Multi-bubbles for a critical wave equation

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DEA 2019, Sept. 19th 2019, Kraków

Energy-critical nonlinear focusing wave equation:

$$(\text{NLW})\quad \partial_t^2 u(t,x) = \Delta_x u(t,x) + |u(t,x)|^{\frac{4}{N-2}} u(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N$$

Lagrangian:

$$\mathcal{L}(u,\partial_t u) = \iint \left(\frac{1}{2}(\partial_t u)^2 - \frac{1}{2}|\nabla u|^2 + \frac{N-2}{2N}|u|^{\frac{2N}{N-2}} \right) dx\,dt$$

Energy:

$$E_c(u,\partial_t u) = \int \frac{1}{2}(\partial_t u)^2\,dx$$

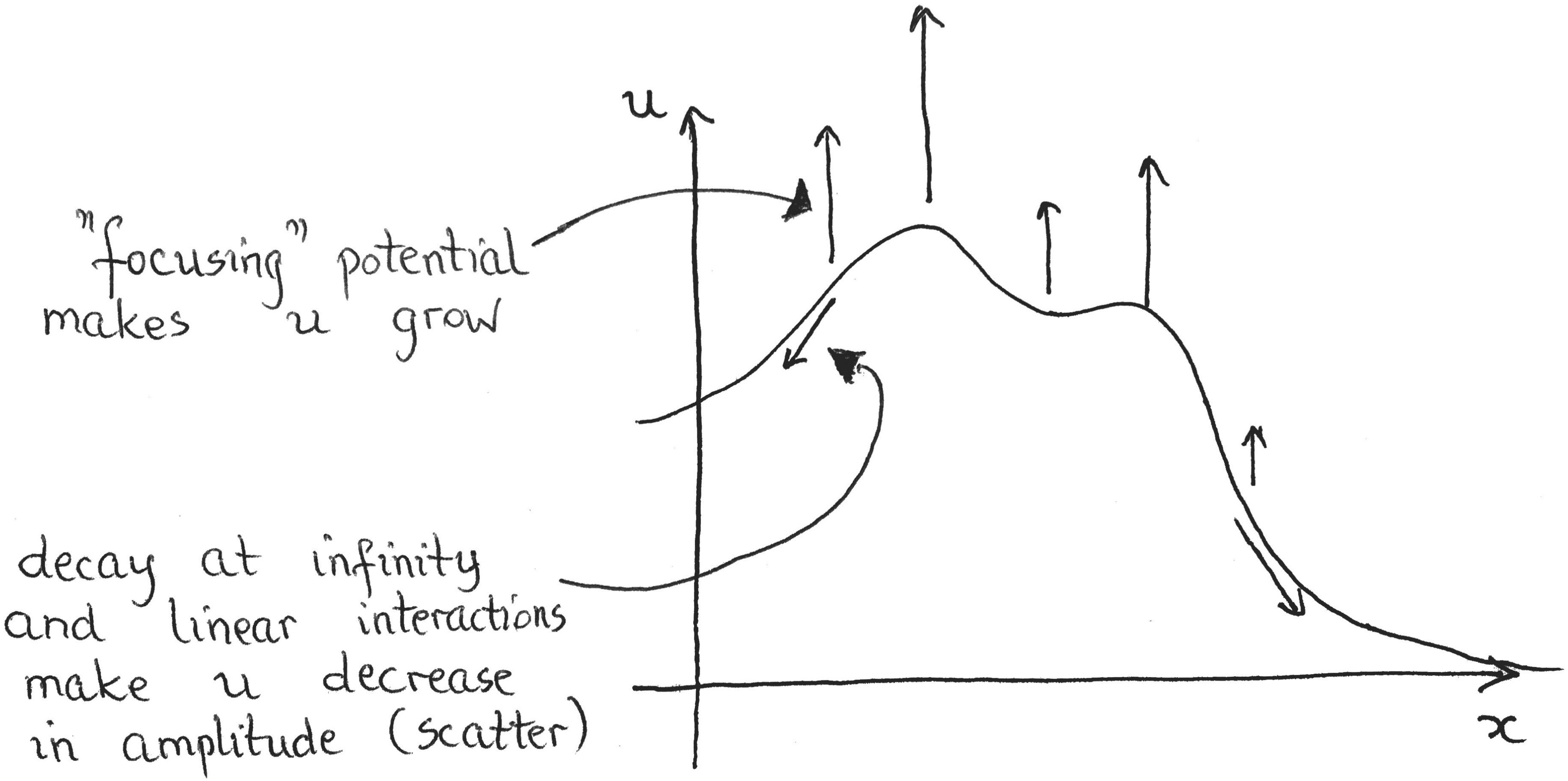
$$E_p(u,\partial_t u) = \int \left(\frac{1}{2}|\nabla u|^2 - \frac{N-2}{2N}|u|^{\frac{2N}{N-2}} \right) dx$$

$E = E_c + E_p$ is conserved in time.

