

Soliton resolution for energy-critical wave maps in the equivariant case

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Seminar on General Relativity, LJLL
January 12th, 2023

Energy-critical wave maps equation

"Definition" An application $\Psi: \mathbb{R}^{1+2} \rightarrow S^2 \subset \mathbb{R}^3$ is a wave map if it is a critical point of the Lagrangian

$$\mathcal{L}(\Psi, \partial_t \Psi, \nabla_x \Psi) := \frac{1}{2} \iint (|\partial_t \Psi|^2 - |\nabla_x \Psi|^2) dx dt.$$

Natural analogs of linear waves in a nonlinear, geometric setting.

Euler-Lagrange equation:

$$\partial_t^2 \Psi(t, x) - \Delta_x \Psi(t, x) = -(|\partial_t \Psi(t, x)|^2 - |\nabla_x \Psi(t, x)|^2) \Psi(t, x).$$

Local well-posedness and global well-posedness for small data:

Klainerman, Selberg, Machedon, Tataru, Tao, ... (1993-2002)

Equivariant wave maps

We study the dynamics (long time behavior) of large solutions, but only in a special case:

$$\Psi(t, r\cos\theta, r\sin\theta) = (\sin\psi(t, r)\cos(k\theta), \sin\psi(t, r)\sin(k\theta), \cos\psi(t, r)).$$

Here, $k \in \{1, 2, \dots\}$ is the equivariance degree, $t \in \mathbb{R}$, $r \in (0, \infty)$

Equation for ψ :

$$(WM_k) \quad \partial_t^2 \psi(t, r) - \partial_r^2 \psi(t, r) - \frac{1}{r} \partial_r \psi(t, r) + \frac{k^2}{2r^2} \sin(2\psi(t, r)) = 0.$$

Lagrangian: $\mathcal{L} := \pi \iint \left((\partial_t \psi)^2 - (\partial_r \psi)^2 - \frac{k^2 \sin^2 \psi}{r^2} \right) r dr dt$

Energy: $E(\psi_0, \dot{\psi}_0) := \pi \int_0^\infty \left(\underbrace{\dot{\psi}_0^2}_{\text{kinetic}} + \underbrace{(\partial_r \psi_0)^2 + \frac{k^2 \sin^2 \psi_0}{r^2}}_{\text{potential}} \right) r dr$

Scaling invariance and criticality

If ψ solves (WM_k) and $\lambda > 0$, then

$\psi_\lambda(t, r) := \psi\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right)$ solves (WM_k) as well.

Moreover, $E(\psi_\lambda, \dot{\psi}_\lambda) = E(\psi, \dot{\psi}) \rightsquigarrow$ energy-critical problem

Note: if $\lambda \ll 1$, then ψ_λ is concentrated and evolves fast.

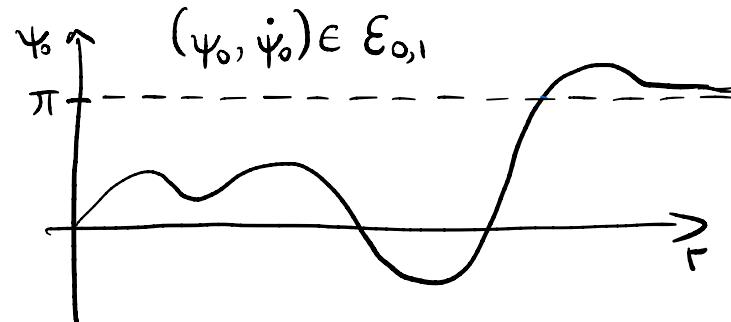
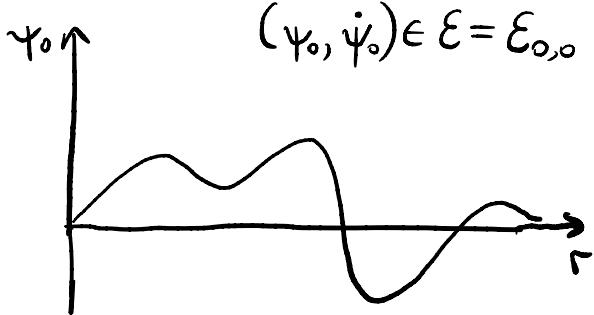
Local theory, small data theory

Energy norm: $\|(\psi_0, \dot{\psi}_0)\|_{\mathcal{E}}^2 := \|\dot{\psi}_0\|_{L^2}^2 + \|\psi_0\|_{H^1}^2$,

$$\|\dot{\psi}_0\|_{L^2}^2 := \int_0^\infty \dot{\psi}_0^2 r dr, \quad \|\psi_0\|_{H^1}^2 := \int_0^\infty \left((\partial_r \psi_0)^2 + \frac{k^2}{r^2} \psi_0^2 \right) r dr.$$

$$\|\psi_0\|_{L^\infty} \text{ small} \Rightarrow \|(\psi_0, \dot{\psi}_0)\|_{\mathcal{E}}^2 \approx E(\psi_0, \dot{\psi}_0).$$

Finite energy sectors : $\mathcal{E}_{m,n} := \left\{ (\psi_0, \dot{\psi}_0) : E(\psi_0, \dot{\psi}_0) < \infty, \lim_{r \rightarrow 0} \psi_0(r) = m\pi, \lim_{r \rightarrow \infty} \psi_0(r) = n\pi \right\}.$



Theorem (Shatah-Struwe 1994, $k=1$
Burg, Planchon-Stalker-Tahvildar Zadeh 2003, $k \geq 2$)

Equation (WM_k) is locally well-posed in each finite energy sector, in the sense of strong solutions.

