Chapter 2

Elements of harmonic analysis

2.1 Riesz-Thorin interpolation theorem

Lemma 2.1.1 (Three-line theorem, Phragmen-Lindelöf principle)**.** *Let $F(z)$ be bounded and continuous on the strip $0 \leq x \leq 1$ and analytic inside. If $|F(it, y)| \leq M_1$ and $F(1 + it, y) \leq M_2$ for all y, then*

$$|F(x, y)| \leq M_1^{1-x} M_2^x, \qquad \textit{for all } x \in [0, 1].$$

Proof. It is sufficient to consider $M_1 = M_2 = 1$. By considering the function $\widetilde{F}(z) := F(z)\mathrm{e}^{\epsilon(z^2-1)}$, we reduce to the case $\lim_{y\to\infty} |F(z)| = 0$, and the conclusion follows from the Maximum Principle. □

Proposition 2.1.2 (Riesz-Thorin interpolation theorem)**.** *Let (X, μ) and $(\widetilde{X}, \widetilde{\mu})$ be measure spaces. Let $1 \leq p_1, p_2 \leq \infty$ and assume that $Y \subset L^{p_1}(X, \mu) \cap L^{p_2}(X, \mu)$ is dense in both $L^{p_1}(X, \mu)$ and $L^{p_2}(X, \mu)$. Let T be a linear operator defined on Y taking its values in measurable functions on $(\widetilde{X}, \widetilde{\mu})$ and assume that $1 \leq q_1, q_2 \leq \infty$, M_1, M_2 are such that*

$$\|Tf\|_{L^{q_j}(\widetilde{X},\widetilde{\mu})} \leq M_j \|f\|_{L^{p_j}}(X, \mu), \qquad \textit{for all } f \in Y \textit{ and } j \in \{1, 2\}.$$

Then for all $\theta \in [0, 1]$

$$\|Tf\|_{L^q(\widetilde{X},\widetilde{\mu})} \leq M_1^\theta M_2^{1-\theta} \|f\|_{L^p(X,\mu)} \qquad \textit{for all } f \in Y,$$

where

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof. The conclusion is obvious if $\theta = 0$ or $\theta = 1$, so assume $0 < \theta < 1$. If $p_1 = p_2 = \infty$, then the theorem follows from the Hölder inequality, thus we may assume $p_1 < \infty$ or $p_2 < \infty$, which allows us to consider only f being a step function with finite set of values. Note that we can assume that Y contains such functions (extending T by density if needed; we could also assume that $Y = L^{p_1} \cap L^{p_2}$).

We need to estimate

$$\sup\{\langle Tf, g\rangle : \|f\|_{L^p} \leq 1, \|g\|_{L^{q'}} \leq 1\},$$

with the supremum taken over step functions with a finite set of values:

$$f = \sum_j a_j I_{A_j}, \qquad g = \sum_k b_k I_{B_k}.$$

(Attention to the case $q = q_1 = q_2 = 1$).

For $0 \le \Re z \le 1$ we set

$$\frac{1}{p(z)} := \frac{1-z}{p_1} + \frac{z}{p_2}, \qquad \frac{1}{q'(z)} := \frac{1-z}{q_1'} + \frac{z}{q_2},$$

$$\phi(z) := \sum_j |a_j|^{\frac{p}{p(z)}} \mathrm{e}^{i \arg a_j} I_{A_j}, \quad \psi(z) := \sum_k |b_k|^{\frac{q'}{q'(z)}} \mathrm{e}^{i \arg b_k} I_{B_k}.$$

We apply the three-line theorem to $\langle T\phi(z), \psi(z)\rangle$. □

2.2 Real analysis

In this section, we follow Chapter 1 from the book [1].

Proposition 2.2.1 (Minkowski inequality)**.** *If* (X, μ), (Y, ν) *measure spaces,* $1 \le p \le q \le \infty$ *and* $f : X \times Y \to \mathbb{R}_+$ *is measurable, then*

$$\left\| y \mapsto \|f(\cdot, y)\|_{L^p(X)} \right\|_{L^q(Y)} \le \left\| x \mapsto \|f(x, \cdot)\|_{L^q(Y)} \right\|_{L^p(X)}.$$

Proof. We can assume that $f \ge 0$ and, upon replacing f by f^p, also that $p = 1$. Let $g \in L^{q'}(Y)$. We have

$$\int_Y g(y) \int_X f(x, y) \, \mathrm{d}x \, \mathrm{d}y \le \int_X \|f(x, \cdot)\|_{L^q} \|g\|_{L^{q'}} \, \mathrm{d}x$$

by Hölder inequality. □

2.2.1 Young inequalities for convolutions

Recall that for f, g functions on $\mathbb{R}^d$ we denote

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y) g(y) \, \mathrm{d}y,$$

whenever this expression makes sense.

Proposition 2.2.2 (Young's inequality)**.** *Let* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. *If*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

then

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. If $q = 1$, this follows from Minkowski inequality. If $q = p'$ and $r = \infty$, this follows from Hölder inequality. The remaining cases follow from Riesz-Thorin. □

2.2.2 Weak L^p spaces

For a measurable function g we define

$$\|g\|_{L^q_w}^q := \sup_{\lambda>0} \lambda^q \mu\{x : |g(x)| \geq \lambda\}.$$

Lemma 2.2.3 (Markov inequality)**.** *For any measurable g, $\|g\|_{L^q_w} \leq \|g\|_{L^q}$.* □

Proposition 2.2.4 (Refined Young's inequality)**.** *Under assumptions of Proposition 2.2.2, if $1 < p, q, r < \infty$, there exists $C > 0$ such that for all measurable f, g*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q_w}.$$

Corollary 2.2.5 (Hardy-Littlewood-Sobolev inequality)**.** *If $\alpha \in (0, d)$ and $(p, r) \in (1, \infty)$ satisfy*

$$\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r},$$

then

$$\| |\cdot|^{-\alpha} * f\|_{L^r} \leq C\|f\|_{L^p}.$$

Proof. The function $|x|^{-\alpha}$ is in the space $L_w^{d/\alpha}(\mathbb{R}^d)$. □

In order to prove the refined Young inequality, we use the following tool.

Proposition 2.2.6 (Atomic decomposition)**.** *Let (X, μ) a measure space, $p \in [1, \infty)$, $f \in L^p(X)$ positive. There exist sequences of positive real numbers $(c_k)_{k\in\mathbb{Z}}$ and functions $(f_k)_{k\in\mathbb{Z}}$ such that*

$$\begin{aligned} \operatorname{supp} f_j \cap \operatorname{supp} f_k = \emptyset, \\ \mu(\operatorname{supp} f_k) \leq 2^{k+1}, \\ \|f_k\|_{L^\infty} \leq 2^{-\frac{k}{p}}, \\ \frac{1}{2}\|f\|_{L^p}^p \leq \sum_{k\in\mathbb{Z}} c_k^p \leq 2\|f\|_{L^p}^p. \end{aligned}$$

Proof. We set

$$\begin{aligned} \lambda_k := \inf\{\lambda : \mu(f > \lambda) < 2^k\}, \\ c_k := 2^{\frac{k}{p}} \lambda_k, \\ f_k := c_k^{-1} I_{\lambda_{k+1} < f \leq \lambda_k} f. \end{aligned}$$

We will check all the requirements. □

Remark 2.2.7. Many other decompositions of this type are used in harmonic analysis. We will encounter at least one more example, the Littlewood-Paley decomposition.

