Bubble decomposition for the harmonic map heat flow in the equivariant case

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Energy-critical harmonic map heat flow

<u>Def.</u> For a map $\Psi_0 : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ its <u>energy</u> is defined by

$$E(\Psi_0) := \frac{1}{2}\int |\nabla \Psi_0|^2 \, dx .$$

The harmonic map heat flow is the gradient flow for E with respect to the L^2 inner product :

$$\Psi : [0, T_+) \times \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3,$$

$$\text{(HMHF)} \quad \begin{cases} \Psi(0, \cdot) = \Psi_0 , \\ \partial_t \Psi = \Delta \Psi + |\nabla \Psi|^2 \Psi \end{cases}$$

(Eells- Sampson, 1964)

Energy identity : $\frac{d}{dt} E(\Psi(t,\cdot)) = -\int |\partial_t \Psi(t,x)|^2 dx$

Equivariant maps

We study the <u>dynamics</u> (long time behavior) of <u>large solutions</u>, but only in a special case:

$$\Psi(t, r\cos\theta, r\sin\theta) = (\sin\psi(t,r)\cos(k\theta), \sin\psi(t,r)\sin(k\theta), \cos\psi(t,r)).$$

Here, $k \in \{1, 2, \dots\}$ is the <u>equivariance degree</u>, $t \in \mathbb{R}$, $r \in (0, \infty)$

Equation for ψ:

$$(\mathrm{HMHF}_k) \qquad \partial_t \psi(t,r) = \overbrace{\partial_r^2 \psi(t,r) + \frac{1}{r}\partial_r \psi(t,r) - \frac{k^2}{2r^2}\sin(2\psi(t,r))}^{T(\psi) \ \text{"tension of } \psi\text{"}}$$

Energy: $\quad E(\Psi_0) = E(\psi_0) := \pi \int_0^\infty \left((\partial_r \psi_0)^2 + \frac{k^2 \sin^2 \psi_0}{r^2} \right) r\,dr$

$$\frac{d}{dt} E(\psi(t,\cdot)) = -2\pi \int (\partial_t \psi(t,r))^2 r\,dr = -2\pi \|T(\psi(t,\cdot))\|_{L^2}^2$$

$\boxed{\text{Scaling invariance and criticality}}$

If ψ solves $(HMHF_k)$ and $\lambda > 0$, then

$$\psi_\lambda(t,r) := \psi\left(\frac{t}{\lambda^2}, \frac{r}{\lambda}\right) \quad \text{solves } (HMHF_k) \text{ as well.}$$

Moreover, $E(\psi_\lambda) = E(\psi)$ $\rightsquigarrow$ energy-critical problem

Note: if $\lambda \ll 1$, then ψ_λ is concentrated and evolves fast.

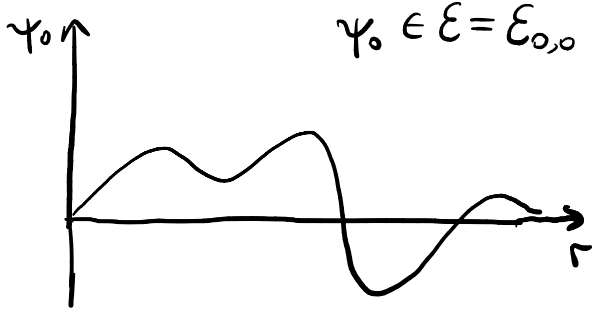
$\boxed{\text{Local theory, small data theory}}$