Linearisation around $\psi=0$: $\partial_t^2 \psi_L - \partial_r^2 \psi_L - \frac{1}{r} \partial_r \psi_L + \frac{k^2}{r^2} \psi_L = 0$.

If ψ is small, the nonlinear effects become negligible for large times:

Theorem If $E(\psi_0, \dot{\psi}_0)$ is small enough, then ψ exists globally and

$$\lim_{t \rightarrow \pm\infty} \|(\psi(t), \partial_t \psi(t)) - (m\pi + \psi_L^\pm(t), \partial_t \psi_L^\pm(t))\|_{\mathcal{E}} = 0 \quad (\text{scattering})$$

These results are consequences of Strichartz estimates.

Stationary solutions ("solitons" or "bubbles")

Minimisers of E :

* on $\mathcal{E}_{m,m}$ \rightsquigarrow constant functions

* on $\mathcal{E}_{m,m+1}$ $\rightsquigarrow (m\pi + 2\arctan(r^k/\lambda^k), 0)$, $\lambda > 0$

* on $\mathcal{E}_{m,m-1}$ $\rightsquigarrow (m\pi - 2\arctan(r^k/\lambda^k), 0)$, $\lambda > 0$

* on other sectors $\rightsquigarrow \emptyset$

We denote $Q(r) := 2\arctan(r^k)$, $Q_\lambda := Q(r/\lambda)$ for $\lambda > 0$.

- Key role in the description of the dynamics of large solutions.

Absence of self-similarity

Theorem (Christodoulou, Tahvildar-Zadeh, Shatah 1992)

If $T_+ < \infty$ the max. time of existence of Ψ ,
then for every $\alpha > 0$

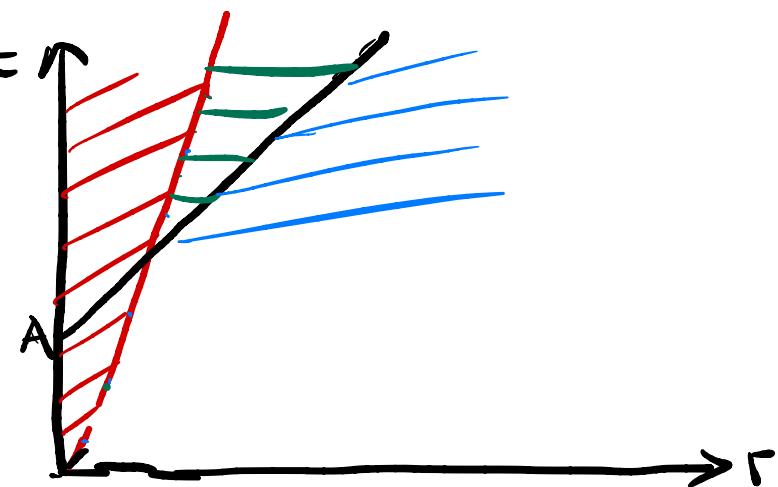
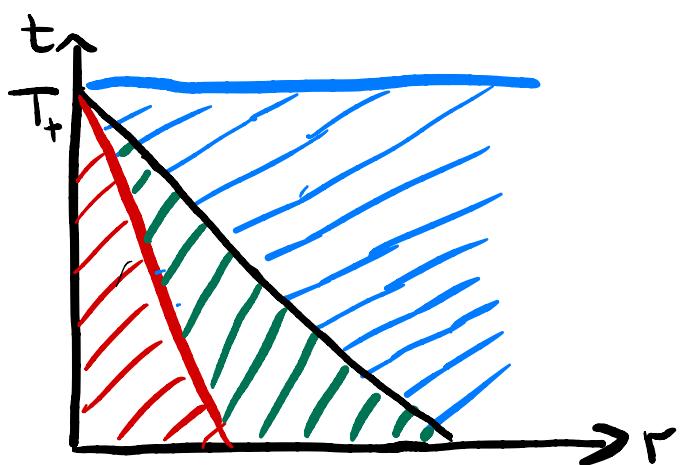
$$\lim_{t \rightarrow T} \int_{\alpha(T_+-t)}^{T_+-t} \left[(\partial_t \Psi)^2 + (\partial_r \Psi)^2 + \frac{k^2}{r^2} \Psi^2 \right] r dr = 0.$$

Theorem (Côte, Kenig, Lawrie, Schlag 2015)

If $T_+ = \infty$ the max. time of existence of Ψ ,

then for every $\alpha > 0$

$$\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \int_{\alpha t}^{t-A} \left[(\partial_t \Psi)^2 + (\partial_r \Psi)^2 + \frac{k^2}{r^2} \Psi^2 \right] r dr = 0.$$



(The energy in the green region $\rightarrow 0$ as $t \rightarrow T_+$.)