I consider solutions in the energy space:

$$(u(t), \partial_t u(t)) \in \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$$

This model is not physically relevant.


"focusing" potential makes u grow
decay at infinity and linear interactions make u decrease in amplitude (scatter)
u
x

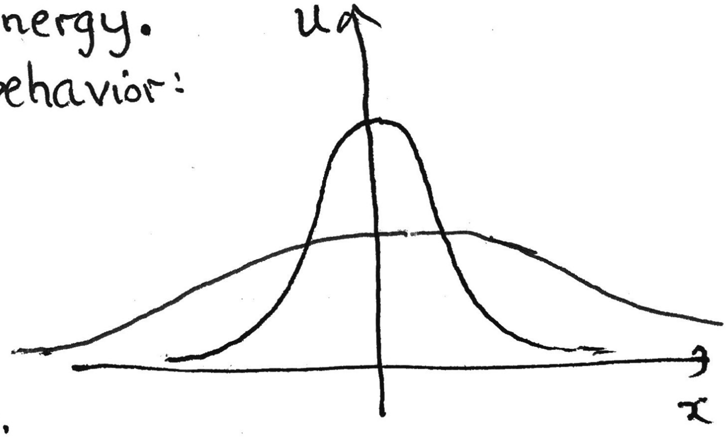
Stationnary solutions

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N-2}{2}}, \qquad W_\lambda(x) = \lambda^{-\frac{N-2}{2}} W\left(\frac{x}{\lambda}\right)$$

We have stationnary solutions $(u, \partial_t u) = (\pm W_\lambda, 0)$.

→ $\pm W_\lambda$ are mountain passes of E_p; they all have the same energy.

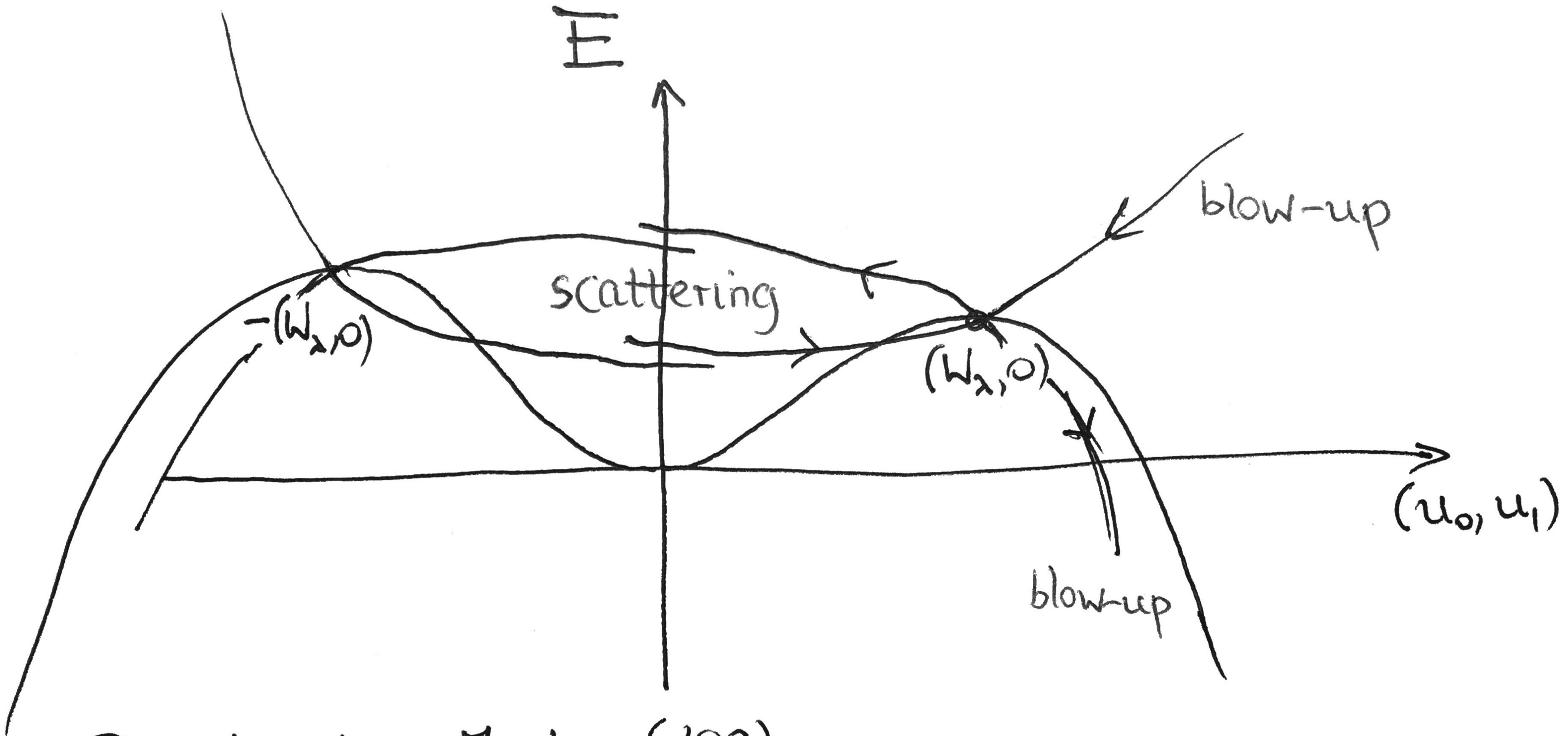
→ "minimal objects" for nonlinear behavior:



