Soliton resolution for energy-critical wave maps in the equivariant case

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Energy-critical wave maps equation

"Definition" An application $\Psi : \mathbb{R}^{1+2} \to S^2 \subset \mathbb{R}^3$ is a wave map if it is a critical point of the Lagrangian

$$\mathcal{L}(\Psi, \partial_t \Psi, \nabla_x \Psi) := \frac{1}{2} \iint \left(|\partial_t \Psi|^2 - |\nabla_x \Psi|^2 \right) dx\, dt .$$

Natural analogs of linear waves in a nonlinear, geometric setting.

Euler-Lagrange equation:

$$\partial_t^2 \Psi(t,x) - \Delta_x \Psi(t,x) = -\left(|\partial_t \Psi(t,x)|^2 - |\nabla_x \Psi(t,x)|^2 \right) \Psi(t,x).$$

Local well-posedness and global well-posedness for small data:

Klainerman, Selberg, Machedon, Tataru, Tao, ... (1993-2002)

Equivariant wave maps

We study the dynamics (long time behavior) of large solutions, but only in a special case:

$$\Psi(t, r\cos\theta, r\sin\theta) = \big(\sin\psi(t,r)\cos(k\theta), \sin\psi(t,r)\sin(k\theta), \cos\psi(t,r)\big).$$

Here, $k \in \{1, 2, \dots\}$ is the equivariance degree, $t \in \mathbb{R}$, $r \in (0,\infty)$

Equation for ψ:

$$\boxed{(\mathrm{WM}_k) \quad \partial_t^2 \psi(t,r) - \partial_r^2 \psi(t,r) - \frac{1}{r}\partial_r \psi(t,r) + \frac{k^2}{2r^2}\sin(2\psi(t,r)) = 0.}$$

Lagrangian: $\displaystyle \mathcal{L} := \pi \iint \Big((\partial_t \psi)^2 - (\partial_r \psi)^2 - \frac{k^2 \sin^2 \psi}{r^2} \Big) r\,dr\,dt$

Energy: $\displaystyle E(\psi_0, \dot\psi_0) := \pi \int_0^\infty \Big(\underbrace{\dot\psi_0^{\,2}}_{\text{kinetic}} + \underbrace{(\partial_r \psi_0)^2 + \frac{k^2 \sin^2 \psi_0}{r^2}}_{\text{potential}} \Big) r\,dr$

Scaling invariance and criticality

If ψ solves (WM_k) and $\lambda > 0$, then

$$\psi_\lambda(t,r) := \psi\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right) \text{ solves } (WM_k) \text{ as well.}$$

Moreover, $E(\psi_\lambda, \dot\psi_\lambda) = E(\psi, \dot\psi)$ $\rightsquigarrow$ energy-critical problem

Note: if $\lambda \ll 1$, then ψ_λ is concentrated and evolves fast.

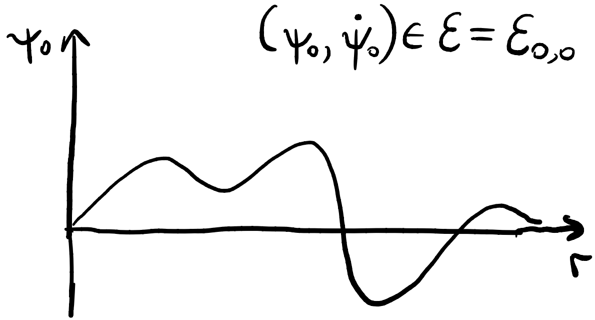
Local theory, small data theory

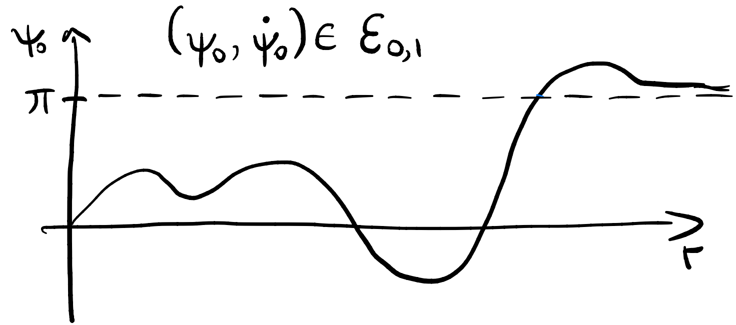
Energy norm: $\|(\psi_0, \dot\psi_0)\|_{\mathcal{E}}^2 := \|\dot\psi_0\|_{L^2}^2 + \|\psi_0\|_{\mathcal{H}}^2$,

$$\|\dot\psi_0\|_{L^2}^2 := \int_0^\infty \dot\psi_0^2 \, r\,dr, \qquad \|\psi_0\|_{\mathcal{H}}^2 := \int_0^\infty \left((\partial_r \psi_0)^2 + \frac{k^2}{r^2}\psi_0^2 \right) r\,dr.$$

$\|\psi_0\|_{L^\infty}$ small $\Rightarrow$ $\|(\psi_0, \dot\psi_0)\|_{\mathcal{E}}^2 \simeq E(\psi_0, \dot\psi_0)$.