Theorem (Struwe 2003)

If $T_+ < \infty$, then there exist $t_n \rightarrow T_+$, $m \in \mathbb{Z}$, $\nu \in \{-1, 1\}$ and $0 < \lambda_n \ll T_+ - t_n$ such that for all $R > 0$

$$\lim_{n \rightarrow \infty} \|(\gamma(t_n, \lambda_n \cdot), \lambda_n \partial_t \gamma(t_n, \lambda_n \cdot)) - (m\pi + \nu Q, 0)\|_{\mathcal{E}(r \leq R)} = 0.$$

Sequential soliton resolution

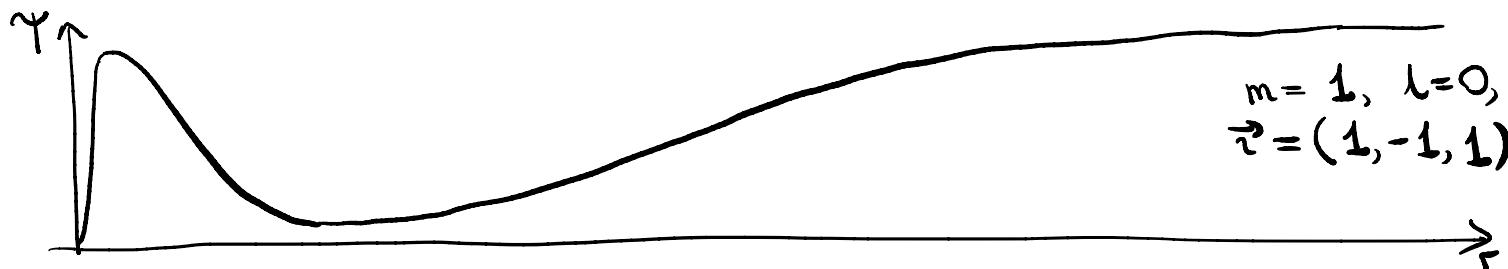
Given $M \in \mathbb{N}$, $m \in \mathbb{Z}$, $\vec{z} = (z_1, \dots, z_M) \in \{-1, 1\}^M$,
 $\vec{\lambda} = (\lambda_1, \dots, \lambda_M) \in (0, \infty)^M$, $\lambda_1 < \lambda_2 < \dots < \lambda_M$,

we denote

$$Q(m, \vec{z}, \vec{\lambda}; r) := m\pi + \sum_{j=1}^M z_j (Q_{\lambda_j} - \pi) \in E_{l,m}$$

$$\text{with } l = m - \sum_{j=1}^M z_j.$$

If $\lambda_1 \ll \lambda_2 \ll \dots \ll \lambda_M$, we call $Q(m, \vec{z}, \vec{\lambda})$
a multi-bubble configuration.



Sequential soliton resolution

Theorem (Côte 2015, Jia-Kenig 2017)

① If $T_+ < \infty$, then there exist $(\psi_*, \dot{\psi}_*)$, $M, m, \vec{z}, \vec{\lambda}_n, t_n \rightarrow T_+$:

$$\lim_{n \rightarrow \infty} \left[\|\psi(t_n) - \psi_* - Q(m, \vec{z}, \vec{\lambda}_n)\|_{\mathcal{H}}^2 + \|\partial_t \psi(t_n) - \dot{\psi}_*\|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\lambda_{n,j} / \lambda_{n,j+1} \right)^k + \left(\lambda_{n,M} / (T_+ - t_n) \right)^k \right] = 0.$$

