Dynamics of bubbling wave maps with prescribed radiation

Jacek Jendrej
(CNRS & université Paris 13)

joint work with Andrew Lawrie (MIT)
and Casey Rodriguez (MIT)

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Critical wave maps

- Wave maps $\mathbb{R}^{1+2} \to \mathbb{S}^2 \subset \mathbb{R}^3$ are critical points of the Lagrangian

$$\mathscr{L}(\Psi, \partial_t\Psi) := \iint \Big(\frac{1}{2}|\partial_t\Psi|^2 - \frac{1}{2}|\nabla\Psi|^2\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

- Energies:

$$E_c(\Psi, \partial_t\Psi) := \int_{\mathbb{R}^2} \frac{1}{2}|\partial_t\Psi|^2\,\mathrm{d}x,$$

$$E_p(\Psi, \partial_t\Psi) := \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\Psi|^2\,\mathrm{d}x,$$

$$\boxed{E := E_c + E_p \quad \text{is a conserved quantity.}}$$

- Scaling and criticality: If Ψ is a wave map and $\lambda > 0$, then so is $\Psi_\lambda(t, x) := \Psi(t/\lambda, x/\lambda)$. Moreover, $E(\Psi_\lambda, \partial_t\Psi_\lambda) = E(\Psi, \partial_t\Psi)$ **energy-critical setting**

Corotational maps

- We consider solution of the form

$$\Psi(t, r\cos\theta, r\sin\theta) = \\ = \big(\sin(\psi(t,r))\cos\theta, \sin(\psi(t,r))\sin\theta, \cos(\psi(t,r))\big).$$

- This class is conserved by the flow.
- $\psi = 2k\pi$ is the north pole, and $\psi = (2k+1)\pi$ the south pole.
- Ψ is a wave map if and only if

$$\begin{cases} \partial_t^2\psi(t,r) = \partial_r^2\psi(t,r) + \frac{1}{r}\partial_r\psi(t,r) - \frac{1}{2r^2}\sin(2\psi(t,r)), \\ (\psi(t_0,r), \partial_t\psi(t_0,r)) = (\psi_0(r), \dot\psi_0(r)). \end{cases} \tag{WM}$$

- $E < \infty$ forces $\lim_{r\to 0}\psi(t,r) \in \pi\mathbb{Z}$ and $\lim_{r\to\infty}\psi(t,r) \in \pi\mathbb{Z}$. We assume $\psi(t,0) = 0$ and $\psi(t,\infty) = n\pi$, with $n \in \mathbb{Z}$ fixed.

Comments

- Well-posedness (Shatah, Tahvildar-Zadeh '94): Given $(\psi_0, \dot{\psi}_0)$ such that $E(\psi_0, \dot{\psi}_0) < \infty$, there exists a unique strong solution to (WM), $\psi : (T_-, T_+) \to H^1 \times L^2$. Moreover, if $T < \infty$ and

$$\liminf_{t \to T} \int_0^{T-t} \Big((\partial_t \psi(t))^2 + (\partial_r \psi(t))^2 + \frac{\sin^2 \psi}{r^2} \Big) rdr \ll 1,$$

  then $T_+ > T$. $\boxed{\text{concentration of energy}}$

- For $\lambda > 0$ we denote $\psi_\lambda(t, x) := \psi(t/\lambda, x/\lambda)$. If ψ is a wave map on the time interval $[0, T_+)$, then ψ_λ is a wave maps as well, but on the time interval $[0, \lambda T_+)$.

Stationary states

- Explicit radially symmetric solutions of $\partial_r^2\psi(r) + \frac{1}{r}\partial_r\psi(r) - \frac{1}{2r^2}\sin(2\psi(r)) = 0$:

$$Q_\lambda(r) := 2\arctan\left(\frac{r}{\lambda}\right), \qquad (Q_\lambda, 0) \in H^1 \times L^2.$$

- $E(Q_\lambda, 0) = 4\pi$ $\boxed{\text{orbital stability}}$
- $(Q_\lambda, 0)$ are, up to sign and translation by π, all the corotational stationary states.
- Threshold elements for nonlinear behavior – (Côte, Kenig, Lawrie and Schlag '15, using ideas of Kenig and Merle '08).