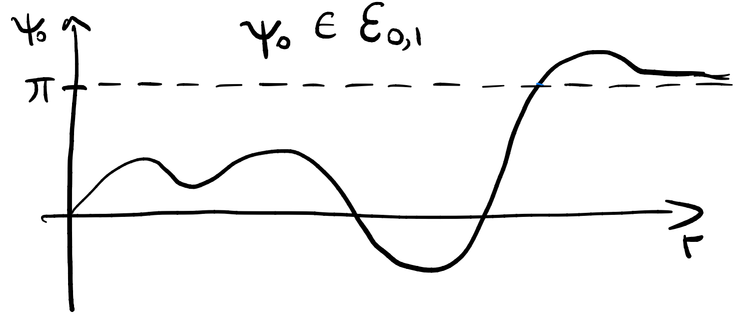
Energy norm: $\|\psi_0\|_{\mathcal{E}}^2 := \int_0^\infty \left((\partial_r \psi_0)^2 + \frac{k^2}{r^2} \psi_0^2 \right) r\,dr$,

$$\|\psi_0\|_{L^\infty} \text{ small} \implies \|\psi_0\|_{\mathcal{E}}^2 \simeq E(\psi_0)$$

Finite energy sectors: $\mathcal{E}_{m,n} := \{ \psi_0 : E(\psi_0) < \infty,$

$$\lim_{r \to 0} \psi_0(r) = m\pi, \quad \lim_{r \to \infty} \psi_0(r) = n\pi \}.$$





<u>Theorem</u> (Struwe 1985)

- $\forall\ \ell, m \in \mathbb{Z}$, $\psi_0 \in \mathcal{E}_{\ell,m}$, $\exists!\ T_+ > 0$ max. time of existence, $\psi : [0, T_+) \to \mathcal{E}_{\ell,m}$ solution of (HMHF_k) such that $\psi(0,\cdot) = \psi_0$.
- If $T_+ < \infty$, then $\exists\ \varepsilon_0 > 0$ such that $\forall\ r_0 > 0$
$$\limsup_{t \to T_+} \|\psi(t)\|^2_{\mathcal{E}(r \leq r_0)} \geq \varepsilon_0 \quad \text{(energy concentration)}$$
- $\forall\ 0 \leq t_1 \leq t_2 < T_+$ we have
$$E(\psi(t_2)) + 2\pi \int_{t_1}^{t_2} \int_0^\infty (\partial_t \psi(t,r))^2\, r\,dr\,dt = E(\psi(t_1)).$$

$\boxed{\text{Stationary solutions ("solitons")}}$

Minimisers of E:

* $*$ on $\mathcal{E}_{m,m}$ $\rightsquigarrow$ constant functions
* $*$ on $\mathcal{E}_{m,m+1}$ $\rightsquigarrow$ $\left(m\pi + 2\arctan\left(r^k/\lambda^k\right), 0\right)$, $\lambda > 0$
* $*$ on $\mathcal{E}_{m,m-1}$ $\rightsquigarrow$ $\left(m\pi - 2\arctan\left(r^k/\lambda^k\right), 0\right)$, $\lambda > 0$
* $*$ on other sectors $\rightsquigarrow$ $\emptyset$

We denote $Q(r) := 2\arctan(r^k)$, $Q_\lambda := Q(r/\lambda)$ for $\lambda > 0$.

- Key role in the description of the dynamics of large solutions.

<u>Remark</u> Stationary solutions of (HMHF) are called <u>harmonic maps</u> $\mathbb{R}^2 \longrightarrow S^2$ and correspond to meromorphic functions and its conjugates (Eells – Wood, 1975)

Bubble decomposition

Main Theorem (J.–Lawrie, 2022)

Let $k \in \mathbb{N}$ and ψ a finite energy solution of $(HMHF_k)$.

