Dynamics of kink clusters for scalar fields in dimension 1+1

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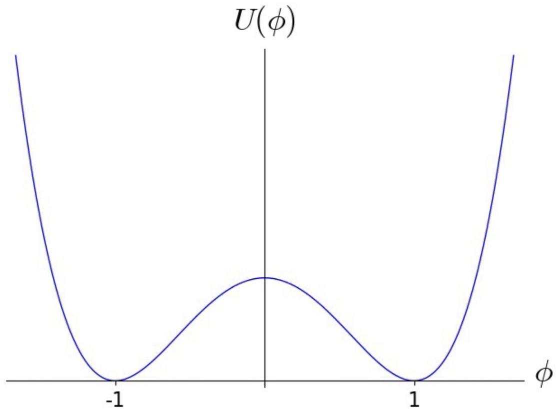
Scalar fields in dimension 1+1

Let $U(\phi) := \frac{1}{8}(1-\phi^2)^2$

and consider the equation

$$(\mathrm{CSF}) \qquad \partial_t^2 \phi - \partial_x^2 \phi + U'(\phi) = 0,$$

$$\phi = \phi(t,x), \qquad (t,x) \in \mathbb{R}\times\mathbb{R},$$



which is the Euler–Lagrange equation for the Lagrangian

$$\mathcal{L} := \iint \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_x \phi)^2 - U(\phi) \right) dx\, dt.$$

$$E(\phi_0, \dot{\phi}_0) = E_k(\dot{\phi}_0) + E_p(\phi_0) = \int_{\mathbb{R}} \Big(\underbrace{\frac{1}{2}\dot{\phi}_0^2}_{\text{kinetic}} + \underbrace{\frac{1}{2}(\partial_x \phi_0)^2 + U(\phi_0)}_{\text{potential}} \Big) dx .$$

The space of states of finite energy is a union of four affine spaces $\mathcal{E}_{-1,-1}$, $\mathcal{E}_{-1,1}$, $\mathcal{E}_{1,-1}$ and $\mathcal{E}_{1,1}$, each parallel to $\mathcal{E} := H^1(\mathbb{R}) \times L^2(\mathbb{R})$

Kinks, antikinks and multi-kink configurations

Minimisers of the energy:

$\mathcal{E}_{-1,-1}$, $\mathcal{E}_{1,1}$ $\rightsquigarrow$ constant functions

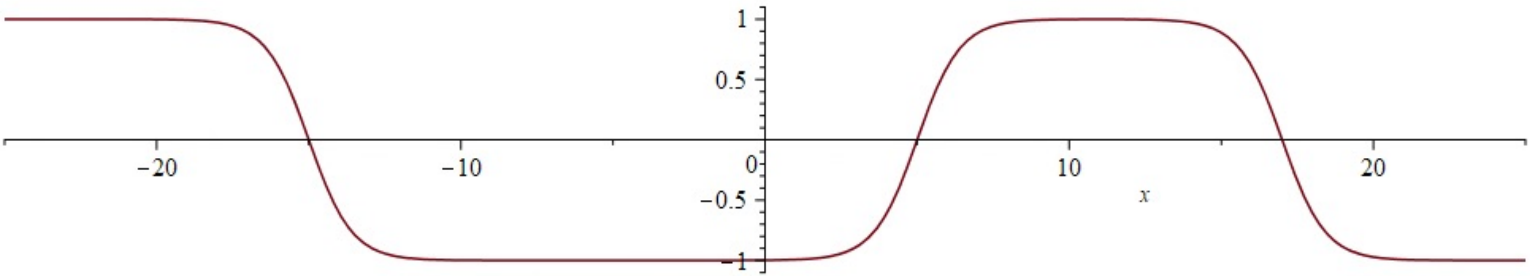
$\mathcal{E}_{-1,1}$ $\rightsquigarrow$ $\tanh\left(\frac{x-a}{2}\right)$, the <u>kinks</u>

$\mathcal{E}_{1,-1}$ $\rightsquigarrow$ $-\tanh\left(\frac{x-a}{2}\right)$, the <u>antikinks</u>

We set $H(x) := \tanh\left(\frac{x}{2}\right)$.