Proof of Proposition 2.2.4. Next lecture. □

2.3 Fourier transform

In this section, the presentation is often close to the one in Chapter 4, Volume 1 of the book by Muscalu and Schlag [2].

Let μ be a complex-valued Borel measure on $\mathbb{R}^d$ of finite total variation. We define its Fourier transform:

$$(\mathcal{F}\mu)(\xi) = \widehat{\mu}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-ix\cdot\xi}\mu(\,\mathrm{d}x), \qquad \forall \xi \in \mathbb{R}^d.$$

We see that $\widehat{\mu}$ is a bounded continuous function.

If $f \in L^1(\,\mathrm{d}x)$, we set $\mathcal{F}f := \mathcal{F}(f\,\mathrm{d}x)$.

2.3.1 Fourier transform on the Schwartz space

It is useful to extend the Fourier transformation on functions which are not in L^1. In order to do this, we introduce the space of tempered distributions.

Definition 2.3.1. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the space of complex-valued functions $f \in C^\infty(\mathbb{R}^d)$ such that for any multi-indices $\alpha, \beta \in \mathbb{N}^d$

$$x^\alpha \partial^\beta f \in L^\infty(\mathbb{R}^d).$$

We say that a sequence $f_n \in \mathcal{S}(\mathbb{R}^d)$ converges to $f \in \mathbb{S}(\mathbb{R}^d)$ if for any multi-indices α, β

$$\lim_{n\to\infty} \|x^\alpha \partial^\beta (f_n - f)\|_{L^\infty} = 0.$$

Proposition 2.3.2. *The Fourier transform $\mathcal{F}$ is continuous $\mathcal{S} \to \mathcal{S}$.*

Proof. This follows from the formulas:

$$\begin{aligned}(i\partial)^\alpha \widehat{f}(\xi) &= \mathcal{F}(x^\alpha f)(\xi),\\ (i\xi)^\alpha \widehat{f}(\xi) &= \mathcal{F}(\partial^\alpha f)(\xi).\end{aligned}$$

□

Proposition 2.3.3 (Fourier inversion theorem)**.** *The Fourier transform takes $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$. For any $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi} \widehat{f}(\xi)\,\mathrm{d}\xi, \qquad \forall x \in \mathbb{R}^d. \tag{2.3.1}$$

Proof. We need the following fact. For any $\epsilon > 0$ we have (see Exercise 2.6.2):

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi - \frac{\epsilon}{2}|\xi|^2}\,\mathrm{d}\xi = \left(\frac{2\pi}{\epsilon}\right)^{\frac{d}{2}} \mathrm{e}^{-\frac{|x|^2}{2\epsilon}}. \tag{2.3.2}$$

Using this, we can write, for any $\epsilon > 0$:

$$\begin{aligned}(2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi}\widehat{f}(\xi)\mathrm{e}^{-\frac{\epsilon}{2}|\xi|^2}\,\mathrm{d}\xi &= (2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi}\int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot\xi} f(y)\mathrm{e}^{-\frac{\epsilon}{2}|\xi|^2}\,\mathrm{d}y\,\mathrm{d}\xi\\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d} \mathrm{e}^{i(x-y)\cdot\xi - \frac{\epsilon}{2}|\xi|^2}\,\mathrm{d}\xi\,\mathrm{d}y\\ &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} f(y)\epsilon^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}|\frac{x-y}{\sqrt{\epsilon}}|^2}\,\mathrm{d}y\end{aligned}$$

When $\epsilon \to 0$, the left hand side tends to the right hand side of (2.3.1), and the right hand side tends to $f(x)$. This finishes the proof. $\square$

Definition 2.3.4. A *tempered distribution* on $\mathbb{R}^d$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^d)$, that is a linear functional $\phi : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ such that $\langle \phi, u_n \rangle \to \langle \phi, u \rangle$ whenever $u_n \to u$ in $\mathcal{S}(\mathbb{R}^d)$.

We say that a sequence $\phi_n \in \mathcal{S}(\mathbb{R}^d)$ converges to $u \in \mathcal{S}(\mathbb{R}^d)$ if $\langle \phi_n, u \rangle \to \langle \phi, u \rangle$ for all $u \in \mathcal{S}(\mathbb{R}^d)$.

Proposition 2.3.5. *If $\phi \in \mathcal{S}'(\mathbb{R}^d)$, then there exists $C, N \geq 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^d)$*

$$|\langle \phi, u \rangle| \leq C \sum_{|\alpha| \leq N, |\beta| \leq N} \|x^\alpha \partial^\beta u\|_{L^\infty(\mathbb{R}^d)}.$$

Proof. Exercise 2.6.4. $\square$

Example 2.3.6. If f is locally integrable and there exists k such that $(1 + |x|)^{-k} f(x) \in L^1(\mathbb{R}^d)$, then we define $T_f \in \mathcal{S}'$ by the formula

$$\langle T_f, u \rangle = \int_{\mathbb{R}^d} f u \, \mathrm{d}x.$$

Note that traditionally we do not use the complex conjugate in this case.

Definition 2.3.7. For any continuous operator $A : \mathcal{S} \to \mathcal{S}$ we define the operator $A^t : \mathcal{S}' \to \mathcal{S}'$ by the formula:

$$\langle A^t \phi, u \rangle = \langle \phi, Au \rangle.$$

The Fubini theorem implies $\mathcal{F}^t u = \mathcal{F} u$ for $u \in L^1(\mathbb{R}^d)$, hence we will write $\mathcal{F}$ instead of $\mathcal{F}^t$. Analogously, we define $\partial^\alpha := (-1)^{|\alpha|}(\partial^\alpha)^t$ itp. If $\theta \in \mathcal{S}$, then we define $\phi * \theta$ by

$$\langle \phi * \theta, u \rangle = \Big\langle \phi, x \mapsto \int_{\mathbb{R}^d} \theta(y - x) u(y) \, \mathrm{d}y \Big\rangle.$$

Proposition 2.3.8. *The usual properties of the Fourier transform continue to hold. For any $u \in \mathcal{S}'$:*

$$\begin{aligned} (i\partial)^\alpha \widehat{u} &= \mathcal{F}(x^\alpha u), \\ (i\xi)^\alpha \widehat{u} &= \mathcal{F}(\partial^\alpha u), \\ \mathrm{e}^{-ia\cdot\xi} \widehat{u}(\xi) &= \mathcal{F}(x \mapsto u(x - a))(\xi), \\ \widehat{u}(\xi - \omega) &= \mathcal{F}(\mathrm{e}^{ix\cdot\omega} u)(\xi), \\ \mathcal{F}(x \mapsto u(\lambda x))(\xi) &= \lambda^{-d} \widehat{u}(\xi/\lambda), \\ \mathcal{F}(u * \theta) &= \widehat{u}\widehat{\theta}, \qquad \text{for all } \theta \in \mathcal{S}. \end{aligned}$$