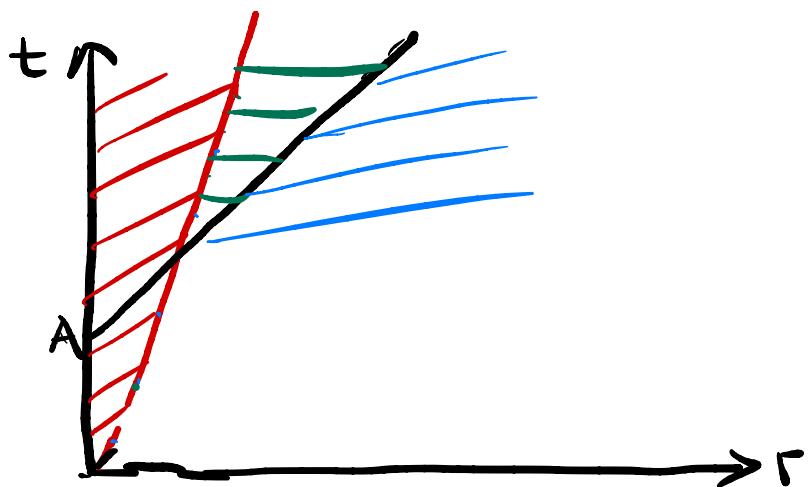
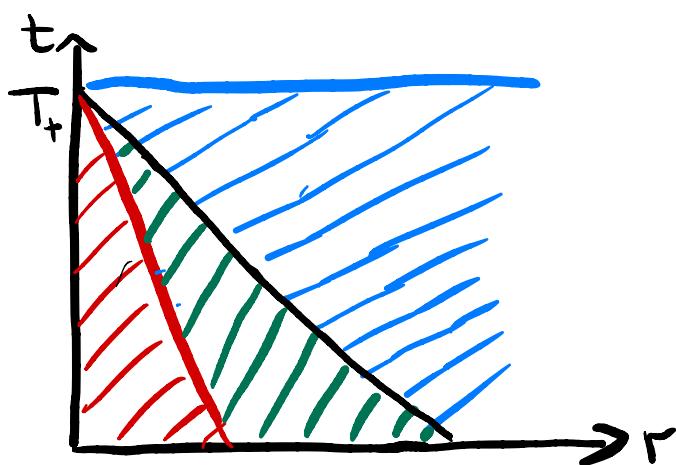
② If $T_+ = \infty$, then there exist $(\psi_L, \partial_t \psi_L)$, $M, m, \vec{z}, \vec{\lambda}_n, t_n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \left[\|\psi(t_n) - \psi_L(t_n) - Q(m, \vec{z}, \vec{\lambda}_n)\|_{\mathcal{H}}^2 + \|\partial_t \psi(t_n) - \partial_t \psi_L(t_n)\|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\lambda_{j,n} / \lambda_{j+1,n} \right)^k + \left(\lambda_{M,n} / t_n \right)^k \right] = 0.$$

Remark Convergence in continuous time outside of cones:

① $\forall \alpha > 0$: $\lim_{t \rightarrow T_+} \left(\|\psi(t) - \psi_*\|_{\mathcal{H}(r \geq \alpha(T_+ - t))}^2 + \|\partial_t \psi(t) - \dot{\psi}_*\|_{L^2(r \geq \alpha(T_+ - t))}^2 \right) = 0$.

② $\forall \alpha > 0$: $\lim_{t \rightarrow \infty} \left(\|\psi(t) - \psi_L(t)\|_{\mathcal{H}(r \geq \alpha t)}^2 + \|\partial_t \psi(t) - \partial_t \psi_L(t)\|_{L^2(r \geq \alpha t)}^2 \right) = 0$.



$$\Xi \rightarrow ME(Q)$$

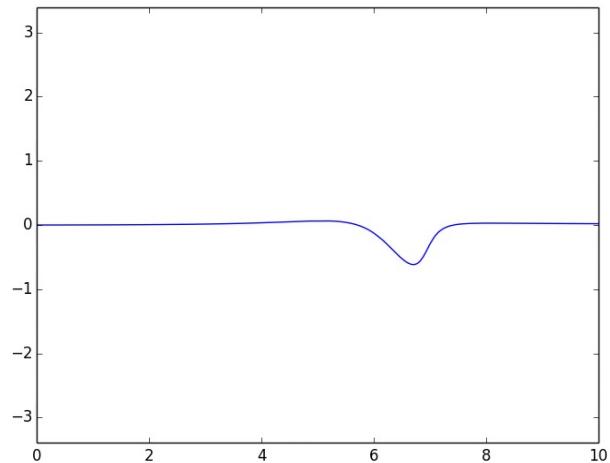
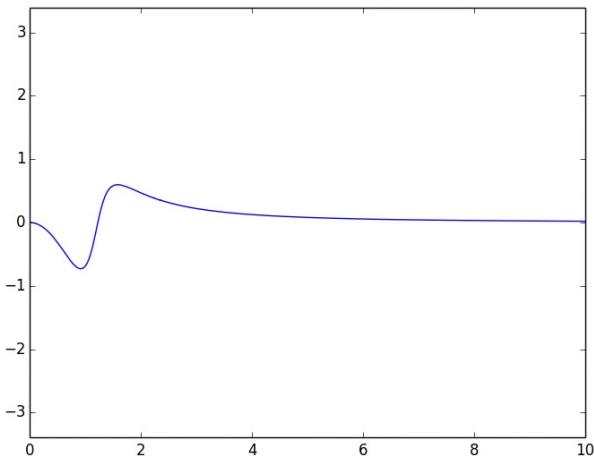
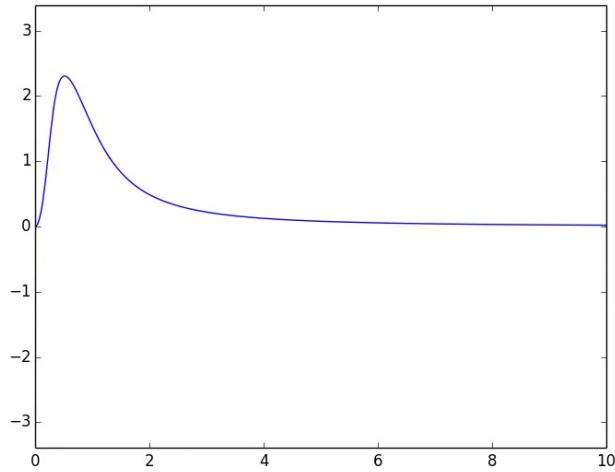
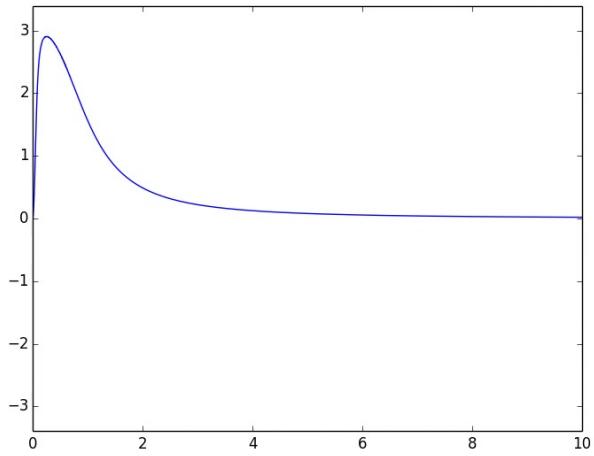
$$\Xi \rightarrow O$$

