Soliton resolution for equivariant wave maps

Jacek Jendrej
(CNRS & Université Sorbonne Paris Nord)
joint work with Andrew Lawrie
(Massachusetts Institute of Technology)

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Energy-critical wave maps equation

"Definition" An application $\Psi : \mathbb{R}^{1+2} \to S^2 \subset \mathbb{R}^3$ is a wave map if it is a critical point of the Lagrangian

$$\mathcal{L}(\Psi) := \frac{1}{2} \iint \left(|\partial_t \Psi|^2 - |\partial_x \Psi|^2 \right) dx\, dt .$$

Natural analogs of linear waves in a nonlinear, geometric setting.

Euler-Lagrange equation:

$$\partial_t^2 \Psi(t,x) - \Delta \Psi(t,x) = -\left(|\partial_t \Psi(t,x)|^2 - |\nabla \Psi(t,x)|^2 \right) \Psi(t,x).$$

Local well-posedness and global well-posedness for small data:

Klainerman, Selberg, Sterbenz, Tataru, Tao, ... (1993-2002)

$\boxed{\text{Equivariant wave maps}}$

We study the dynamics (long time behavior) of large solutions, but only in a special case:

$$\Psi(t, r\cos\theta, r\sin\theta) = \big(\sin\psi(t,r)\cos(k\theta), \sin\psi(t,r)\sin(k\theta), \cos\psi(t,r)\big).$$

Here, $k \in \{1, 2, \dots\}$ is the equivariance degree, $t \in \mathbb{R}$, $r \in (0, \infty)$

Equation for ψ:

$$\boxed{(\mathrm{WM}_k) \quad \partial_t^2 \psi(t,r) - \partial_r^2 \psi(t,r) - \frac{1}{r}\partial_r \psi(t,r) + \frac{k^2}{2r^2}\sin(2\psi(t,r)) = 0.}$$

Lagrangian: $\displaystyle \mathcal{L} := \pi \iint \Big((\partial_t \psi)^2 - (\partial_r \psi)^2 - \frac{k^2 \sin^2 \psi}{r^2} \Big) r\,dr\,dt$

Energy: $\displaystyle E(\psi_0, \dot\psi_0) := \pi \int_0^\infty \Big(\underbrace{\dot\psi_0^{\,2}}_{\text{kinetic}} + \underbrace{(\partial_r \psi_0)^2 + \frac{k^2 \sin^2 \psi_0}{r^2}}_{\text{potential}} \Big) r\,dr$

Scaling invariance and criticality

If ψ solves (WM_k) and $\lambda > 0$, then

$$\psi_\lambda(t,r) := \psi\left(\tfrac{t}{\lambda}, \tfrac{r}{\lambda}\right) \text{ solves } (WM_k) \text{ as well.}$$

Moreover, $E(\psi_\lambda, \dot\psi_\lambda) = E(\psi, \dot\psi)$ $\rightsquigarrow$ energy-critical problem

Note: if $\lambda \ll 1$, then ψ_λ is concentrated and evolves fast.

Local theory, small data theory

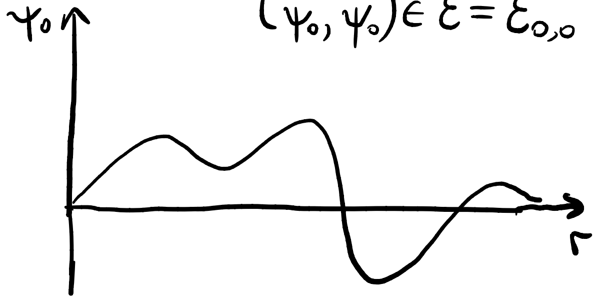
Energy norm: $\|(\psi_0, \dot\psi_0)\|_{\mathcal{E}}^2 := \|\dot\psi_0\|_{L^2}^2 + \|\psi_0\|_{\mathcal{H}}^2$,

$$\|\dot\psi_0\|_{L^2}^2 := \int_0^\infty \dot\psi_0^2 \, r\,dr, \qquad \|\psi_0\|_{\mathcal{H}}^2 := \int_0^\infty \left((\partial_r \psi_0)^2 + \frac{k^2}{r^2}\psi_0^2 \right) r\,dr.$$

