sequence which admits a profile decomposition* $U^{(j)}$ *with remainders* $w_n^{(J)}$*, then for all* j *the sequence* $\big(g_n^{(j)}\big)^{-1}u_n$ *converges weakly to* $U^{(j)}$ *and for all* $J \in \{1, 2, \ldots\}$ *we have the* Pythagorean expansion

$$\|u_n\|_H^2 = \sum_{j=1}^{J} \|U^{(j)}\|_H^2 + \|w_n^{(J)}\|_H^2 + o(1), \qquad \textit{when } n \to \infty. \tag{5.1.2}$$

Proof. By the hypothesis u_n is a bounded sequence, so to show the first part it suffices to verify that for all j we have $\lim_{n\to\infty} \big\|big(g_n^{(j)}\big)^{-1}u_n - U^{(j)}\big\|_W = 0$. Let j be fixed and let $\varepsilon > 0$. We first find $J \geq j$ such that

$$\limsup_{n\to\infty} \big\|\big(g_n^{(j)}\big)^{-1} w_n^{(J)}\big\|_W \leq \limsup_{n\to\infty} \big\|w_n^{(J)}\big\|_G \leq \varepsilon.$$

Then, for $k \in \{1, 2, \ldots, J\} \backslash \{j\}$ we have $\lim_{n\to\infty} \big\|big(g_n^{(j)}\big)^{-1} g_n^{(k)} U^{(k)}\big\|_W = 0$, therefore (5.1.1) implies

$$\limsup_{n\to\infty} \big\|big(g_n^{(j)}\big)^{-1}u_n - U^{(j)}\big\|_W \leq \epsilon.$$

To show (5.1.2), fix J, take $j \in \{1, \ldots, J\}$, apply $\big(g_n^{(j)}\big)^{-1}$ to (5.1.1) and take the weak limit when $n \to \infty$. From what we have just shown, we obtain that the left-hand side as well as the right-hand side converge to the same element $U^{(j)}$, which implies that $\big(g_n^{(j)}\big)^{-1} w_n^{(J)}$ converges weakly to 0, in particular

$$\lim_{n\to\infty} \langle g_n^{(j)} U^{(j)}, w_n^{(J)} \rangle = 0, \quad \text{for all } j \in \{1, \ldots, J\}.$$

We also have $\langle g_n^{(j)} U^{(j)}, g_n^{(k)} U^{(k)} \rangle = \big\langle \big(g_n^{(k)}\big)^{-1} g_n^{(j)} U^{(j)}, U^{(k)} \big\rangle \to 0$, so by taking $\|\cdot\|_H^2$ from (5.1.1) we get (5.1.2). □

Theorem 5.1.8 (Schindler and Tintarev). *For any bounded sequence* $u_n \in H$ *possède a subsequence that admits a profile decomposition. If* $\|u_n|_G \to 0$*, then its only profile decomposition is the trivial decomposition* $U^{(j)} = 0$ *for all* j*.*

Proof. To demonstrate the first part, we construct the $U^{(j)}$ profiles one by one. Set $w_n^{(0)} := u_n$. Assume that the sequences $w_n^{(0)}, w_n^{(J-1)}$, the displacements $g_n^{(1)}, g_n^{(J-1)}$ and the profiles $U^{(j)}$ are the same. and the profiles $U^{(1)}, \ldots, U^{(J-1)}$ are defined, satisfying the following conditions:

- the sequences $g_n^{(1)}, \ldots, g_n^{(J-1)}$ are orthogonal,
- $\big(g_n^{(j)}\big)^{-1} w_n^{(J-1)} \rightharpoonup 0$ for $j = 1, \ldots, J-1$,
- $\|U^{(j)}\|_H \geq \limsup_{n\to\infty} \|w_n^{(j-1)}\|_G$ for $j = 1, \ldots, J-1$.

We'll find $g_n^{(J)}$ and $U^{(J)}$, and the sequence $w_n^{(J)}$ will be defined by the relation

$$w_n^{(J-1)} = g_n^{(J)} U^{(J)} + w_n^{(J)}.$$

If $\lim_{n\to\infty} \|w_n^{(J-1)}\|_G = 0$, then we posit $U^{(J)} = U^{(J+1)} = \ldots = 0$ and the proof is complete.