Proof. Exercise 2.6.5. $\square$

Proposition 2.3.9. *Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\langle \phi, u \rangle$ for all $u \in \mathcal{S}(\mathbb{R}^d)$ with* $\operatorname{supp} u \subset \mathbb{R}^d \setminus \{0\}$. *Then $\widehat{\phi}$ is a polynomial, in other words ϕ is a finite linear combination of the Dirac delta and its derivatives.*

Proof. Exercise 2.6.6. $\square$

Proposition 2.3.10. *For any $\alpha \in (0, d)$ there exists $C(\alpha, d)$ such that*

$$\mathcal{F}(|x|^{-\alpha}) = C(\alpha, d)|\xi|^{\alpha - d}.$$

Remark 2.3.11. The functions $|x|^{-\alpha}$ are called *Riesz potentials.*

Proof. Exercise 2.6.7. □

Proposition 2.3.12 (Plancherel formula)**.** *For all $f \in L^2(\mathbb{R}^d)$,*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}}\|f\|_{L^2(\mathbb{R}^d)}.$$

Proof. Exercise 2.6.8. □

Proposition 2.3.13 (Hausdorff-Young inequality)**.** *For all $p \in [1, 2]$ and $f \in L^p(\mathbb{R}^d)$ the inequality $\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^d)} \leq (2\pi)^{\frac{d}{p'}}\|f\|_{L^p(\mathbb{R}^d)}$ is true.*

Proof. This is clear for $p = 1$, for $p = 2$ follows from Proposition 2.3.12, and for the remaining values from the Riesz-Thorin theorem. □

Lemma 2.3.14 (Bernstein inequality)**.** *There exists $C_d \geq 0$ such that if $f \in \mathcal{S}(\mathbb{R}^d)$ is such that* $\operatorname{supp} \widehat{f} \subset \{|\xi| \leq R\}$*, then for any multi-index α*

$$\|\partial^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C(\alpha, d)R^{|\alpha|+d(1/p-1/q)}\|f\|_{L^p(\mathbb{R}^d)}, \qquad \textit{for all } 1 \leq p \leq q \leq \infty.$$

Proof. Considering $g(x) := f(x/R)$, we reduce the proof to the case $R = 1$. Indeed, $\widehat{g}$ is supported in the unit ball (see Proposition 2.3.8), $\|g\|_{L^p(\mathbb{R}^d)} = R^{d/p}\|f\|_{L^p(\mathbb{R}^d)}$ and $\|\partial^\alpha g\|_{L^q(\mathbb{R}^d)} = R^{-|\alpha|+d/q}\|\partial^\alpha f\|_{L^q(\mathbb{R}^d)}$.

In order to prove the lemma for $R = 1$, we write $\widehat{\partial^\alpha f}(\xi) = (i\xi)^\alpha\widehat{\chi}(\xi)\widehat{f}(\xi)$, where $\widehat{\chi} \in C^\infty$ is identically 1 on $\{|\xi| \leq 1\}$ and $\operatorname{supp}\chi \subset \{|\xi| \leq 2\}$. In particular $\chi \in \mathcal{S}(\mathbb{R}^d)$. Taking the inverse Fourier transform we obtain $\partial^\alpha f = f * \mathcal{F}^{-1}((i\xi)^\alpha\widehat{\chi})$. Let $r := (1/(p') + 1/q)^{-1} \geq 1$ (the last inequality follows from $q \geq p$). Proposition 2.2.2 yields

$$\|\partial^\alpha f\|_{L^q} \leq \|\mathcal{F}^{-1}((i\xi)^\alpha\widehat{\chi})\|_{L^r}\|f\|_{L^p} \leq C(\alpha, d)\|f\|_{L^p},$$

with

$$C(\alpha, d) := \max(\|\mathcal{F}^{-1}((i\xi)^\alpha\widehat{\chi})\|_{L^1}, \|\mathcal{F}^{-1}((i\xi)^\alpha\widehat{\chi})\|_{L^\infty}).$$

□

Remark 2.3.15. A more careful analysis shows that one can take $C(\alpha, d) = C_d^{1+|\alpha|}$, where C_d depends only on d.

2.4 Sobolev spaces

Definition 2.4.1. For any $s \in \mathbb{R}$ the *Sobolev space* H^s is defined as the completion of $\mathcal{S}$ in $\mathcal{S}'$ for the topology defined by the norm

$$\|f\|_{H^s} := \Big(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \Big)^{\frac{1}{2}}.$$

Lemma 2.4.2. . *For any $s > \frac{d}{2}$ there is the inclusion $H^s(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ and there exists $C_s \geq 0$ such that for all $f \in H^s(\mathbb{R}^d)$*

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq C_s \|f\|_{H^s(\mathbb{R}^d)}.$$

Proof. Exercise 2.6.9 □

We denote $\mathcal{S}_0$ the set of functions $u \in \mathcal{S}$ such that $\operatorname{supp} \widehat{u} \subset \mathbb{R}^d \setminus \{0\}$.

Definition 2.4.3. For any $s < \frac{d}{2}$ the *homogeneous Sobolev space* $\dot{H}^s$ is defined as the completion of $\mathcal{S}_0$ in $\mathcal{S}'$ for the topology defined by the norm

$$\|f\|_{\dot{H}^s} := \Big(\int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \Big)^{\frac{1}{2}}.$$

Proposition 2.4.4 (Sobolev embedding)**.** *Let $s < \frac{d}{2}$ and let $p > 0$ be determined by the relation*

$$\frac{1}{2} - \frac{1}{p} = \frac{s}{d} \;\Leftrightarrow\; p = \frac{2d}{d-2s}.$$

There exists a constant $C = C(s,d)$ such that

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \qquad \text{for all } f \in \dot{H}^s(\mathbb{R}^d).$$

Proof. We can assume $f \in \mathcal{S}(\mathbb{R}^d)$ (for $f \in \dot{H}^s(\mathbb{R}^d)$ will follow by density). Let $g := \mathcal{F}^{-1}(|\xi|^s \widehat{f}(\xi))$, so that $f = \mathcal{F}^{-1}(|\xi|^{-s}) * g$. Now we use the Hardy-Littlewood-Sobolev inequality. □

2.4.1 Stationary and non-stationary phase

We now study *oscillatory integrals*, that is integrals of the form

$$I(\lambda) = \int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\phi(\xi)} a(\xi) \, \mathrm{d}\xi,$$

where $a \in C_0^\infty(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathbb{R}^d)$. We are interested in the asymptotic behaviour of $I(\lambda)$ as $\lambda \to +\infty$. Notice that if ϕ is a non-trivial affine function, then Proposition 2.3.2 implies that $|I(\lambda)|$ decays faster than any power of λ. The lemma below generalises this fact.

