Dynamics of multi-solitons for Klein-Gordon equations

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Klein-Gordon equations

Consider the nonlinear Klein-Gordon equation

$$\partial_t^2 \psi - \Delta \psi + \psi - \psi^p = 0, \ (t, x) \in \mathbb{R}^{1+d}. \tag{1}$$

Stationary solutions satisfy

$$-\Delta Q + Q - Q^p = 0 \tag{2}$$

and decay exponentially.
A unique radial positive ground state with the least energy

$$E(Q) := \int_{\mathbb{R}^d} \left(\frac{|\nabla Q|^2 + Q^2}{2} - \frac{Q^{p+1}}{p+1} \right) \mathrm{d}x$$

among all non-zero solutions.

This solution Q is linearly unstable in the sense that the linearized operator near Q

$$L := -\Delta + 1 + V = -\Delta + 1 - 3Q^2 \tag{3}$$

has a negative eigenvalue $-\nu^2$.

Due to the translational invariance of the equation, one also has

$$L\phi_m^0 = -\Delta\phi_m^0 + \phi_m^0 - 3Q^2\phi_m^0 = 0 \tag{4}$$

where $\phi_m^0 = \partial_{x_m} Q,\ m = 1, 2, 3.$

Lorentz boost

Besides space-time translational symmetries, there is invariance by Lorentz transform: Let $\beta \in \mathbb{R}^d$, $|\beta| < 1$, be a velocity vector. For a function $\phi : \mathbb{R}^d \to \mathbb{R}^d$, the Lorentz boost of ϕ with respect to β is given by

$$\phi_\beta(x) := \phi(\Lambda_\beta x), \quad \Lambda_\beta x := x + (\gamma - 1)\frac{(\beta \cdot x)\beta}{|\beta|^2}, \quad \gamma := \frac{1}{\sqrt{1 - |\beta|^2}}. \tag{5}$$

The Lorentz transformation is given by

$$(t', x') = \big(\gamma(t - \beta \cdot x),\ \Lambda_\beta x - \gamma\beta t\big) = \big(\gamma(t - \beta \cdot x),\ \Lambda_\beta(x - \beta t)\big).$$

$\forall \beta \in \mathbb{R}^d$, $|\beta| \in [0, 1)$, if u is a solution of (1) then $u_\beta\left(x - \beta t\right)$ is also a solution.

It is convenient to rewrite the equation (1) a dynamical system in

$$\mathcal{H} := H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$$

using its Hamiltonian form. Denoting

$$\boldsymbol{\psi}(t) = \begin{pmatrix} \psi \\ \psi_t \end{pmatrix}$$

we can write

$$\partial_t \boldsymbol{\psi}(t) = J H_0 \boldsymbol{\psi}(t) + \boldsymbol{F}(\boldsymbol{\psi}), \tag{6}$$

where $\boldsymbol{F}(\boldsymbol{\psi}) = \begin{pmatrix} 0 \\ \psi^3 \end{pmatrix}$ and

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H_0 := \begin{pmatrix} -\Delta + 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Focus on $d = 3, p = 3$. Dynamical questions

- Energy below the ground state. Global existence vs blowup (Payne and Sattinger 1975)
- Energy below the ground state. Global existence implies scattering i.e. asymptotically linear behaviour. (Ibrahim-Masmoudi-Nakanishi 2010. Kenig-Merle concentration compactness 2006)
- Energy slightly above the ground state. Classification of global dynamics near Q, construction of the center manifold. (Nakanishi-Schlag 2011)

Our goal: study the dynamics of solutions near superpositions of a finite number of Lorentz-transformed solitons, moving with distinct speeds (multi-solitons).