Finite energy sectors: $\mathcal{E}_{m,n} := \left\{ (\psi_0, \dot\psi_0) : E(\psi_0,\dot\psi_0) < \infty, \lim_{r\to 0} \psi_0(r) = m\pi, \lim_{r\to\infty} \psi_0(r) = n\pi \right\}$.





Theorem (Shatah–Struwe 1994, $k=1$
Burq, Planchon – Stalker – Tahvildar Zadeh 2003, $k \geq 2$)

Equation (WM_k) is locally well-posed in each finite energy sector, in the sense of strong solutions.

Linearisation around $\psi = 0$: $\partial_t^2 \psi_L - \partial_r^2 \psi_L - \frac{1}{r}\partial_r \psi_L + \frac{k^2}{r^2}\psi_L = 0$.

If ψ is small, the nonlinear effects become negligible for large times:

Theorem If $E(\psi_0, \dot\psi_0)$ is small enough, then ψ exists globally and

$$\lim_{t \to \pm\infty} \left\| (\psi(t), \partial_t \psi(t)) - (m\pi + \psi_L^{\pm}(t), \partial_t \psi_L^{\pm}(t)) \right\|_{\mathcal{E}} = 0 \qquad \text{(scattering)}$$

These results are consequences of Strichartz estimates.

$\boxed{\text{Stationary solutions ("solitons" or "bubbles")}}$

Minimisers of E:

* on $\mathcal{E}_{m,m}$ $\rightsquigarrow$ constant functions
* on $\mathcal{E}_{m,m+1}$ $\rightsquigarrow$ $(m\pi + 2\arctan(r^k/\lambda^k), 0)$, $\lambda > 0$
* on $\mathcal{E}_{m,m-1}$ $\rightsquigarrow$ $(m\pi - 2\arctan(r^k/\lambda^k), 0)$, $\lambda > 0$
* on other sectors $\rightsquigarrow$ $\emptyset$

We denote $Q(r) := 2\arctan(r^k)$, $Q_\lambda := Q(r/\lambda)$ for $\lambda > 0$.

- Key role in the description of the dynamics of large solutions.

<u>**Remark**</u> In the non-equivariant case, the stationary solutions are called <u>harmonic maps</u> $\mathbb{R}^2 \longrightarrow S^2$ and correspond to meromorphic functions and its conjugates (Eells – Wood, 1975).

Absence of self-similarity

Theorem (Christodoulou, Tahvildar-Zadeh, Shatah 1992)

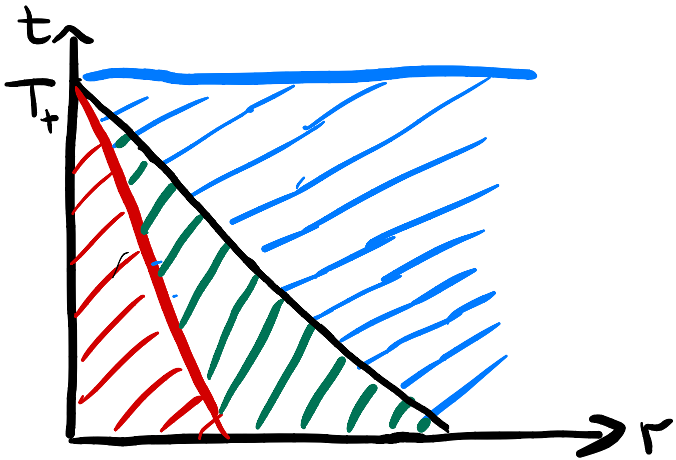
If $T_+ < \infty$ the max. time of existence of ψ, then for every $\alpha > 0$

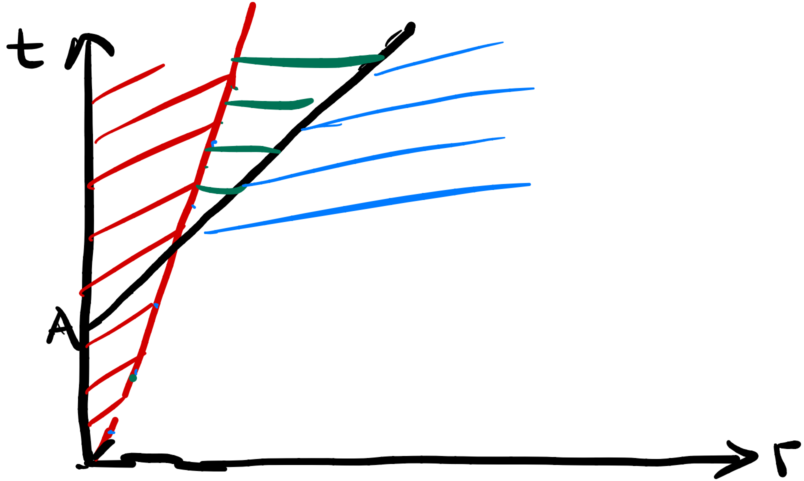
$$\lim_{t \to T} \int_{\alpha(T_+ - t)}^{T_+ - t} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + \frac{k^2}{r^2} \psi^2 \right] r\, dr = 0.$$

Theorem (Côte, Kenig, Lawrie, Schlag 2015)

If $T_+ = \infty$ the max. time of existence of ψ, then for every $\alpha > 0$

$$\lim_{A \to \infty} \limsup_{t \to \infty} \int_{\alpha t}^{t - A} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + \frac{k^2}{r^2} \psi^2 \right] r\, dr = 0.$$





(The energy in the green region $\longrightarrow 0$ as $t \to T_+$.)