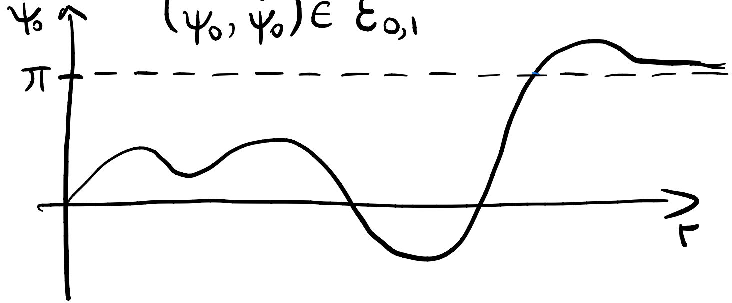
$$\|\psi_0\|_{L^\infty} \text{ small} \quad \Rightarrow \quad \|(\psi_0, \dot\psi_0)\|_{\mathcal{E}}^2 \simeq E(\psi_0, \dot\psi_0).$$

Finite energy sectors: $\mathcal{E}_{m,n} := \left\{ (\psi_0, \dot\psi_0) : E(\psi_0, \dot\psi_0) < \infty, \ \lim\limits_{r \to 0} \psi_0(r) = m\pi, \ \lim\limits_{r \to \infty} \psi_0(r) = n\pi \right\}$.

$(\psi_0, \dot\psi_0) \in \mathcal{E} = \mathcal{E}_{0,0}$



$(\psi_0, \dot\psi_0) \in \mathcal{E}_{0,1}$



Theorem (Shatah–Struwe 1994, $k=1$
Planchon–Stalker–Tahvildar Zadeh 2003, $k \geq 2$)

Equation (WM_k) is locally well-posed in each finite energy sector, in the sense of strong solutions.

Linearisation around $\psi = 0$: $\partial_t^2 \psi_L - \partial_r^2 \psi_L - \frac{1}{r}\partial_r \psi_L + \frac{k^2}{r^2}\psi_L = 0$.

If ψ is small, the nonlinear effects become negligible for large times:

Theorem If $E(\psi_0, \dot\psi_0)$ is small enough, then ψ exists globally and

$$\lim_{t \to \pm\infty} \left\| (\psi(t), \partial_t \psi(t)) - (m\pi + \psi_L^{\pm}(t), \partial_t \psi_L^{\pm}(t)) \right\|_{\mathcal{E}} = 0 \qquad \text{(scattering)}$$

These results are consequences of Strichartz estimates.

Stationary solutions (solitons)

Minimisers of E:

* on $\mathcal{E}_{m,m}$ $\rightsquigarrow$ constant functions
* on $\mathcal{E}_{m,m+1}$ $\rightsquigarrow$ $\left(m\pi + 2\arctan\left(r^k/\lambda^k\right), 0\right)$, $\lambda>0$
* on $\mathcal{E}_{m,m-1}$ $\rightsquigarrow$ $\left(m\pi - 2\arctan\left(r^k/\lambda^k\right), 0\right)$, $\lambda>0$
* on other sectors $\rightsquigarrow$ $\emptyset$

We denote $Q(r) := 2\arctan(r^k)$, $Q_\lambda := Q(r/\lambda)$ for $\lambda>0$.

- Key role in the description of the dynamics of large solutions.