Strichartz for KG

One of essential tools are Strichartz estimates. The following is a complete list of the standard Strichartz estimates for the Klein–Gordon equation in three dimensions:

Theorem

The free Klein–Gordon flow in three dimensions $e^{it\sqrt{-\Delta+1}}$ *satisfies the Strichartz estimates*

$$\|e^{it\sqrt{-\Delta+1}} f\|_{L^p_t W^{1/q-1/p-1/2,q}_x} \lesssim \|f\|_{L^2_x}$$

whenever $2 \leq p, q \leq \infty$, $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$ *(Schrödinger-admissible exponents).*

We use $p = \infty, q = 2$ and $p = 3, q = \frac{18}{5}$. By Sobolev embedding,

$$\|\psi\|_{L^3_t L^6_x \cap L^\infty_t H^1_x} \lesssim \|\psi_0\|_{H^1} + \|\dot{\psi}_0\|_{L^2_x} + \|(\partial_t^2 - \Delta + 1)\psi\|_{L^1_t L^2_x}. \qquad (8)$$

Applications of Strichartz estimates

Small data global existence and long time behavior.
Consider the cubic KG in 3d

$$\partial_t^2\psi - \Delta\psi + \psi \pm \psi^3 = 0 \tag{9}$$

with $\|\psi_0\|_{H^1} + \|\dot\psi_0\|_{L^2} \ll 1$.
Applying Strichartz estimates

$$\begin{aligned}\|\psi\|_{L_t^3L_x^6\bigcap L_t^\infty H_x^1[0,T]} &\lesssim \|\psi_0\|_{H^1} + \|\dot\psi_0\|_{L_x^2} + \|\psi^3\|_{L_t^1L_x^2[0,T]} \\ &\lesssim \|\psi_0\|_{H^1} + \|\dot\psi_0\|_{L_x^2} + \|\psi\|^3_{L_t^3L_x^6[0,T]}\end{aligned}$$

$\|\psi\|_{L_t^3L_x^6\bigcap L_t^\infty H_x^1[0,T]} \ll 1$, absorb the nonlinear term to LHS, bootstrap and pass $T \to \infty$. Gobal existence.

Scattering

It is more convenient to use the Hamiltonian formalism

$$\partial_t \psi(t) = JH_0 \psi(t) + \boldsymbol{F}(\psi). \tag{10}$$

One can write

$$\psi(t) = e^{JH_0 t}\psi(0) + \int_0^t e^{JH_0(t-s)} \boldsymbol{F}(\psi)\, ds. \tag{11}$$

Scattering means asymptotically like the free evolution

$$\lim_{t\to\infty} \left\| \psi(t) - e^{JH_0 t}\psi_+ \right\|_{\mathcal{H}} = 0.$$

How to find ψ_+: check that

$$\psi_+ = \psi(0) + \int_0^\infty e^{-JH_0 s} \boldsymbol{F}(\psi)\, ds. \tag{12}$$

This follows from:

$$\left\| \int_0^\infty e^{-JH_0 s} \boldsymbol{F}(\psi)\, ds \right\|_{\mathcal{H}} \lesssim \|\psi^3\|_{L^1_t L^2_x} \lesssim \|\psi\|^3_{L^3_t L^6_x}. \tag{13}$$

So finite Strichartz norms==>Scattering.

Multi-solitons

Applying Lorentz transforms and the translational symmetry: a family of traveling waves

$$Q_\beta\left(x-\beta t+x_0\right).$$

Multi-soliton:

$$R\left(t,x\right)=\sum_{j=1}^{N}\sigma_j Q_{\beta_j}\left(x-\beta_j t-y_j\right),\ \sigma_j\in\{\pm 1\} \tag{14}$$

Using these traveling waves as building blocks, for $\beta_j\neq\beta_k$ and arbitrary x_j's, one can construct a pure multi-soliton:

$$\psi\to R(t,x), \tag{15}$$

Côte-Muñoz, Bellazzini-Ghimenti-Le Coz and Côte-Martel.

Motivated the Soliton Resolution Conjecture, we study dynamics of multi-solitons.

- We show conditional asymptotic stability of multi-solitons and construct centre-stable manifold around them.
- We derive refined information on pure multi-solitons and classify all possible pure multi-solitons.

Stability

The study of the (conditional) stability problem of solitons in nonlinear dispersive PDEs has a long history.

- For example, Weinstein established orbital stability: starting with a soliton plus a small perturbation, the solution remains in this form for all time.
- The asymptotic stability problem which is a stronger property-the situation in which small perturbations not only remain small, but in fact disperse. Soffer-Weinstein, Beceanu, Cuccagna, Krieger-Schlag, Nakanishi-Schlag, Perelman, Rodnianski-Schlag-Soffer, Schlag etc.
- Overall, the stability problem near one single soliton has been studied extensively. For the stability problem around a multi-soliton, Perelman, and Rodnianski-Schlag-Soffer, use pointwise decay estimates to obtain the asymptotic stability in $L^2 \bigcap L^1$.