Theorem

Let $E(\psi_0, \dot\psi_0) < 4\pi$. Then the solution ψ of (WM) with initial data $(\psi(0), \partial_t\psi(0)) = (\psi_0, \dot\psi_0)$ exists globally and scatters in both time directions.

Bubbling (Part I)

- (Struwe '03, using results on decay of energy on the light cone by Shatah and Tahvildar-Zadeh) If ψ is smooth and $T_- = 0$, then there exist $t_n \to 0$, $\lambda_n > 0$ with $\lambda_n \ll t_n$ such that $\psi(t_n + \lambda_n t, \lambda_n r)$ converges to $Q(r)$ in terms of local energy. In particular,

$$\liminf_{t \to 0} \int_0^t \left((\partial_t \psi(t))^2 + (\partial_r \psi(t)) + \frac{\sin^2 \psi(t)}{r^2} \right) r dr \geq E(Q).$$

- Radiation (Côte, Kenig, Lawrie, Schlag '15): If ψ is a wave map with $T_- = 0$ and $E(\psi, \partial_t \psi) < E(Q) + \epsilon$, then there exist $(\psi_0^*, \dot{\psi}_0^*) \in H^1 \times L^2$ and a continuous function λ such that

$$\lim_{t \to 0} \|(\psi(t), \partial_t \psi(t)) - (\psi_0^*, \dot{\psi}_0^*) - (Q_{\lambda(t)}, 0)\|_{(H^1 \times L^2)} = 0.$$

Bubbling (Part II)

- Existence results: There exists a degree-1 wave map ψ such that $T_- = 0$ and there exist $\lambda_c(t) > 0$ and $(\psi_0^*, \dot\psi_0^*) \in H^1 \times L^2$ such that

$$\left\| \psi(t), \partial_t \psi(t)) - (\psi_0^*, \dot\psi_0^*) - (Q_{\lambda_c(t)}, 0) \right\|_{H^1 \times L^2} \to 0, \qquad \text{as } t \to 0,$$

  Here
  - $\lambda_c(t) = t^{1+\nu}$ with $\nu > 0$ (Kriger, Schlag, Tataru '08).
  - $\lambda_c(t) = t \exp(-\sqrt{|\log t|} + O(1))$ (Raphaël, Rodnianski '09).
  - $\lambda_c(t) \simeq t^2 |\log t|^{-1}$ (Rodriguez '18). For this result $(\psi_0^*, \dot\psi_0^*) = (-Q, 0)$.
- Numerics: Bizoń, Chmaj, Tabor '01.

Main results (Part I)

- We consider the problem of "attaching" a blow-up bubble to a given radiation $(\psi_0^*, \dot\psi_0^*)$.
- We restrict to two situations: $(\psi_0^*, \dot\psi_0^*) \in H^1 \times L^2$ such that either

$$\begin{aligned} \psi_0^*(r) &= qr^\nu + o(r^\nu) \quad \text{as } r \to 0, \\ \dot\psi_0^*(r) &= 0, \end{aligned} \tag{EVEN}$$

or

$$\begin{aligned} \psi_0^*(r) &= 0, \\ \dot\psi_0^*(r) &= qr^{\nu-1} + o(r^{\nu-1}) \quad \text{as } r \to 0, \end{aligned} \tag{ODD}$$

where $\nu > \frac{9}{2}$ and $q \in \mathbb{R} \setminus \{0\}$.

Main results (Part II)

If $q < 0$ in (EVEN), there exist $T_+ > 0$ and a solution ψ to (WM) blowing up at $T_- = 0$ such that

$$\left\|(\psi(t), \partial_t \psi(t)) - (\psi_0^*, \dot{\psi}_0^*) - (Q_{\lambda_c(t)}, 0)\right\|_{H^1 \times L^2} \to 0, \quad \text{as } t \to 0^+,$$

with

$$\lambda_c(t) = \frac{p|q|}{\nu^2(\nu+1)} \frac{t^{\nu+1}}{|\log t|},$$

where

$$p = p(\nu) := \frac{\nu(\nu+2)\sqrt{\pi}\Gamma\left(\frac{3+\nu}{2}\right)}{4\Gamma\left(\frac{4+\nu}{2}\right)}.$$