<u>**Theorem**</u> (Struwe 2003)

If $T_+ < \infty$, then there exist $t_n \to T_+$, $m \in \mathbb{Z}$, $\iota \in \{-1, 1\}$ and $0 < \lambda_n \ll T_+ - t_n$ such that for all $R > 0$

$$\lim_{n\to\infty} \left\| \left(\psi(t_n, \lambda_n \cdot), \lambda_n \partial_t \psi(t_n, \lambda_n \cdot) \right) - \left(m\pi + \iota Q, 0 \right) \right\|_{\mathcal{E}(r \leq R)} = 0.$$

Sequential soliton resolution

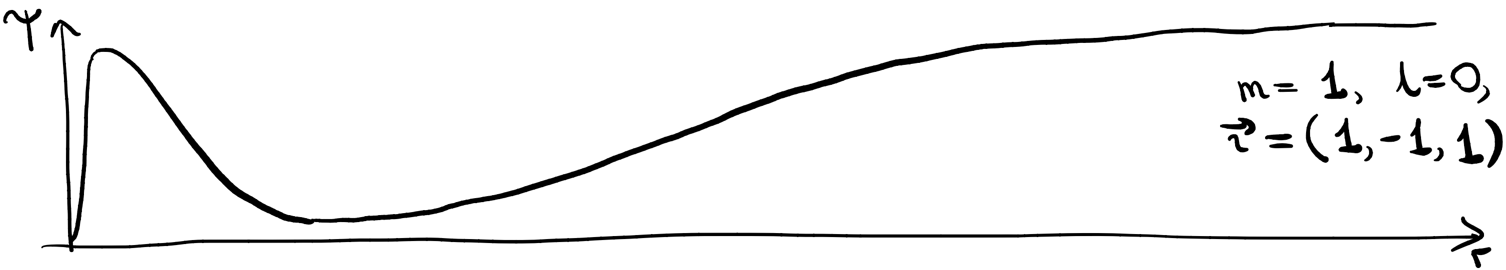
Given $M \in \mathbb{N}$, $m \in \mathbb{Z}$, $\vec{\iota} = (\iota_1, \dots, \iota_M) \in \{-1, 1\}^M$, $\vec{\lambda} = (\lambda_1 \dots \lambda_M) \in (0, \infty)^M$, $\lambda_1 < \lambda_2 < \dots < \lambda_M$,

we denote

$$Q(m, \vec{\iota}, \vec{\lambda}; r) := m\pi + \sum_{j=1}^{M} \iota_j \left(Q_{\lambda_j} - \pi \right) \in \mathcal{E}_{\ell, m}$$

with $\ell = m - \sum_{j=1}^{M} \iota_j$.

If $\lambda_1 \ll \lambda_2 \ll \dots \ll \lambda_M$, we call $Q(m, \vec{\iota}, \vec{\lambda})$ a <u>multi-bubble configuration</u>.



Sequential soliton resolution

Theorem (Côte 2015, Jia–Kenig 2017)

① If $T_+<\infty$, then there exist $(\psi_*,\dot\psi_*)$, M, m, $\vec{\iota}$, $\vec{\lambda}_n$, $t_n\to T_+$:

$$\lim_{n\to\infty}\Big[\|\psi(t_n)-\psi_*-Q(m,\vec{\iota},\vec{\lambda}_n)\|_{\mathcal{H}}^2+\|\partial_t\psi(t_n)-\dot\psi_*\|_{L^2}^2$$
$$+\sum_{j=1}^{M-1}\left(\lambda_{n,j}/\lambda_{n,j+1}\right)^k+\left(\lambda_{n,M}/(T_+-t_n)\right)^k\Big]=0.$$