Otherwise, by Lemma reflem:big-profile, after eventually extracting a sub-sequence, there exists a sequence $g_n^{(J)} \in G$ and $U^{(J)} \in H$ such that

$$(g_n^{(J)})^{-1} w_n^{(J-1)} \rightharpoonup U^{(J)}, \qquad \|U^{(J)}\|_H \geq \limsup_{n\to\infty} \|w_n^{(J-1)}\|_G > 0. \tag{5.1.3}$$

Let's assume that $g_n^{(J)}$ is not orthogonal to $g_n^{(j)}$ for some $j \in \{1, \ldots, J-1\}$. After the extraction of a sub-suite we will then have $(g_n^{(j)})^{-1} g_n^{(J)} \to g \in G$. If $\phi \in H$, then $(g_n^{(j)})^{-1} g_n^{(J)} \phi \to g\phi$ (by the hypothesis of strong continuity), $(g_n^{(j)})^{-1} w_n^{(J-1)} \rightharpoonup 0$ and $(g_n^{(j)})^{-1} g_n^{(J)} U^{(J)} \to gU^{(J)}$, so

$$\big\langle (g_n^{(j)})^{-1} g_n^{(J)} \phi, (g_n^{(j)})^{-1} w_n^{(J-1)} - (g_n^{(j)})^{-1} g_n^{(J)} U^{(J)} \big\rangle \to \langle g\phi, gU^{(J)}\rangle = \langle \phi, U^{(J)}\rangle.$$

On the other hand,

$$\big\langle (g_n^{(j)})^{-1} g_n^{(J)} \phi, (g_n^{(j)})^{-1} w_n^{(J-1)} - (g_n^{(j)})^{-1} g_n^{(J)} U^{(J)} \big\rangle = \big\langle \phi, (g_n^{(J)})^{-1} w_n^{(J-1)} - U^{(J)} \big\rangle \to 0.$$

This is impossible because $U^{(J)} \neq 0$, which demonstrates orthogonality.

We see that for all J there is the Pythagorean expansion (5.1.2), in particular $\lim_{J\to\infty} \|U^{(J)}|_H = 0$, so (5.1.3) implies $\lim_{J\to\infty} \limsup_{n\to\infty} \|w_n^{(J)}\|_G = 0$.

The second point is a consequence of the first conclusion of the Proposition 5.1.7. ☐

Proposition 5.1.9. *Weak convergence with concentration is characterized by the preceding theorem, i.e. it is the only topology for which the theorem is true.*

Proof. Exercise (we won't use it later). ☐

5.2 Description of topology

To use the theorem we've just demonstrated, it's necessary to have, in each case, a description of the topology of convergence, a description of the convergence topology with concentration in terms of Lebesgue, Sobolev, Besov and other norms. Here's the simplest example of such a description.

Proposition 5.2.1. *Let* $H := H^1(\mathbb{R})$ *and* $G := \mathbb{R}$ *act through translations:* $T_x f := f(\cdot - x)$. *Then, if* u_n *is a bornée sequence,*

$$\|u_n\|_G \to 0 \iff \|u_n\|_{L^\infty} \to 0.$$

Remark 5.2.2. Both norms are invariant by translations, which bodes well.

Proof. If $\|u_n\|_{L^\infty} \to 0$, then for any sequence x_n we get $\|T_{x_n} u_n\|_{L^\infty} = \|u_n\|_{L^\infty} \to 0$, so $T_{x_n} u_n \rightharpoonup 0$.

Conversely, suppose that u_n is a bounded sequence in $H^1(\mathbb{R})$ and that for any real sequence x_n we have $T_{x_n} u_n \rightharpoonup 0$. In particular, let $x_n \in \mathbb{R}$ be such that $|u_n(-x_n)| \geq \frac{1}{2}\|u_n\|_{L^\infty}$, in other words

$$\|u_n\|_{L^\infty} \leq 2|v_n(0)|, \qquad \text{où } v_n := T_{x_n} u_n.$$

Since $v_n \rightharpoonup 0$, Rellich's theorem implies $|v_n(0)| \to 0$, so $\|u_n\|_{L^\infty} \to 0$. ☐

Note that this action verifies the hypotheses of continuity and evanescence of matrix coefficients. We return to the framework of the previous chapter.