For $\vec{a} = (a_1, a_2, \ldots, a_n)$, $a_1 \leq a_2 \leq \ldots \leq a_n$, we denote

$$H(\vec{a}\,; x) := 1 + \sum_{k=1}^{n} (-1)^k \left(H(x-a_k) + 1\right)$$



n- body approximation

- View $H(\cdot - a)$ as an "extended particle" of mass $M := E_p(H) = \frac{2}{3}$.
- Potential energy of multi-kink configurations

$$\widetilde{E}_p(\vec{a}) := E_p(H(\vec{a})) = nM - 8\sum_{k=1}^{n-1} e^{-(a_{k+1}-a_k)} + \dots$$

- Force acting on the k-th "particle"

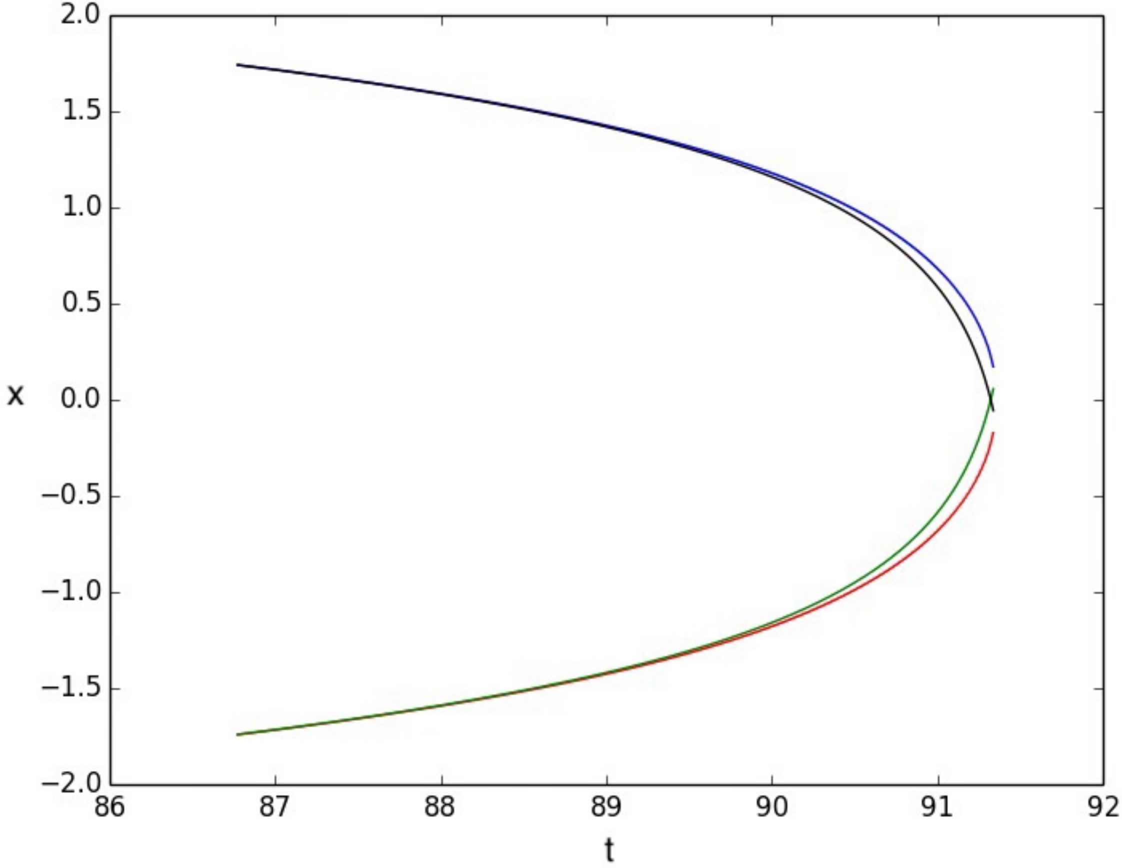
$$F_k(\vec{a}) = -\partial_{a_k}\widetilde{E}_p(\vec{a}) = 8\left(e^{-(a_{k+1}-a_k)} - e^{-(a_k-a_{k-1})}\right) + \dots$$