Stability

Stability in natural spaces.

- Using appropriate Strichartz estimates, one can analyze the stability problem in some natural topology like the energy space, see e.g. Beceanu, Nakanishi-Schlag for one-soliton problem.
- On the other hand, for the energy and modulation methods to study asymptotic stability of KdV type problems, see Merle-Martel, Martel-Merle-Tsai in the soliton region. These papers use specific monotonicity formulas of the KdV type equations and do not apply to wave equations.

We obtained necessary Strichartz estimates to study multi-solitons in the energy space.

Pure multi-solitons

Constructing pure multi-solitons in various models has a long history:

- Bellazzini-Ghimenti-Le Coz , Côte-Muñoz , Côte-Martel, Martel.
- For gKdV, the problem of existence and uniqueness of pure multi-solitons was solved by Martel, and Combet (monotonicity formulas)
- In the works by Côte-Friederich and Friederich, assuming certain algebraic decay rates in time, they are able to prove uniqueness for various models.
- C-J 21, a fixed point argument, which naturally results in the uniqueness, to construct pure multi-kink (soliton) solutions for $1+1$ scalar field models in the class of exponential multi-kink solutiosn.

For $\beta \in \mathbb{R}^3$ such that $|\beta| < 1$, we consider the corresponding Lorentz boosts of the ground state Q_β. Taking the vector version of the Lorentz boost of the ground state, we have

$$\boldsymbol{Q}_\beta := (Q_\beta, -\beta \cdot \nabla Q_\beta). \tag{16}$$

The traveling wave is given by $\boldsymbol{Q}_\beta(\mathrm{x}-\beta t - y)$.
Given a fixed natural number N, *distinct* Lorentz parameters

$$\boldsymbol{\beta} = (\beta_1, \dots \beta_N) \in \mathbb{R}^{3N}, \tag{17}$$

a set of shifts

$$\boldsymbol{y} = (y_1, \dots, y_N) \in \mathbb{R}^{3N} \tag{18}$$

and a set of signs

$$\boldsymbol{\sigma} = (\sigma_1, \dots \sigma_N),\ \sigma_j \in \{\pm 1\}, \tag{19}$$

we consider the multi-soliton structure given by

$$\boldsymbol{Q}(\boldsymbol{\beta},\boldsymbol{y}) = \sum_{j=1}^{N} \sigma_j \boldsymbol{Q}_{\beta_j}(\cdot - y_j). \tag{20}$$

We are interested in multi-solitons which satisfy a *separation condition* (avoid collisions now): $y_j : [0, \infty) \to \mathbb{R}^3$, $j = 1, \ldots, N$ satisfy the *separation condition* with parameters $\rho, \delta > 0$ if

$$|y_j(t) - y_k(t)| \geq \delta t + \rho, \qquad \text{for all } t \geq 0 \text{ and } j \neq k. \tag{21}$$

The vectors $y_1^{in}, \ldots, y_N^{in}, \beta_1^{in}, \ldots, \beta_N^{in} \in \mathbb{R}^3$ satisfy the *separation condition* with parameters $\rho, \delta > 0$ if $y_j(t) = y_j^{in} + \beta_j^{in} t$ satisfy the separation condition with the same parameters.

Theorem (C-J 2022)

For every $\delta > 0$ there exist $\rho, \eta > 0$ such that the following holds. Let the initial parameters $(y_j^{in}, \beta_j^{in})_{j=1}^{J}$ satisfy the separation condition, and let

$$\|\psi_0 - \boldsymbol{Q}(\boldsymbol{\beta}^{in}, \boldsymbol{y}^{in})\|_{\mathcal{H}} \leq \eta. \tag{22}$$