1) If $T_+ < \infty$, then there exist $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$,

$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll \sqrt{T_+ - t}$, $\iota_1, \iota_2, \dots, \iota_N \in \{-1, 1\}$

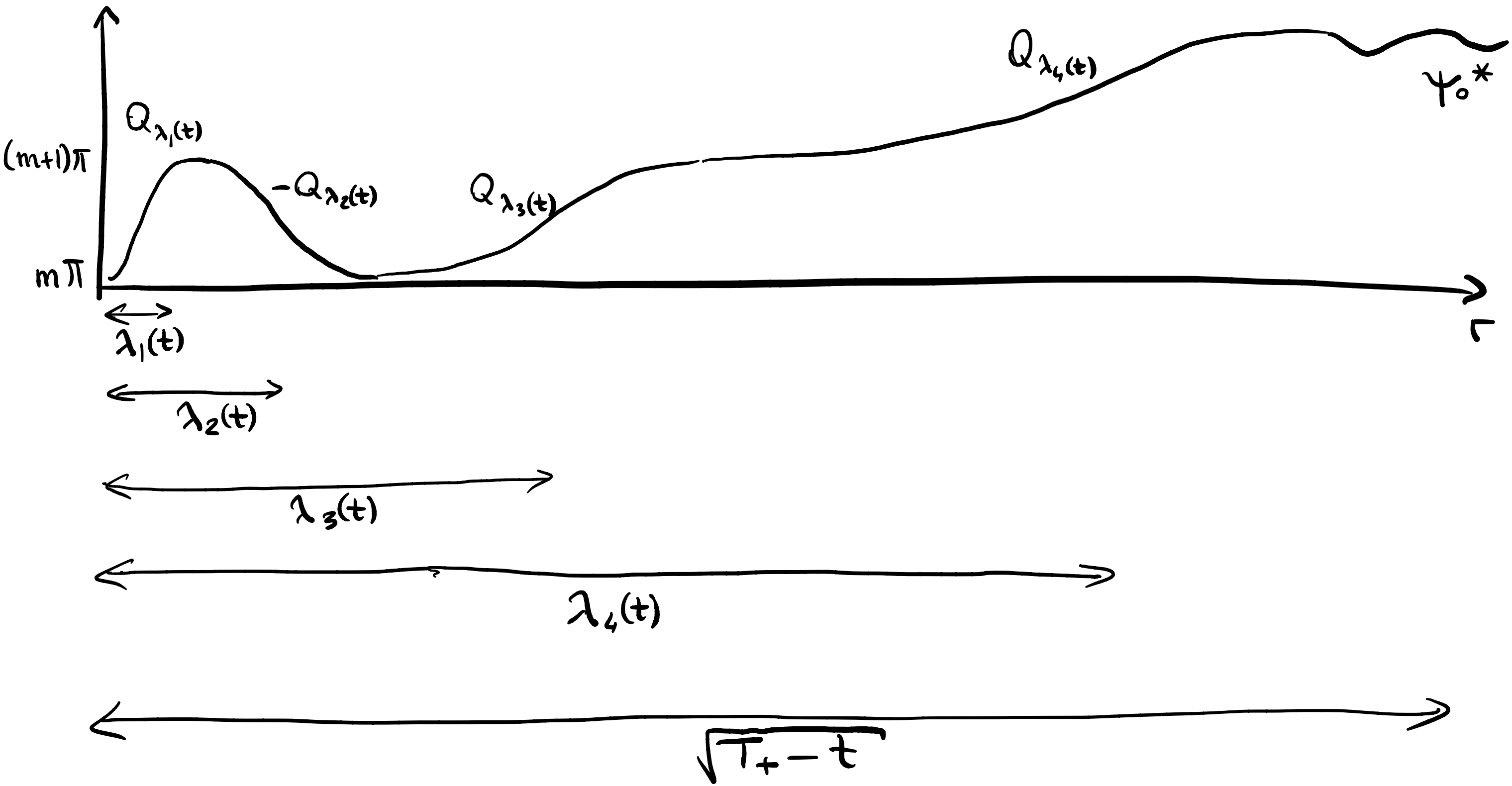
and $\psi_0^* \in \mathcal{E}_{0,l}$ such that

$$\lim_{t \to T_+} \left\| \psi(t) - \left(m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_j(t)} + \psi_0^* \right) \right\|_{\mathcal{E}} = 0.$$