Proposition 5.2.3. *Let* $G = (0, \infty)$ *be the multiplication group acting on* $\mathcal{H}$ *(cf. previous chapter) by change of scale, i.e.*

$$(T_\lambda u)(r) := u_\lambda(r) = u(r/\lambda), \qquad \textit{for any } \lambda \in (0, \infty),\ u \in \mathcal{H}.$$

Then, if u_n *is a bounded sequence,*

$$\|u_n\|_G \to 0 \iff \|u_n\|_{L^\infty} \to 0.$$

Proof. We are brought back to the previous result by the change of variable $r^k = e^x$, which defines an isometry $\mathcal{H} \simeq H^1(\mathbb{R})$, as we had seen. It's easy to check that the change of scale by a coefficient λ corresponds, through this change of variable, to translation by $y = k \log r$. □

Observe that this group action, being isomorphic to the action in the preceding example, also verifies the hypotheses of strong continuity and evanescence of matrix coefficients, also verifies the hypotheses of strong continuity and evanescence of matrix coefficients.

If $(u_0, \dot{u}_0) \in \mathcal{E}$ and $\lambda > 0$, we write

$$(u_0, \dot{u}_0)_\lambda := r \mapsto (u_0(\lambda^{-1} r), \lambda^{-1} \dot{u}_0(\lambda^{-1} r)).$$

We see that $E((u_0, \dot{u}_0)_\lambda) = E((u_0, \dot{u}_0))$ and $\|(u_0, \dot{u}_0)_\lambda|_{\mathcal{E}} = \|(u_0, \dot{u}_0)\|_{\mathcal{E}}$.

Consider the (non-commutative) group $\mathbb{R} \times (0, \infty)$ acting on $\mathcal{E}$ by

$$T_{(t,\lambda)}(u_0, \dot{u}_0) := S(-t)\big((u_0, \dot{u}_0)_\lambda\big) = r \mapsto (u_L(-\lambda^{-1} t, \lambda^{-1} r), \lambda^{-1} \partial_t u_L(-\lambda^{-1} t, \lambda^{-1} r)),$$

où $u_L(t) = S(t)(u_0, \dot{u}_0)$.

Remark 5.2.4. It's easy to verify that the group law is given by

$$(t_2, \lambda_2) \cdot (t_1, \lambda_1) = (t_2 + \lambda_2 t_1, \lambda_2 \lambda_1),$$

so

$$(t_2, \lambda_2)^{-1} cdot (t_1, \lambda_1) = (-t_2/\lambda_2 + t_1/\lambda_2, \lambda_1/\lambda_2),$$

which means that two sequences $g_n = (t_n, \lambda_n)$ and $\widetilde{g}_n = (\widetilde{t}_n, \widetilde{\lambda}_n)$ are orthogonal if and only if

$$\lim_{n \to \infty} \frac{\lambda_n}{\widetilde{\lambda}_n} + \frac{\widetilde{\lambda}_n}{\lambda_n} + \frac{|t_n - \widetilde{t}_n|}{\lambda_n} = \infty.$$

Proposition 5.2.5. *If* $(u_{0,n}, \dot{u}_{0,n}) \in \mathcal{E}$ *is a bounded sequence, then*

$$\|(u_{0,n}, \dot{u}_{0,n})\|_G \to 0 \iff \|u_{L,n}\|_{L^\infty L^\infty} \to 0 \iff u_{L,n}\|_S \to 0.$$

Proof. The second condition implies the third by Lemma 4.1.4.

Suppose $\|S(t)(u_{0,n}, \dot{u}_{0,n})\|_S \to 0$. This implies that for any sequence (t_n, λ_n) we have

$$\big\|S(t)\big(T_{(t_n, \lambda_n)}(u_{0,n}, \dot{u}_{0,n})\big)\big\|_S \to 0$$

(since the S norm is invariant to the change of scale and by the linéaire flow). In particular, $T_{(t_n,\lambda_n)}(u_{0,n}, \dot{u}_{0,n}) \rightharpoonup 0$. (in fact, the function $(v_0, \dot{v}_0) \mapsto \|S(t)(v_0, \dot{v}_0)\|_S$ is continuous and convex).