Lemma 2.4.5 (Non-stationary phase)**.** *If $\nabla\phi \neq 0$ on $\operatorname{supp} a$, then for any $N \geq 1$ there exists $C(N,a,\phi) \geq 0$ such that*

$$|I(\lambda)| \leq C(N,a,\phi)\lambda^{-N}, \qquad \text{as } \lambda \to \infty.$$

Proof. Consider the differential operators

$$Lu := \frac{1}{i\lambda}\frac{\nabla\phi\cdot\nabla u}{|\nabla\phi|^2}, \qquad L^*u := \frac{i}{\lambda}\nabla\cdot\Big(\frac{u\nabla\phi}{|\nabla\phi|^2}\Big).$$

We have $L\mathrm{e}^{i\lambda\phi} = \mathrm{e}^{i\lambda\phi}$, hence integration by parts yields

$$|I(\lambda)| = \Big|\int_{\mathbb{R}^d} L^N \mathrm{e}^{i\lambda\phi(\xi)}a(\xi)\,\mathrm{d}\xi\Big| = \Big|\int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\phi(\xi)}(L^*)^N a(\xi)\,\mathrm{d}\xi\Big| \le \int_{\mathbb{R}^d}\big|(L^*)^N a(\xi)\big|\,\mathrm{d}\xi \le C(N,a,\phi)\lambda^{-N}.$$

□

Lemma 2.4.6 (Stationary phase)**.** *Assume that all the critical points of ϕ belonging to* $\operatorname{supp} a$ *are non-degenerate, in other words*

$$\xi_0 \in \operatorname{supp} a \quad \textit{and} \quad \nabla\phi(\xi_0) = 0 \quad \Rightarrow \quad \det\big(\nabla^2\phi(\xi_0)\big) \neq 0.$$

Then there exists $C(a,\phi) \ge 0$ such that

$$|I(\lambda)| \le C(a,\phi)\lambda^{-\frac{d}{2}}, \qquad \textit{as } \lambda\to\infty.$$

Proof. Let $\chi \in C^\infty$ be a cut-off function, that is $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Since non-degenerate critical points are isolated, in $\operatorname{supp} a$ there is a finite number of them. Call them $\xi_1,\dots,\xi_m$. For each critical point ξ_j, let

$$I_j(\lambda) := \int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\phi(\xi)}a(\xi)\chi\big(\sqrt{\lambda}(\xi-\xi_j)\big)\,\mathrm{d}\xi.$$

Obviously $|I_j(\lambda)| \le C(a)\lambda^{-\frac{d}{2}}$. Set

$$I_0(\lambda) := I(\lambda) - \sum_{j=1}^m I_j(\lambda) = \int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\phi(\xi)}\widetilde{a}(\xi)\,\mathrm{d}\xi,$$

where $\widetilde{a}(\xi) := \big(1 - \sum_{j=1}^m \chi\big(\sqrt{\lambda}(\xi-\xi_j)\big)\big)a(\xi)$. From the non-degeneracy condition, there exists $c(a,\phi) > 0$ such that

$$|\nabla\phi(\xi)| \ge c(a,\phi)\sqrt{\lambda}, \qquad \forall\xi\in\operatorname{supp}\widetilde{a}.$$

□

We also have the following improved version.

Lemma 2.4.7. *Assume that ξ_0 is the only critical point of ϕ in* $\operatorname{supp} a$ *and that it is non-degenerate. Then for any $k\in\mathbb{N}$ there exists $C(k,a,\phi)$ such that*

$$\Big|\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\big(\mathrm{e}^{-i\lambda\phi(\xi_0)}I(\lambda)\big)\Big| \le C(k,a,\phi)\lambda^{-\frac{d}{2}-k}, \qquad \textit{as } \lambda\to\infty.$$

Proof. □

Corollary 2.4.8. *Let $\sigma_{S^{d-1}}(\xi)$ be the surface measure of the unit sphere $S^{d-1}\subset\mathbb{R}^d$. Then*

$$\mathcal{F}^{-1}\sigma_{S^{d-1}}(x) = \mathrm{e}^{i|x|}\omega_+(|x|) + \mathrm{e}^{-i|x|}\omega_-(|x|), \qquad |x|\ge 1,$$

where $\omega_\pm$ are smooth and for all $k\in\mathbb{N}$ there exists $C_k \ge 0$ such that

$$|\partial_r^k\omega_\pm| \le C_k r^{-\frac{d-1}{2}-k}, \qquad \textit{for all } r\ge 1.$$

Proof. □

2.5 Littlewood-Paley theory

Lemma 2.5.1 (Partition of unity over a geometric scale)**.** *There exists a radial nonnegative function* $\psi \in C^\infty(\mathbb{R}^d)$ *such that* $\operatorname{supp}\psi \subset \{\frac{1}{2} \leq x \leq 2\}$ *and*

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1, \qquad \forall x \neq 0.$$

Proof. We take $\chi \in C^\infty$ a radial non-increasing cut-off function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. We set $\psi(x) := \chi(x) - \chi(2x)$. □

Definition 2.5.2. For $j \in \mathbb{Z}$ we define the *homogeneous dyadic block* $\dot{\Delta}_j$ and the *homogeneous low-frequency cut-off operator* $\dot{S}_j$:

$$\dot{\Delta}_j u := \psi(2^{-j}D)u := \mathcal{F}^{-1}(\psi(2^{-j}\xi)\widehat{u}(\xi)) = 2^{jd}\int_{\mathbb{R}^d}(\mathcal{F}^{-1}\psi)(2^j y)u(x-y)\,\mathrm{d}y,$$
$$\dot{S}_j u := \sum_{j'<j}\Delta_j u = \mathcal{F}^{-1}(\chi(2^{-j}\xi)\widehat{u}(\xi)) = 2^{jd}\int_{\mathbb{R}^d}(\mathcal{F}^{-1}\chi)(2^j y)u(x-y)\,\mathrm{d}y.$$

Lemma 2.5.3. *The operators* $\dot{\Delta}_j$ *and* $\dot{S}_j$ *are bounded* $L^p \to L^p$ *for all* $p \in [1,\infty]$*, with bounds independent of* j*.*

Proof. Exercise 2.6.12. □

Note that $\dot{\Delta}_j$ and $\dot{S}_j$ are *Fourier multipliers*, and as such they commute with other Fourier multipliers, like convolutions, derivatives, ...

The *formal* homogeneous Littlewood-Paley decomposition is

$$\mathrm{Id} = \sum_{j\in\mathbb{Z}}\dot{\Delta}_j,$$

but in what sense the series converges is, for now, unclear.

Definition 2.5.4 (Homogeneous Besov norms)**.** Let $s \in \mathbb{R}$ and $p, r \in [1,\infty]$. For any $u \in \mathcal{S}_0$ we define

$$\|u\|_{\dot{B}^s_{p,r}} := \Big(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

We call $\|\cdot\|_{\dot{B}^s_{p,r}}$ the *homogeneous Besov norm.*

Remark 2.5.5. We can think of the homogeneous Besov norms as follows. For each $j \in \mathbb{Z}$, take the L^p norm of $\dot{\Delta}_j u$, multiply it by 2^{js} and take the l^r norm of the resulting sequence.