Sequential soliton resolution

Theorem (Côte–Kenig–Lawrie–Schlag 12, Côte 13, Jia–Kenig 15)

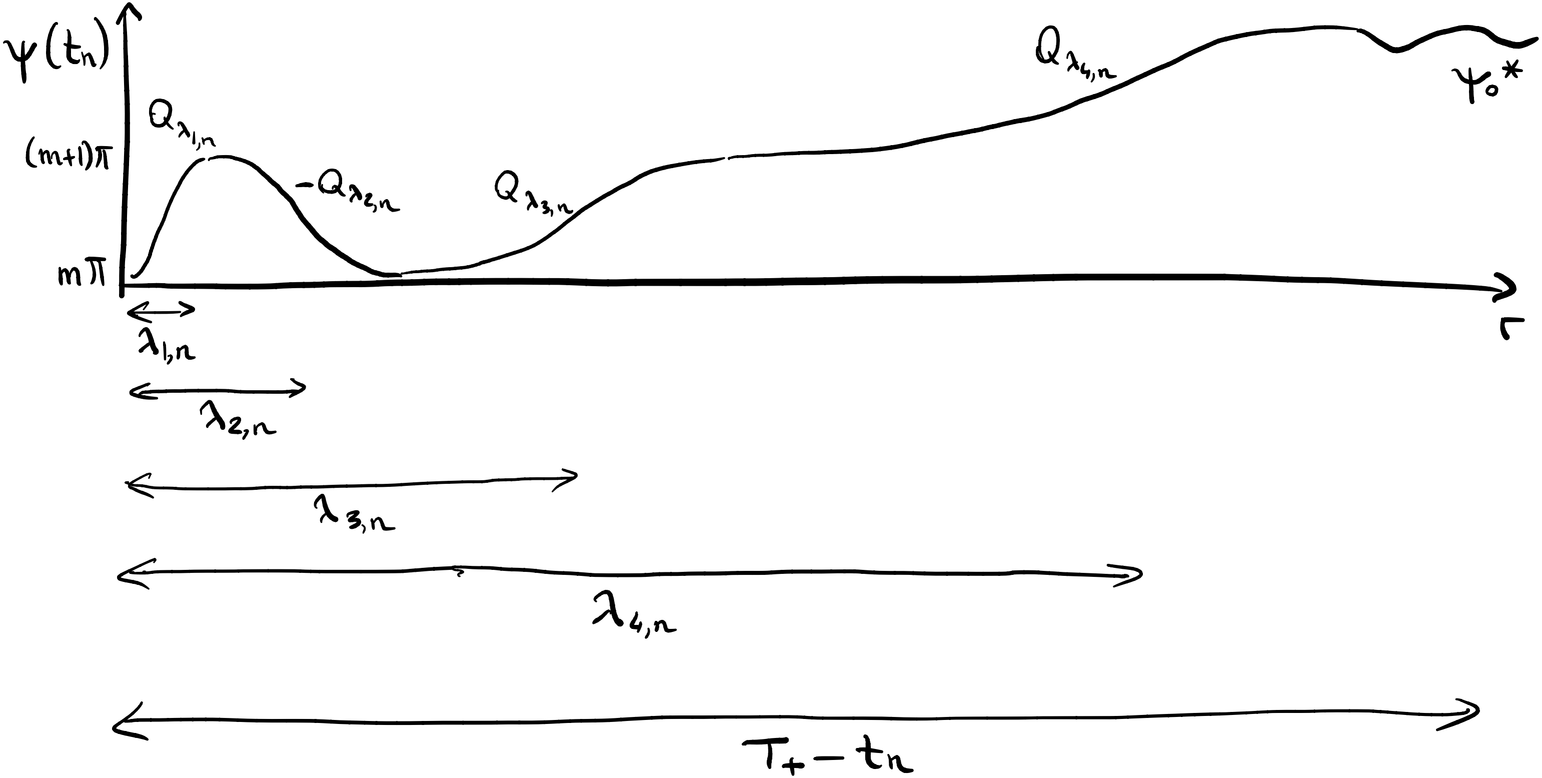
Let ψ a finite energy wave map for $k \in \{1,2\}$.