Finally, suppose $\|(u_{0,n}, \dot{u}_{0,n})\|_G \to 0$. Let t_n be a sequence such that

$$\|S(t)(u_{0,n}, \dot{u}_{0,n})\|_{L^\infty L^\infty} \le 2\|S(t_n)(u_{0,n}, \dot{u}_{0,n})\|_{L^\infty}. \tag{5.2.1}$$

Let $(v_n, \dot{v}_n) := S(t_n)(u_{0,n}, \dot{u}_{0,n})$. We have $(v_n, \dot{v}_n)_{lambda_n} \rightharpoonup 0$. for any sequence $\lambda_n \in]0, \infty[$. By conséquent, Proposition 5.2.3 implies $\|v_n\|_{L^\infty} \to 0$, so (5.2.1) gives $\|S(t)(u_{0,n}, \dot{u}_{0,n})\|_{L^\infty L^\infty} \to 0$. □

Remark 5.2.6. In other words, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|(u_0, \dot{u}_0)\|_{\mathcal{E}} \le 1$ and $\|(u_0, \dot{u}_0)\|_G \le \delta$ implies $\|u_{L,n}\|_S \le \epsilon$, and conversely $\|(u_0, \dot{u}_0)\|_{\mathcal{E}} \le 1$ and $\|u_{L,n}\|_S \le \delta$ implies $\|(u_0, \dot{u}_0)\|_G \le \epsilon$.

Combining the Theorem 5.1.8 with the Proposition 5.2.5 we have the following result.

Proposition 5.2.7. *Let $(u_{0,n}, \dot{u}_{0,n}) \in \mathcal{E}$ be a bounded sequence. Then, after possibly extracting a sub-sequence,*

$$(u_{0,n}, \dot{u}_{0,n}) = \sum_{j=1}^{J} \left(S(-t_{j,n}/\lambda_{j,n})(U^{(j)}, \dot{U}^{(j)})\right)_{\lambda_{j,n}} + (w_n^{(J)}, \dot{w}_n^{(J)}),$$

with

$$\lim_{J\to\infty} \|S(t)(w_n^{(J)}, \dot{w}_n^{(J)})\|_S = 0.$$

Proof. We need to verify the hypotheses of the Theorem 5.1.8. Strong continuity is immediate. To demonstrate the decay of matrix coefficients, let's take $(u_0, \dot{u}_0) \in \mathcal{E}$ and a sequence $(t_n, \lambda_n) \to \infty$. It suffices to show that every sub-suite of $T_{(t_n,\lambda_n)}(u_0, \dot{u}_0)$ has a sub-sequence that converges weakly to 0 in $\mathcal{E}$. We consider three cases.

Case 1: $t_n/\lambda_n \to -\infty$. Let $(v_n, \dot{v}_n) := T_{(t_n,\lambda_n)}(u_0, \dot{u}_0)$. We see that $\|S(t)(v_n, \dot{v}_n)\|_{S([0,\infty[)} \to 0$, which implies $(v_n, \dot{v}_n) \rightharpoonup 0$ by Fatou's propriété. (The functional $(v, \dot{v}) \mapsto \|S(t)(v_n, \dot{v}_n)\|_{S([0,\infty[)}$ is continuous $\mathcal{E} \to \mathbb{R}$, convex and cancels only at $v = 0$).

Case 2: $t_n/\lambda_n \to \infty$. Let $(v_n, \dot{v}_n) := T_{(t_n,\lambda_n)}(u_0, \dot{u}_0)$. We see that $\|S(t)(v_n, \dot{v}_n)\|_{S(]-\infty,0])} \to 0$, which implies $(v_n, \dot{v}_n) \rightharpoonup 0$.