2) If $T_+ = \infty$, then there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$,

$0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t) \ll \sqrt{t}$, $\iota_1, \iota_2, \dots, \iota_N \in \{-1, 1\}$

$$\lim_{t \to \infty} \left\| \psi(t) - \left(m\pi + \sum_{j=1}^{N} \iota_j Q_{\lambda_j(t)} \right) \right\|_{\mathcal{E}} = 0$$



$Q_{\lambda_1(t)}$
$(m+1)\pi$
$-Q_{\lambda_2(t)}$
$Q_{\lambda_3(t)}$
$Q_{\lambda_4(t)}$
ψ_0^*
$m\pi$
r
$\lambda_1(t)$
$\lambda_2(t)$
$\lambda_3(t)$
$\lambda_4(t)$
$\sqrt{T_+ - t}$

Comments

* Such a decomposition, but <u>for a time sequence</u>, is known even for (HMHF): Qing 95, Ding–Tian 95, Wang 96, Qing–Tian 97, Topping 97

* The question of existence and description of solutions containing a given number of bubbles remains largly open. Related results include:
  - infinite time bubble tree construction for the critical heat equation by Del Pino, Musso and Wei, 2021
  - finite time blow-up construction for $k=1$ by Raphaël and Schweyer, 2013
  - stability of Q in the energy norm for $k \geq 3$ by Gustafson, Nakanishi and Tsai, 2010

* The case $T_+ < \infty$ of the Main Theorem was essentially settled by van der Hout in 2003, who proved using max. principle that in this case $N = 1$

* It is possible that the case $T_+ = \infty$ could also be attacked using max. principle

* Our approach does not rely on it, hence gives hope of generalisation to (HMHF).

Compactness Lemma

Multi-bubble: $\quad Q(m,\vec{\iota},\vec{\lambda}) := m\pi + \sum_{j=1}^{M} \iota_j \left(Q_{\lambda_j} - \pi\right)$

Localised distance to a multi-bubble:

$$\delta_R(\psi) := \inf_{M,m,\vec{\iota},\vec{\lambda}} \left(\|\psi - Q(m,\vec{\iota},\vec{\lambda})\|^2_{\mathcal{E}(r\le R)} + \sum_{j=1}^{M-1} \left(\lambda_j/\lambda_{j+1}\right)^k + \left(\lambda_M/R\right)^k \right)^{1/2}$$

<u>Lemma</u> Let $\psi_n \in \mathcal{E}_{\ell,m}$, $\sup_{n\to\infty} E(\psi_n) < \infty$, $0 < \rho_n < \infty$, and assume that $\lim_{n\to\infty} \rho_n \|T(\psi_n)\|_{L^2} = 0$.
Then there exists a subsequence of (ψ_n), $M, m, \vec{\iota}, \vec{\lambda}_n$, $R_0 > 0$ and $R_n \to \infty$ such that $\lambda_{n,M} \le R_0\rho_n$ and

$$\lim_{n\to\infty} \left(\|\psi_n - Q(m,\vec{\iota},\vec{\lambda}_n)\|^2_{\mathcal{E}(r\le R_n\rho_n)} + \sum_{j=1}^{M-1} \left(\lambda_{n,j}/\lambda_{n,j+1}\right)^k \right) = 0,$$

in particular $\quad \lim_{n\to\infty} \delta_{R_n\rho_n}(\psi_n) = 0.$

$\boxed{\text{Propagation speed}}$

Energy density : $e(\psi_0) := \frac{1}{2}\left((\partial_r \psi)^2 + \frac{k^2}{r^2}\sin^2\psi\right)$