1) If $T_+ < \infty$, then there exist $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$, $t_n \to T_+$, $0 < \lambda_{1,n} \ll \lambda_{2,n} \ll \dots \ll \lambda_{N,n} \ll T_+ - t_n$, $\iota_1, \iota_2, \dots, \iota_N \in \{-1,1\}$ and $(\psi_0^*, \dot\psi_0^*) \in \mathcal{E}_{0,l}$ such that

$$\lim_{n\to\infty} \Big\| (\psi(t_n), \dot\psi(t_n)) - \Big(m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_{j,n}} + \psi_0^*, \dot\psi_0^*\Big)\Big\|_{\mathcal{E}} = 0.$$

2) If $T_+ = \infty$, then there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$, $t_n \to \infty$, $0 < \lambda_{1,n} \ll \lambda_{2,n} \ll \dots \ll \lambda_{N,n} \ll t_n$, $\iota_1, \iota_2, \dots, \iota_N \in \{-1,1\}$ and ψ_L solution of the linear wave equation such that

$$\lim_{n\to\infty} \Big\| (\psi(t_n), \dot\psi(t_n)) - \Big(m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_{j,n}} + \psi_L(t_n), \dot\psi_L(t_n)\Big)\Big\|_{\mathcal{E}} = 0.$$

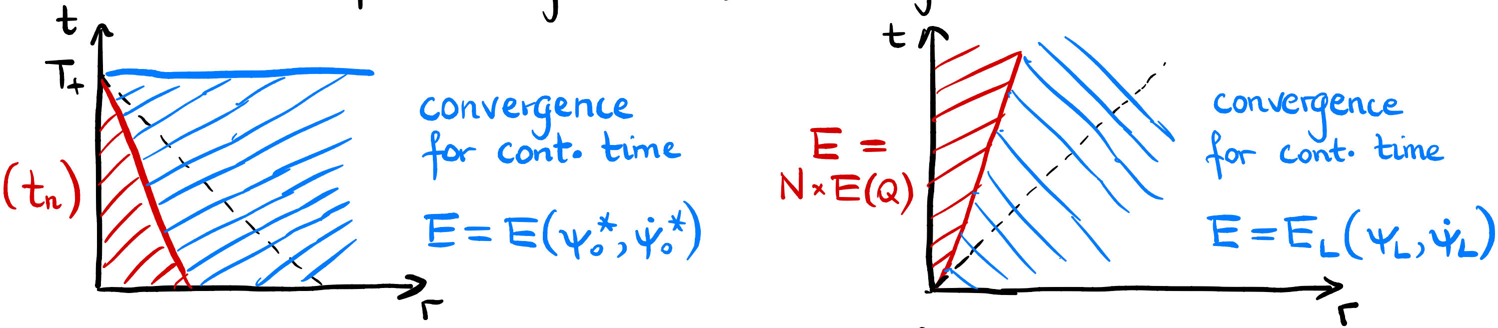


Corollary (CKLS)

If $(\psi, \dot{\psi}) \in \mathcal{E}$ and $E(\psi, \dot{\psi}) < 8k\pi$, then ψ scatters.

Remarks

* In the blow-up case, fundamental earlier results of Christodoulou, Shatah–Tahvildar Zadeh and Struwe
* Outside of the light cone, convergence in continuous time



* Generalizes to any $k \in \{1, 2, \dots\}$.

Proposition (Côte)

If, in the above result, $N = 1$, or $\iota_1 = \iota_2 = \dots = \iota_N$, then the decomposition holds for continuous time.

Do these decompositions hold in continuous time unconditionally?

Soliton Resolution Conjecture

<u>Theorem</u> (Duyckaerts – Kenig – Martel – Merle, March 2021)

Let ψ a finite energy wave map for $k=1$.

1) If $T_+ < \infty$, then there exist $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$,

$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll T_+ - t$, $\iota_1, \iota_2, \dots, \iota_N \in \{-1, 1\}$

and $(\psi_0^*, \dot\psi_0^*) \in \mathcal{E}_{0,l}$ such that

$$\lim_{t \to T_+} \left\| (\psi(t), \dot\psi(t)) - \left(m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_j(t)} + \psi_0^*, \dot\psi_0^*\right) \right\|_{\mathcal{E}} = 0.$$

2) If $T_+ = \infty$, then there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$,