$$\Xi \rightarrow E(\psi_*, \dot{\psi}_*) \quad (\text{resp. } E_L(\psi_L, \partial_t \psi_L))$$

Question : Does the decomposition hold in continuous time?

Ennemy : Collisions in the red region.

(Excluded if all bubbles have the same sign.)



Continuous time soliton resolution

Theorem (J.-Lawrie 2021)

① If $T_+ < \infty$, then there exists $\vec{\lambda} : [0, T_+) \rightarrow (0, \infty)^M$ such that

$$\lim_{t \rightarrow T_+} \left[\|\psi(t) - \psi_* - Q(m, \vec{z}, \vec{\lambda}(t))\|_H^2 + \|\partial_t \psi(t) - \dot{\psi}_*\|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \right)^k + \left(\frac{\lambda_M(t)}{T_+ - t} \right)^k \right] = 0.$$

② If $T_+ = \infty$, then there exists $\vec{\lambda} : [0, \infty) \rightarrow (0, \infty)^M$ such that

$$\lim_{t \rightarrow \infty} \left[\|\psi(t) - \psi_L(t) - Q(m, \vec{z}, \vec{\lambda}(t))\|_H^2 + \|\partial_t \psi(t) - \partial_t \psi_L(t)\|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \right)^k + \left(\frac{\lambda_M(t)}{t} \right)^k \right] = 0.$$

Some related results

- * Duyckaerts, Kenig, Merle 2010 — sequential soliton resolution for $\partial_t^2 u - \Delta_x u - u^5 = 0$ in 3D.
- * Duyckaerts, Kenig, Merle 2012 — continuous time; energy channels
- * J-Lawrie 2017 — classification for $E \leq 2E(Q)$, $k \geq 2$; crucially used estimates of interactions between solitons (reduction to a "2-body problem")
- * Rodriguez 2019 ————— || ————— $k=1$
- * D,K,M 2019 : generalisation of 2012 result to all odd dim.
- * Duyckaerts, Kenig, Martel, Merle 2021 — settled $k=1$ using energy channels

Existence of bubbles and multi-bubbles

The answer depends on the equivariance class:

- * $k=1,2,3,\dots$: blow-up solutions with $M=1$ bubble
 (Krieger-Schlag-Tataru '08, Rodnianski-Sterbenz 2010
 Raphaël-Rodnianski 2011)
 - blow-up with $M \geq 2$ unknown
 - * many other constructions for $M=1$ exist.
 - * $k=2,3,\dots$: existence, uniqueness and asymptotic description
 of pure two-bubble solutions (J'16, J-Lawrie 17-20)
 - * $k=1$: no solutions with $M \geq 2$ are known;
 no pure two-bubbles exist (Rodríguez '18)
 no pure multi-bubbles exist (J.-Lawrie, unpublished)

Threshold behavior : the two-bubbles

Simplest case not settled by the sequential result :