If the corresponding solution ψ stays close in a neighborhood of multi-soliton family,

$$\sup_{t \in \mathbb{R}^+} \inf_{\boldsymbol{\beta} \in \mathbb{R}^{3N}, \boldsymbol{y} \in \mathbb{R}^{3N}} \|\psi(t) - \boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{y})\|_{\mathcal{H}} \lesssim \eta, \tag{23}$$

then ψ scatters to the multi-soliton family: there exist $\beta_j \in \mathbb{R}^3$, paths $y_j(t) \in \mathbb{R}^3$ and $\psi_+ \in \mathcal{H}$ with the property that $\dot{y}_j(t) \to \beta_j$ and

$$\lim_{t \to \infty} \left\| \psi(t) - \boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{y}(t)) - e^{JH_0 t}\psi_+ \right\|_{\mathcal{H}} = 0.$$

We say that a solution ψ to the equation (6) is a pure multi-soliton if there exist $\boldsymbol{\beta} \in \mathbb{R}^{3N}$ satisfying $|\beta_j| < 1$, $\beta_j \neq \beta_k$ for $j \neq k$, and $\boldsymbol{y}_p(t) \in \mathbb{R}^{3N}$ satisfying $|y_{p,j}(t) - y_{p,k}(t)| \geq L \gg 1$ for all $t \geq 0$ such that

$$\lim_{t\to\infty} \left\| \psi(t) - \boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{y}_p(t)) \right\|_{\mathcal{H}} = 0 \tag{24}$$

Next two theorems are concerned with pure multi-solitons.

Theorem (C-J 2022)

Suppose $\psi(t)$ is a pure multi-soliton. Then actually there exists $\boldsymbol{x}_0 \in \mathbb{R}^{3N}$ such that one has

$$\|\psi(t) - \boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{\beta} t + \boldsymbol{x}_0)\|_{\mathcal{H}} \lesssim e^{-\rho_0 t} \tag{25}$$

for small $\rho_0 > 0$ which is independent of ψ.

Finally, regarding the classification of pure multi-solitons, we have the following theorem:

Theorem (C-J 2022)

For fixed $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^{3N}$ *satisfying* $|\beta_j| < 1$ *and* $\beta_j \neq \beta_k$ *for* $j \neq k$ *and* $\boldsymbol{x}_0 = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$*, the set of solution* ψ *to* (6) *satisfying*

$$\lim_{t\to\infty} \|\psi(t) - \boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{\beta} t + \boldsymbol{x}_0)\|_{\mathcal{H}} = 0 \tag{26}$$

forms a dimension N *manifold.*

We need some notation.
The linear operator associated with each potential has a generalized kernel and stable/unstable modes.

$$\mathcal{Y}^0_{m,\beta}(x) = \partial_{x_m}\boldsymbol{Q}_\beta(x), \mathcal{Y}^1_{m,\beta}(x) = \frac{1}{\gamma^2}\partial_{\beta^m}\boldsymbol{Q}_\beta(x),$$
$$\mathcal{Y}^-_\beta(x) = e^{\gamma\nu\beta\cdot x}(\phi - \gamma\beta\cdot\nabla\phi - \gamma\nu\phi)_\beta(x),$$
$$\mathcal{Y}^+_\beta(x) = e^{-\gamma\nu\beta\cdot x}(\phi, -\gamma\beta\cdot\nabla\phi + \gamma\nu\phi)_\beta(x),$$

For the moving setting:

$$\mathcal{Y}^-_j(t) := \mathcal{Y}^-_{\beta_j(t)}(\cdot - y_j(t)), \mathcal{Y}^+_j(t) := \mathcal{Y}^+_{\beta_j(t)}(\cdot - y_j(t)),$$
$$\mathcal{Y}^0_{j,m}(t) = \mathcal{Y}^0_{m,\beta_j(t)}(\cdot - y_j(t))\,\mathcal{Y}^1_{j,m}(t) := \mathcal{Y}^1_{m,\beta_j(t)}(\cdot - y_j(t)).$$

Given a multi-soliton $\boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{y})$, we define

$$\mathcal{N}_L(\boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{y})) := \left\{ \begin{array}{c} \boldsymbol{R} \in \mathcal{H} | \left\langle \alpha^i_{j,m,\beta_j}(\cdot - y_j), \boldsymbol{R} \right\rangle = \left\langle \alpha^+_{j,\beta}(\cdot - y_j), \boldsymbol{R} \right\rangle = 0 \\ i = 0, 1, j = 1, \ldots, N, \ m = 1, 2, 3, \ \|\boldsymbol{R}\|_{\mathcal{H}} < \eta \end{array} \right\} \tag{27}$$

which is of codimension $7N$.