$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll t$, $\iota_1, \iota_2, \dots, \iota_N \in \{-1, 1\}$

and ψ_L solution of the linear wave equation such that

$$\lim_{t \to \infty} \left\| (\psi(t), \dot\psi(t)) - \left(m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_j(t)} + \psi_L(t), \dot\psi_L(t)\right) \right\|_{\mathcal{E}} = 0.$$

Remarks

* The first result of this type for a model which is not completely integrable was obtained by Duyckaerts, Kenig and Merle in 2012 for the energy-critical focusing wave equation with the power nonlinearity, introducing the method of energy channels.
* Generalized to all odd dimension in 2019 by the same authors
* The proof of the last theorem : energy channels

Our contribution is to prove, using rather different ideas, that the last result holds for any $k \in \{1, 2, 3, \dots\}$

Soliton Resolution Conjecture

Theorem (J. – Lawrie, June 2021)

Let ψ a finite energy wave map for $k \in \{1, 2, 3, \ldots\}$

1) If $T_+ < \infty$, then there exist $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$,

$0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_N(t) \ll T_+ - t$, $\iota_1, \iota_2, \ldots, \iota_N \in \{-1, 1\}$

and $(\psi_0^*, \dot{\psi}_0^*) \in \mathcal{E}_{0,l}$ such that

$$\lim_{t \to T_+} \left\| (\psi(t), \dot{\psi}(t)) - \left(m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_j(t)} + \psi_0^*, \dot{\psi}_0^*\right) \right\|_{\mathcal{E}} = 0.$$

2) If $T_+ = \infty$, then there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$,

$0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_N(t) \ll t$, $\iota_1, \iota_2, \ldots, \iota_N \in \{-1, 1\}$

and ψ_L solution of the linear wave equation such that

$$\lim_{t \to \infty} \left\| (\psi(t), \dot{\psi}(t)) - \left(m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_j(t)} + \psi_L(t), \dot{\psi}_L(t)\right) \right\|_{\mathcal{E}} = 0.$$

Remarks

* The radial critical Yang–Mills equation is closely related to the 2-equivariant wave maps equation and could also be treated by our method.
* No solutions with $N > 1$ have been constructed for $k = 1$.
  For $k \geq 2$, J. and J.–Lawrie constructed and studied the "two-bubble solutions".
  Prior constructions include Krieger–Schlag–Tataru for $k \in \{1, 2\}$ and Raphaël–Rodnianski (blow-up with $N = 1$).
* The soliton resolution problem is inspired by the theory of completely integrable systems and numerical simulations.

Main ideas of the proof

The desired decomposition holds for a time sequence.

We need to prevent the following scenario:

close to multi-bubble configuration $\rightsquigarrow$ ("collision") far from multi-bubble configuration $\rightsquigarrow$ close to multi-bubble configuration

"collision (time) interval"

We need a <u>no-return lemma</u>.

* Inspired by Duyckaerts–Merle, Nakanishi–Schlag, Krieger–Nakanishi–Schlag for <u>single</u> soliton which is linearly unstable.
* Here, inter-soliton interactions play a similar role as linear instability in those works (cf. J – Lawrie '17)

Virial identity

If ψ is a smooth wave map, then

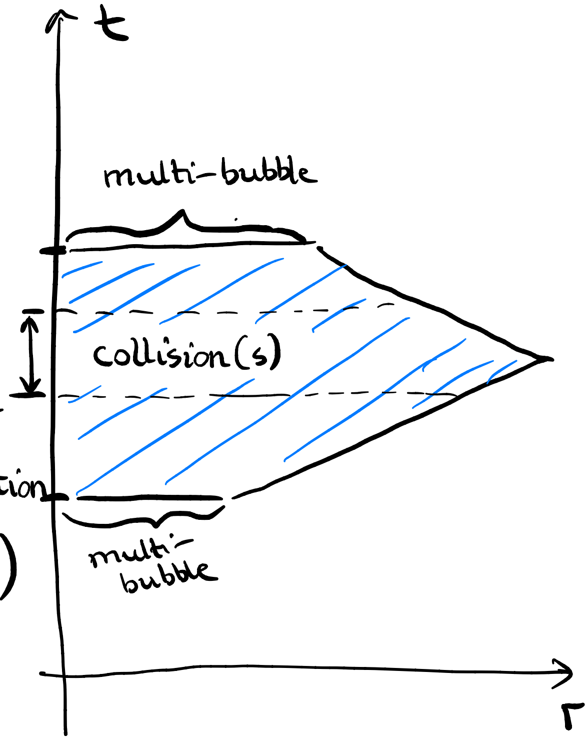
$$\mathrm{div}_{t,r}\left(\partial_t\phi\, r^2\partial_r\phi,\ -\tfrac{1}{2}r^2(\partial_t\phi)^2-\tfrac{1}{2}r^2(\partial_r\phi)^2+\tfrac{k^2}{2}\sin^2\phi\right)=-r(\partial_t\phi)^2$$