Case 3: t_n/λ_n borné, $|\log \lambda_n| \to \infty$. Then $\{(u_L(-t_n/\lambda_n), \partial_t u_L(-t_n/\lambda_n))\}$ is a compact set in $\mathcal{E}$, so the évanescence of the matrix coefficients for the change of scale implies

$$T_{(t_n,\lambda_n)}(u_0, \dot{u}_0) = (u_L(-t_n/\lambda_n), \partial_t u_L(-t_n/\lambda_n))_{\lambda_n} \rightharpoonup 0.$$

(We have seen the evanescence of matrix coefficients for the change of scale in $\mathcal{H}$; this is also true in L^2, as can be verified either directly, or by reducing to the group of translations by the change of variable $r^k = e^x$.) □

Remark 5.2.8. In particular, this proof shows that for all $(u_0, \dot{u}_0) \in \mathcal{E}$ we have

$$\lim_{t\to\pm\infty} |S(t)(u_0, \dot{u}_0)\|_{L^\infty} = 0.$$

By extracting a sub-suite once again, we can assume that for all j, we have one of the following cases:

$$\begin{aligned} &t_{j,n} = 0,\ \forall n && \textit{centered wave},\\ &\frac{t_{j,n}}{\lambda_{j,n}} \text{to}\infty && \textit{incoming wave},\\ &\frac{t_{j,n}}{\lambda_{j,n}} \to -\infty && \textit{outgoing wave}. \end{aligned}$$

This dénomination is intuitive if we think of $(u_{0,n}, \dot{u}_{0,n})$ as a sequence of initial data which we evolve by $S(t)$.

5.3 Decomposition into non-linear profiles

Profile-nonlinear decomposition is essentially a tool for calculating variations. We'll now turn our attention to questions of *propagation* by non-linear flow. There is no general theory for this type of question, but for the equation (4.1.1) we can find satisfactory results.

Let $(u_{0,n}, \dot{u}_{0,n})$ be a sequence of initial data having a decomposition into profiles given by the Proposition 5.2.7. For each profile $(U^{(j)}, \dot{U}^{(j)})$, we associate the corresponding non-linear profile $V^{(j)} = V^{(j)}(t, r)$, which is the solution to the equation (4.1.1) that satisfies

$$\lim_{n\to\infty} \|(V^{(j)}(-t_{j,n}/\lambda_{j,n}), \partial_t V^{(j)}(-t_{j,n}/\lambda_{j,n})) - S(-t_{j,n}/\lambda_{j,n})(U^{(j)}, \dot{U}^{(j)})\|_{\mathcal{E}} = 0.$$

The existence and uniqueness of $V^{(j)}$ results from the Lemma reflem:cauchy and the Remark refrem:wave-op.

We will write

$$V_n^{(j)}(t) := V^{(j)}((t - t_{j,n})/\lambda_{j,n})_{\lambda_{j,n}}, \qquad U_n^{(j)}(t) := \big(S((t-t_{j,n})/\lambda_{j,n})(U^{(j)}, \dot{U}^{(j)})\big)_{\lambda_{j,n}}.$$

It is useful to note that for j sufficiently large $\|(U^{(j)}, \dot{U}^{(j)})\|_{\mathcal{E}}$ is small, so the Lemma reflem:cauchy implies in particular that for j large $V^{(j)}$ exists for all $t \in \mathbb{R}$ and

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|(V_n^{(j)}(t), \partial_t V_n^{(j)}(t)) - (U_n^{(j)}(t), \partial_t U_n^{(j)}(t))\|_{\mathcal{E}} + \|(V_n^{(j)}(t), \partial_t V_n^{(j)}(t)) - (U_n^{(j)}(t), \partial_t U_n^{(j)}(t))\|_S \\ \lesssim \|(U^{(j)}, \dot{U}^{(j)})\|_{\mathcal{E}}^3 \lesssim \|(U^{(j)}, \dot{U}^{(j)})\|_{\mathcal{E}}^2. \end{aligned} \tag{5.3.1}$$

Lemma 5.3.1. *Let $(u_{0,n}, \dot{u}_{0,n})$ be a sequence that admits profile decomposition, and suppose that for all j the non-linear profile $V^{(j)}$ is defined for all times and $\|V^{(j)}\|_S < \infty$. Then for n sufficiently large the solution of (4.1.1) for the initial data $(u_{0,n}, \dot{u}_{0,n})$ exists for all times and disperses.*

Proof. The idea is to consider an approximate solution

$$v_n^{(J)}(t) := \sum_{j=1}^{J} V_n^{(j)}(t) + S(t)(w_n^{(J)}, \dot{w}_n^{(J)})$$

and show that for J sufficiently large, it's possible to use Lemma 4.1.9.