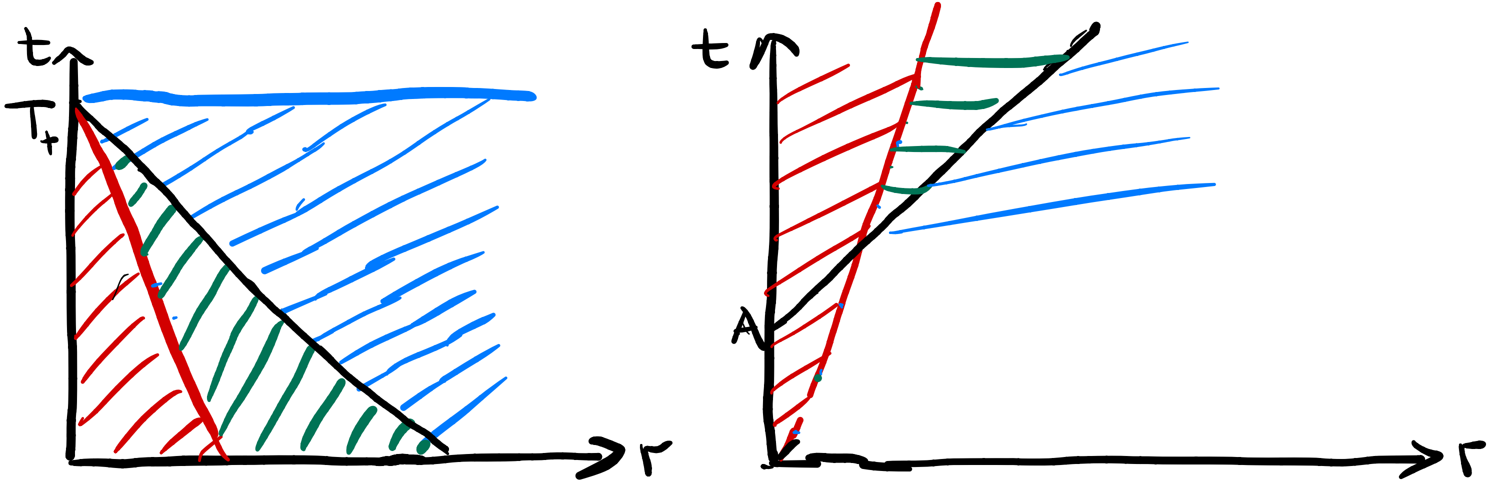
② If $T_+=\infty$, then there exist $(\psi_L,\partial_t\psi_L)$, M, m, $\vec{\iota}$, $\vec{\lambda}_n$, $t_n\to\infty$:

$$\lim_{n\to\infty}\Big[\|\psi(t_n)-\psi_L(t_n)-Q(m,\vec{\iota},\vec{\lambda}_n)\|_{\mathcal{H}}^2+\|\partial_t\psi(t_n)-\partial_t\psi_L(t_n)\|_{L^2}^2$$
$$+\sum_{j=1}^{M-1}\left(\lambda_{j,n}/\lambda_{j+1,n}\right)^k+\left(\lambda_{M,n}/t_n\right)^k\Big]=0.$$

Remark Convergence in continuous time outside of cones:

① $\forall\alpha>0$: $\displaystyle\lim_{t\to T_+}\left(\|\psi(t)-\psi_*\|^2_{\mathcal{H}(r\geq\alpha(T_+-t))}+\|\partial_t\psi(t)-\dot\psi_*\|^2_{L^2(r\geq\alpha(T_+-t))}\right)=0.$

② $\forall\alpha>0$: $\displaystyle\lim_{t\to\infty}\left(\|\psi(t)-\psi_L(t)\|^2_{\mathcal{H}(r\geq\alpha t)}+\|\partial_t\psi(t)-\partial_t\psi_L(t)\|^2_{L^2(r\geq\alpha t)}\right)=0.$



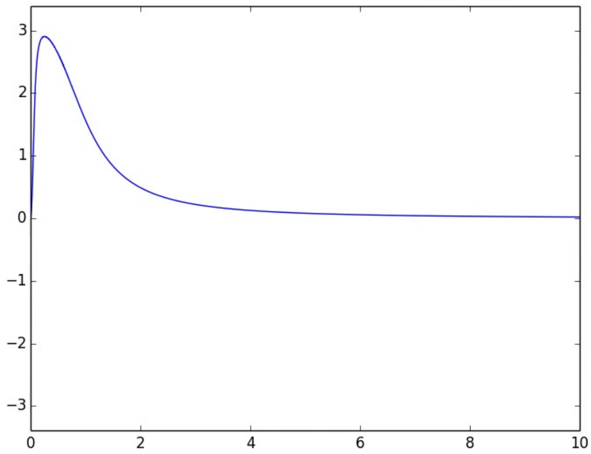
$E \longrightarrow ME(Q)$

$E \longrightarrow 0$

$E \longrightarrow E(\psi_*, \dot\psi_*) \qquad (\text{resp. } E_L(\psi_L, \partial_t \psi_L))$

Question : Does the decomposition hold in continuous time?