We estimate the boundary terms from above and the space-time integral of $r(\partial_t\phi)^2$ from below.

"horizontal" boundary: $\ll$ spatial scale

$\iint(\partial_t\phi)^2\, r\,dr\,dt$: $\gtrsim$ collision duration

("Compactness Lemma")



remaining boundary: ignore for now

collision duration $\gtrsim$ spatial scale ?

Interior and exterior bubbles

Let $K \in \{1, 2, \dots, N\}$, $[a,b] \subset [T_0, T_+)$, $0 < \varepsilon < \eta$.

We say $[a,b]$ is a collision interval with $N-K$ exterior bubbles if:

* $d(a) \leq \varepsilon$, $d(b) \leq \varepsilon$, $\exists\, c \in [a,b] : d(c) \geq \eta$

  where $d(t)$ is the distance of $\psi(t)$ to (multi-bubble + radiation)

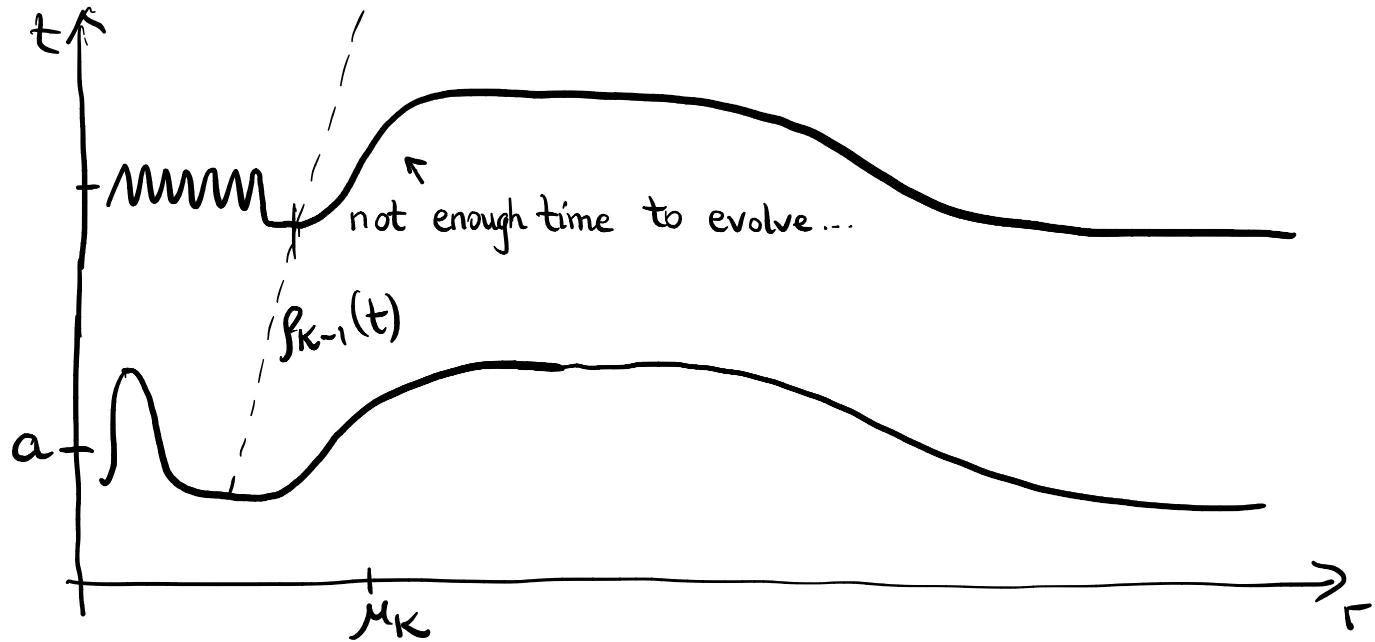
* $\exists$ curve $r = \rho_K(t)$ such that in the region $r \geq \rho_K(t)$