(by convention $a_0 = -\infty$, $a_{n+1} = \infty$)

- n-body problem with attractive exponential nearest-neighbour interactions

$$\text{(NBP)} \qquad M\, a_k''(t) = 8\left(e^{-(a_{k+1}(t)-a_k(t))} - e^{-(a_k(t)-a_{k-1}(t))}\right).$$

Validity of the n- body approximation



$\boxed{\text{Kink clusters}}$

Definition A solution ϕ of (CSF) is a <u>kink n-cluster</u> (for positive times) if there exists $\vec{a} : \mathbb{R} \to \mathbb{R}^n$ such that

$$\lim_{t\to\infty} \left(\|\partial_t \phi(t)\|_{L^2}^2 + \|\phi(t) - H(\vec{a}(t))\|_{H^1}^2 + \sum_{k=1}^{n-1} e^{-(a_{k+1}(t) - a_k(t))} \right) = 0.$$

Characterisation 1 : chain of transitions between vacua of minimal possible energy

ϕ is a kink cluster $\iff$ $\exists\, x_0(t) \leq x_1(t) \leq \ldots \leq x_n(t)$ such that

- $\displaystyle\lim_{t\to\infty} \phi(t, x_k(t)) = (-1)^k \qquad \forall k \in \{0, 1, \ldots, n\}$
- $E(\phi, \partial_t \phi) \leq n E_p(H)$

Characterisation 2 : asymptotically static solutions (analogues of parabolic motions in classical mechanics)

$$\lim_{t\to\infty} \|\partial_t \phi(t)\|_{L^2}^2 = 0 \iff \phi \text{ or } -\phi \text{ is a kink cluster}$$

Asymptotic behaviour of kink clusters

Explicit parabolic motion of (NBP)

$$a_{k+1}(t) - a_k(t) = 2\log(2t) - \log\frac{k(n-k)}{3}, \quad a_k'(t) = \frac{n+1-2k}{t}.$$

<u>Theorem 1.</u> If ϕ is a kink n-cluster, then there exist continuously differentiable functions $a_1, a_2, \ldots, a_n : \mathbb{R} \to \mathbb{R}$ such that $g(t) := \phi(t) - H(\vec{a}(t))$ satisfies

$$\lim_{t\to\infty}\Big(\max_{1\le k<n}\Big|a_{k+1}(t) - a_k(t) - \Big(2\log(2t) - \log\frac{k(n-k)}{3}\Big)\Big|$$
$$+ \max_{1\le k\le n}\big|t\,a_k'(t) - (n+1-2k)\big| + t\,\|\partial_t g(t)\|_{L^2} + t\,\|g(t)\|_{H^1}\Big) = 0.$$

Orthogonality conditions and modulation

The decomposition $\phi(t) = H(\vec{a}(t)) + g(t)$ is made <u>unique</u> by imposing the orthogonality conditions

$$(\mathrm{ORTH}) \quad \int_{-\infty}^{\infty} \partial_x H(x - a_k(t))\, g(t,x)\, dx = 0 \quad \text{for } k \in \{1, \dots, n\}.$$

The map $\phi \mapsto (\vec{a}, g)$ is a diffeomorphism from a neighborhood of the set of widely separated multi-kink configurations to a manifold in $\mathbb{R}^n \times H^1(\mathbb{R})$ of codimension n.

"Modulation method":

(CSF) + Time differentiation of (ORTH) $\rightsquigarrow$

$\rightsquigarrow$ coupled system of differential equations for $\vec{a}$ and g.