Remark 2.5.6. One can check that, up to a constant, the definition of the Besov norm does not depend on the choice of the function ψ.

Remark 2.5.7. One also defines *homogeneous Besov spaces*, but there are some functional-theoretic subtleties which we would like to avoid here.

Proposition 2.5.8 (Duality for Besov norms). *For any $s \in \mathbb{R}$ and $p, r \in [1, \infty]$ there exists $C \geq 0$ such that*

$$\langle \phi, u \rangle \leq C \|\phi\|_{\dot{B}^{-s}_{p',r'}} \|u\|_{\dot{B}^s_{p,r}}, \qquad \forall u, \phi \in \mathcal{S}_0$$

and

$$\|u\|_{\dot{B}^s_{p,r}} \leq C \sup_{\phi \in Q^{-s}_{p',r'}} \langle \phi, u \rangle, \qquad \forall u \in \mathcal{S}_0,$$

where $Q^{-s}_{p',r'}$ is the set of $\phi \in \mathcal{S}_0$ such that $\|\phi\|_{\dot{B}^{-s}_{p',r'}} \leq 1$.

Proof. □

Proposition 2.5.9. *For any $p \in [2, \infty)$ there exists C_p such that for all $u \in \mathcal{S}_0$*

$$\|u\|_{L^p} \leq C_p \|u\|_{\dot{B}^0_{p,2}}.$$

For any $p \in (1, 2]$ there exists C_p such that for all $u \in \mathcal{S}_0$

$$\|u\|_{\dot{B}^0_{p,2}} \leq C_p \|u\|_{L^p}.$$

Remark 2.5.10. This result is a part of the Littlewood-Paley theorem, a fundamental result in harmonic analysis, which is more difficult and hopefully we will not need it.

Proof of Proposition 2.5.9. □

Proposition 2.5.11 (Refined Sobolev inequality). *Let $0 < s < \frac{d}{2}$ and $p = \frac{2d}{d-2s}$. Then*

$$\|f\|_{L^p} \leq C \|f\|^{\frac{p-2}{p}}_{\dot{B}^s_{2,\infty}} \|f\|^{\frac{2}{p}}_{\dot{H}^s}.$$

Proof. □

2.6 Exercises

Exercise 2.6.1. Prove the following special case of the Marcinkiewicz interpolation theorem. Let (X, μ) be a measure space and let T be a sublinear positive operator, that is an operator satisfying

$$f \geq 0 \Rightarrow Tf \geq 0, \qquad \text{for all measurable } f$$
$$T(af + bg) \leq aTf + bTg, \qquad \forall a, b \geq 0 \text{ and measurable positive } f, g.$$

Suppose moreover that T is bounded $L^1 \to L^1_w$ and $L^\infty \to L^\infty$, in other words there exist constants $C_1, C_\infty > 0$ such that

$$\sup_{\lambda > 0} \lambda \cdot \mu\{Tf > \lambda\} \leq C_1 \|f\|_{L^1},$$
$$\|Tf\|_{L^\infty} \leq C_\infty \|f\|_{L^\infty}.$$

Then T is bounded $L^p \to L^p$ for all $p \in (1, \infty)$.
Hint.

- Show that $T(0) = 0$ and $T(\lambda f) = \lambda T f$ for all $\lambda > 0$.
- It suffices to prove that there exists $C > 0$ such that for all f, g satisfying $\|f\|_{L^p} \le 1$ and $\|g\|_{L^{p'}} \le 1$ there is $\langle Tf, g\rangle \le C$.
- Let $f = \sum_j c_j f_j$ and $g = \sum_k d_k g_k$ be atomic decompositions of f and g. Let $a_{jk} := \langle Tf_j, g_k\rangle$. Thus $\langle Tf, g\rangle \le \sum_{(j,k)\in\mathbb{Z}^2} a_{jk} c_j d_k$. Using the Young inequality for the counting measure, prove that it is sufficient to show that $a_{jk} \le A(j-k)$ for some summable function $A : \mathbb{Z} \to \mathbb{R}_+$.
- We will prove that there exist $\widetilde{C}, \epsilon > 0$ (depending on C_1 and C_∞) such that $A(n) = \widetilde{C} 2^{-\epsilon|n|}$ works. We treat separately $j \ge k$ and $k \ge j$.
- If $j \ge k$, use $\|g_k\|_{L^1} \le 2^{-\frac{k}{p'}} 2^{k+1}$, $\|Tf_j\|_{L^\infty} \le C_\infty 2^{-\frac{j}{p}}$ and conclude.
- In the case $j \le k$, choose some $a \in (1, p)$, and then prove and use the following bounds:

$$\|g_k\|_{L^{a'}} \le 2^{-\frac{k}{p'}} 2^{\frac{k+1}{a'}},$$

$$\|Tf_j\|_{L^a}^a \le \frac{a}{a-1} C_1 C_\infty^{a-1} \|f_j\|_{L^1} \|f_j\|_{L^\infty}^{a-1} \lesssim 2^{-\frac{j}{p}+(j+1)-\frac{j}{p}(a-1)} \;\Rightarrow\; \|Tf_j\|_{L^a} \lesssim 2^{-\frac{j}{p}} 2^{\frac{j+1}{a}}.$$

Exercise 2.6.2. Prove (2.3.2).
Hint. Reduce to $\epsilon = 1$ and $d = 1$. Define $I(x) := \int_{\mathbb{R}} \mathrm{e}^{ix\xi - \frac{1}{2}|\xi|^2}\,\mathrm{d}\xi$. The value of $I(0)$ is well-known. There are at least two ways to conclude the proof:

- either use complex analysis to show that $e^{\frac{|x|^2}{2}} I(x)$ is independent of x,
- or check that $I'(x) = -xI(x)$ for all $x \in \mathbb{R}$.

Exercise 2.6.3. Prove the following generalisation of (2.3.2). Let $z \in \mathbb{C} \setminus \{0\}$ with $\Re z \ge 0$. Then

$$\mathcal{F}\big(\mathrm{e}^{-\frac{z}{2}|x|^2}\big) = \Big(\frac{2\pi}{z}\Big)^{\frac{d}{2}} \mathrm{e}^{-\frac{|\xi|^2}{2z}},$$

where $z^{-\frac{d}{2}} := |z|^{-\frac{d}{2}} \mathrm{e}^{-i\frac{d}{2}\theta}$ for $z = |z|\mathrm{e}^{i\theta}$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
Hint. For $\Re z > 0$, this follows from the unique continuation principle in complex analysis. In order to treat the case $\Re z$, use the fact that $\mathcal{F}$ is continuous $\mathcal{S}' \to \mathcal{S}'$.

Exercise 2.6.4. Prove Proposition 2.3.5.
Hint. Assuming the conclusion is false, construct a sequence $u_n \in \mathcal{S}(\mathbb{R}^d)$ such that $u_n \to 0$ in $\mathcal{S}(\mathbb{R}^d)$ and $\langle \phi, u_n\rangle \ge 1$ for all n.

Exercise 2.6.5. Prove Proposition 2.3.8.