Ennemy : Collisions in the red region.
(Excluded if all bubbles have the same sign.)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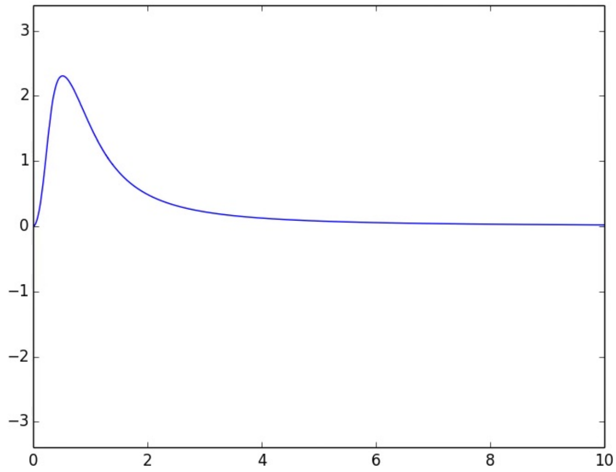
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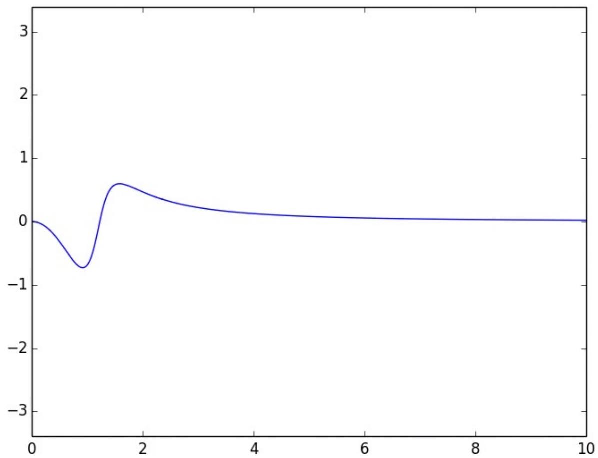
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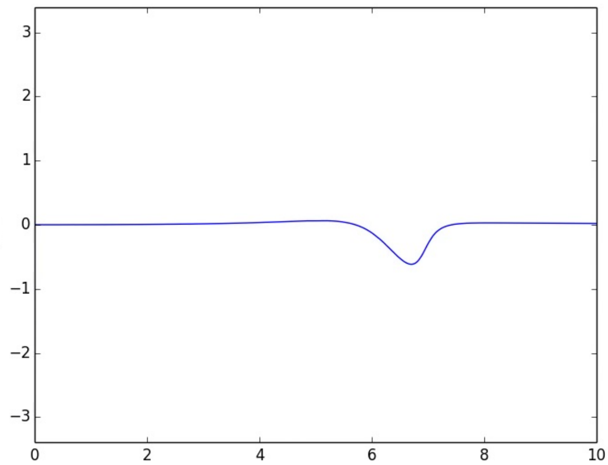
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Continuous time soliton resolution

Theorem (J.–Lawrie 2021)

① If $T_+ < \infty$, then there exists $\vec{\lambda} : [0, T_+) \to (0,\infty)^M$ such that

$$\lim_{t \to T_+} \Big[\| \psi(t) - \psi_* - Q(m, \vec{\iota}, \vec{\lambda}(t)) \|_H^2 + \| \partial_t \psi(t) - \dot{\psi}_* \|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\lambda_j(t) / \lambda_{j+1}(t) \right)^k + \left(\lambda_M(t) / (T_+ - t) \right)^k \Big] = 0.$$

② If $T_+ = \infty$, then there exists $\vec{\lambda} : [0, \infty) \to (0,\infty)^M$ such that

$$\lim_{t \to \infty} \Big[\| \psi(t) - \psi_L(t) - Q(m, \vec{\iota}, \vec{\lambda}(t)) \|_H^2 + \| \partial_t \psi(t) - \partial_t \psi_L(t) \|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\lambda_j(t) / \lambda_{j+1}(t) \right)^k + \left(\lambda_M(t) / t \right)^k \Big] = 0.$$

Some related results

* Duyckaerts, Kenig, Merle 2010 — sequential soliton resolution for $\partial_t^2 u - \Delta_x u - u^5 = 0$ in 3D.
* Duyckaerts, Kenig, Merle 2012 — continuous time; energy channels
* J – Lawrie 2017 – classification for $E \leqslant 2E(Q)$, $k \geqslant 2$; crucially used estimates of interactions between solitons (reduction to a "2-body problem")
* Rodriguez 2019 ——— " ——— $k=1$
* D, K, M 2019 : generalisation of 2012 result to all odd dim.
* Duyckaerts, Kenig, Martel, Merle 2021 — settled $k=1$ using energy channels

Existence of bubbles and multi-bubbles

The answer depends on the equivariance class:

* $k=1,2,3,\dots$: blow-up solutions with $M=1$ bubble
  (Krieger–Schlag–Tataru '08, Rodnianski–Sterbenz 2010
  Raphaël–Rodnianski 2011)

  blow-up with $M\geq 2$ unknown

* many other constructions for $M=1$ exist.

* $k=2,3,\dots$: existence, uniqueness and asymptotic description
  of pure two-bubble solutions (J'16, J-Lawrie 17-20)

* $k=1$: no solutions with $M\geq 2$ are known;
  no pure two-bubbles exist (Rodriguez '18)
  no pure multi-bubbles exist (J.-Lawrie, unpublished)

Main ideas of the proof of soliton resolution

The desired decomposition holds for a time sequence.

We need to prevent the following scenario:

$$\underbrace{\text{close to multi-bubble configuration} \xrightarrow{\text{"collision"}} \text{far from multi-bubble configuration} \rightsquigarrow \text{close to multi-bubble configuration}}_{\text{"collision (time) interval"}}$$

occuring an infinite number of times (a <u>no-return lemma</u>).

* Inspired by Duyckaerts-Merle, Nakanishi-Schlag, Krieger-Nakanishi-Schlag for <u>single</u> soliton which is linearly unstable.