Existence of kink clusters

<u>Definition</u> (distance to multi-kink configurations)

$$\delta(\phi_0, \dot{\phi}_0) := \inf_{\vec{b} \in \mathbb{R}^n} \left(\|\dot{\phi}_0\|_{L^2}^2 + \|\phi_0 - H(\vec{b})\|_{H^1}^2 + \sum_{k=1}^{n-1} e^{-(b_{k+1}-b_k)} \right).$$

Thus, ϕ is a kink n-cluster $\Leftrightarrow$ $\lim_{t\to\infty} \delta(\phi(t), \partial_t \phi(t)) = 0$.

The position vector $\vec{a}$ given by (ORTH) does not necessarily reach the inf, but it does up to a constant:

$$\|\partial_t \phi(t)\|_{L^2}^2 + \|\phi(t) - H(\vec{a}(t))\|_{H^1}^2 + \sum_{k=1}^{n-1} e^{-(a_{k+1}(t) - a_k(t))} \simeq \delta(\phi(t), \partial_t \phi(t))$$

<u>Theorem 2.</u> If L is sufficiently large and $\vec{a}_0 = (a_{0,1}, \dots, a_{0,n})$ such that $a_{0,k+1} - a_{0,k} \geq L$, then there exists a kink cluster $\phi(t) = H(\vec{a}(t)) + g(t)$ such that $\vec{a}(0) = \vec{a}_0$ and $\delta(\phi(t), \partial_t \phi(t)) \lesssim (e^L + t^2)^{-1}$ for all $t \geq 0$.

$\boxed{\text{The case } n=2 \quad (\text{kink-antikink pairs})}$

Theorem (J.- Kowalczyk- Lawrie 2022, preprint 2019)

There exist a C^1 function $a(t)$ and a solution ϕ_c of (CSF) such that, for all $t \geq T_0 = T_0(\varepsilon)$

$$|a(t) - \log(2\sqrt{3}\,t)| \leq t^{-2+\varepsilon}, \quad |a'(t) - t^{-1}| \lesssim t^{-3+\varepsilon},$$

$$\| \phi_c(t) - (1 - H(\cdot + a(t)) + H(\cdot - a(t))) \|_{H^1}$$
$$+ \| \dot{\phi}_c(t) + a'(t)(\partial_x H(\cdot + a(t)) + \partial_x H(\cdot - a(t))) \|_{L^2} \lesssim t^{-2+\varepsilon}.$$

Moreover, if ϕ is any kink-antikink pair, then there exist $t_0, x_0 \in \mathbb{R}$ such that

$$\phi(t,x) = \phi_c(t - t_0, x - x_0).$$

Remark For $n=2$, there is <u>uniqueness</u> in Theorem 2.

We expect uniqueness for general n.

Related results

- Wadati, Ohkuma : "Multiple-pole solutions of mKdV" 1982 (2-solitons and 3-solitons with asymptotically equal velocities).
- Krieger, Martel, Raphaël : "Two-soliton-solutions to Hartree" 2009 (construction of 2-solitons with velocities converging to 0)
- Theorem 1 for $n=2$ was obtained by J. '18 for gKdV and by J.-Kowalczyk-Lawrie '19 for (CSF).
- Analysis of the gravitational n-body problem
  Pollard, Saari, ... '60-'70
- Hénon '74 found n independent conserved quantities for (NBP)
- Lan and Wang '22 constructed soliton n-clusters for any n for generalised Benjamin-Ono.

$\boxed{\text{Refined modulation}}$

- Modulation method : decompose

$$\phi(t,x) = H(\vec{a}(t), x) + g(t,x), \qquad \vec{a} : \mathbb{R} \to \mathbb{R}^n$$

$$\int_{-\infty}^{\infty} \partial_x H(x - a_k(t))\, g(t,x)\, dx = 0.$$

- Introduce the localised momenta

$$p_k(t) := -\int_{-\infty}^{\infty} \left[(-1)^k \partial_x H(x - a_k(t)) + \chi_k(t,x)\partial_x g(t,x)\right] \partial_t \phi(t,x)\, dx,$$

χ_k — cut-off in the region of k^{th} kink

<u>Proposition</u> Let $\rho(t) := \sum_{k=1}^{n-1} e^{-(a_{k+1}(t) - a_k(t))}$.