Exercise 2.6.6. Prove Proposition 2.3.9.

Exercise 2.6.7. Prove Proposition 2.3.10.
Hint. Denote $\phi := \mathcal{F}(|x|^{-\alpha})$. Prove that ψ is homogeneous of degree $\alpha - d$ (which means $\langle \phi, u(\lambda\cdot)\rangle = \lambda^{-\alpha}\langle \phi, u\rangle$ for any $\lambda > 0$ and $u \in \mathcal{S}$) and rotationally symmetric (which means $\langle \phi, u(R\cdot)\rangle = \langle \phi, u\rangle$ for any rotation R and $u \in \mathcal{S}$). Set $\psi(\xi) := |\xi|^{d-\alpha}\phi(\xi)$ and deduce that ψ is homogeneous of degree 0 and rotationally symmetric. Show that $x \cdot \nabla\psi = 0$ and $(x_j\partial_k - x_k\partial_j)\psi = 0$ for $j \ne k$ in the distributional sense. Taking an appropriate linear combination deduce that $\langle \nabla\psi, u\rangle = 0$ for all $u \in \mathcal{S}_0$. Deduce that $\partial_j\psi$ is a polynomial for all j. One should be able to conclude from here, but to be honest at the moment I'm not sure how.

Exercise 2.6.8. Prove Proposition 2.3.12.
Hint. First prove, using Fubini's theorem, that for any $f, g \in \mathcal{S}$ we have $\int_{\mathbb{R}^d} f(\xi)\widehat{g}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \widehat{f}(x)g(x)\,\mathrm{d}x$.

Exercise 2.6.9. Prove Lemma 2.4.2.

Exercise 2.6.10. Show that $\mathcal{S} \subset \dot{H}^s$ if and only if $s > -\frac{d}{2}$. What about $\dot{B}^s_{p,r}$ instead of $\dot{H}^s$?

Exercise 2.6.11. Complete the proof of Proposition 2.4.4.

Exercise 2.6.12. Prove Lemma 2.5.3.

Exercise 2.6.13. Show that for any $u \in \mathcal{S}_0$ and any $s \in \mathbb{R}, p \in [1, \infty], r \in [1, \infty]$ the $\dot{B}^s_{p,r}$ norm of u is finite.

Chapter 3

Strichartz estimates

3.1 The linear wave propagator

We consider the linear wave equation without potential from $\mathbb{R}^d$ to $\mathbb{R}$:

$$\partial_t^2 u(t,x) = \Delta u(t,x), \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d,\ u(t,x)\in\mathbb{R}.$$

Notice that there is no loss of generality in considering $u(t,x)\in\mathbb{R}$ instead of $u(t,x)\in\mathbb{R}^m$, because in the vector-valued case the components $u^{(j)}$ are decoupled.

We rewrite this equation in a standard way as a first-order in time system:

$$\partial_t \begin{pmatrix} u(t,x) \\ \dot u(t,x)\end{pmatrix} = \begin{pmatrix} \dot u(t,x) \\ \Delta u(t,x)\end{pmatrix}, \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d,\ u(t,x),\dot u(t,x)\in\mathbb{R}. \tag{3.1.1}$$

We will write $\boldsymbol{u} := (u,\dot u)$.

Definition 3.1.1. Let $\boldsymbol{u} = (u,\dot u)\in C([0,T],\mathcal{S}'\times\mathcal{S}')$. We say that $\boldsymbol{u}$ is a *weak solution* of (3.1.1) if for all $\boldsymbol{\phi} = (\phi,\dot\phi)\in C^\infty([0,T],\mathcal{S})$

$$\int_0^T \langle\dot\phi, u - \dot u\rangle + \langle\phi,$$

Proposition 3.1.2. *Let $s < \frac{d}{2}$. Denote $\mathcal{H}^s := \dot H^s\times\dot H^{s-1}$ and $\|\boldsymbol{u}_0\|_{\mathcal{H}^s} := \sqrt{\|u_0\|_{\dot H^s}^2 + \|\dot u_0\|_{\dot H^{s-1}}^2}$. For all $\boldsymbol{u}_0 = (u_0,\dot u_0)\in\mathcal{H}^s$ and $t_0\in\mathbb{R}$ there exists a unique weak solution $\boldsymbol{u}\in C(\mathbb{R},\mathcal{S}'\times\mathcal{S}')$ of* (3.1.1) *such that $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. This solution satisfies:*

$$\boldsymbol{u}\in C(\mathbb{R},\mathcal{H}^s),$$
$$\|\boldsymbol{u}(t)\|_{\mathcal{H}^s} = \|\boldsymbol{u}_0\|_{\mathcal{H}^s}, \qquad \text{for all } t\in\mathbb{R}.$$

Let γ

We write $\boldsymbol{u}(t) = S(t,t_0)\boldsymbol{u}_0$. Thus $S(t,t_0)$ is an isometry of $\mathcal{H}^s$ for all t,t_0 and $s<\frac{d}{2}$.

We also consider the non-homogeneous equation

$$\partial_t^2 u(t,x) = \Delta u(t,x) + f(t,x), \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d,\ u(t,x)\in\mathbb{R}. \tag{3.1.2}$$

Proposition 3.1.3 (Energy estimate)**.** *Let $s < \frac{d}{2}$. For all $\boldsymbol{u}_0 = (u_0, \dot{u}_0) \in \mathcal{H}^s$, $f \in L^1(I, \dot{H}^{s-1})$ and $t_0 \in I$ there exists a unique weak solution $\boldsymbol{u} \in C(\mathbb{R}, \mathcal{S}' \times \mathcal{S}')$ of* (3.1.2) *such that $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. This solution satisfies:*

$$\boldsymbol{u} \in C(I, \mathcal{H}^s),$$
$$\|\boldsymbol{u}(t)\|_{\mathcal{H}^s} \le \|\boldsymbol{u}_0\|_{\mathcal{H}^s} + \Big| \int_{t_0}^t \|f(t')\|_{\dot{H}^{s-1}} \, \mathrm{d}t' \Big|, \qquad \text{for all } t \in I.$$

Moreover, if $\boldsymbol{u}_0(x) = 0$ for $|x - x_0| \le |t - t_0| + R$ and $f(t', x) = 0$ for $|x - x_0| \le |t - t'| + R$, then $\boldsymbol{u}(t, x) = 0$ for $|x - x_0| \le R$.

Remark 3.1.4. The last property is the *finite speed of propagation.*

Proof. We only treat the case of smooth data.