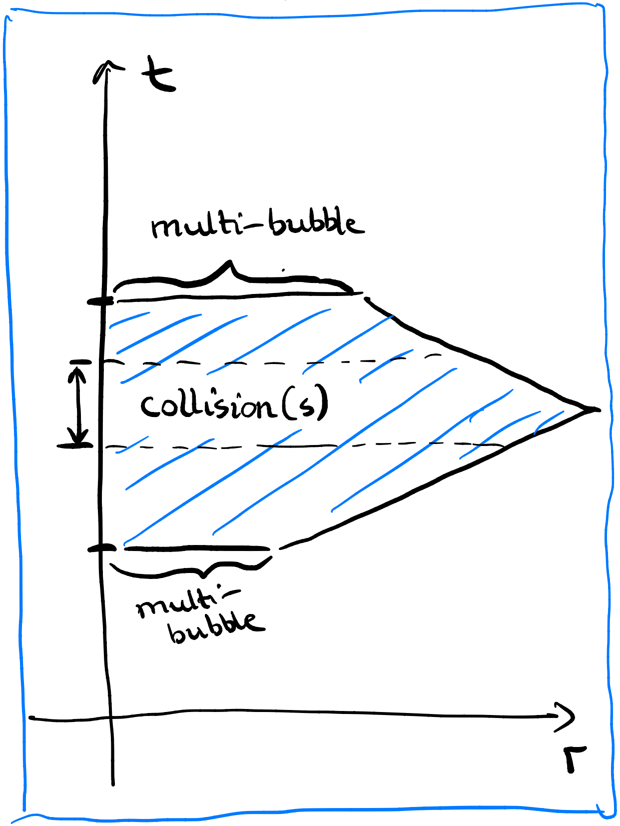
* Here, inter-soliton interactions play a similar role as linear instability in those works (cf. J.-Lawrie '17)

$\boxed{\text{Virial identity}}$

If ψ is a smooth wave map, then

$$\operatorname{div}_{t,r}\left(\partial_t\psi\, r^2\partial_r\psi,\ -\tfrac{1}{2}r^2(\partial_t\psi)^2-\tfrac{1}{2}r^2(\partial_r\psi)^2+\tfrac{k^2}{2}\sin^2\psi\right)=-r(\partial_t\psi)^2$$

We estimate the boundary terms from above and the space-time integral $\iint(\partial_t\psi)^2\, r\,dr\,dt$ from below



* collision duration $\gtrsim$ spatial scale
* "horizontal" boundary $\ll$ spatial scale
* $\iint(\partial_t\psi)^2\, r\,dr\,dt \gtrsim$ collision duration ("Compactness Lemma")
* remaining boundary : reduction to n-body

Collision intervals

Let ψ a wave map, $t\in[0,T_+)$, $0<\rho<\infty$, $0\leq K\leq N$. We set

$$d_K(t;\rho) := \inf_{\vec{\iota},\vec{\lambda}} \Big(\|\partial_t \psi(t) - \dot{\psi}_0^*\|^2_{L^2(\geq\rho)} + \|\psi(t) - \psi_0^* - Q(m,\vec{\iota},\vec{\lambda})\|^2_{\mathcal{E}(\geq\rho)} + \sum_{j=K}^{N} \big(\lambda_j/\lambda_{j+1}\big)^k \Big)^{1/2}$$

where $\lambda_K := \rho$ and $\lambda_{N+1} := \begin{cases} T_+ - t & \text{blow-up case} \\ t & \text{global case} \end{cases}$

(Proximity to a multi-bubble in the exterior region $r\geq\rho$).

We set $d(t) := d_0(t;0)$.

We know that $\lim\limits_{n\to\infty} d(t_n) = 0$, and want to prove $\lim\limits_{t\to T_+} d(t) = 0$.

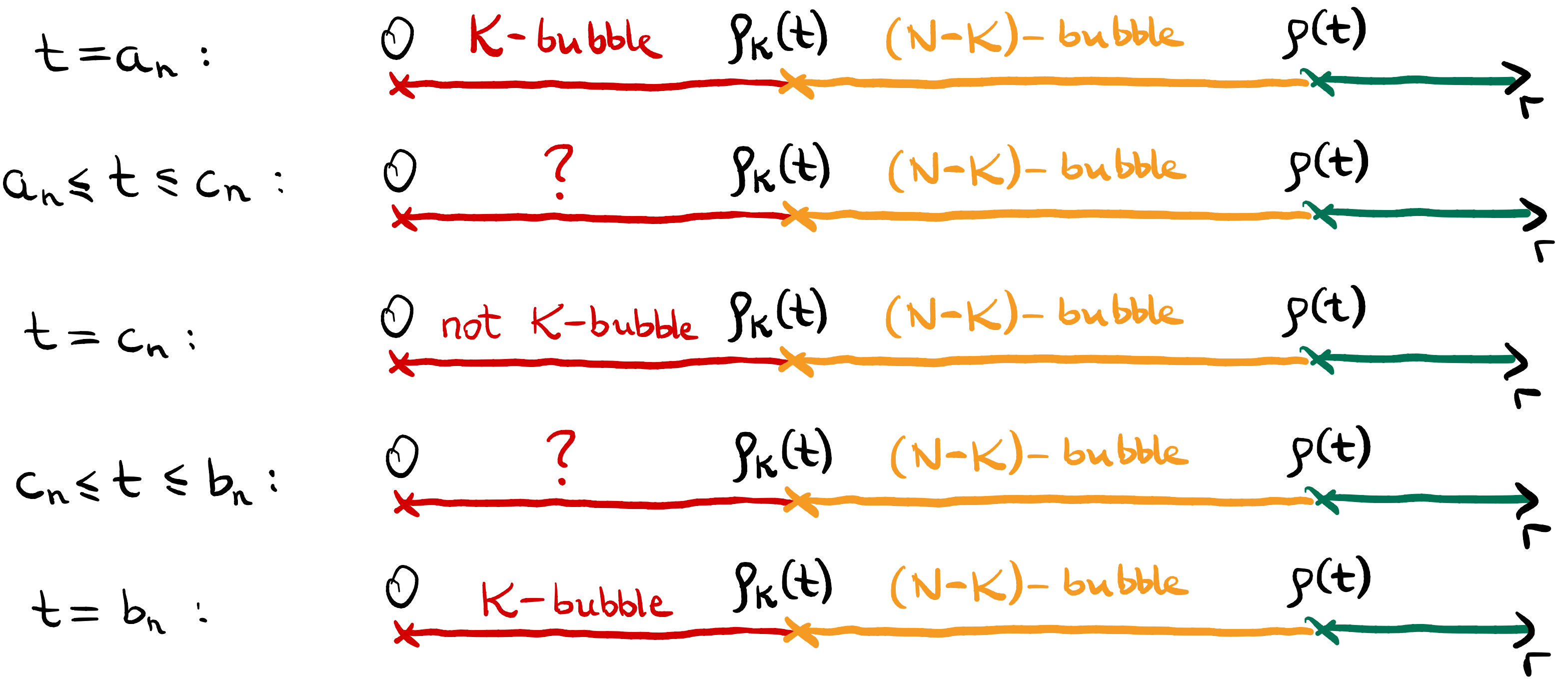
Def. Let $0\leq K\leq N$, $0<\varepsilon<\eta$. We say $[a,b]\subset(0,T_+)$ is a collision interval with parameters ε,η,K if