The function $v_n^{(J)}(t)$ solves the equation

$$\partial_t^2 v_n - \partial_r^2 v_n - r^{-1}\partial_r v_n + \frac{k^2}{2r^2}\sin(2v_n) = h_n :=$$
$$\frac{k^2}{2r^2}\sin\left(2\Big(\sum_{j=1}^{J} V_n^{(j)}(t) + S(t)(w_n^{(J)}, \dot{w}_n^{(J)})\Big)\right)$$
$$-\frac{k^2}{2r^2}\sum_{j=1}^{J}\sin\left(2V_n^{(j)}(t)\right) - \frac{k^2}{r^2}S(t)(w_n^{(J)}, \dot{w}_n^{(J)}).$$

To apply the Lemma reflem:perturbation, we need to verify that

- $\limsup_n \|(v_n^{(J)}, \partial_t v_n^{(J)})\|_{L^\infty(\mathbb{R};\mathcal{E})} + \|v_n^{(J)}\|_{S(\mathbb{R})}$ is bounded uniformly in J,
- $\lim_{J\to\infty}\limsup_{n\to\infty}\|h_n\|_{L^1L^2} = 0.$

The first point flows from (5.3.1). Indeed, the Pythagorean expansion (5.1.2) we obtain

$$\sup_J \limsup_n \left(\sup_{t\in\mathbb{R}}\Big\|\sum_{j=1}^{J} V_n^{(j)}(t) - \sum_{j=1}^{J} U_n^{(j)}(t)\Big\|_{\mathcal{E}} + \Big\|\sum_{j=1}^{J} V_n^{(j)}(t) - \sum_{j=1}^{J} U_n^{(j)}(t)\Big\|_S\right) < \infty,$$

but

$$\sup_J \limsup_n \left(\sup_{t\in\mathbb{R}}\Big\|\sum_{j=1}^{J} U_n^{(j)}(t)\Big\|_{\mathcal{E}} + \Big\|\sum_{j=1}^{J} U_n^{(j)}(t)\Big\|_S\right) \lesssim \sup_J \limsup_n \Big\|\sum_{j=1}^{J}\left(U_n^{(j)}(0), \partial_t U_n^{(j)}(0)\right)\Big\|_{\mathcal{E}}$$
$$\leq \sup_J \limsup_n \left(\|(u_{n,0}, \dot{u}_{n,0})\|_{\mathcal{E}} + \|(w_n^{(J)}, \dot{w}_n^{(J)})\|_{\mathcal{E}}\right) < \infty.$$

To show the second point, using the inequality

$$\left|\sin(\alpha+\beta) - \sin(\alpha) - \beta\right| \lesssim \alpha^2\beta + \beta^3$$

with $\alpha = 2\sum_{j=1}^{J} V_n^{(j)}(t)$ and $\beta = 2S(t)(w_n^{(J)}, \dot{w}_n^{(J)})$, we see that it's sufficient to verify that for J fixed

$$\limsup_n \left\|\frac{1}{r^2}\sin\left(2\sum_{j=1}^{J} V_n^{(j)}(t)\right) - \frac{1}{r^2}\sum_{j=1}^{J}\sin\left(2V_n^{(j)}(t)\right)\right\|_{L^1L^2} = 0.$$

We forget that $V^{(j)}$ is a solution of (4.1.1) and show this convergence for any sequence of functions $V^{(1)}, \dots, V^{(J)} \in S$ (this trick is due to Bahouri and Gérard). By density, we can assume $V^{(j)} \in C_0^\infty(\mathbb{R}\times(0,\infty))$. But then, for n large, the supports (in space-time) of the functions $V_n^{(j)}$ for $j = 1, \dots, J$ are disjoint. □

Sometimes you don't want to assume that all profiles are globally defined and spread out. The situation then becomes a little more complicated. We'll consider propagation for positive times, which doesn't restrict the generality thanks to the time-reversibility.

Definition 5.3.2. Let u be a solution of (4.1.1). We'll say that a time sequence T_n is *regular* for solution u if there exists an interval $I \subset \mathbb{R}$ such that $T_n \in I$ for all n and $\|u\|_{S(I)} < \infty$. We say that a sequence of interval $[a_n, b_n]$ is regular if any sequence $T_n \in [a_n, b_n]$ is regular.