  $\psi(t,r)$ is ε-close to $[(N-K)$-bubble + radiation$]$.

We now set K to be the smallest number such that there exist $\eta > 0$, a sequence $\varepsilon_n \to 0$ and a sequence of collision intervals $[a_n, b_n]$ corresponding to these parameters ε_n, η and K.

Lemma $\exists\, C = C(\psi) > 0$ and $\varepsilon > 0$ such that if $[a,b]$ collision interval with parameters (ε, η, K), then $b - a \geq C \min(\mu_K(a), \mu_K(b))$, where μ_K is the scale of the K-th bubble.

Proof If not, K would not be smallest possible.



Compactness Lemma

Let $\rho_n > 0$ and ψ_n a sequence of wave maps of bounded energy, ψ_n defined on the time interval $[0, \rho_n]$.

Suppose $\exists\, R_n \to +\infty$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} \int_0^{R_n \rho_n} (\partial_t \psi_n(t,r))^2 \, r \, dr \, dt = 0.$$

Then, up to passing to a subsequence, $\exists\, t_n \in [0, \rho_n]$ and $1 \ll r_n \leq R_n$ such that $\psi_n(t_n)$ converges to a multi-bubble for $0 \leq r \leq r_n \rho_n$.

<u>Proof</u> Quite related to the existing proofs of sequential soliton resolution (Côte, Jia-Kenig), based on the Bahouri-Gérard <u>profile decomposition</u>.

Modulation

* Near a_n and b_n, ψ is close to a multi-bubble and the analysis above does not apply.
* In this case, the main dynamical information are the scales of the bubbles
* We obtain differential inequalities on these scales.

Informally: $$\lambda_j' \simeq -\iota_j \iota_{j+1} \omega^2 \frac{\lambda_j^{k-1}}{\lambda_{j+1}^k} + \iota_j \iota_{j-1} \omega^2 \frac{\lambda_{j-1}^k}{\lambda_j^{k+1}}.$$

* Error bounded by the energy of attractive interactions.
* The influence of the exterior bubbles and radiation can essentially be neglected by enlarging ε_n.
* Refined modulation parameters: Raphaël-Szeftel '11, J.-Lawrie '17.