Localized energy : $E(\psi_0; R_1, R_2) := \int_{R_1}^{R_2} e(\psi_0)\, r\,dr$

<u>Lemma</u> (Struwe) $\exists\, C$ such that $\forall\, \psi : [0,t] \to \mathcal{E}_{\ell,m}$ solution of $(HMHF_k)$ and $R, R_1, R_2, r > 0$

$$E(\psi(t); 0, R) \leq E(\psi(0); 0, R+r) + C\frac{t}{r^2}E(\psi(0))$$

$$E(\psi(t); R_1+r, R_2-r) \leq E(\psi(t); R_1, R_2) + C\frac{t}{r^2}E(\psi(0))$$

<u>Proof</u> Let ϕ a smooth function.

$$\int_0^t \int_0^\infty (\partial_t \psi)^2 \phi^2\, r\,dr\,dt + \int_0^\infty e(\psi(t_2))\,\phi^2\, r\,dr$$

$$= \int_0^\infty e(\psi(t_1))\,\phi^2\, r\,dr - 2\int_{t_1}^{t_2}\int_0^\infty \partial_t\psi\,\partial_r\psi\,\phi\,\partial_r\phi\, r\,dr\,dt \implies$$

$$\int_0^\infty e(\psi(t_2))\,\phi^2\, r\,dr + \frac{1}{2}\int_{t_1}^{t_2}\int_0^\infty (\partial_t\psi)^2\phi^2\, r\,dr\,dt$$

$$\leq \int_0^\infty e(\psi(t_1))\,\phi^2\, r\,dr + 2\int_{t_1}^{t_2}\int_0^\infty (\partial_r\psi)^2(\partial_r\phi)^2\, r\,dr\,dt .$$

Sequential bubble decomposition

Proposition Let ψ a finite energy solution of $(HMHF_k)$ such that $T_+ < \infty$. There exist $m, l \in \mathbb{Z}$, $N \in \mathbb{N}$, $\iota_1, \iota_2, \dots, \iota_N \in \{-1, 1\}$ $t_n \to T_+$, $0 < \lambda_{1,n} \ll \dots \ll \lambda_{N,n} \ll \sqrt{T_+ - t_n}$ and $\psi_0^* \in \mathcal{E}_{0,L}$:

(i) $\lim_{n\to\infty} \| \psi(t_n) - \psi_0^* - Q(m, \vec{\iota}, \vec{\lambda_n}) \|_{\mathcal{E}} = 0.$

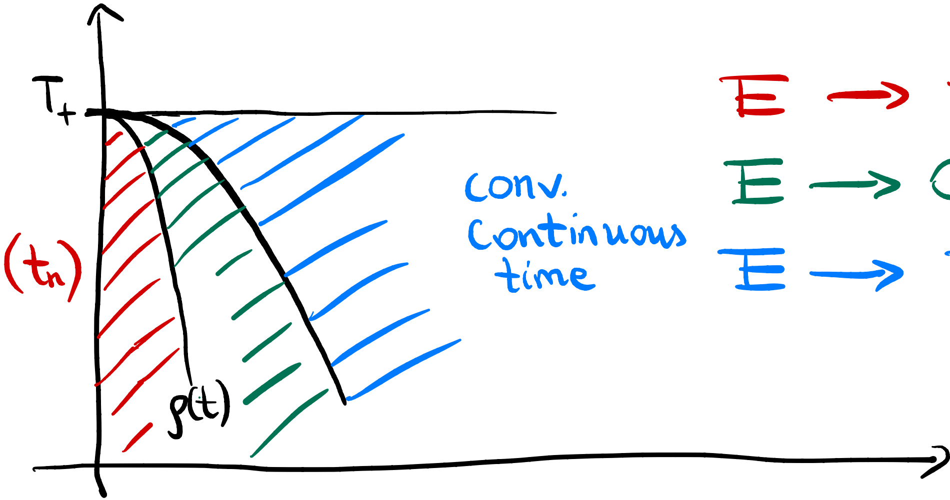
(ii) $\lim_{t\to T_+} E(\psi(t)) = N E(Q) + E(\psi_0^*)$