If $q < 0$ in (ODD) then the exact same result holds with the explicit constant

$$p(\nu) = \frac{(\nu+1)\sqrt{\pi}\Gamma\left(\frac{2+\nu}{2}\right)}{4\Gamma\left(\frac{3+\nu}{2}\right)}.$$

Main results (Part III)

Let ψ be *any* finite energy solution to (WM) that blows up by concentrating *one* bubble backwards-in-time at $T_- = 0$ while radiating $(\psi_0^*, \dot\psi_0^*)$ satisfying (EVEN), i.e. ψ admits a decomposition

$$\left\|(\psi(t), \partial_t \psi(t)) - (\psi_0^*, \dot\psi_0^*) - (Q_{\lambda(t)}, 0)\right\|_{H^1 \times L^2} \to 0, \quad \text{as } t \to 0^+,$$

with $\lambda(t) \to 0$ as $t \to 0^+$. Then $q < 0$ and the rate $\lambda(t)$ satisfies,

$$\lambda(t) = \left(\frac{p|q|}{\nu^2(\nu+1)} + o(1)\right) \frac{t^{\nu+1}}{|\log t|} \quad \text{as } t \to 0^+,$$

where $p(\nu)$ is as before, in other words

$$\lim_{t \to 0} \left|\lambda(t)/\lambda_c(t) - 1\right| = 0.$$

If instead the radiation takes the form (ODD) then the same result holds, with appropriate $p(\nu)$.

Comments

- We could easily produce other blow-up rates by imposing a different asymptotic behavior at 0 of the radiation.
- The requirement $\nu > \frac{9}{2}$ could be improved, but some condition of this type is unavoidable with our methods.
- We treat *unstable* solutions; the solutions of Raphaël and Rodnianski are not covered by our approach.
- We expect that not only the sign of the bubble and the blow-up rate are determined by the radiation, but also the whole solution is unique. This would mean that the solution can be *uniquely reconstructed* from the data outside of the light cone with the tip at the singularity.
  unique continuation after blow-up
- This type of question is inspired by nonlinear scattering.

Formal computation (Part I)

- Let ψ^* the solution of (WM) with initial data $(\psi^*(0), \partial_t \psi^*(0)) = (\psi_0^*, \dot\psi_0^*)$. We seek solutions of the form $\psi(t) \simeq \psi^*(t) + Q_{\lambda(t)}$.
- Because of a slow decay of Q at spatial infinity, it is preferable to consider $\psi(t) \simeq \psi^*(t) + \chi_t Q_{\lambda(t)}$, where $\chi_t(r) := \chi(r/t)$ and χ is a cut-off function.
- Consider the case (EVEN). Then, at main order and inside the light cone, $\psi^*(t, r) \simeq pqrt^{\nu-1}$.
- Using this and $\partial_t \psi(t) \simeq \partial_t \psi^*(t) - \chi_t \frac{\lambda'(t)}{\lambda(t)} \Lambda Q_{\lambda(t)}$, we can compute the reduced Lagrangian

$$\widetilde{\mathscr{L}}(t, \lambda, \lambda') := \mathscr{L}\Big(\psi^*(t) + \chi_t Q_{\lambda(t)}, \partial_t \psi^*(t) - \chi_t \frac{\lambda'(t)}{\lambda(t)} \Lambda Q_{\lambda(t)}\Big)$$
$$\simeq 2(\lambda')^2 |\log(\lambda/t)| + 4pq\lambda t^{\nu-1} - E(Q).$$

Formal computation (Part II)

- The reduced Lagrangian

$$\widetilde{\mathscr{L}}(t, \lambda, \lambda') \simeq 2(\lambda')^2 |\log(\lambda/t)| + 4pq\lambda t^{\nu-1} - E(Q).$$

- The Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(4\lambda' |\log(\lambda/t)|\big) = 4pqt^{\nu-1}$$

  has a solution $\lambda(t) \simeq \frac{pq}{\nu^2(\nu+1)} \frac{t^{\nu+1}}{|\log t|}$.
- The reduced system does not conserve the reduced energy (this is expected, since the Lagrangian explicitly depends on t).
- The case (ODD), as well as non-polynomial expansions at $r = 0$, can be treated analogously.