Lemma 5.3.3. *Let u be a solution of (4.1.1), $(t_n, \lambda_n) \to \infty$ and τ_n a bounded sequence such that the sequence $T_n := \frac{\tau_n - t_n}{\lambda_n}$ is regular. Then $(u(T_n), \partial_t u(T_n))_{lambda_n} \rightharpoonup 0$.*

Proof. By extracting a sub-sequence, we can assume that the limits $\lim_{n\to\infty} \log \lambda_n \in [-\infty, \infty]$ and $\lim_{n\to\infty} T_n \in [-\infty, \infty]$ exist. If $\lambda_n \to 0$ or $\lambda_n \to \infty$, then we obtain the conclusion by considering separately the three cases $T_n \to -\infty$, $T_n \to \infty$ or $T_n \to T \in \mathbb{R}$.

If $\log \lambda_n$ is bounded, then $|t_n| \to \infty$, which implies $|T_n| \to \infty$, since τ_n and bornée, so we also get $(u(T_n), \partial_t u(T_n))_{lambda_n} \rightharpoonup 0$. □

Lemma 5.3.4. *Let $f \in inC_0^\infty(\mathbb{R}\times, (0, \infty))$, u be a solution of (4.1.1), $(t_n, \lambda_n) \to \infty$, $u_n(t, r) := u((t-t_n)/\lambda_n, r/\lambda_n)$ and I_n a sequence of intervals such that the sequence $(I_n - t_n)/\lambda_n$ is regular for u. Then*

$$\lim_{n\to\infty} \int_{I_n} \int_0^\infty (\partial_t u_n) f \, r dr \, dt = 0.$$

Proof. We integrate by parts in t. The boundary terms converge to 0 according to the preceding lemma. □

Proposition 5.3.5. *Let $(u_{0,n}, \dot{u}_{0,n})$ be a sequence in $\mathcal{E}$ which admits a profile decomposition and let T_n be a sequence such that for all j the sequence $(T_n - t_{j,n})/\lambda_{j,n}$ is régulière for $V^{(j)}$. Let u_n be the solution of (4.1.1) for the initial data $(u_{0,n}, \dot{u}_{0,n})$ and let*

$$v_n^{(J)}(t) := \sum_{j=1}^{J} V_n^{(j)}(t) + S(t)(w_n^{(J)}, \dot{w}_n^{(J)}).$$

Then for n sufficiently large u_n is defined for $t \in I_n := [0, T_n[$ and

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left(\|(u_n, \partial_t u_n) - (v_n^{(J)}, \partial_t v_n^{(J)})\|_{L^\infty(I_n;\mathcal{E})} + \|u_n - v_n^{(J)}\|_{S(I_n)} \right) = 0.$$

In addition, we have the Pythagorean expansion

$$\lim_{J\to\infty} \limsup_{n\to\infty} \sup_{t\in[0,T_n[} \left| \|(u_n(t), \partial_t u_n(t))\|_{\mathcal{E}}^2 - \sum_{j=1}^{J} \|(V_n^{(j)}(t), \partial_t V_n^{(j)}(t))\|_{\mathcal{E}}^2 - \|S(t)(w_n^{(J)}, \dot{w}_n^{(J)})\|_{\mathcal{E}}^2 \right| = 0. \tag{5.3.2}$$

Proof. For the first part, the argument is the same as for Lemma reflem:tout-disperse, but at the end, the $V^{(j)}$ profiles are approximated by functions whose supports are compact subsets of $I^{(j)}(0, \infty)$.