Thm (Kenig–Merle '06)

If $E(u_0, u_1) < E(W, 0)$ and (u_0, u_1) is in the potential well of E, then it asymptotically behaves like a linear wave (scatters).

("Ground state conjecture")



Duyckaerts–Merle ('08)
Krieger–Nakanishi–Schlag ('12)

Q: Study solutions on the central manifold of $\{(W_\lambda,0)\}$, close to $\{(W_\lambda,0)\}$, in particular the behavior of λ.

Bubbling

Energy conservation forces $(u, \partial_t u)$ to stay close to the family $\{(W_\lambda, 0)\}$, but λ can change in time.

Bizoń – Chmaj – Tabor
Krieger – Schlag – Tataru + subsequent works
Rodnianski – Sterbenz, Raphël – Rodnianski

$\lim_{t \to T_+} \lambda(t) = 0$, $T_+ < \infty$

"bubbling blow-up"

Donninger – Krieger

$\lim_{t \to \infty} \lambda(t) = 0$

"infinite time bubbling"

$$\| (u(t), \partial_t u(t)) - \underbrace{(W_{\lambda(t)}, 0)}_{\text{bubble}} - \underbrace{(u^*(t), \partial_t u^*(t))}_{\text{radiation}} \| \to 0$$

Many bubbles

→ $(u_0, u_1) \simeq \sum_{k=1}^{K} \left(\pm W_{\lambda_k}(\cdot - z_k), 0\right)$, $\quad \frac{\lambda_j}{\lambda_k} + \frac{\lambda_k}{\lambda_j} + \frac{|z_j - z_k|}{\lambda_k} \gg 1.$

→ Studying such configurations is motivated by the "Soliton resolution conjecture"

→ The only way to destroy the configuration is through a collision of bubbles

→ Can we compute how the bubbles interact before they collide ?

→ One concrete problem is to study pure multi-bubbles:

$$\left\| (u(t), \partial_t u(t)) - \sum_{k=1}^{K} \left(\pm W_{\lambda_k(t)}(\cdot - z_k(t)), 0\right) \right\| \to 0, \qquad \text{as } t \to T_+ .$$

$$\lambda_j(t)/\lambda_k(t) + \lambda_k(t)/\lambda_j(t) + |z_j(t) - z_k(t)|/\lambda_k(t) \to \infty$$

Approximation by restriction

We consider the case of K bubbles concentrating at fixed points $z_1, z_2, \dots, z_K \in \mathbb{R}^5 = \mathbb{R}^N$, as $t \to \infty$.

$$\mathcal{M} := \left\{ \sum_{k=1}^{K} W_{\lambda_k}(\cdot - z_k), \quad \lambda_k \ll 1 \right\}$$

(K-dimensional manifold with coordinates $\lambda_1, \dots, \lambda_K$).