Modulation method – Part I

- We want to understand the evolution of solutions *close to a bubble*, that is

$$\inf_{\lambda>0}\Big(\|(\psi(t),\partial_t\psi(t)) - (\psi^*(t),\dot\psi^*(t)) - (\chi_t Q_\lambda, 0)\|_{H^1\times L^2} + \lambda\Big) \leq \eta \ll 1.$$

- We decompose

$$\begin{aligned}\psi(t) &= \chi_t Q_{\lambda(t)} + \psi^*(t) + g(t),\\ \partial_t\psi(t) &= \partial_t\psi^*(t) + \dot g(t).\end{aligned}$$

- The choice of $\lambda(t)$ is determined by an orthogonality condition $\langle \mathcal{Z}_\lambda, g\rangle = 0$.
- Differentiating the orthogonality condition yields differential equations for $\lambda(t)$ involving $(g(t),\dot g(t))$. The goal is to reduce this to an ODE.

Modulation method – Part II

- We try to estimate $(g, \dot g)$ in terms of t and λ using the conservation of energy:

$$
\begin{aligned}
E(Q) + E(\psi^*, \partial_t \psi^*) &= E(\psi(t), \partial_t\psi(t)) \\
&= E(\chi_t Q_{\lambda(t)} + \psi^*(t) + g(t), \partial_t\psi^*(t) + \dot g(t)) \\
&= E(\chi_t Q_{\lambda(t)} + \psi^*(t), \partial_t\psi^*(t)) \\
&\quad + \mathrm{D}E(\cdots)(g, \dot g) + \frac{1}{2}\mathrm{D}^2 E(\cdots)(g, \dot g)^2.
\end{aligned}
$$

- The last term is $\gtrsim \|(g, \dot g)\|^2_{H^1\times L^2}$ $\boxed{\text{energy coercivity}}$
- In order to compute the first term of the last line (which is <u>not</u> negligible), we use the observation (due to Struwe '99) that the time derivative of this term is *quadratic in* g.

Modulation method – Part III

- We define auxiliary functions

$$\zeta(t) := 4\lambda(t)\log(t/\lambda(t)) - \int \chi_t \frac{1}{\lambda(t)}\Lambda Q_{\lambda(t)} g(t) r\,\mathrm{d}r,$$

$$b(t) := -\int \frac{1}{\lambda(t)}\Lambda Q_{\lambda(t)}, \dot g(t) r\,\mathrm{d}r - \int \dot g(t)\text{``}\frac{1}{\lambda(t)}\Lambda_0\text{``} g(t) r\,\mathrm{d}r.$$

localised virial correction

- We obtain the bounds

$$|b(t)| \leq (4+\delta)^{\frac{1}{2}}\Big(\log\frac{t}{\lambda(t)}\Big)^{\frac{1}{2}}\|\dot g(t)\|_{L^2} + C\|(g(t),\dot g(t))\|_{H^1\times L^2}^2,$$

$$|\zeta'(t) - b(t)| \lesssim \|\dot g(t)\|_{L^2} + \lambda(t)/t,$$

$$b'(t) \geq (4p|q| - \delta)t^{\nu-1} - C\frac{\lambda(t)}{t^2} - \frac{\delta}{\lambda(t)}\|(g(t),\dot g(t))\|_{H^1\times L^2}^2.$$

- Note that, a posteriori, the contribution of the terms involving $(g,\dot g)$ is not of main order.

Some references

- We use a *backward in time* construction due to Merle '90 and Martel '05.
- The virial correction (of an energy functional) in the blow-up setting is due to Raphaël and Szeftel '11.
- Relating the blow-up rate to the radiation $(\psi^*, \partial_t \psi^*)$ is analogous to the approach of Martel, Merle and Raphaël to "exotic blow-up" for gKdV.
- Using this approach in the energy-critical setting is due to J '17.
- Using Struwe's observation in order to obtain *upper bounds* on the blow-up rate for energy-critical NLW is due to J '15.
- Correcting the modulation parameter with a virial functional was used by J '17.
- Using this to obtain also *lower bounds* for the concentration rate is due to J and Lawrie '18.
- Corresponding results for corotational wave maps are due to Rodriguez (preprint '18).

Thank you for your attention.