In order to prove the finite speed of propagation, we consider the vector field in $\mathbb{R}^{1+d}$

$$G(t, x) := \Big(\frac{1}{2}\big((\partial_t u)^2 + |\nabla u|^2\big), -\partial_t u \nabla u\Big).$$

We compute

$$\mathrm{div}_{\mathbb{R}^{1+d}} G(t, x) = \partial_t^2 u \partial_t u + \partial_t \nabla u \cdot \nabla u - \partial_t \nabla u \cdot \nabla u - \partial_t u \Delta u = f \partial_t u.$$

Without loss of generality take $t_0 = 0$, $x_0 = 0$ and $t \ge 0$. We apply the space-time divergence theorem to the cone bounded by the disks $D((0, x), R + |t|)$ and $D((t, x), R)$. We obtain the so-called *energy identity*:

$$\int_K f \partial_t u \, \mathrm{d}x = -\int_{|x| \le R + |t|} \frac{1}{2}\big((\dot{u}_0)^2 + |\nabla u_0|^2\big) \, \mathrm{d}x + \int_{|x| \le R} \frac{1}{2}\big((\partial_t u(t))^2 + |\nabla u(t)|^2\big) \, \mathrm{d}x + \frac{1}{2\sqrt{2}} \int_M |\nabla^\perp u|^2 \, \mathrm{d}\sigma,$$

where K is the cone, M is the "side" of the cone and $\nabla^\perp$ is the tangential derivative. If the first two terms are identically zero, then the other two as well, which proves the claim. ☐

We can solve explicitly (3.1.2) by taking the Fourier transform in space variables. We obtain

$$\widehat{u}(t, \xi) = \widehat{u}_0(\xi) \cos(|\xi|(t - t_0)) + \widehat{\dot{u}_0} \frac{\sin(|\xi|(t - t_0))}{|\xi|} + \int_{t_0}^t \widehat{f}(s, \xi) \frac{\sin(|\xi|(t - s))}{|\xi|} \, \mathrm{d}s,$$

or equivalently

$$u(t) = \cos((t - t_0)|D|) u_0 + \frac{\sin((t - t_0)|D|)}{|D|} \dot{u}_0 + \int_{t_0}^t \frac{\sin((t - s)|D|)}{|D|} f(s) \, \mathrm{d}s.$$

We are led to study dispersive properties of the *half wave propagators* $\mathrm{e}^{\pm it|D|}$. Note that we can transform the wave equation to the half-wave equation formally by taking $u_+(t) := u(t) + \frac{1}{i|D|} \partial_t u(t)$. Denote

$$\langle x \rangle := \sqrt{1 + x^2}.$$

Proposition 3.1.5. *There exists $C \ge 0$ such that for all complex-valued $f \in \mathcal{S}$ such that* $\mathrm{supp}\, \widehat{f} \subset \{\frac{1}{2} \le |\xi| \le 2\}$ *and all $t \in \mathbb{R}$*

$$\|\mathrm{e}^{\pm it|D|} f\|_{L^\infty} \le C \langle t \rangle^{-\frac{1}{2}} \|f\|_{L^1}. \tag{3.1.3}$$

Proof. Without loss of generality take the sign "+". Let $\chi(\xi) = \chi(|\xi|) \in C^\infty$ be equal to 1 for $\frac{1}{2} \le |\xi| \le 2$ and to 0 for $|\xi| \le \frac{1}{4}$ or $\xi \ge 4$. By our assumption, we have

$$\mathrm{e}^{it|D|} f = \mathrm{e}^{it|D|} \chi(|D|) f.$$

Taking the inverse Fourier transform, up to a normalising factor we get

$$\big(\mathrm{e}^{it|D|} f\big)(x) = (K_t * f)(x),$$

where

$$K_t(x) := \int_{\mathbb{R}^d} \mathrm{e}^{it|\xi| + i\xi\cdot x} \chi(\xi)\, \mathrm{d}\xi.$$

Thus it suffices to show that

$$\|K_t\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{d-1}{2}}.$$

Changing to polar coordinates, we find

$$K_t(x) = \int_0^\infty \mathrm{e}^{itr} \chi(r) r^{d-1} \mathcal{F}^{-1}\sigma(rx)\, \mathrm{d}r = \int_0^\infty \mathrm{e}^{ir(t \pm |x|)} \chi(r) r^{d-1} \omega_\pm(rx)\, \mathrm{d}r,$$

where σ is the surface measure of $\mathbb{S}^{d-1}$ and we have used Corollary 2.4.8. If $\frac{1}{2}t \le |x| \le 2t$, the conclusion follows directly from Corollary 2.4.8. If not, we integrate by parts. □

From Plancherel we have $\|\mathrm{e}^{\pm it|D|}\gamma\|_{L^2} = \|\gamma\|_{L^2}$, so (3.1.3) and Riesz-Thorin theorem yield

$$\|\mathrm{e}^{\pm it|D|}\gamma\|_{L^q} \le C\langle t \rangle^{-\frac{1}{2}(\frac{1}{p'} - \frac{1}{p})t} \|\gamma\|_{L^{p'}}, \qquad \forall f \in \mathcal{S},\ p \in [2,\infty].$$

3.2 The TT^* method

In this section, we prove general Strichartz estimates. For f a measurable function on $\mathbb{R} \times \mathbb{R}^d$ and $p, q \in [1,\infty]$ we define

$$\|f\|_{L^pL^q} := \left(\int_{\mathbb{R}} \|f(t,\cdot)\|_{L^q}^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

Measurability.

Lemma 3.2.1. *Let $(p_j, q_j) \in [1,\infty]^2$ and $\theta_j \ge 0$ with $\sum_{j=1}^m \theta_j = 1$. Suppose that*

$$\frac{1}{p} = \sum_{j=1}^m \frac{\theta_j}{p_j}, \qquad \frac{1}{q} = \sum_{j=1}^m \frac{\theta_j}{q_j}.$$

Then

$$\|f\|_{L^pL^q} \le \prod_{j=1}^m \|f\|_{L^{p_j}L^{q_j}}^{\theta_j}, \qquad \forall f \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d).$$

Proof. Exercise. □

Definition 3.2.2. Let $\sigma > 0$. We say that a pair (p, q) is *σ-admissible* if

$$\frac{1}{p} + \frac{\sigma}{q} = \frac{\sigma}{2}, \qquad (p, q, \sigma) \neq (2, \infty, 1).$$

If σ is known from the context, we can call such a pair *admissible.*

Remark 3.2.3. It is easy to see that in the case $\sigma = 0$ we do not obtain anything interesting. We would be forced to admit $(\infty, 2)$ is the only 0-admissible pair.

Theorem 3.2.4. *Let $U(t)$ be a bounded family of continuous operators such that*

$$\|U(t)U^*(t')f\|_{L^\infty} \leq C|t - t'|^{-\sigma}\|f\|_{L^1}, \qquad \forall t, t' \in \mathbb{R},\ f \in \mathcal{S}. \tag{3.2.1}$$

Let $\chi : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function such that $|\chi(t, t')| \leq 1$ for all t, t'. Then for all σ-admissible pairs (p, q)

$$\left\| \int_{\mathbb{R}} \chi(t, t')U(t)U^*(t')f(t')\,\mathrm{d}t' \right\|_{L^{p_1}L^{q_1}} \leq C\|f\|_{L^{p_2'}L^{q_2'}}, \tag{3.2.2}$$

with C independent of χ.