$$M = 2, \quad z_1 = 1, \quad z_2 = -1.$$

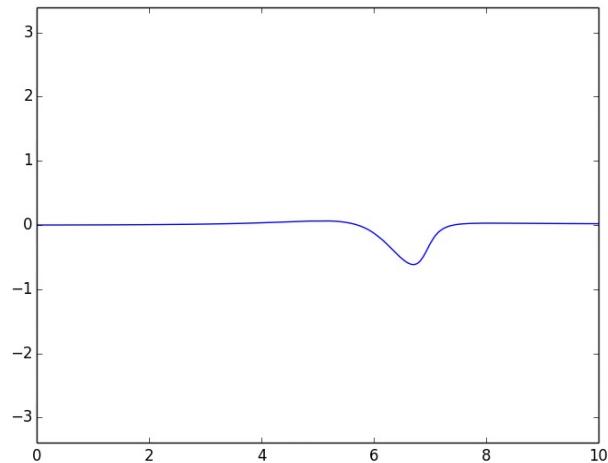
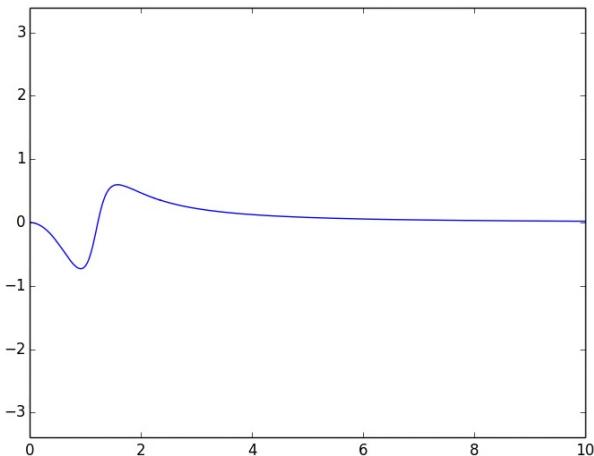
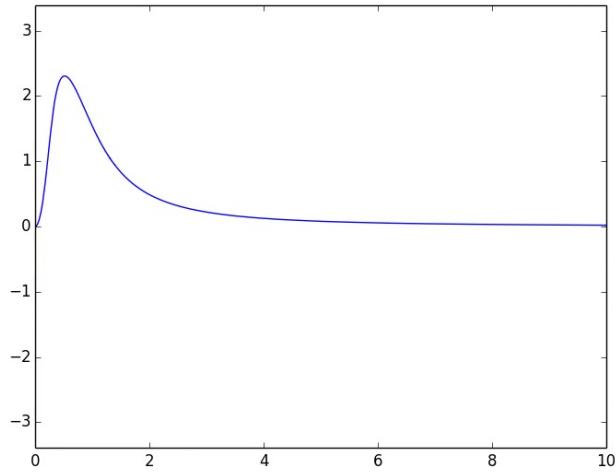
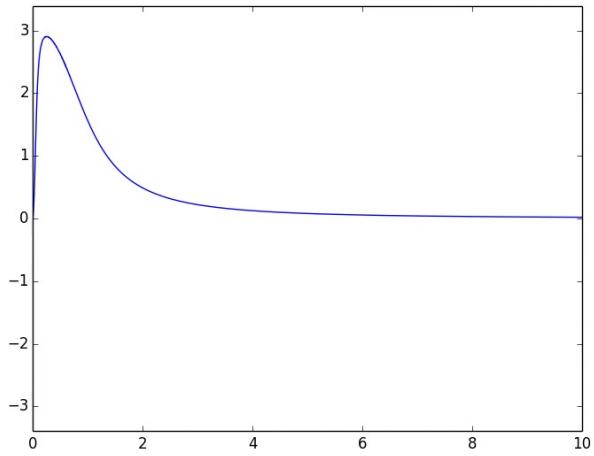
Minimal possible energy : $2E(Q) = 8k\pi$.

Theorem (J.-PhD thesis 2016)

Let $k \geq 2$. There exists a solution $(\psi_c, \dot{\psi}_c)$ of (WM_k) such that

$$\lim_{t \rightarrow \infty} \|(\psi_c, \dot{\psi}_c) - (Q_{kt-\frac{2}{k-2}} - Q, 0)\|_\varepsilon = 0,$$

where K is an explicit constant depending on k .



Theorem (J.-Lawrie 2018)

Let $k \geq 2$. If $(\psi, \dot{\psi})$ is a solution of (WM_k) in the sector $\mathcal{E} = \mathcal{E}_{0,0}$ of energy $E(\psi, \dot{\psi}) \leq 8k\pi$,
then either $(\psi, \dot{\psi})$ scatters in both time directions, or
forms a two-bubble in one time direction (in cont. time)
and scatters in the other time direction.

Theorem (J.-Lawrie 2020)

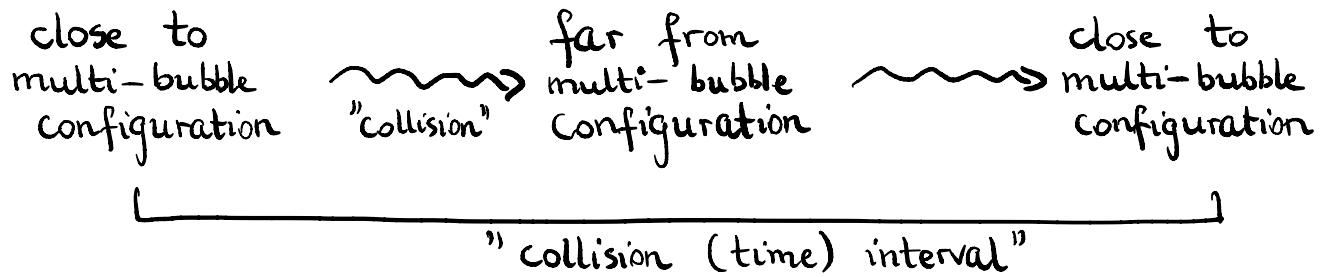
Let $k \geq 4$. If $(\psi, \dot{\psi})$ is a solution of (WM_k) in the sector $\mathcal{E} = \mathcal{E}_{0,0}$ of energy $E(\psi, \dot{\psi}) \leq 8k\pi$,
then either $(\psi, \dot{\psi})$ scatters in both time directions,
or there exist $\gamma, \sigma \in \{-1, 1\}$, $\lambda > 0$ and $t_0 \in \mathbb{R}$ such that

$$\psi(t, r) = \gamma \psi_c\left(\sigma \frac{t-t_0}{\lambda}, \frac{r}{\lambda}\right), \quad \forall (t, r) \in \mathbb{R} \times (0, \infty)$$

Main ideas of the proof of soliton resolution

The desired decomposition holds for a time sequence.

We need to prevent the following scenario:



occurring an infinite number of times (a no-return lemma).

- * Inspired by Duyckaerts–Merle, Nakanishi–Schlag,
Krieger–Nakanishi–Schlag for single soliton which is linearly unstable.
- * Here, inter-soliton interactions play a similar role
as linear instability in those works (cf. J.–Lawrie '17)

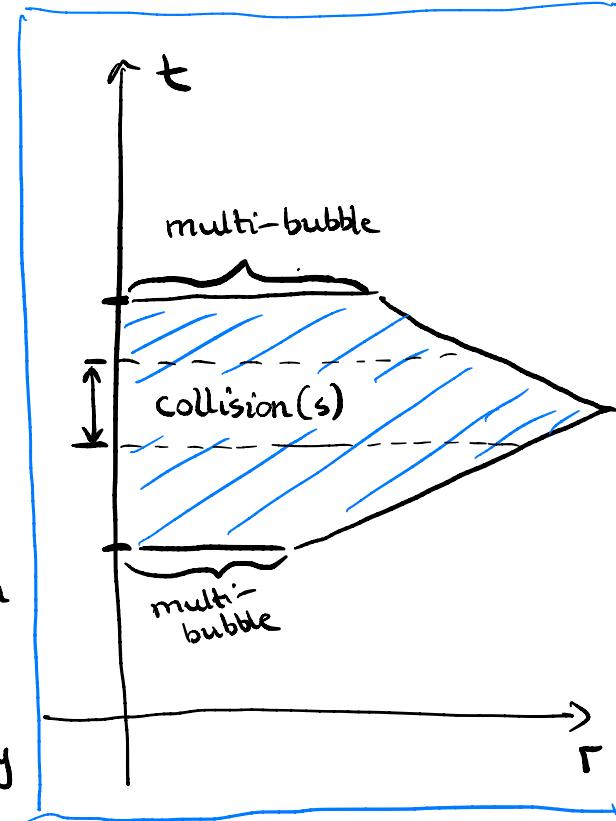
Virial identity

If ψ is a smooth wave map, then

$$\operatorname{div}_{t,r} \left(\partial_t \psi r^2 \partial_r \psi, -\frac{1}{2} r^2 (\partial_t \psi)^2 - \frac{1}{2} r^2 (\partial_r \psi)^2 + \frac{k^2}{2} \sin^2 \psi \right) = -r (\partial_t \psi)^2$$

We estimate the boundary terms from above and the space-time integral $\iint (\partial_t \psi)^2 r dr dt$ from below

- * Collision duration \gtrsim spatial scale
- * "horizontal" boundary \ll spatial scale
- * $\iint (\partial_t \psi)^2 r dr dt \gtrsim$ collision duration
("Compactness Lemma")
- * remaining boundary : reduction to n-body



Interior and exterior bubbles

Let $K \in \{1, 2, \dots, N\}$, $[a, b] \subset [T_0, T_+)$, $0 < \varepsilon < \eta$.

We say $[a, b]$ is a collision interval with $N-K$ exterior bubbles if:

- * $d(a) \leq \varepsilon$, $d(b) \leq \varepsilon$, $\exists c \in [a, b] : d(c) \geq \eta$

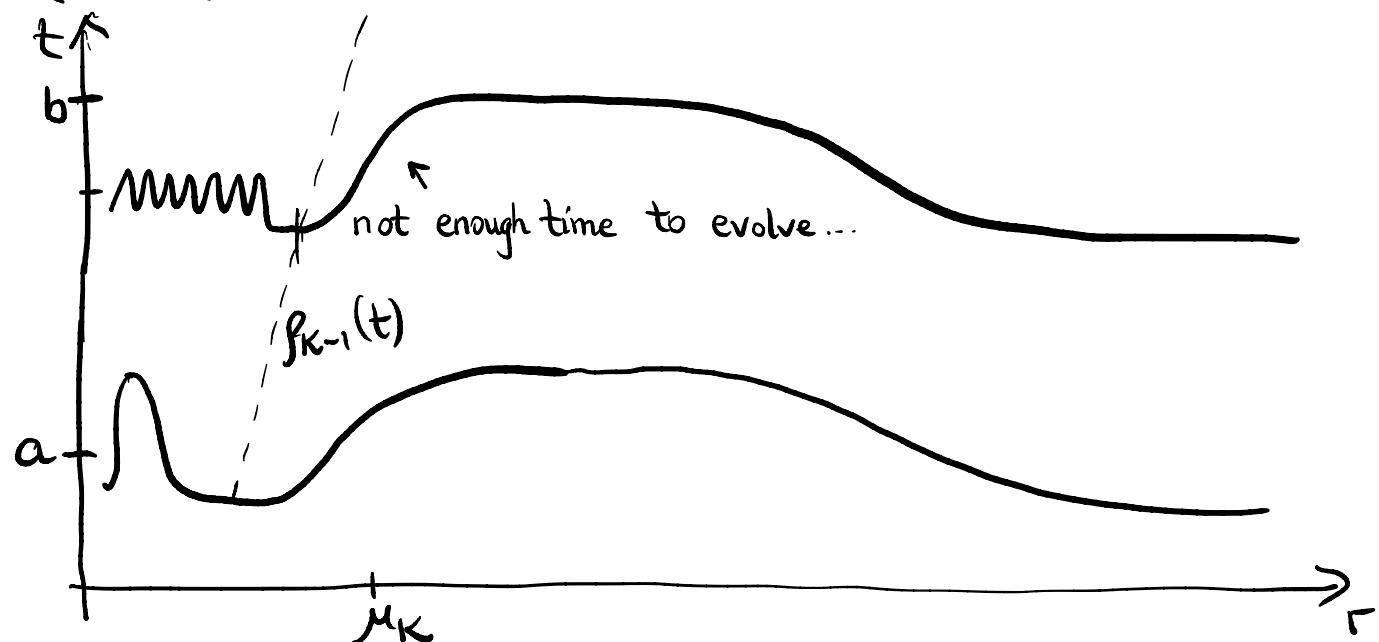
where $d(t)$ is the distance of $\gamma(t)$ to (multi-bubble + radiation)

- * \exists curve $r = g_K(t)$ such that in the region $r \geq g_K(t)$
 $\gamma(t, r)$ is ε -close to $[(N-K)\text{-bubble} + \text{radiation}]$.

We now set K to be the smallest number such that
there exist $\eta > 0$, a sequence $\varepsilon_n \rightarrow 0$
and a sequence of collision intervals $[a_n, b_n]$
corresponding to these parameters ε_n, η and K .

Lemma $\exists C = C(\gamma) > 0$ and $\varepsilon > 0$ such that
if $[a, b]$ collision interval with parameters (ε, η, K) ,
then $b - a \geq C \min(\mu_K(a), \mu_K(b))$,
where μ_K is the scale of the K -th bubble.

Proof If not, K would not be smallest possible.



Wave maps with small kinetic energy

Lemma ("Compactness Lemma") Let $f_n > 0$, $R_n \rightarrow \infty$, ψ_n defined for $t \in [0, f_n]$ of bounded energy such that

$$\lim_{n \rightarrow \infty} \frac{1}{f_n} \int_0^{f_n} \int_0^{R_n f_n} (\partial_t \psi_n)^2 r dr dt = 0.$$

Then, up to extraction of a subsequence, there exist $r_n \rightarrow \infty$, $t_n \in [0, f_n]$, $M, m, \vec{z}, \vec{\lambda}_n$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} & \left[\| \psi_n(t_n) - Q(m, \vec{z}, \vec{\lambda}_n) \|_{H(r \leq r_n f_n)}^2 + \| \partial_t \psi_n(t_n) \|_{L^2(r \leq r_n f_n)}^2 \right. \\ & \left. + \sum_{j=1}^{M-1} \left(\lambda_{n,j} / \lambda_{n,j+1} \right)^k + \left(\lambda_{n,M} / r_n \right)^k \right] = 0 \end{aligned}$$

Modulation

- * Near a_n and b_n , ψ is close to a multi-bubble and the analysis above does not apply.
- * In this case, the main dynamical information are the scales of the bubbles
- * We obtain differential inequalities on these scales.
Informally: $\lambda_j'' \approx -r_j r_{j+1} \omega^2 \frac{\lambda_j^{k-1}}{\lambda_{j+1}^k} + r_j r_{j-1} \omega^2 \frac{\lambda_{j-1}^k}{\lambda_j^{k+1}}$.
- * Error bounded by the energy of attractive interactions.
- * The influence of the exterior bubbles and radiation can essentially be neglected by enlarging E_n .
- * Refined modulation parameters: Raphaël-Szeftel '11, J.-Lauwé '17.

Lemma If d starts growing at t_0 , then $\forall t^* \geq t_0$

$$\int_{t_0}^{t_*} d(t) dt \leq C_0 d(t_*)^{2/k} \mu_k(t_0) \quad \text{if } k \geq 2$$

$$\int_{t_0}^{t_*} d(t) dt \leq C_0 d(t_*)^2 \sqrt{-\log d(t_*)} \mu_k(t_0) \quad \text{if } k=1$$

The final step is to partition $[a_n, b_n]$ into