* $d(a)\leq\varepsilon$, $d(b)\leq\varepsilon$, $\exists\, c\in[a,b]$ such that $d(c)\geq\eta$,
* $\exists\, \rho_K : [a,b]\to(0,\infty)$: $d_K(t;\rho_K(t))\leq\varepsilon \quad \forall t\in[a,b]$.

Interior and exterior bubbles

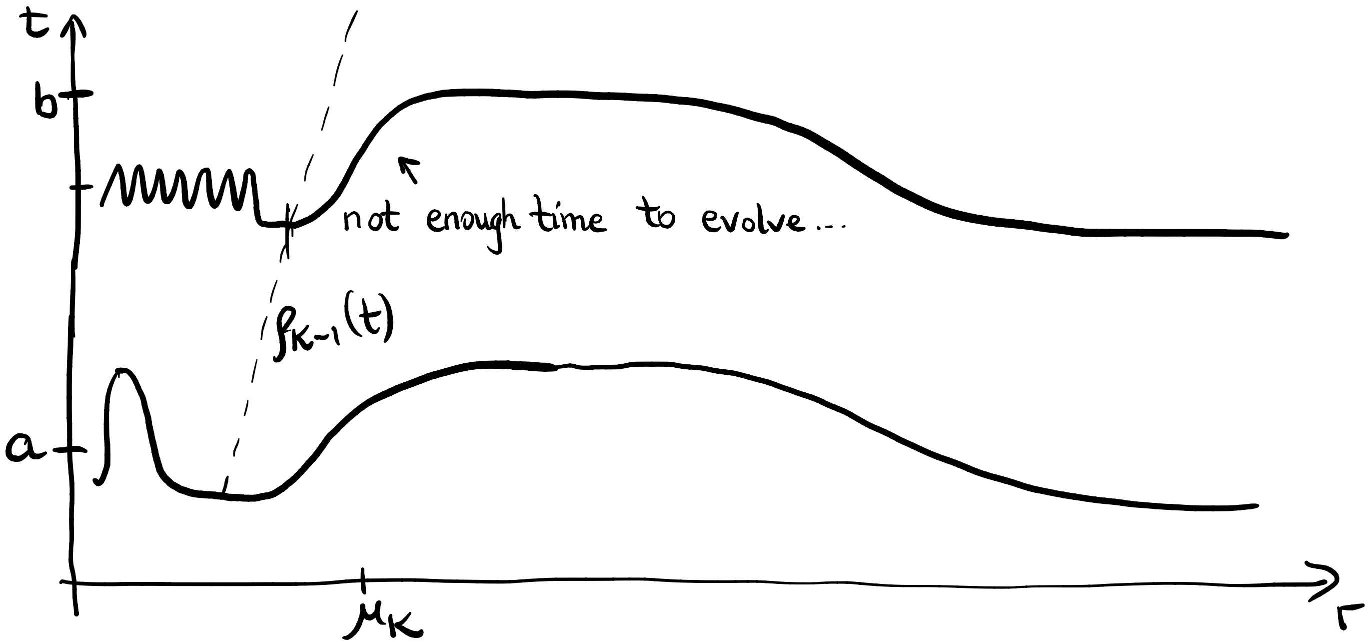
Let K be <u>minimal</u> such that $\exists\, \eta > 0$, $\varepsilon_n \to 0$ and collision intervals $[a_n, b_n]$ with parameters ε_n, η, K.

$$d(t) \not\to 0 \implies K \text{ is well-defined, } 1 \le K \le N.$$



Lemma $\exists\, C = C(\psi) > 0$ and $\varepsilon > 0$ such that if $[a,b]$ collision interval with parameters (ε, η, K), then $b - a \geq C \min(\mu_K(a), \mu_K(b))$, where μ_K is the scale of the K-th bubble.