Then $\|\partial_t \phi(t)\|_{L^2}^2 + \|g(t)\|_{H^1}^2 \simeq \rho(t)$ and

$$|M a_k'(t) - p_k(t)| \lesssim \rho(t), \qquad |p_k'(t) - F_k(\vec{a}(t))| \lesssim \frac{\rho(t)}{-\log \rho(t)}.$$

Ejection mechanism

Let $V(\vec{a}) := \widetilde{E}_p(\vec{a}) - nM = -8\sum_{k=1}^{n-1} e^{-(a_{k+1}-a_k)} + \dots$

$$M\vec{a}\,'(t) = \vec{p}(t) + \dots \qquad \vec{p}\,'(t) = -\nabla V(\vec{a}(t)) + \dots$$

$$\frac{d}{dt}\left(-V(\vec{a}(t))\right) = -\frac{1}{M}\,\vec{p}(t)\cdot\nabla V(\vec{a}(t)) + \dots$$

$$\frac{d^2}{dt^2}\left(-V(\vec{a}(t))\right) = \frac{1}{M}\left|\nabla V(\vec{a}(t))\right|^2 - \frac{1}{M^2}\,\vec{p}(t)\cdot\nabla^2 V(\vec{a}(t))\,\vec{p}(t) + \dots$$

$$\rightsquigarrow \quad \rho''(t) \gtrsim \rho(t)^2 \quad \rightsquigarrow \quad \rho(t) \simeq t^{-2}.$$

$$\rightsquigarrow \quad \limsup_{t\to\infty}\left|\min_{1\le k<n}\left(a_{k+1}(t)-a_k(t)\right) - 2\log t\right| < \infty.$$

<u>Remark</u> Bounds on $\max_{1\le k<n}\left(a_{k+1}(t)-a_k(t)\right)$ are more difficult.

Construction of kink clusters

<u>Theorem 2.</u> If L is sufficiently large and $\vec{a}_0 = (a_{0,1}, \dots, a_{0,n})$ such that $a_{0,k+1} - a_{0,k} \geq L$, then there exists a kink cluster $\phi(t) = H(\vec{a}(t)) + g(t)$ such that $\vec{a}(0) = a_0$ and $\delta(\phi(t), \partial_t \phi(t)) \lesssim (e^L + t^2)^{-1}$ for all $t \geq 0$.

Overall proof scheme (due to Martel '05):

- $\forall\, T > 0\ \exists\, \phi_T$ solution of (CSF) satisfying the conditions above for $t \in [0, T]$
- pass to a weak limit as $T \to \infty$.

The first step is done by imposing initial data at $t = T$ and using Poincaré–Miranda theorem (multi-dim intermediate value theorem).

Kink clusters as profiles of kink formation / collapse

Theorem 3. Let $\eta > 0$ be sufficiently small and let $(\phi_m)_m$ solutions of (CSF) on time intervals $[0, T_m]$ satisfying

i) $\lim_{m\to\infty} \delta(\phi_m(T_m), \partial_t \phi_m(T_m)) = 0$

ii) $\delta(\phi_m(t), \partial_t \phi_m(t)) \leq \eta$ for all $t \in [0, T_m]$

iii) $\delta(\phi_m(0), \partial_t \phi_m(0)) = \eta$.

Then, up to extracting a subsequence, there exist $0 = n^{(0)} < n^{(1)} < \dots < n^{(\ell)} = n$, for $j \in \{1, \dots, \ell\}$ a cluster $P^{(j)}$ of $n^{(j)} - n^{(j-1)}$ kinks and $(X_m^{(j)})_m$ such that

$$\lim_{m\to\infty} \left(X_m^{(j+1)} - X_m^{(j)}\right) = \infty \quad \text{for } j \in \{1, \dots, \ell-1\},$$

$$\left\| \phi_m(t) - \left[1 + \sum_{j=1}^{\ell} (-1)^{n^{(j-1)}} \left(P^{(j)}(t, \cdot - X_m^{(j)}) - 1\right)\right] \right\|_{H^1} \to 0 \quad \text{loc. unif. in } t.$$