"**Lemma**" If d starts growing at t_0, then $\forall\, t^* \geq t_0$

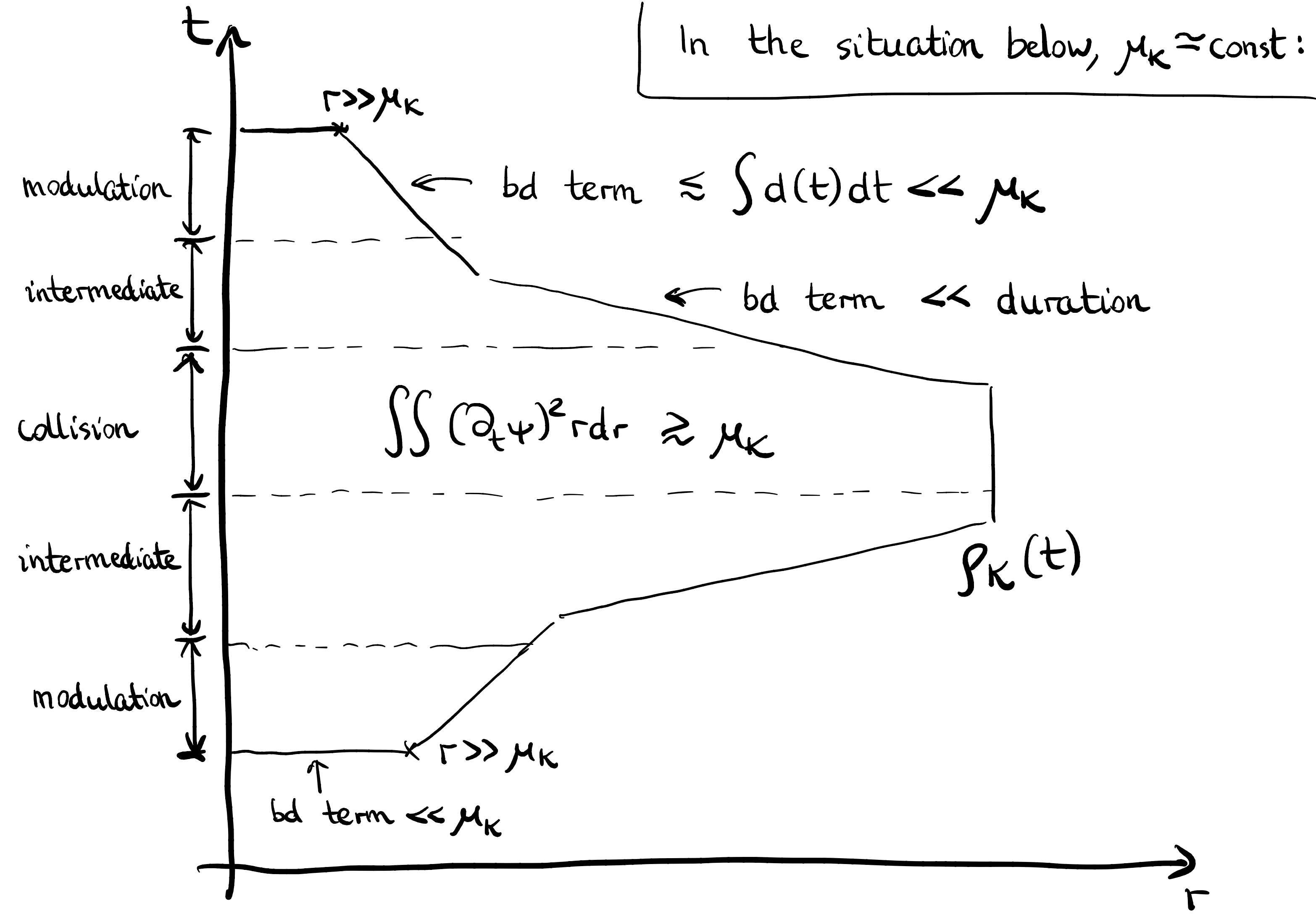
$$\int_{t_0}^{t_*} d(t)\,dt \leq C_0\, d(t_*)^{2/k} \mu_k(t_0) \quad \text{if } k \geq 2$$

$$\int_{t_0}^{t_*} d(t)\,dt \leq C_0\, d(t_*)^{2} \sqrt{-\log d(t_*)}\, \mu_k(t_0) \quad \text{if } k = 1.$$

The final step is to partition $[a_n, b_n]$ into

modulation	intermediate	collision	intermediate	modulation	...

boundary term $\ll \mu_K$	$d \geq \theta_n \to 0$ boundary term absorbed by $\iint (\partial_t \psi)^2 r\,dr$	$d \geq \varepsilon$ $\iint (\partial_t \psi)^2 r\,dr \gtrsim \mu_K$	...


In the situation below, $\mu_K \simeq$ const:
t
$r \gg \mu_K$
modulation
bd term $\lesssim \int d(t)\,dt \ll \mu_K$
intermediate
bd term $\ll$ duration
collision
$\iint (\partial_t \psi)^2 r\,dr \gtrsim \mu_K$
intermediate
$\rho_K(t)$
modulation
$r \gg \mu_K$
bd term $\ll \mu_K$
r