Proof If not, K would not be smallest possible.



Wave maps with small kinetic energy

Lemma ("Compactness Lemma") Let $\rho_n > 0$, $R_n \to \infty$, ψ_n defined for $t \in [0, \rho_n]$ of bounded energy such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} \int_0^{\rho_n} \int_0^{R_n \rho_n} (\partial_t \psi_n)^2 \, r \, dr \, dt = 0 .$$

Then, up to extraction of a subsequence, there exist $r_n \to \infty$, $t_n \in [0, \rho_n]$, $M, m, \vec{\iota}, \vec{\lambda}_n$ such that

$$\lim_{n\to\infty} \Big[\| \psi_n(t_n) - Q(m, \vec{\iota}, \vec{\lambda}_n) \|^2_{\mathcal{H}(r \le r_n \rho_n)} + \|\partial_t \psi_n(t_n)\|^2_{L^2(r \le r_n \rho_n)} + \sum_{j=1}^{M-1} \left({\lambda_{n,j}}/{\lambda_{n,j+1}} \right)^k + \left({\lambda_{n,M}}/{r_n \rho_n} \right)^k \Big] = 0$$

Proof We follow Jia-Kenig.
Main idea: Bahouri-Gérard profile decomposition.

Modulation

* Near a_n and b_n, ψ is close to a multi-bubble and the analysis above does not apply.
* In this case, the main dynamical information are the scales of the bubbles
* We obtain differential inequalities on these scales.

Informally: $$\lambda_j'' \simeq -\iota_j \iota_{j+1} \omega^2 \frac{\lambda_j^{k-1}}{\lambda_{j+1}^{k}} + \iota_j \iota_{j-1} \omega^2 \frac{\lambda_{j-1}^{k}}{\lambda_j^{k+1}}.$$

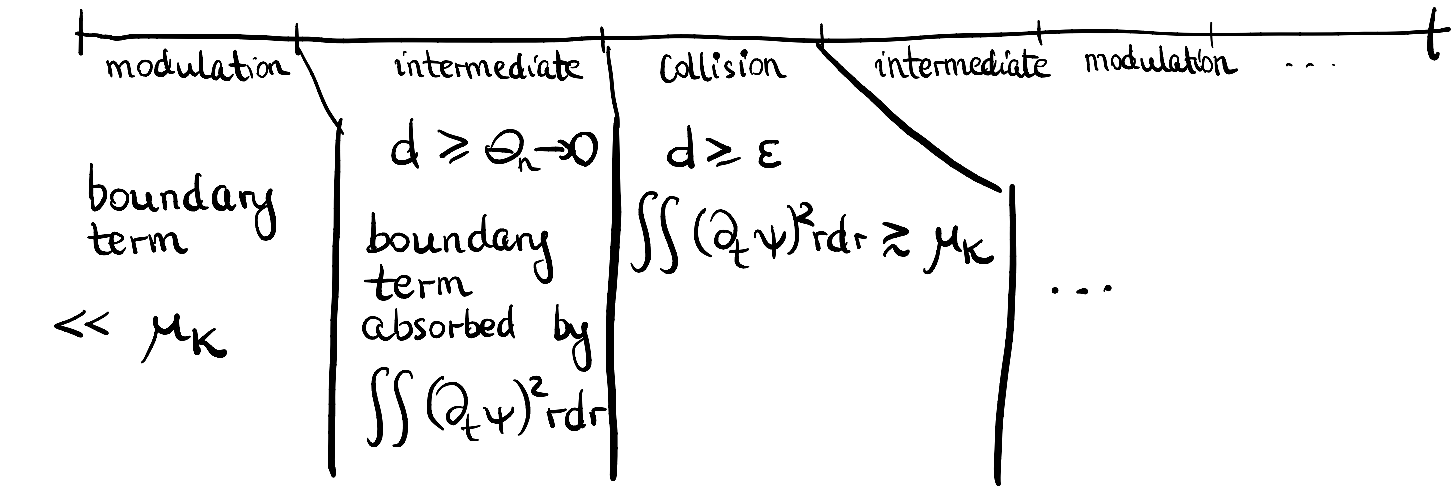
* Error bounded by the energy of attractive interactions.
* The influence of the exterior bubbles and radiation can essentially be neglected by enlarging ε_n.
* Refined modulation parameters: Raphaël-Szeftel '11, J.-Lawrie '17.

Lemma If d starts growing at t_0, then $\forall\, t^* \geq t_0$

$$\int_{t_0}^{t_*} d(t)\,dt \leq C_0\, d(t_*)^{2/k} \mu_k(t_0) \quad \text{if } k \geq 2$$

$$\int_{t_0}^{t_*} d(t)\,dt \leq C_0\, d(t_*)^2 \sqrt{-\log d(t_*)}\, \mu_k(t_0) \quad \text{if } k=1.$$

The final step is to partition $[a_n, b_n]$ into



In the situation below, $\mu_K \simeq$ const: