Dynamics of multi-solitons for Klein-Gordon equations

Jacek Jendrej
(CNRS & Sorbonne Paris Nord)

joint work with Gong Chen (Georgia Tech)

Institute of Theoretical Physics
Jagiellonian University, March 13th 2024

Scalar field equation

- We consider nonlinear scalar fields $\mathbb{R}^{1+1} \to \mathbb{R}$ which are critical points of the Lagrangian

$$\mathscr{L}(\phi, \partial_t \phi) := \iint \Big(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - U(\phi)\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

  where $U : \mathbb{R} \to \mathbb{R}$ is a positive function.
- The Euler-Lagrange equation reads

$$\begin{gathered}\partial_t^2 \phi(t,x) - \partial_x^2 \phi(t,x) + U'(\phi(t,x)) = 0, \\ (t,x) \in \mathbb{R} \times \mathbb{R}, \quad \phi(t,x) \in \mathbb{R}.\end{gathered} \tag{CSF}$$

- "The simplest" nonlinear wave equation.
- This equation and its quantisation are used as toy models in Quantum Field Theory.

Scalar field equation

- Energy:

$$E(\phi, \partial_t \phi) = E_k(\partial_t \phi) + E_p(\phi)$$
$$:= \int_{\mathbb{R}} \Big[\frac{1}{2}(\partial_t \phi)^2 + \Big(\frac{1}{2}(\partial_x \phi)^2 + U(\phi)\Big)\Big]\,\mathrm{d}x.$$

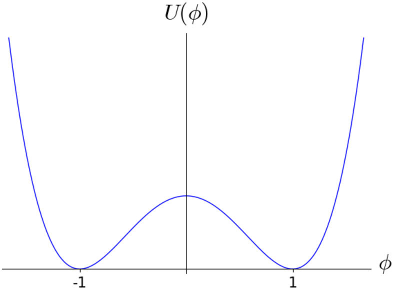
- Momentum:

$$P(\phi, \partial_t \phi) := \int_{\mathbb{R}} -\partial_t \phi \partial_x \phi \,\mathrm{d}x.$$

- $E := E_k + E_p$ and P are conserved quantities.
- If ω is a non-degenerate minimum of U (a "vacuum"), $U(\omega) = 0$ and $U''(\omega) > 0$, then $\phi(t, x) \equiv \omega$ is a trivial stable solution of (CSF). The flow linearised around this solution is a linear Klein-Gordon equation with mass $\sqrt{U''(\omega)}$.

Two vacua and transitions between them

Consider U having many vacua. For the sake of simplicity, we consider a double-well potential, say $U(\phi) := \frac{1}{8}(1-\phi^2)^2$, corresponding to the "$\phi^4$ model".



- The space of finite energy states $(\phi_0, \dot{\phi}_0)$ is the union of four affine spaces (topological classes) $\mathcal{E}_{-1,-1}$, $\mathcal{E}_{1,1}$, $\mathcal{E}_{-1,1}$ and $\mathcal{E}_{1,1}$ characterised by:
  1. $\mathcal{E}_{-1,-1}$: $\lim_{x\to-\infty}\phi_0(x) = -1$ and $\lim_{x\to\infty}\phi_0(x) = -1$,
  2. $\mathcal{E}_{1,1}$: $\lim_{x\to-\infty}\phi_0(x) = 1$ and $\lim_{x\to\infty}\phi_0(x) = 1$,
  3. $\mathcal{E}_{-1,1}$: $\lim_{x\to-\infty}\phi_0(x) = -1$ and $\lim_{x\to\infty}\phi_0(x) = 1$,
  4. $\mathcal{E}_{1,-1}$: $\lim_{x\to-\infty}\phi_0(x) = 1$ and $\lim_{x\to\infty}\phi_0(x) = -1$.

Kinks, antikinks and traveling waves

Minimizers of the energy in each topological class turn out to be:

- $\mathcal{E}_{1,1}, \mathcal{E}_{-1,-1} \rightsquigarrow$ constant fields
- $\mathcal{E}_{-1,1} \rightsquigarrow \tanh((x-a)/2)$, the *kinks*
- $\mathcal{E}_{1,-1} \rightsquigarrow -\tanh((x-a)/2)$, the *antikinks*.

We set $H(x) := \tanh(x/2)$. Thus, $\phi(t,x) = \pm H(x-a)$ for any $a \in \mathbb{R}$ are (stationary) solutions of (CSF). Examples of *topological solitons* – particle-like solutions of mass $M := E(H) = \frac{2}{3}$ (Skyrme '62).

Lorentz invariance: $\phi(t,x)$ solves (CSF) if and only if $(t,x) \mapsto \phi(t',x')$ does, where

$$(t',x') = (\gamma_v(t - vx), \gamma_v(x - vt)), \quad \gamma_v := (1 - v^2)^{-\frac{1}{2}}.$$

In particular, we obtain *traveling waves* $\phi(t,x) = H(\gamma_v(x - vt))$. We have $(E,P) = (\gamma_v M, \gamma_v v M)$, in accordance with Special Relativity.

Stability of the kink

Orbital stability: If $E\big(\phi(t,x) - H(\gamma_v(x)), \partial_t\phi(t,x) + v\gamma_v\partial_x H(\gamma_v(x))\big)$ is small, then for all t there exists $a(t) \in \mathbb{R}$ such that

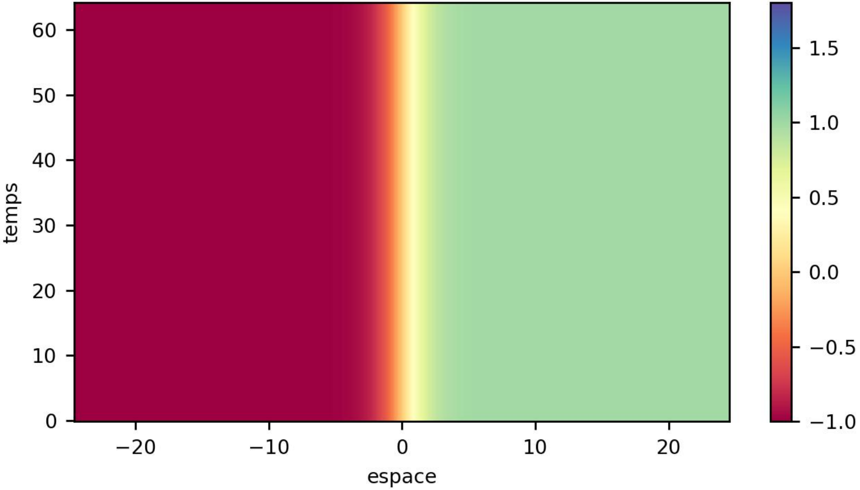
$$E\big(\phi(t,x) - H(\gamma_v(x - a(t))), \partial_t\phi(t,x) + v\gamma_v\partial_x H(\gamma_v(x - a(t)))\big) \text{ is small.}$$

(Follows from the fact that the kinks are energy minimizers.)
Conjecture: Under the same assumption, then there exist w close to v and $b(t) \in \mathbb{R}$ such that

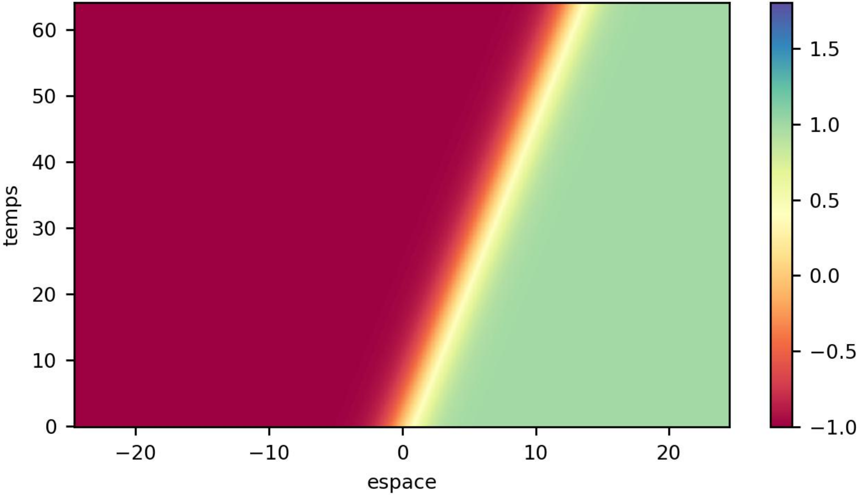
$$\phi(t,x) \to H\big(\gamma_w(x - wt - b(t))\big) \quad \text{as } t \to \infty.$$

- In the case of orbital stability, we have no description of how the remainder behaves, except for the fact that it has small energy (for example, turbulence is not excluded).
- Works by Henry-Perez-Wreszinski, Weinstein, . . .
- The meaning of the convergence to the traveling wave has to be made precise. One expects at least uniform convergence. A finer description of the *radiation* can also be asked for.
- Works by Buslaev-Perelman, Soffer-Weinstein, Beceanu, Cuccagna, Krieger-Schlag, Rodnianski-Schlag-Soffer, Kopylova-Komech, Romańczukiewicz, . . .

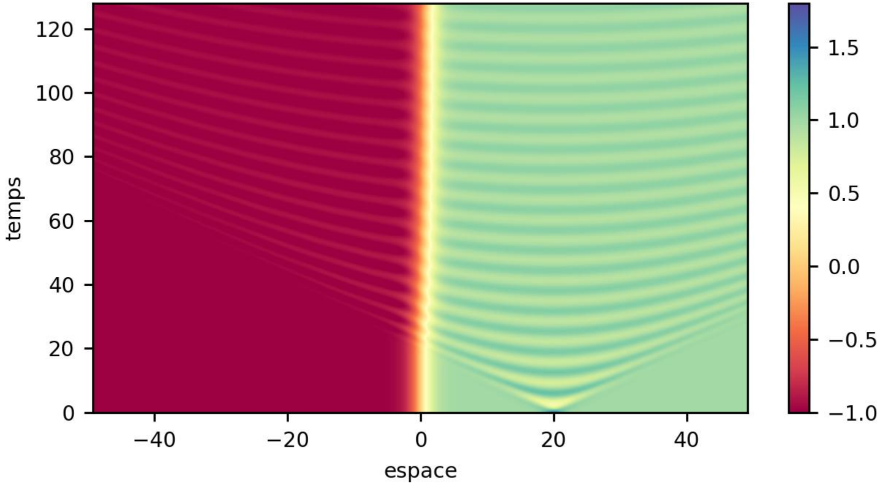
A kink



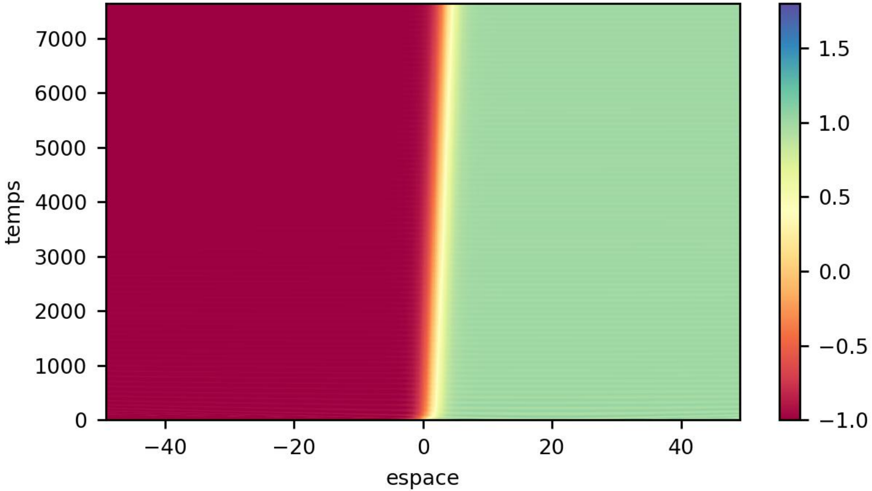
A traveling wave



A perturbed kink



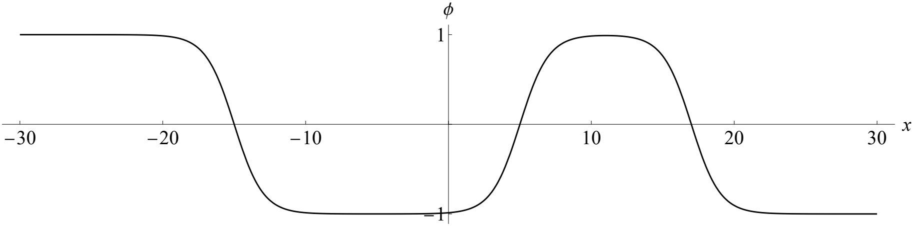
A perturbed kink



Multi-solitons

For $\vec{v} = (v_1, \ldots, v_n) \in (-1, 1)^n$ strictly increasing and $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, denote

$$H(\vec{v}, \vec{a}; t, x) := 1 + \sum_{k=1}^{n} (-1)^k \big(H(\gamma_{v_k}(x - v_k t - a_k)) + 1\big).$$



Multi-solitons

We have *existence and uniqueness of pure multi-solitons.*
Theorem (Chen and J. 2022): For any $\vec{v}$ and $\vec{a}$ there exists a unique ϕ such that

$$\phi \to H(\vec{v}, \vec{a}) \quad \text{as } t \to \infty$$

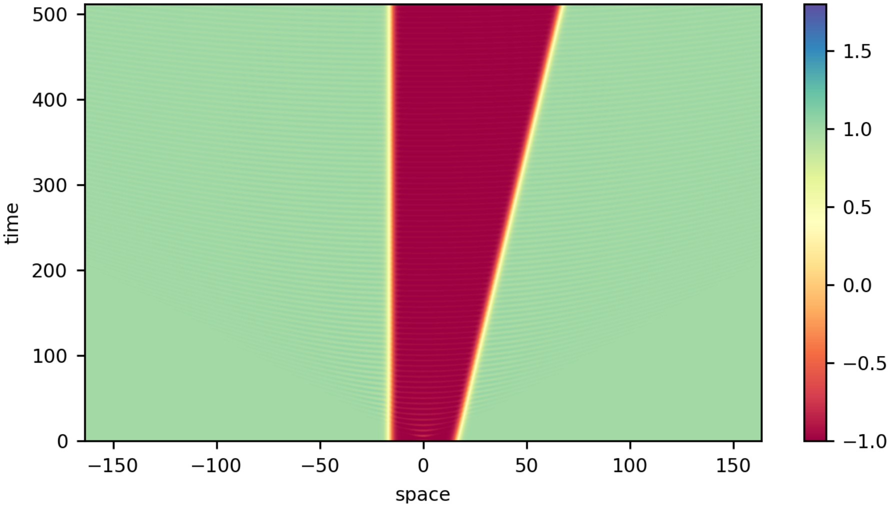
exponentially fast.

The following conjecture generalizes the problem of the stability of the kink.
Conjecture: If at time $t = 0$, $E\big(\phi - H(\vec{v}, \vec{a}), \partial_t \phi - \partial_t H(\vec{v}, \vec{a})\big)$ is small, then there exist $\vec{w}$ close to $\vec{v}$ and $\vec{b}(t) \in \mathbb{R}^n$ such that

$$\phi \to H(\vec{w}, \vec{b}(t)) \quad \text{as } t \to \infty.$$

The next problem is still more general.
Soliton Resolution Conjecture: Any solution of (CSF) converges as $t \to \infty$ to a sum of a multi-soliton and radiation.

A perturbed multi-soliton



Klein-Gordon equations

Stability is too difficult for the ϕ^4 model, due to slow dispersion.
We consider instead the nonlinear Klein-Gordon equation

$$\partial_t^2\psi - \Delta\psi + \psi - \psi^3 = 0,\ (t,x) \in \mathbb{R}^{1+3}. \tag{NLKG}$$

Stationary solutions satisfy

$$-\Delta Q + Q - Q^3 = 0$$

and decay exponentially.
There exists a unique radial positive ground state with the least energy

$$E(Q) := \int_{\mathbb{R}^d} \left(\frac{|\nabla Q|^2 + Q^2}{2} - \frac{Q^4}{4}\right) \mathrm{d}x$$

among all non-zero solutions. *The ground state is linearly unstable.*

Results about the long-time behavior

- Energy below the ground state. Global existence vs blowup (Payne and Sattinger 1975)
- Energy below the ground state. Global existence implies scattering i.e. asymptotically linear behaviour. (Ibrahim-Masmoudi-Nakanishi 2010. Kenig-Merle concentration compactness 2006)
- Energy slightly above the ground state. Classification of global dynamics near Q, construction of the center manifold. (Nakanishi-Schlag 2011)

Our goal: study the dynamics of solutions near superpositions of a finite number of Lorentz-transformed solitons, moving with distinct speeds (multi-solitons).