Theorem (C-J 2022)

For every $\delta > 0$, there exist $\rho > 0$ and $\eta > 0$ such that the following holds. Let $(\boldsymbol{\beta}_0, \boldsymbol{y}_0)$ satisfy the separation condition. There exists a map such that

$$\boldsymbol{\Phi} : \mathcal{N}_L(\boldsymbol{Q}(\boldsymbol{\beta}_0, \boldsymbol{y}_0)) \to \mathbb{R}^N$$

$$|\boldsymbol{\Phi}(\boldsymbol{R}_0)| \lesssim \frac{1}{\delta} e^{-\rho} + \frac{1}{\delta} e^{-\rho} \|\boldsymbol{R}_0\|_{\mathcal{H}} + \|\boldsymbol{R}_0\|^2_{\mathcal{H}}, \ \boldsymbol{R}_0 \in \mathcal{N}_L(\boldsymbol{Q}(\boldsymbol{\beta}_0, \boldsymbol{y}_0))$$

$$\left|\boldsymbol{\Phi}(\boldsymbol{R}_0) - \boldsymbol{\Phi}(\tilde{\boldsymbol{R}}_0)\right| \lesssim \frac{1}{\delta} e^{-\rho} \left\|\boldsymbol{R}_0 - \tilde{\boldsymbol{R}}_0\right\|_{\mathcal{H}} + \eta \left\|\boldsymbol{R}_0 - \tilde{\boldsymbol{R}}_0\right\|_{\mathcal{H}},$$

$$\boldsymbol{R}_0, \tilde{\boldsymbol{R}}_0 \in \mathcal{N}_L(\boldsymbol{Q}(\boldsymbol{\beta}_0, \boldsymbol{y}_0))$$

Theorem (C-J 2022 conti.)

so that $\forall \boldsymbol{R}_0 \in \mathcal{N}_L\left(\boldsymbol{Q}(\boldsymbol{\beta}_0, \boldsymbol{y}_0)\right)$, *the solution* $\psi(t)$ *to* (6) *with initial data*

$$\psi(0) = \boldsymbol{Q}\left(\boldsymbol{\beta}_0, \boldsymbol{y}_0\right) + \boldsymbol{R}_0 + \boldsymbol{\Phi}(\boldsymbol{R}_0) \cdot \boldsymbol{\mathcal{Y}}^+(\boldsymbol{\beta}_0, \boldsymbol{y}_0)$$

where $\boldsymbol{\Phi}(\boldsymbol{R}_0) \cdot \boldsymbol{\mathcal{Y}}^+(\boldsymbol{\beta}_0, \boldsymbol{y}_0) = \sum_{j=1}^{N} \phi_j(\boldsymbol{R}_0)\mathcal{Y}^+_{\beta_{0,j}}(\cdot - y_{0,j})$ *exists globally, and it scatters to the multi-soliton family: there exist* $\beta_j \in \mathbb{R}^3$, *paths* $y_j(t) \in \mathbb{R}^3$ *and* ψ_+ *with the property that* $\dot{y}_j(t) \to \beta_j$ *and*

$$\lim_{t\to\infty} \left\| \psi(t) - \boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{y}(t)) - e^{JH_0 t}\psi_+ \right\|_{\mathcal{H}} = 0.$$

Locally gains $6N$ dimensions back.

Proposition

There exist a small η and a Lipschitz manifold $\mathcal{N}\left(\boldsymbol{Q}(\beta_0^{in}, \boldsymbol{y}_0^{in})\right)$ inside the space $B_\eta\left(\boldsymbol{Q}(\beta_0^{in}, \boldsymbol{y}_0^{in})\right) \subset \mathcal{H}$ of codimension N so that the following property holds: for any choice of initial data $\psi(0) \in \mathcal{N}\left(\boldsymbol{Q}(\beta_0^{in}, \boldsymbol{y}_0^{in})\right)$, $\psi(t)$ with initial data $\psi(0)$ exists globally, and it scatters to the multi-soliton family, namely there exist $\beta_j \in \mathbb{R}^3$, paths $y_j(t) \in \mathbb{R}^3$ and ψ_+ with the property that $\dot{y}_j(t) \to \beta_j$ and

$$\lim_{t\to\infty} \left\| \psi(t) - \boldsymbol{Q}(\beta, \boldsymbol{y}(t)) - e^{JH_0 t}\psi_+ \right\|_{\mathcal{H}} = 0.$$

Implicit function theorem...

Finally, the local charts are compatible, so we can patch all these local constructions together. As a result, there is a centre stable manifold for the well-separated multi-soliton family.

Theorem

Fixed a natural number N, given $\delta > 0$, there exists $\rho > 0$ large, such that there exists a codimension N centre-stable manifold $\mathcal{N}$ around the well-separated multi-soliton family $\mathfrak{S}_{\delta,\rho}$ which is invariant for $t \geq 0$.

By construction, for any initial data $\psi \in \mathcal{N}$, the corresponding solution $\psi(t)$ enjoys the orbital stability. The converse is also sure.

Corollary

For every $\delta > 0$ there exist $\rho, \eta > 0$ such that the following holds. Let the initial parameters $(y_j^{in}, \beta_j^{in})_{j=1}^{J}$ satisfy the separation condition, and let

$$\|\psi_0 - \boldsymbol{Q}(\beta^{in}, \boldsymbol{y}^{in})\|_{\mathcal{H}} \leq \eta. \tag{28}$$

Suppose that the solution ψ to (6) with initial data ψ_0 stays close in a neighborhood of multi-soliton family:

$$\sup_{t\in\mathbb{R}^+} \inf_{\boldsymbol{\beta}\in\mathbb{R}^{3N}, \boldsymbol{y}\in\mathbb{R}^{3N}} \|\psi(t) - \boldsymbol{Q}(\boldsymbol{\beta}, \boldsymbol{y})\|_{\mathcal{H}} \lesssim \eta. \tag{29}$$

Then $\psi(t) \in \mathcal{N}$.

Linear models and Strichartz estimates

The linearization around a multi-soliton results in

$$\partial_t \psi(t) = JH(t)\psi(t) + \boldsymbol{F} \tag{30}$$

where

$$H(t) := \begin{pmatrix} -\Delta + 1 + \sum_{j=1}^{N} V_{\beta_j(t)}(\cdot - y_j(t)) & 0 \\ 0 & 1 \end{pmatrix}.$$

and from the view of modulation equation

$$\left\|\beta_j'(t)\right\|_{L_t^1 \cap L_t^\infty} + \left\|y_j'(t) - \beta_j(t)\right\|_{L_t^1 \cap L_t^\infty} \ll 1. \tag{31}$$

Theorem (C-J 2022)

Denote the Strichartz norm

$$S = L_t^\infty L_x^2 \bigcap L_t^2 B_{6,2}^{-5/6}.$$

Consider the system

$$\partial_t \psi(t) = JH(t)\psi(t) + \boldsymbol{F} \tag{32}$$

such that

$$\pi_0(t)\psi(t) = 0, \ \forall t \in \mathbb{R}.$$

Using the notations above, one has Strichartz estimates

$$\|\mathcal{D}(\pi_{cs}(t)\psi(t))_1\|_S + \|(\pi_{cs}(t)\psi(t))_2\|_S \lesssim \|\psi(0)\|_{\mathcal{H}} + \|\mathcal{D}F_1\|_{S^*} + \|F_2\|_{S^*}. \tag{33}$$

Moreover, $\pi_s(t)\psi(t)$ *scatters to a free wave. There exists* $\psi_+ \in \mathcal{H}$ *such that*

$$\left\|\pi_{cs}(t)\boldsymbol{u}(t) - e^{JH_0 t}\psi_+\right\|_{\mathcal{H}} \to 0, \ t \to \infty. \tag{34}$$

The linear theory

Difficulties:

- instabilities from negative eigenvalues.
- time-dependent trajectories: can not apply Lorentz transforms which mix space and time.

Consider

$$\partial_{tt}\psi - \Delta\psi + \psi + V_\beta(x - \beta t)\psi = F \tag{35}$$

Rewrite the equation in the matrix form

$$\frac{d}{dt}\boldsymbol{\psi} = \begin{pmatrix} 0 & 1 \\ \Delta - V_\beta(x - \beta t) - 1 & 0 \end{pmatrix}\boldsymbol{\psi} + \boldsymbol{F}. \tag{36}$$

Using the moving frame $x - \beta t \to x$, the equation above can be written as

$$\frac{d}{dt}\boldsymbol{\psi} = \begin{pmatrix} \beta \cdot \nabla & 1 \\ \Delta - V_\beta(x) - 1 & \beta \cdot \nabla \end{pmatrix}\boldsymbol{\psi} + \boldsymbol{F} \tag{37}$$

$$:= \mathcal{L}_\beta\boldsymbol{\psi} + \boldsymbol{F}. \tag{38}$$

Basic setting of the contraction map: Given a set of data $(\beta(t), \boldsymbol{y}(t), \psi(t))$, we consider the following map

$$F((\beta(t), \boldsymbol{y}(t), \psi(t))) = \left(\tilde{\beta}(t), \tilde{\boldsymbol{y}}(t), \tilde{\psi}(t)\right) \tag{39}$$

where $\left(\tilde{\beta}(t), \tilde{\boldsymbol{y}}(t), \tilde{\psi}(t)\right)$ are defined by solving

$$\begin{aligned}\frac{d}{dt}\tilde{\psi}(t) &= JH(t)\tilde{\psi}(t) + \mathcal{I}(Q) + \mathfrak{I}_1(Q^2, \psi) + \mathfrak{I}_2(Q, \psi^2) + \boldsymbol{F}(\psi) \\ &- \dot{\tilde{\beta}}(t)\partial_\beta \boldsymbol{Q}(\beta(t), \boldsymbol{y}(t)) - \left(\dot{\tilde{\boldsymbol{y}}}(t) - \tilde{\beta}(t)\right)\partial_y \boldsymbol{Q}(\beta(t), \boldsymbol{y}(t)) \\ &=: JH(t)\tilde{\psi}(t) + \boldsymbol{W}(\psi(t), \beta(t), \boldsymbol{y}(t)) + \widetilde{\mathrm{Mod}'}(t)\nabla_M \boldsymbol{Q}(\beta(t), \boldsymbol{y}(t))\end{aligned}$$

and the modulation equations

$$\frac{d}{dt}\left\langle \alpha^0_{m,\beta_j(t)}(\cdot - y_j(t)), \tilde{\psi}(t)\right\rangle = \frac{d}{dt}\left\langle \alpha^1_{m,\beta_j(t)}(\cdot - y_j(t)), \tilde{\psi}(t)\right\rangle = 0 \tag{40}$$

Subject to stabilization conditions

$$
\begin{aligned}
\tilde{a}_j^+(t) &= \left\langle \alpha^+_{\beta_j(t)}(\cdot - y_j(t)), \tilde{\psi}(t) \right\rangle \qquad (41)\\
&= -\int_t^\infty \exp\left(\int_s^t \frac{\nu}{\gamma_j(\tau)}\, d\tau\right) \left\langle \alpha^+_{\beta_j(s)}(\,\cdot - y_j(s)), \mathcal{W}(\cdot, s) \right\rangle\, ds\\
&- \sum_{k\neq j} \int_t^\infty \exp\left(\int_s^t \frac{\nu}{\gamma_j(\tau)}\, d\tau\right) \left\langle \alpha^+_{\beta_j(s)}(\,\cdot - y_j(s)), \mathcal{V}_{\beta_k(s)}(\,\cdot - y_k(s))\tilde{\psi}(s) \right\rangle\, ds\\
&- \int_t^\infty \exp\left(\int_s^t \frac{\nu}{\gamma_j(\tau)}\, d\tau\right) \left\langle (y_j'(s) - \beta_j(s)) \cdot \nabla \alpha^+_{\beta_j(s)}(\,\cdot - y_j(s)), \tilde{\psi}(s) \right\rangle ds\\
&- \int_t^\infty \exp\left(\int_s^t \frac{\nu}{\gamma_j(\tau)}\, d\tau\right) \left\langle \beta_j'(s)\partial_\beta \alpha^+_{\beta_j(s)}(\,\cdot - y_j(s)), \tilde{\psi}(s) \right\rangle\, ds.
\end{aligned}
$$

Thank you for your attention.