(iii) $\forall \alpha > 0$ $\lim_{t\to T_+} E(\psi(t); 0, \alpha\sqrt{T_+ - t}) = N E(Q)$

$\lim_{t\to T_+} E(\psi(t) - \psi_0^*; \alpha\sqrt{T_+ - t}, \infty) = 0,$

(iv) $\exists$ $\rho : [0, T_+) \to (0, \infty)$ such that

$$\lim_{t\to T_+} \left(\rho(t)/\sqrt{T_+ - t} + \| \psi(t) - \psi_0^* - m\pi \|_{\mathcal{E}(r \geq \rho(t))} \right) = 0.$$



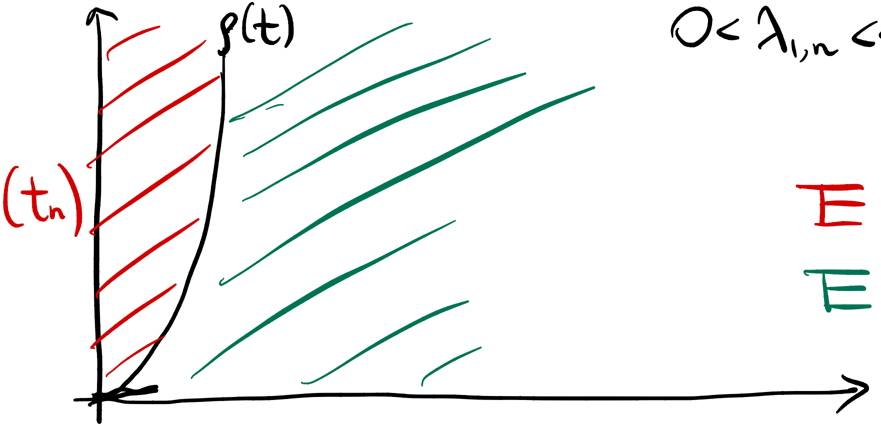
$E \longrightarrow NE(Q)$

$E \longrightarrow 0$

$E \longrightarrow E(\psi_0^*)$

Corresponding result in the case $T_+ = \infty$: $\rho(t) \ll \sqrt{t}$

$0 < \lambda_{1,n} \ll \dots \ll \lambda_{N,n} \ll \sqrt{t_n}$



$E \longrightarrow NE(Q)$

$E \longrightarrow 0$

Collision intervals

Let ψ a solution of $(HMHF_k)$, $t\in[0,T_+)$, $0<\rho<\infty$, $0\leq K\leq N$.

We set $d_K(t;\rho) := \inf_{\vec{\iota},\vec{\lambda}} \left(\| \psi(t) - \psi_0^* - Q(m,\vec{\iota},\vec{\lambda}) \|^2_{\mathcal{E}(r\geq\rho)} + \sum_{j=K}^{N} \left({}^{\lambda_j}/_{\lambda_{j+1}} \right)^k \right)^{1/2}$

where $\lambda_K := \rho$ and $\lambda_{N+1} := \sqrt{T_+-t}$ or $\sqrt{t}$.

(Proximity to a multi-bubble in the <u>exterior</u> region $r\geq\rho$).

We set $d(t) := d_0(t;0)$.

We know that $\lim_{n\to\infty} d(t_n)=0$, and want to prove $\lim_{t\to T_+} d(t)=0$.

<u>Def.</u> Let $0\leq K\leq N$, $0<\varepsilon<\eta$. We say $[a,b]\subset(0,T_+)$ is a collision interval with parameters ε,η,K if

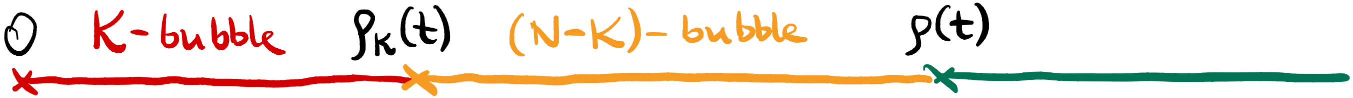
* $d(a)\leq\varepsilon$, $d(b)\geq\eta$
* $\exists\, \rho_K : [a,b]\to(0,\infty)$: $d_K(t;\rho_K(t))\leq\varepsilon \quad \forall t\in[a,b]$.