Proof of Theorem 3.2.4 in the non-endpoint case. The proof is considerably easier in the non-endpoint case $p > 2$, so we present it first.
Step 1. For $f, g \in C^\infty(\mathbb{R}, \mathcal{S})$ we define

$$T_\chi(f, g) := \int_{\mathbb{R}^2} \chi(t, t')\langle U(t)U^*(t')f(t'), g(t)\rangle\,\mathrm{d}t\,\mathrm{d}t',$$

where $\langle\cdot,\cdot\rangle$ is the L^2 inner product. By duality, (3.2.2) is equivalent to

$$|T_\chi(f, g)| \leq C\|f\|_{L^{p_2'}L^{q_2'}}\|g\|_{L^{p_1'}L^{q_1'}}. \tag{3.2.3}$$

Step 2. We show (3.2.3) with $(p_2, q_2) = (p_1, q_1)$. Interpolating between (3.2.1) and the $L^2 \to L^2$ bound we have

$$\|U(t)U^*(t')f(t')\|_{L^q} \leq |t - t'|^{-\sigma\left(1-\frac{2}{q}\right)}\|f(t')\|_{L^{q'}},$$

thus

$$\langle U(t)U^*(t')f(t'), g(t)\rangle \leq C|t - t'|^{-\sigma\left(1-\frac{2}{q}\right)}\|f(t')\|_{L^{q'}}\|g(t)\|_{L^{q'}} = C|t - t'|^{-\frac{2}{p}}\|f(t')\|_{L^{q'}}\|g(t)\|_{L^{q'}},$$

and we conclude using Hardy-Littlewood-Sobolev inequality, using the fact that $2 < p < \infty$.
Step 3. We prove that

$$\left\| \int_{\mathbb{R}} U^*(t)f(t)\,\mathrm{d}t \right\|_{L^2} \leq C\|f\|_{L^{p'}L^{q'}}. \tag{3.2.4}$$

Denote $T = T_\chi$ with $\chi(t, t') = 1$ for all t, t'. Directly from the definition of T_χ we obtain

$$T(f, f) = \left\| \int_{\mathbb{R}} U^*(t)f(t)\,\mathrm{d}t \right\|_{L^2}^2,$$

so (3.2.4) follows from Step 1.
Step 4. We prove (3.2.3) for any σ-admissible pairs (p_1, q_1) and (p_2, q_2). By symmetry, without loss

of generality we can assume $q_1 \leq q_2$. Fixing t and using (3.2.4) with t' instead of t and $\chi(t,t')f(t')$ instead of $f(t)$ we get

$$\Big\| \int_{\mathbb{R}} \chi(t,t')U(t)U^*(t')f(t')\,\mathrm{d}t' \Big\|_{L^\infty L^2} \leq C\|f\|_{L^{p_2'}L^{q_2'}}.$$

Lemma 3.2.1 and (3.2.2) for $(p_1,q_1) = (p_2,q_2)$ thus imply (3.2.2) in the general case. □

The endpoint case $p = 2$ is much more difficult. It was first settled by Keel and Tao [6]. The Hardy-Littlewood-Sobolev inequality is not directly applicable. Instead, we will revisit its proof in our particular setting. First, we write

$$T_\chi(f,g) = \sum_{j\in\mathbb{Z}} T_j(f,g) := \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^2} \chi_j(t,t')\langle U(t)U^*(t')f(t'), g(t)\rangle\,\mathrm{d}t\,\mathrm{d}t',$$

where $\chi_j(t,t') := I_{2^j \leq |t-t'| < 2^{j+1}}\chi(t,t')$. Our main goal is to prove (3.2.3) with $p_1 = p_2 = 2$ and $q_1 = q_2 = q = \frac{2\sigma}{\sigma-1} < \infty$.

Lemma 3.2.5. *There exists an open neighbourhood V of (q,q) in $\mathbb{R}^2$ such that for all $(a,b) \in V$ and $j \in \mathbb{Z}$*

$$|T_j(f,g)| \leq C2^{-j\beta(a,b)}\|f\|_{L^2L^{a'}}\|g\|_{L^2L^{b'}}, \quad \beta(a,b) := \sigma - 1 - \frac{\sigma}{a} - \frac{\sigma}{b}. \tag{3.2.5}$$

Proof of Theorem 3.2.4 in the endpoint case, assuming Lemma 3.2.5. The proof is based on the atomic decomposition lemma. □

Proof of Lemma 3.2.5. Considering $\widetilde{U}(t) := U(2^j t)$, $\widetilde{f}(t,x) := f(2^j t, 2^{\sigma j}x)$ and $\widetilde{g}(t,x) := g(2^j t, 2^{\sigma j}x)$ we reduce to $j = 0$.
Step 1. We prove (3.2.5) for $a = b = \infty$. This easily follows from (3.2.1).
Step 2. Using the non-endpoint case, we show that (3.2.5) holds for $b = 2$ and $2 \leq a < q$, as well as for $a = 2$ and $2 \leq b < q$. **Step 3.** We use interpolation. What exactly interpolation theorem are we using?

□

3.3 Strichartz estimates for the wave equation

Definition 3.3.1. We say that a pair (p,q) is *wave-admissible* if there exists $2 \leq \widetilde{q} \leq q$ such that

$$\frac{2}{p} + \frac{d-1}{\widetilde{q}} = \frac{d-1}{2}, \qquad (p,\widetilde{q},d) \neq (2,\infty,3).$$

Theorem 3.3.2. *Suppose that (p,q) and (a,b) are wave-admissible and*

$$\frac{1}{p} + \frac{d}{q} = \frac{1}{a'} + \frac{d}{b'} - 2 = \frac{d}{2} - \sigma.$$

Let u be the solution of (3.1.2)*. Then*

$$\|u\|_{L^pL^q} \leq C\big(\|\boldsymbol{u}_0\|_{\mathcal{H}^\sigma} + \|f\|_{L^{a'}L^{b'}}\big).$$

We first prove that the theorem is true if all the functions involved have spatial Fourier transforms contained in $\{\frac{1}{2} \le |\xi| \le 2\}$. This is done using Theorem 3.2.4.

By scaling invariance, this implies that the conclusion holds if all the functions involved have spatial Fourier transforms contained in $\{2^{j-1} \le |\xi| \le 2^{j+1}\}$ for some $j \in \mathbb{Z}$.

The third step is to "glue the Littlewood-Paley pieces", which we are now going to explain.

Note that $\dot{\Delta}_j$ commutes with $\mathrm{e}^{it|D|}$. Thus

$$\|\dot{\Delta}_j \mathrm{e}^{it|D|} f\|_{L^p L^q} \le C \|\dot{\Delta}_j f\|_{\dot{H}^\sigma}. \tag{3.3.1}$$

For fixed t we can write:

$$\|\mathrm{e}^{it|D|} f\|^2_{\dot{B}^0_{q,2}} = \sum_{j \in \mathbb{Z}} \|\dot{\Delta}_j \mathrm{e}^{it|D|} f\|^2_{L^q},$$

so the Minkowski inequality and (3.3.1) yield

$$\|\mathrm{e}^{it|D|} f\|_{L^p \dot{B}^0_{q,2}} \le C \|f\|_{\dot{H}^\sigma}.$$

Finally, we use $\dot{B}^0_{q,2} \subset L^q$.

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