Reduced (effective ?) Lagrangian :

$$\tilde{\mathcal{L}}(\lambda_1, \dots, \lambda_K) := \mathcal{L}\left(\sum_{k=1}^{K} W_{\lambda_k}(\cdot - z_k), \sum_{k=1}^{K} -\frac{\lambda_k'}{\lambda_k} \Lambda W_{\lambda_k}(\cdot - z_k) \right)$$

Formally predicted modulation equations

$$\frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k'}\right) = \frac{\partial \mathcal{L}}{\partial \lambda_k} \quad \text{leads to}$$

$$\text{(ODE)} \qquad \lambda_k''(t) = \frac{128\sqrt{15}}{7\pi}\sqrt{\lambda_k(t)}\sum_{j\neq k}\lambda_j(t)^{3/2}\,|z_j - z_k|^{-3}$$

→ Taking into account the homogeneity, it is natural to seek solutions $\lambda_k(t) = c_k t^{-2}$.

→ We can find c_k thanks to the variational structure inherited from the Lagrangian

Lemma Let $S_+^{k-1} := \{ \vec{\theta} = (\theta_1 \dots \theta_k) \in (0,\infty)^k : |\vec{\theta}|^2 = 1 \}$,

$$V : S_+^{k-1} \to \mathbb{R}, \quad V(\vec{\theta}) := - \frac{126\sqrt{15}}{21\pi} \sum_{j \neq k} \theta_j^{2/3} \theta_k^{2/3} |z_j - z_k|^{-3}.$$

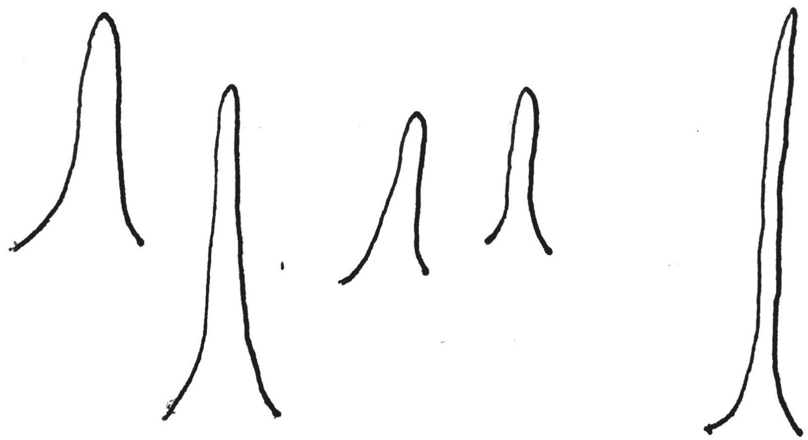
The function V reaches its global minimum at at least one point $\vec{\theta}_0 \in S_+^{k-1}$.

Let $\vec{c} := \dfrac{6\vec{\theta}_0}{-\vec{\theta}_0 \cdot \nabla V(\vec{\theta}_0)}$. Then $\lambda_k(t) = c_k t^{-2}$ solves (ODE).

Theorem (J–Martel '19)

Let $K \geq 2$ and $z_1 \cdots z_K \in \mathbb{R}^5$ be K distinct points. There exists a solution $u(t,x)$ of (NLW), defined for all $t \geq 0$, such that

$$\Big\| u(t) - \sum_{k=1}^{K} W_{\lambda_k(t)}(\cdot - z_k) \Big\|_{\dot H^1(\mathbb{R}^5)} + \|\partial_t u(t)\|_{L^2(\mathbb{R}^5)} \leq C t^{-\frac{1}{3}}.$$



Proof

- → Start with initial data corresponding to concentrated bubbles and show that they deconcentrate according to (ODE)
- → For this, control the (growing in time) error by a mixed energy-Morawetz functional
- → pass to a limit and reverse the time direction.

Remarks

- → attractive interaction of bubbles of the same sign is necessary to get a solution
- → analogous results for the nonlinear heat equation were proved by Cortazar-del Pino-Musso ('17).

Thank you for your attention.