To show the second part, simply estimate $\|(v_n^{(J)}(t), \partial_t v_n^{(J)}(t))\|_{\mathcal{E}}^2$ then use the first part. Calculate the time distribution of the terms crossed. We write $w_n^{(J)}(t) := S(t)(w_n^{(J)}, \dot{w}_n^{(J)})$.

$$\frac{d}{dt}\left(\langle \partial_t V_n^{(j)}, \partial_t S(t)(w_n^{(J)}, \dot{w}_n^{(J)})\rangle + \langle V_n^{(j)}, L_k S(t)(w_n^{(J)}, \dot{w}_n^{(J)})\rangle\right) = \langle Z(V_n^{(j)})(V_n^{(j)})^3/(r^2), \partial_t w_n^{(J)}\rangle.$$

So we need to show that for any $s_n \in [0, T_n[$

$$\int_0^{s_n} \langle Z(V_n^{(j)})(V_n^{(j)})^3/(r^2), \partial_t w_n^{(J)}\, dt \to 0.$$

To do this, we integrate by parts in time and obtain:

$$-\int_0^{s_n} \left\langle \partial_t\big(Z(V_n^{(j)})(V_n^{(j)})^3/(r^2)\big), w_n^{(J)} \right\rangle + \text{edge terms.}$$

The first integral is small using $V^{(j)} \in S$, $\partial_t V^{(j)} \in L^i nfty L^2$ and $\|w_n^{(J)}\|_S \ll 1$. The boundary terms are small thanks to the fact that $(V^{(j)})^3/r^2 \in L^\infty L^1$ and $\|w_n^{(J)}\|_{L^\infty L^\infty} \ll 1$.

Regarding the cross terms $\langle Z(V_n^{(j)})(V_n^{(j)})^3/(r^2), \partial_t V_n^{(k)}\rangle$, Lemma 5.3.4 shows that they converge to 0 as $n \to \infty$. □

Remark 5.3.6. Thomas Duyckaerts pointed out to me that there was a faster way of proving (5.3.2). Namely, we can show that if the sequence $(T_n - t_n^{(j)})/\lambda_n^{(j)}$ is regular for $V^{(j)}$ for all j, then we can construct a *linear* decomposition of the sequence $(u_n(T_n), \partial_t u_n(T_n))$ (where the new linear profiles will be close to the non-linear profiles $(V_n^{(j)}, \partial_t V_n^{(j)})$ in $\mathcal{E}$ norm), and conclude using the linear Pythagorean expansion.

To conclude this chapter, we give another application, also considered in the article by Bahouri and Gérard, namely the weak continuity of the flow in energy space. We need the following lemma.

Lemma 5.3.7. *Let $V^{(j)}$ be a non-linear profile whose norm is small and suppose that the sequence $(t_n^{(j)}, \lambda_n^{(j)})$ is orthogonal to the constant sequence $(0, 1)$. Then $(V_n^{(j)}(t), \partial_t V_n^{(j)}(t)) \rightharpoonup 0$ for all $t \in \mathbb{R}$.*

Proof. Without loss of generality, we can assume that $\frac{t - t_n^{(j)}}{\lambda_n^{(j)}} \to t_0 \in [-\infty, \infty]$, and treat the three cases separately, noting that the sequence $(t_n^{(j)} - t, \lambda_n^{(j)})$ is orthogonal to $(0, 1)$. □

Proposition 5.3.8. *There exists $\eta > 0$ such that the following is true. Let $u : [0, T] \to \mathcal{E}$ be a strong solution of (4.1.1), and $(u_{0,n}, \dot{u}_{0,n}) \in \mathcal{E}$ a sequence such that $\|(u_{0,n}, \dot{u}_{0,n}) - (u(0), \partial_t u(0))\|_{\mathcal{E}} \le \eta$ for all n and $(u_{0,n}, \dot{u}_{0,n}) \rightharpoonup (u(0), \partial_t u(0))$. Then for n large the solution u_n of (4.1.1) exists for $t \in [0, T]$ and*

$$(u_n(t), \partial_t u_n(t)) \rightharpoonup (u(t), \partial_t u(t)), \qquad \text{for all } t \in [0, T].$$

Proof. A sub-sequence of the sequence $(u_{0,n}, \dot{u}_{0,n})$ admits a décomposition in profiles. The first profile, which corresponds to the trivial displacement $(0, 1)$, is $(u(0), \partial_t u(0))$, and all the others are small. By an argument similar to that used in the proof of Lemma 5.3.1, the preceding lemma yields the conclusion. □

Bibliography

[1] H. Bahouri, J.-Y. Chemin and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer.

[2] C. Muscalu and W. Schlag. *Classical and Multilinear Harmonic Analysis*, Cambridge University Press.

[3] E. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press.