Interior and exterior bubbles

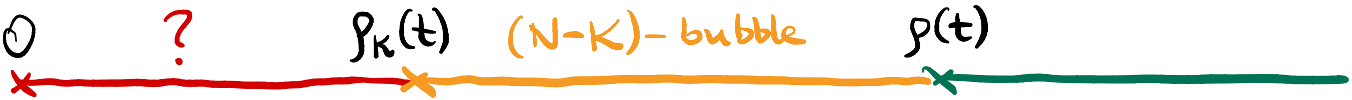
Let K be <u>minimal</u> such that $\exists \eta > 0$, $\varepsilon_n \to 0$ and collision intervals $[a_n, b_n]$ with parameters ε_n, η, K.

$$d(t) \not\to 0 \implies K \text{ is well-defined, } 1 \le K \le N.$$

$t = a_n$:



$a_n \le t \le b_n$:



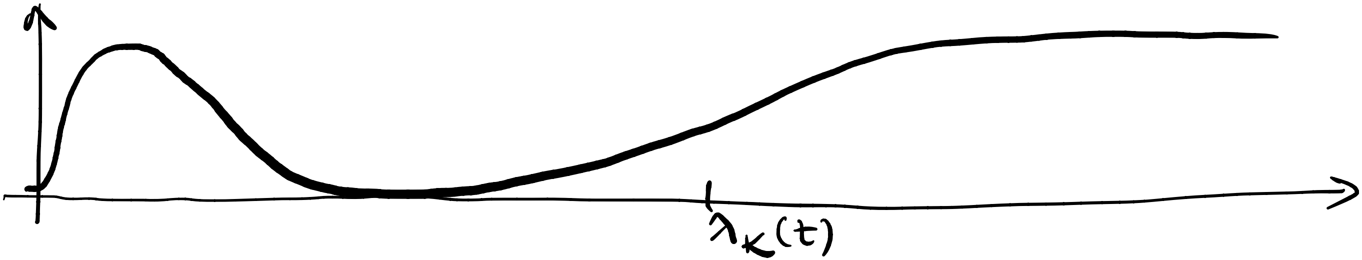
$t = b_n$:

Length of the collision intervals

Lemma $\exists\, C = C(\psi) > 0$ and $\varepsilon > 0$ such that if $[a,b]$ collision interval with parameters (ε, η, K), then $b - a \geq C^{-1} \lambda_K(a)^2$.

Proof If not, K would not be smallest possible.

$t = a$:



Propagation speed $\Rightarrow$ K-th bubble stays coherent

$t = b$:

End of the proof

We have $\varepsilon > 0$ and an infinite sequence $[c_n, d_n] \subset [a_n, b_n]$ with $d_n - c_n \gtrsim \lambda_K(t)^2$ for $t \in [c_n, d_n]$ and $\forall\, r_n \to \infty$

$$\liminf_{n\to\infty} \inf_{s\in[c_n,d_n]} \delta_{r_n \lambda_K(s)} (\psi(s_n)) \geqslant \varepsilon .$$

By the "Compactness Lemma",

$$\limsup_{n\to\infty} \inf_{s\in[c_n,d_n]} \lambda_K(s) \|T(\psi(s))\|_{L^2} > 0,$$

$$\int_{c_n}^{d_n} \|T(\psi(s))\|_{L^2}^2 \, ds \gtrsim \lambda_K^2 \times \frac{1}{\lambda_K^2},$$

which is impossible since $\displaystyle\int_0^{T_+} \|T(\psi(t))\|_{L^2}^2 \, dt < \infty.$