Traveling waves

The equation (NLKG) has *Lorentz invariance*, just like (CSF). For $v \in \mathbb{R}^3$, $|v| < 1$, we set

$$\Lambda_v x := x + (\gamma_v - 1)\frac{(v \cdot x)v}{|v|^2}, \quad \gamma_v := \frac{1}{\sqrt{1 - |v|^2}}.$$

Then $\psi(t, x)$ solves (NLKG) if and only if $(t, x) \mapsto \psi(t', x')$ does, where

$$(t', x') = (\gamma_v(t - v \cdot x), \Lambda_v(x - vt)).$$

In particular, we obtain *traveling waves* $\psi(t, x) = Q(\Lambda_v(x - vt))$. We have $(E, P) = (\gamma_v M, \gamma_v v M)$, where $M := E(Q)$.

Multi-solitons

For $\vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^{3n}$ distinct, $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}^{3n}$ and $\vec{\sigma} = (\sigma_1, \dots, \sigma_n) \in \{-1, 1\}^n$, we denote

$$Q(\vec{v}, \vec{a}; t, x) := \sum_{k=1}^{n} \sigma_k Q(\Lambda_{v_k}(x - v_k t - a_k)).$$

Existence and uniqueness of pure multi-solitons (with an exponential decay of the remainder) was proved by R. Côte and C. Muñoz.

Main result

Theorem (C-J 2022)

Let $\vec{v}$ *and* $\vec{a}$ *be such that the lines* $x = v_k t + a$ *stay sufficiently far away from each other. Let* ψ *be such that*

$$E(\psi - Q(\vec{v}, \vec{a}), \partial_t \psi - \partial_t Q(\vec{v}, \vec{a})) \quad \text{is small for } t = 0.$$

If ψ *stays close the multi-soliton family, that is if*

$$\inf_{\vec{w} \in \mathbb{R}^{3n}, \vec{b} \in \mathbb{R}^{3n}} E(\psi - Q(\vec{w}, \vec{b}), \partial_t \psi - \partial_t Q(\vec{w}, \vec{b}))$$

is small for all $t \geq 0$*, then* ψ *scatters to the multi-soliton family: there exist* $\vec{w} \in \mathbb{R}^{3n}$ *and* $\vec{b}(t) \in \mathbb{R}^{3n}$ *such that*

$$\psi \to Q(\vec{w}, \vec{b}(t)) \qquad \text{as } t \to \infty.$$

Comments on the main result

- We obtain in fact that ψ is a sum of $Q(\vec{w}, \vec{b}(t))$, a solution of the linearized equation (which is the free Klein-Gordon equation) and a term whose energy decays to 0 as $t \to \infty$. In principle, one can also estimate other, stronger norms (so that for instance turbulence is excluded).
- We also show that the set of initial data for which the assumption of the theorem holds is a manifold of codimension n.
- The main technical tool are bounds on space-time norms called *Strichartz estimates*, for solutions of linear Klein-Gordon equations with moving potentials, which is inspired by *scattering theory*.

Results on pure multi-solitons

Definition: We say that ψ is a pure multi-soliton if there exists $\vec{w} \in \mathbb{R}^{3n}$ with $w_j \neq w_k$ such that

$$\lim_{t\to\infty} \inf_{\vec{b}\in\mathbb{R}^{3n}} E(\psi - Q(\vec{w},\vec{b}), \partial_t\psi - \partial_t Q(\vec{w},\vec{b})) = 0.$$

Theorem (C-J 2022)

If ψ is a pure multi-soliton, then in fact the convergence towards a multi-soliton is exponentially fast. For any $\vec{w} \in \mathbb{R}^{3n}$ and $\vec{b} \in \mathbb{R}^{3n}$ there exists an n-parameter family of pure multi-solitons converging to $Q(\vec{w},\vec{b})$ and these are all the pure multi-solitons.

Thank you for your attention.